note : A simple derivation of Lindblad equation

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S(principal) and B(bath) are 2 system in $\mathcal{H}_S$ and $\mathcal{H}_B$ (L:41)

$$\hat{H}(t) = \hat{H}_s \otimes \mathbb{1}_B + \mathbb{1}_S \otimes \hat{H}_B + \alpha \hat{H}_{SB} \tag{1}$$

for the sake of simplicity : (L:50)

$$\hat{H}(t) = \hat{H}_S + H_B + \alpha \hat{H}_{SB} \tag{2}$$

there is : (L:55)

$$\frac{\mathbf{d}\hat{\rho}_{SB}}{\mathbf{d}t} = -\frac{\mathbb{i}}{\hbar}\left[\hat{H}_S + \hat{H}_B + \alpha \hat{H}_{SB}, \hat{\rho}_{SB}\right] \tag{3}$$

Hamiltonian is timed as : (L:64)

$$\hat{H}(t) = \mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\hat{H}_{SB}\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t} \tag{4}$$

while the : (L:74)

$$\begin{aligned} \hat{\rho}(t) &= \mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\hat{\rho}_{SB}\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t} \\ \hat{\rho}_{SB(t)} &= \mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\hat{\rho}\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t} \end{aligned} \tag{5}$$

note that on the rear end of an equation , central elements are not timed , they are the things at a certain constant time .
(L:91)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\frac{\mathbb{i}}{\hbar}\left[\hat{H}(t), \hat{\rho}(t)\right] \tag{6}$$

noting that : (L:98)

$$\hat{\rho}_S(t) = \mathrm{Tr}_B\left[\hat{\rho}_{SB(t)}\right] \tag{7}$$

and then we have that (L:106)

$$\hat{\rho}(t) = \hat{\rho}(0) - \frac{\mathbb{i}}{\hbar}\alpha\int_0^t \left[\hat{H}(t'), \hat{\rho}(t')\right]\mathbf{d}t' \tag{8}$$

(L:114)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\mathbb{i}\frac{\alpha}{\hbar}\left[\hat{H}(t),\hat{\rho}(0)\right] - \frac{\alpha^2}{\hbar^2}\left[\hat{H}(t),\int_0^t \left[\hat{H}(t'),\hat{\rho}(t')\right]\mathbf{d}t'\right] \tag{9}$$

considering the relationship between the SB and the S we know that : being aware that : so here what we care about is that :

(L:136)

$$\begin{aligned}&\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\left(\left[\hat{H},\hat{\rho}\right]\right)\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\\ &= \mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\left(\hat{H}\hat{\rho}\right)\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t} - \mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\left(\hat{\rho}\hat{H}\right)\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\end{aligned} \tag{10}$$

and in this contex what I know is that : (L:151)

$$\begin{aligned}&\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\left(\hat{H}\hat{\rho}\right)\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\\ &= \left(\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\hat{H}\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\right)\left(\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\hat{\rho}\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t}\right)\end{aligned} \tag{11}$$

by the (11) , we know that this can be done so by this we can say that (L:171)

$$\mathbb{e}^{-i/\hbar(\hat{H}_S+\hat{H}_B)t}\left[\hat{H},\hat{\rho}\right]\mathbb{e}^{i/\hbar(\hat{H}_S+\hat{H}_B)t} = \left[\hat{H}(t),\hat{\rho}(t)\right] \tag{12}$$

note that here the first expression starts from a moment there is no time on it , the time is given by the evaluation we can have the **Born Approximation** applied to (9)
now we like to cut the bath part out of the picture , so we are going to do an trace over it : according to the auther : $\hat{H}$ is defined by the $\hat{H}_S, \hat{H}_B$ and the interaction term as shown as (2) , so there is a way to define $\hat{H}_S, \hat{H}_B$ without losing generality , so that (L:215)

$$\mathrm{Tr}_{B\frac{-\mathbb{i}\alpha}{\hbar}\left[\hat{H}(t),\hat{\rho}(0)\right]}[=]\,0 \tag{13}$$

if that is true then we could have that : (L:224)

$$\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} = -\frac{\alpha^2}{\hbar^2}\mathrm{Tr}_B\left[\hat{H}(t),\int_0^t \left[\hat{H}(t'),\hat{\rho}(t')\right]\mathbf{d}t'\right] \tag{14}$$

integrating 14 from t to the t' yelds: (L:239)

$$\hat{\rho}_S(t') - \hat{\rho}_S(t) = -\frac{\alpha^2}{\hbar^2}\int_t^{t'}\mathrm{Tr}_B\left[\hat{H}(n'),\int_0^{n'}\left[\hat{H}(t''),\hat{\rho}(t'')\right]\mathbf{d}t''\right]\mathbf{d}n' \tag{15}$$

or integing t' to the t yelds: (L:255)

$$\hat{\rho}_S(t) - \hat{\rho}_S(t') = -\frac{\alpha^2}{\hbar^2}\int_{t'}^{t}\mathrm{Tr}_B\left[\hat{H}(n'),\int_0^{n'}\left[\hat{H}(t''),\hat{\rho}(t'')\right]\mathbf{d}t''\right]\mathbf{d}n' \tag{16}$$

difference of $\hat{\rho}(t)$ and $\hat{\rho}(t')$ is of magnitude of α^2 so primiary time equation could be writen in the folowing ways without violationg **Born approximation** (L:273)

$$\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} = -\frac{\alpha^2}{\hbar^2}\mathrm{Tr}_B\left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}(t)\right]\mathbf{d}t'\right] \tag{17}$$

lets say Bath and System are in full interaction($\alpha = 1$) , and this is an approximation , so there is : (L:286)

$$\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}(t)\right]\mathbf{d}t'\right] \tag{18}$$

writing that, (ignoreing Bath Hamiltionain time as well) (L:299)

$$\begin{aligned}\hat{\rho}(t) &= \hat{\rho}_S(t) \otimes \hat{\rho}_{B(t)} \\ &= \hat{\rho}_S(t)\hat{\rho}_{B(t)}\end{aligned} \tag{19}$$

and take the time to be infinitly long , then : (L:304)

$$\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left[\hat{H}(t), \int_0^\infty \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B\right]\mathbf{d}t'\right] \tag{20}$$

1 LINDBLAD EQUATION

1.1 Hamiltionians and Operators

so there is : (L:320)

$$\hat{H}_{SB} = \hbar\left(\hat{S}\hat{B}^\dagger + \hat{S}^\dagger\hat{B}\right) \tag{21}$$

of course with multiple expreseion of $\hat{S}$ it could be expressed as : so : (L:331)

$$\hat{H}_{SB} = \hbar\sum\left(\hat{L}\hat{B}^\dagger + \hat{L}^\dagger\hat{B}\right) \tag{22}$$

and then this $\hat{L}$ is system and Bath related , how does it have in the system is not interested (L:341)

$$\hat{S}(t) = \hat{S} \tag{23}$$

is not aggected by the interaction-picture frame , defining the bath is a bath of bosons : (L:346)

$$\hat{H}_B = \hbar\sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \tag{24}$$

so the bath have k modes , and $\hat{a}^\dagger, \hat{a}$ creates or annihilates them but the operator (L:354)

$$\hat{B} = \sum_k g_k^* \hat{a}_k \tag{25}$$

is not $\hat{H}_B$ and here g_k is said to be complex coefficients represing coupling constants then , in the interaction picture,(of course here $\hat{H}_S + \hat{H}_B$ is not used but only that the $\hat{H}_B$ is used) (L:366)

$$\hat{B}(t) = \mathbb{e}^{\mathbb{i}/\hbar \hat{H}_B t} \hat{B} \mathbb{e}^{-\mathbb{i}/\hbar \hat{H}_B t} \tag{26}$$

in (26) , taylor slicing it gives the exp parts eventually fill my hand with : (L:375)

$$\begin{aligned} \hat{B}(t) &= \mathbb{e}^{\mathbb{i}/\hbar \hat{H}_B t} \hat{B} \mathbb{e}^{-\mathbb{i}/\hbar \hat{H}_B t} \\ &= \sum_k \mathbb{e}^{\sum_{k'} \mathbb{i}\left(\omega_{k'} \hat{a}_{k'}^\dagger, \hat{a}_{k'}\right)t} g_k^* \hat{a}_k \mathbb{e}^{-\sum_{k''} \mathbb{i}\left(\omega_{k''} \hat{a}_{k''}^\dagger, \hat{a}_{k''}\right)t} \end{aligned} \tag{27}$$

for the boson we know that : (L:392)

$$\begin{aligned} \left[\hat{a}_k, \hat{a}_{k'}\right] &= 0 \\ \left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] &= \delta_{k,k'} \\ \left[\hat{a}_{k'}^\dagger, \hat{a}_k\right] &= -\delta_{k,k'} \\ \left[\hat{a}_{k'}^\dagger \hat{a}_{k'}, \hat{a}_k\right] &= \left[\hat{a}_{k'}^\dagger, \hat{a}_k\right] \hat{a}_{k'} + \hat{a}_{k'}^\dagger \left[\hat{a}_{k'}, \hat{a}_k\right] = -\delta_{k,k'} \hat{a}_{k'} \end{aligned} \tag{28}$$

however there is : (L:418)

$$\begin{aligned} \left[\hat{a}_n^\dagger \hat{a}_n \hat{a}_m^\dagger \hat{a}_m, \hat{a}_k\right] &= \left[\hat{a}_n^\dagger \hat{a}_n, \hat{a}_k\right] \hat{a}_m^\dagger \hat{a}_m + \hat{a}_n^\dagger \hat{a}_n \left[\hat{a}_m^\dagger \hat{a}_m, \hat{a}_k\right] \\ &= \delta_{n,k} \hat{a}_k \hat{a}_m^\dagger \hat{a}_m + \delta_{m,k} \hat{a}_n^\dagger \hat{a}_n \hat{a}_k \end{aligned} \tag{29}$$

taking it another step further , we have : we also remind ourself that here we could really just take that things, and ofcourse this S is an arbitary, prod operation will do it in a sorted fashion given partial ordered Set, each element of it is impicitly corresponded to a Natural number (L:436)

$$\begin{aligned} \left[\prod_{z \in S} \hat{a}_z^\dagger \hat{a}_z, \hat{a}_k\right] &= \sum_{z \in S} \delta_{z,k} \left(\prod_{m \in S \wedge m \subset z} \hat{a}_m^\dagger \hat{a}_m\right) \hat{a}_k \left(\prod_{n \in S \wedge n \supset z} \hat{a}_n^\dagger \hat{a}_n\right) \\ &= \sum_{z \in S} \delta_{z,k} \hat{a}_k \left(\prod_{m \in S \wedge m \subset z} \left(\hat{a}_m^\dagger \hat{a}_m - \delta_{m,k} \hat{a}_k\right)\right) \left(\prod_{n \in S \wedge n \supset z} \hat{a}_n^\dagger \hat{a}_n\right) \end{aligned} \tag{30}$$

switching the perspective gives : (L:466)

$$\begin{aligned}
\left[\hat{H}_B, \hat{a}_k\right] &= -\hbar\omega_k \hat{a}_k \\
\left[\hat{H}_B^2, \hat{a}_k\right] &= -\hbar\omega_k \left(\hat{a}_k \hat{H}_B + \hat{H}_B \hat{a}_k\right) \\
\left[\hat{H}_B^n, \hat{a}_k\right] &= -\hbar\omega_k \sum_i^n \left(\prod_{m=0}^{i-1} \hat{H}_B\right) \hat{a}_k \left(\prod_{l=i+1}^{n} \hat{H}_B\right) \\
&= -\hbar\omega_k \hat{a}_k \sum_i^n \left(\prod_{m=0}^{i-1} \hat{H}_B - \hbar\omega_k \hat{a}_k\right) \left(\prod_{l=i+1}^{n} \hat{H}_B\right)
\end{aligned} \tag{31}$$

now : (L:499)

$$\begin{aligned}
\left[\mathbb{e}^{\mathbb{i}\hat{H}_B t/\hbar}, \hat{a}_k\right] &= \sum_n \frac{\left[(\mathbb{i}\hat{H}_B t/\hbar)^n, \hat{a}_k\right]}{n!} \\
&= \sum_n \frac{(\mathbb{i}t/\hbar)^n \left[\hat{H}_B^n, \hat{a}_k\right]}{n!} \\
&= \sum_n \frac{(\mathbb{i}t/\hbar)^n \left(-\hbar\omega_k \sum_i^n \hat{H}_B^{i-1} \hat{a}_k \hat{H}_B^{n-i}\right)}{n!}
\end{aligned} \tag{32}$$

and (L:536)

$$\begin{aligned}
\left[\mathbb{e}^{\mathbb{i}\hat{H}_B t/\hbar}, \hat{a}_k\right] \mathbb{e}^{-\mathbb{i}\hat{H}_B t/\hbar} &= \sum_n \sum_m \frac{(-1)^m (\mathbb{i}t/\hbar)^{n+m} \left(-\hbar\omega_k \sum_i^n \hat{H}_B^{i-1} \hat{a}_k \hat{H}_B^{n-i}\right) \hat{H}_B^m}{n!m!} \\
&= \sum_n \sum_m \frac{(-1)^m (\mathbb{i}t/\hbar)^{n+m} \left(-\hbar\omega_k \hat{a}_k \sum_i^n (\hat{H}_B^{i-1} - \hbar\omega_k \hat{a}_k) \hat{H}_B^{n-i}\right) \hat{H}_B^m}{n!m!}
\end{aligned} \tag{33}$$

for its nth compond ,it has alot of (30) ok , lets just use the CBH formula here which states that: (L:581)

$$\begin{aligned}
\mu(t) &= \mathbb{e}^{\lambda t} \mu \mathbb{e}^{-\lambda t} \\
\frac{\mathbf{d}\mu(t)}{\mathbf{d}t} &= \frac{\mathbf{d}\mathbb{e}^{\lambda t}}{\mathbf{d}t} \mu \mathbb{e}^{-\lambda t} + \mathbb{e}^{\lambda t} \mu \frac{\mathbf{d}\mathbb{e}^{-\lambda t}}{\mathbf{d}t} \\
&= \mathbb{e}^{\lambda t} \left(\lambda\mu - \mu\lambda\right) \mathbb{e}^{-\lambda t} \\
&= \mathbb{e}^{\lambda t} \left[\lambda, \mu\right] \mathbb{e}^{-\lambda t}
\end{aligned} \tag{34}$$

expanding at (L:591)

$$\frac{\mathbf{d}^2\mu(t)}{\partial \mathbf{d}t2} = \mathbb{e}^{\lambda t} \left[\lambda \left[\lambda, \mu\right]\right] \mathbb{e}^{-\lambda t} \tag{35}$$

now expanding the $\mu(t)$ from the time 0 gives (L:602)

$$\mu(t) = \mu + [\lambda, \mu]t + \frac{1}{2!}[\lambda, [\lambda, \mu]]t^2 + ... \tag{36}$$

well now , we know that if (L:607)

$$\begin{aligned} \lambda &= \mathbb{i}\hat{H}/\hbar \\ \mu &= \hat{a}_k \end{aligned} \tag{37}$$

so (L:612)

$$\begin{aligned} [\lambda, \mu] &= -\mathbb{i}\omega_k \hat{a}_k \\ [\lambda, [\lambda, \mu]] &= (-\mathbb{i}\omega_k)^2 \hat{a}_k \end{aligned} \tag{38}$$

so that : (L:617)

$$\begin{aligned} \mathbb{e}^{\mathbb{i}\hat{H}_B t/\hbar} \hat{a}_k \mathbb{e}^{-\mathbb{i}\hat{H}_B t/\hbar} &= \hat{a}_k \left(\sum_n \frac{(-\mathbb{i}\omega_k t)^n}{n!} \right) \\ &= \hat{a}_k \mathbb{e}^{-\mathbb{i}\omega_k t} \end{aligned} \tag{39}$$

so then we had (L:630)

$$\hat{B}(t) = \sum_k g_k^* \hat{a}_k \mathbb{e}^{-\mathbb{i}\omega_k t} \tag{40}$$

now with that (L:637)

$$\hat{H}_{SB} = \hbar \left(\hat{S}\hat{B}^\dagger + \hat{S}^\dagger \hat{B} \right) \tag{41}$$

this $\hat{B}$ is a resemble of Jaynes-Cummings model , which is like : (L:644)

$$\hat{H} = \hbar \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) + g \left(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} \right) \tag{42}$$

and then lokoing back to the denity matrix (20) now we want to study that the comutation part : (L:652)

$$\left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \mathbf{d}t' \right] = \int_0^t \left[\hat{H}(t), \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \right] \mathbf{d}t' \tag{43}$$

the part that cared about is : by the way , is there : $\hat{\rho}_S(t)\hat{\rho}_B = \hat{\rho}_B\hat{\rho}_S(t)$? well I dont think so , becaue $\hat{\rho}_S(t)\hat{\rho}_B = \hat{\rho}_S(t) \otimes \hat{\rho}_B$ $\hat{\rho}_S(t) = \hat{\rho}_S(t) \otimes \mathbb{1}_B$ and $\hat{\rho}_B = \mathbb{1}_S \otimes \hat{\rho}_B$ is what it means (L:675)

$$\begin{aligned} \left[\hat{H}(t), \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \right] &= \hbar \left[\hat{S}\hat{B}^\dagger + \hat{S}^\dagger, \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \right] \\ &= \hbar \left[\hat{S}\hat{B}^\dagger, \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \right] + \hbar \left[\hat{S}^\dagger \hat{B}, \left[\hat{H}(t'), \hat{\rho}_S(t)\hat{\rho}_B \right] \right] \end{aligned} \tag{44}$$

lets say for the part that (L:699)

$$
\begin{aligned}
&\left[\hat{S}\hat{B}^{\dagger}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] \\
&=\hbar\left[\hat{S}\hat{B}^{\dagger}(t),\left[\hat{S}\hat{B}^{\dagger}(t')+\hat{S}^{\dagger}\hat{B}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] \\
&=\hbar\left(\hat{S}\hat{B}^{\dagger}(t)\left[\hat{S}\hat{B}^{\dagger}(t')+\hat{S}^{\dagger}\hat{B}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]-\left[\hat{S}\hat{B}^{\dagger}(t')+\hat{S}^{\dagger}\hat{B}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\hat{S}\hat{B}^{\dagger}(t)\right) \\
&=\hbar\left(\hat{S}\hat{B}^{\dagger}(t)\hat{S}\hat{B}^{\dagger}(t')\hat{\rho}_S(t)\hat{\rho}_B+\hat{S}\hat{B}^{\dagger}(t)\hat{S}^{\dagger}\hat{B}(t')\hat{\rho}_S(t)\hat{\rho}_B-\hat{S}\hat{B}^{\dagger}(t)\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t')-\hat{S}\hat{B}^{\dagger}(t)\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t')\right) \\
&-\hbar\left(\hat{S}\hat{B}^{\dagger}(t')\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t)+\hat{S}^{\dagger}\hat{B}(t')\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t)-\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t')\hat{S}\hat{B}^{\dagger}(t)-\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t')\hat{S}\hat{B}^{\dagger}(t)\right)
\end{aligned}
\tag{45}
$$

grouping similiar terms here gives: so there is $[\hat{\rho}_b,\hat{S}]=0$, $[\hat{\rho}_S(t),\hat{S}]\neq 0$, $[\hat{\rho}_b,\hat{B}]\neq 0$, $[\hat{\rho}_S(t),\hat{B}]=0$ we can then have (L:756)

$$
\begin{aligned}
&\left[\hat{S}\hat{B}^{\dagger}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] \\
&=\hbar\left(\hat{S}\hat{S}\hat{\rho}_S(t)\hat{B}^{\dagger}(t)\hat{B}^{\dagger}(t')\hat{\rho}_B+\hat{S}\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{B}^{\dagger}(t)\hat{B}(t')\hat{\rho}_B-\hat{S}\hat{\rho}_S(t)\hat{S}\hat{B}^{\dagger}(t)\hat{\rho}_B\hat{B}^{\dagger}(t')-\hat{S}\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{B}^{\dagger}(t)\hat{\rho}_B\hat{B}(t')\right) \\
&-\hbar\left(\hat{S}\hat{\rho}_S(t)\hat{S}\hat{B}^{\dagger}(t')\hat{\rho}_B\hat{B}^{\dagger}(t)+\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{S}\hat{B}(t')\hat{\rho}_B\hat{B}^{\dagger}(t)-\hat{\rho}_S(t)\hat{S}\hat{S}\hat{\rho}_B\hat{B}^{\dagger}(t')\hat{B}^{\dagger}(t)-\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{S}\hat{\rho}_B\hat{B}(t')\hat{B}^{\dagger}(t)\right)
\end{aligned}
\tag{46}
$$

and on the other wing we have some similiar : (L:783)

$$
\begin{aligned}
&\left[\hat{S}^{\dagger}\hat{B}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] \\
&=\hbar\left(\hat{S}^{\dagger}\hat{B}(t)\hat{S}\hat{B}^{\dagger}(t')\hat{\rho}_S(t)\hat{\rho}_B+\hat{S}^{\dagger}\hat{B}(t)\hat{S}^{\dagger}\hat{B}(t')\hat{\rho}_S(t)\hat{\rho}_B-\hat{S}^{\dagger}\hat{B}(t)\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t')-\hat{S}^{\dagger}\hat{B}(t)\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t')\right) \\
&-\hbar\left(\hat{S}\hat{B}^{\dagger}(t')\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t)+\hat{S}^{\dagger}\hat{B}(t')\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t)-\hat{\rho}_S(t)\hat{\rho}_B\hat{S}\hat{B}^{\dagger}(t')\hat{S}^{\dagger}\hat{B}(t)-\hat{\rho}_S(t)\hat{\rho}_B\hat{S}^{\dagger}\hat{B}(t')\hat{S}^{\dagger}\hat{B}(t)\right)
\end{aligned}
\tag{47}
$$

re arranging it to : (L:808)

$$
\begin{aligned}
&=\hbar\left(\hat{S}^{\dagger}\hat{S}\hat{\rho}_S(t)\hat{B}(t)\hat{B}^{\dagger}(t')\hat{\rho}_B+\hat{S}^{\dagger}\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{B}(t)\hat{B}(t')\hat{\rho}_B-\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{S}\hat{B}(t)\hat{\rho}_B\hat{B}^{\dagger}(t')-\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{B}(t)\hat{\rho}_B\hat{B}(t')\right) \\
&-\hbar\left(\hat{S}\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{B}^{\dagger}(t')\hat{\rho}_B\hat{B}(t)+\hat{S}^{\dagger}\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{B}(t')\hat{\rho}_B\hat{B}(t)-\hat{\rho}_S(t)\hat{S}\hat{S}^{\dagger}\hat{\rho}_B\hat{B}^{\dagger}(t')\hat{B}(t)-\hat{\rho}_S(t)\hat{S}^{\dagger}\hat{S}^{\dagger}\hat{\rho}_B\hat{B}(t')\hat{B}(t)\right)
\end{aligned}
\tag{48}
$$

1.2 the particle trace

assume there is no particle in the bath : (L:827)

$$
\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}(t')\hat{\rho}_B\right]=\mathrm{Tr}_B\left[\hat{B}^{\dagger}(t)\hat{B}^{\dagger}(t')\hat{\rho}_B\right]=0\quad,\quad\forall t,t' \tag{49}
$$

with this only selective are left over, like (L:839)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{S}B(t)^\dagger,\left[\hat{H}(t'),\hat{\rho}_B\hat{\rho}_S(t)\right]\right] &= \hbar\hat{S}\hat{S}^\dagger\hat{\rho}_S(t)\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right]\\ &- \hbar\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{\rho}_B\hat{B}(t')\right]\\ &- \hbar\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\mathrm{Tr}_B\left[\hat{B}(t')\hat{\rho}_B\hat{B}^\dagger(t)\right]\\ &+ \hbar\hat{\rho}_S(t)\hat{S}^\dagger\hat{S}\mathrm{Tr}_B\left[\hat{\rho}_B\hat{B}(t')\hat{B}^\dagger(t)\right]\end{aligned} \tag{50}$$

and on the other side there is : (L:854)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{S}^\dagger\hat{B}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] &= \hbar\hat{S}^\dagger\hat{S}\hat{\rho}_S(t)\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\\ &- \hbar\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\mathrm{Tr}_B\left[\hat{B}(t)\hat{\rho}_B\hat{B}^\dagger(t')\right]\\ &- \hbar\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{\rho}_B\hat{B}(t)\right]\\ &+ \hbar\hat{\rho}_S(t)\hat{S}\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{\rho}_B\hat{B}^\dagger(t')\hat{B}(t)\right]\end{aligned} \tag{51}$$

now it says that there is ciclic properties of trace : which paiscally means that : (L:874)

$$\mathrm{Tr}\left[\hat{A}\hat{B}\hat{C}\hat{D}\right] = \mathrm{Tr}\left[\hat{B}\hat{C}\hat{D}\hat{A}\right] = \mathrm{Tr}\left[\hat{C}\hat{D}\hat{A}\hat{B}\right] = \mathrm{Tr}\left[\hat{D}\hat{A}\hat{B}\hat{C}\right] \tag{52}$$

rearangement shows : (L:881)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{S}B(t)^\dagger,\left[\hat{H}(t'),\hat{\rho}_B\hat{\rho}_S(t)\right]\right] &= \hbar\hat{S}\hat{S}^\dagger\hat{\rho}_S(t)\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right] - \hbar\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right]\\ &+ \hbar\hat{\rho}_S(t)\hat{S}^\dagger\hat{S}\mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right] - \hbar\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right]\\ \\ &= \hbar\left(\hat{S}\hat{S}^\dagger\hat{\rho}_S(t) - \hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right]\\ &+ \hbar\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S} - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right]\end{aligned} \tag{53}$$

and (L:899)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{S}^\dagger\hat{B}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] &= \hbar\hat{S}^\dagger\hat{S}\hat{\rho}_S(t)\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] - \hbar\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\\ &+ \hbar\hat{\rho}_S(t)\hat{S}\hat{S}^\dagger\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right] - \hbar\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right]\\ \\ &= \hbar\left(\hat{S}^\dagger\hat{S}\hat{\rho}_S(t) - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\\ &+ \hbar\left(\hat{\rho}_S(t)\hat{S}\hat{S}^\dagger - \hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right]\end{aligned} \tag{54}$$

since we like to see how this works with the system hamiltonian , we would like to continue jobs at (44) and then what it give us is that : (L:927)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{H}(t),\left[\hat{H}(t'),\hat{\rho}_S(t)\hat{\rho}_B\right]\right] &= \hbar^2\left(\hat{S}\hat{S}^\dagger\hat{\rho}_S(t)-\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)\mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right]\\ &+\hbar^2\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S}-\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right]\\ &+\hbar^2\left(\hat{S}^\dagger\hat{S}\hat{\rho}_S(t)-\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\\ &+\hbar^2\left(\hat{\rho}_S(t)\hat{S}\hat{S}^\dagger-\hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right]\end{aligned} \tag{55}$$

so that is it

1.3 The expansion of the integrand of the master equation

now we look back to the integration mentioned at the(20) for convenince , we could have that : (L:951)

$$\begin{aligned}F(t) &= \int_0^t \mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\mathbf{d}t'\\ G(t) &= \int_0^t \mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right]\mathbf{d}t'\end{aligned} \tag{56}$$

and if we take the conjugate , it some how just exchange the t' and t I think the density matrix has its trace elements summed to 1 but conjugation of the operator $\hat{B}$ which is not promissed to be hermitian will look like : (L:967)

$$\begin{aligned}\hat{B}^*(t) &= \sum_k g_k\hat{a}_k^*\mathbb{e}^{\mathbb{i}\omega_k t}\\ (\hat{B}^\dagger(t))^* &= \sum_k g_k^*(\hat{a}_k^\dagger)^*\mathbb{e}^{-\mathbb{i}\omega_k t}\end{aligned} \tag{57}$$

we know that for the system (L:980)

$$\begin{aligned}\hat{B}(t)\hat{B}^\dagger(t') &= \sum_k\sum_{k'} g_k^* g_{k'}\mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\hat{a}_k\hat{a}_{k'}^\dagger\\ \left(\hat{B}(t)\hat{B}^\dagger(t')\right)^* &= \sum_k\sum_{k'} g_k g_{k'}^*\mathbb{e}^{\mathbb{i}(\omega_k t-\omega_{k'}t')}(\hat{a}_k\hat{a}_{k'}^\dagger)^*\\ &= \sum_{k'}\sum_k g_k^* g_k\mathbb{e}^{-\mathbb{i}(\omega_{k'}t'-\omega_k t)}(\hat{a}_k)^*(\hat{a}_{k'}^\dagger)^*\end{aligned} \tag{58}$$

so I didn't get this . in the perspective of trace : (L:1004)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] &= \sum_n \langle n|\sum_k\sum_{k'} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\hat{a}_k\hat{a}_{k'}^\dagger \sum_i p_i|i\rangle\langle i||n\rangle \\ &= \sum_n \langle n|\sum_k\sum_{k'} g_k^* g_{k'} p_n \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\hat{a}_k\hat{a}_{k'}^\dagger|n\rangle \\ &= \sum_n \langle n|\sum_k g_k^* g_k p_n \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\hat{a}_k\hat{a}_k^\dagger|n\rangle\end{aligned} \tag{59}$$

this expessionis kind of all real(except the expotential part) if there the , $\hat{\rho}_B = \sum_i |i\rangle\langle i|p_i$ is true , if so turning it around would be like (this might not be true , need to recheck): (L:1038)

$$\begin{aligned}\left(\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\right)^* &= \sum_n \langle n|\sum_k g_k^* g_k p_n \mathbb{e}^{-\mathbb{i}(\omega_{k'}t'-\omega_k t)}\hat{a}_k\hat{a}_k^\dagger|n\rangle \\ &= \mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right]\end{aligned} \tag{60}$$

and this gives me a little acceptability to the expression : (L:1057)

$$\begin{aligned}F(t)^* &= \int_0^t \left(\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right]\right)^* \mathbf{d}t' \\ &= \int_0^t \left(\mathrm{Tr}_B\left[\hat{B}(t')\hat{B}^\dagger(t)\hat{\rho}_B\right]\right) \mathbf{d}t' \\ G(t)^* &= \int_0^t \left(\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right]\right)^* \mathbf{d}t' \\ &= \int_0^t \mathrm{Tr}_B\left[\hat{B}^\dagger(t)\hat{B}(t')\hat{\rho}_B\right] \mathbf{d}t'\end{aligned} \tag{61}$$

so we have that : (L:1079)

$$\begin{aligned}\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} &= -\left(\hat{S}\hat{S}^\dagger\hat{\rho}_S(t) - \hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)G(t)^* \\ &\quad -\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S} - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)F(t) \\ ^* &\quad -\left(\hat{S}^\dagger\hat{S}\hat{\rho}_S(t) - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)F(t) \\ &\quad -\left(\hat{\rho}_S(t)\hat{S}\hat{S}^\dagger - \hat{S}^\dagger\hat{\rho}_S(t)\hat{S}\right)G(t)\end{aligned} \tag{62}$$

1.4 The bath specification

the initial vacuum state of a bath is like : (L:1098)

$$\hat{\rho}_B = (|0\rangle|0\rangle......)\otimes(\langle 0|\langle 0|......) \tag{63}$$

so that : (L:1109)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] &= \mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\left(|0\rangle|0\rangle......\right)\otimes\left(\langle 0|\langle 0|......\right)\right]\\ \mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right] &= \mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\left(|0\rangle|0\rangle......\right)\otimes\left(\langle 0|\langle 0|......\right)\right]\end{aligned} \tag{64}$$

calling some bath states as $\{|b\rangle\}$ then we can say : (L:1141)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] &= \sum_b \langle b|\hat{B}(t)\hat{B}^\dagger(t')\left(|0\rangle|0\rangle......\right)\otimes\left(\langle 0|\langle 0|......\right)|b\rangle\\ &= \left(\langle 0|\langle 0|...\right)\sum_b |b\rangle\langle b|\hat{B}(t)\hat{B}^\dagger(t')\left(|0\rangle|0\rangle...\right)\\ &= \left(\langle 0|\langle 0|...\right)\hat{B}(t)\hat{B}^\dagger(t')\left(|0\rangle|0\rangle...\right)\end{aligned} \tag{65}$$

as well as : (L:1176)

$$\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right] = \left(\langle 0|\langle 0|...\right)\hat{B}^\dagger(t')\hat{B}(t)\left(|0\rangle|0\rangle...\right) \tag{66}$$

so lets shoot them up : (L:1190)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] &= \left(\langle 0|\langle 0|...\right)\sum_{k,k'} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\hat{a}_k\hat{a}_{k'}^\dagger\left(|0\rangle|0\rangle...\right)\\ &= \sum_{k,k'} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\left(\langle 0|\langle 0|...\right)\hat{a}_k\hat{a}_{k'}^\dagger\left(|0\rangle|0\rangle...\right)\end{aligned} \tag{67}$$

(L:1216)

$$\mathrm{Tr}_B\left[\hat{B}^\dagger(t')\hat{B}(t)\hat{\rho}_B\right] = \sum_{k',k} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\left(\langle 0|\langle 0|...\right)\hat{a}_{k'}^\dagger\hat{a}_k\left(|0\rangle|0\rangle...\right) \tag{68}$$

with $\hat{a}_k\hat{a}_{k'}^\dagger = \hat{a}_{k'}^\dagger\hat{a}_k + \delta_{k,k'}$ (L:1236)

$$\begin{aligned}\mathrm{Tr}_B\left[\hat{B}(t)\hat{B}^\dagger(t')\hat{\rho}_B\right] &= \sum_{k,k'} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\delta_{k,k'} + \sum_{k,k'} g_k^* g_{k'} \mathbb{e}^{-\mathbb{i}(\omega_k t-\omega_{k'}t')}\left(\langle 0|\langle 0|...\right)\hat{a}_{k'}^\dagger\hat{a}_k\left(|0\rangle|0\rangle...\right)\\ &= \sum_k |g_k|^2 \mathbb{e}^{-\mathbb{i}\omega_{k(t-t')}}\end{aligned} \tag{69}$$

so it gives the things to the $F(t)$ and we have : (L:1263)

$$\begin{aligned}F(t) &= \sum_k |g_k|^2 \int_0^t \mathbb{e}^{-\mathbb{i}\omega_{k(t-t')}}\mathbf{d}t'\\ G(t) &= 0\end{aligned} \tag{70}$$

1.5 Transition to the continuum

now say (L:1278)

$$J(\omega) = \sum_l |g_{l|}^2 \delta(\omega - \omega_l) \tag{71}$$

reditriubute the k index to the $J(\omega)$ int the function $F(t)$ gives (L:1288)

$$F(t) = \int_0^\infty \mathbf{d}\omega J(\omega) \int_0^t \mathbf{d}t' \mathbb{e}^{-\mathbb{i}\omega(t-t')} \tag{72}$$

now lets use $\tau = t - t'$ so $\mathbf{d}\tau = -\mathbf{d}t'$ then it gives (L:1301)

$$\int_0^t \mathbf{d}t' = -\int_t^0 \mathbf{d}\tau = \int_0^t \mathbf{d}\tau \tag{73}$$

now it writes that (L:1308)

$$F(t) = \int_0^\infty \mathbf{d}\omega J(\omega) \int_0^t \mathbf{d}\tau \mathbb{e}^{-\mathbb{i}\omega\tau} \tag{74}$$

1.6 The Markov approximation

so targeting the good old equation (20) we know that we like to have the t to be infinite on the (74),so (L:1320)

$$\begin{aligned}
\int_0^\infty \mathbf{d}\tau \mathbb{e}^{-\mathbb{i}\omega\tau} &= \lim_{\eta\to 0^+} \int_0^\infty \mathbf{d}\tau \mathbb{e}^{-\mathbb{i}\omega\tau - \eta\tau} \\
&= \lim_{\eta\to 0^+} \frac{1}{\eta + \mathbb{i}\omega} \\
&= \lim_{\eta\to 0^+} \frac{\eta - \mathbb{i}\omega}{\eta^2 + \omega^2} \\
&= \lim_{\eta\to 0^+} \frac{\eta}{\eta^2 + \omega^2} - \lim_{\eta\to 0^+} \frac{\mathbb{i}\omega}{\eta^2 + \omega^2} \\
&= \pi\delta(\omega) - \mathbb{i}\mathrm{P}\ \frac{1}{\omega}
\end{aligned} \tag{75}$$

this P stands for Cauchy pricipal part, then we have: (L:1342)

$$F = \pi \int_0^\infty \mathbf{d}\omega J(\omega)\delta(\omega) - \mathbb{i}\mathrm{P}\ \int_0^\infty \mathbf{d}\omega J(\omega)\frac{1}{\omega} \tag{76}$$

1.7 Final form

(L:1352)

$$\begin{aligned} F &= \frac{\gamma + \mathbb{i}\varepsilon}{2} \\ \gamma &\equiv 2\pi \int_0^\infty \mathbf{d}\omega J(\omega)\delta(\omega) \\ \varepsilon &\equiv -2\mathrm{P} \int_0^\infty \mathbf{d}\omega J(\omega)\frac{1}{\omega} \end{aligned} \tag{77}$$

with $G = 0$ in mind we have that equation (62) evolves into (L:1360)

$$\begin{aligned} \frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} &= -\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S} - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\frac{\gamma - \mathbb{i}\varepsilon}{2} \\ &\quad - \left(\hat{S}^\dagger\hat{S}\hat{\rho}_S(t) - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right)\frac{\gamma + \mathbb{i}\varepsilon}{2} \\ &= -\frac{\gamma}{2}\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S} - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger + \hat{S}^\dagger\hat{S}\hat{\rho}_S(t) - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right) \\ &\quad + \frac{\mathbb{i}\varepsilon}{2}\left(\hat{\rho}_S(t)\hat{S}^\dagger\hat{S} - \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger - \hat{S}^\dagger\hat{S}\hat{\rho}_S(t) + \hat{S}\hat{\rho}_S(t)\hat{S}^\dagger\right) \end{aligned} \tag{78}$$

some how it says that $\varepsilon = 0$ which can be achived by a good choice of denisty state choosing , for example , we can extend the lower limit of integration to $-\infty$ noticing that : and eventually : (L:1386)

$$\frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} = \gamma\left(\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}_S(t)\}\right) \tag{79}$$

so back in the original (L:1396)

$$\hat{\rho}_S(t) = \mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\hat{\rho}_S\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} \tag{80}$$

and derivationg it will be looking like (L:1406)

$$\begin{aligned} \frac{\mathbf{d}\hat{\rho}_S(t)}{\mathbf{d}t} &= \frac{\mathbb{i}}{\hbar}\hat{H}_S\mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\hat{H}_S\hat{\rho}_S\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} + \mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\frac{\mathbf{d}\hat{\rho}_S}{\mathbf{d}t}\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} - \frac{\mathbb{i}}{\hbar}\mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\hat{\rho}_S\hat{H}_S\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} \\ &= \frac{\mathbb{i}}{\hbar}\mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\left[\hat{H}_S, \hat{\rho}_S\right]\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} + \mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\frac{\mathbf{d}\hat{\rho}_S}{\mathbf{d}t}\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} \end{aligned} \tag{81}$$

also , there is that : (L:1434)

$$\left(\hat{S}\hat{\rho}_S(t)\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}_S(t)\}\right) = \mathbb{e}^{\mathbb{i}\hat{H}_S t/\hbar}\left(\hat{S}\hat{\rho}_S\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}_S\}\right)\mathbb{e}^{-\mathbb{i}\hat{H}_S t/\hbar} \tag{82}$$

so connecting this with the explicit timed (79) and the no explicit timed (82) we have : (L:1450)

$$\gamma\left(\hat{S}\hat{\rho}_S\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}_S\}\right) = \frac{\mathbb{i}}{\hbar}\left[\hat{H}_S, \hat{\rho}_S\right] + \frac{\mathbf{d}\hat{\rho}_S}{\mathbf{d}t} \tag{83}$$

and arranging it gives : (L:1462)

$$\frac{\mathbf{d}\hat{\rho}_S}{\mathbf{d}t} = -\frac{\mathbb{i}}{\hbar}\left[\hat{H}_S, \hat{\rho}_S\right] + \gamma\left(\hat{S}\hat{\rho}_S\hat{S}^\dagger - \frac{1}{2}\{\hat{S}^\dagger\hat{S}, \hat{\rho}_S\}\right) \tag{84}$$

2 conclusion

so just with several $\hat{L}$ instead of $\hat{S}$ we can have (L:1479)

$$\frac{\mathbf{d}\hat{\rho}_S}{\mathbf{d}t} = -\frac{\mathbb{i}}{\hbar}\left[\hat{H}_S, \hat{\rho}_S\right] + \gamma\sum_j\left(\hat{L}_j\hat{\rho}_S\hat{L}_j^\dagger - \frac{1}{2}\{\hat{L}_j^\dagger\hat{L}_j, \hat{\rho}_S\}\right) \tag{85}$$