Homology Algebra

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1 A little comment on Algebraic Topology

Algebraic topology studies topology by algebraic method. In topology, what we want to find are the topological invariants. Algebraic topology tells us these topological invariants can be constructed by algebraic method. For example, the fundamental group (which is an algebaric structure) of a topological space is topological invariant. The homology group in this lecture is another example. That is to say, every class of topological spaces has "its own groups". In the view of category theory, we can construct functors from topological categories (such as Tops) to algebraic categories (such as Grps). We know that functors not noly map topological objects to algebraic objects, but also map the arrows between topoligical objects to the arrows between algebraic objects. That's exactly what we want: By investigating the relationship between algebraic objects, we can know the relationship between topological objects. And usually, the algebraic problem is easier than its topological origin.[1]

2 Simplicial Homology

[2] We have the following chain

$$... \xmapsto{\partial_{p+2}} C_{p+1} \xmapsto{\partial_{p+1}} C_p \xmapsto{\partial_p} C_{p-1} ...$$

where $\partial \circ \partial = \partial^2 = 0$, which implies $\mathbf{im}\ \partial_{p+1} \subset \mathbf{ker}\ \partial_p$.

We define

$$H_p(C_*) = \mathbf{ker}\ \partial_p / \mathbf{im}\ \partial_{p+1}$$

Here C_p is a free Abelian group that we will define later. $H_p(C_*)$ is called the homology group.

2.1 An example

First let's consider the following example.

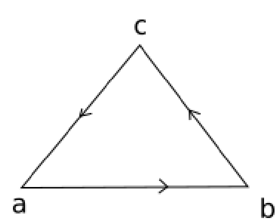


Figure 1: Triangular without an interior

[1]Sometimes, translating the topological problem to the algebraic problem does not result in an easier solution. In fact, a useful translation usually should be done "coarsely" rather than "too intricately".

[2]The reader can refer to Chapter 7 of *An introduction to algebraic topology* by Joseph J.Rotman for a more detailed learn of simplicial homology.

A triangular (without an interior) has there vertices and three edges. We define C_0 to be the vector space with basis three vertices and C_1 to be the vector space with basis three edges. So

$$\begin{aligned} C_1 = \ & \underset{(ab)}{\mathbb{R}} \oplus \underset{(bc)}{\mathbb{R}} \oplus \underset{(ca)}{\mathbb{R}} \\ C_0 = & \underset{a}{\mathbb{R}} \oplus \underset{b}{\mathbb{R}} \oplus \underset{c}{\mathbb{R}} \end{aligned}$$

Because there is no object whose dimension higer than 1 in this triangular, we set $C_2 = C_3 = C_4 = ... = 0$.
Now we have a chain

$$0 \longmapsto \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \longmapsto \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \longmapsto 0$$

We define $\partial : C_1 \longmapsto C_0$ by

$$\partial[x(ab) + y(bc) + z(ca)] = x(b-a) + y(c-b) + z(a-c)$$

The right hand can be written as

$$x(b-a) + y(c-b) + z(a-c) = (z-x)a + (x-y)b + (y-z)c$$

Then the map can be expressed in matrix form

$$\begin{pmatrix} (ab), (bc), (ca) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} a \ , \ b \ , \ c \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It is easy to see that ∂ is exactly the matrix:

$$\partial = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

The reader can verify that

$$\mathbf{ker}\ \partial = \{(\ x \ , \ x \ , \ x \ )^T\}$$

and

$$\mathbf{im}\ \partial = \{\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0\}$$

ker ∂ is a diagonal 1-D subspace of $\mathbb{R}^3$ and **im** ∂ is a 2-D subspace of $\mathbb{R}^3$, that is, a plane with normal vector $\vec{n} = (1, 1, 1)^T$.

By definition of $H_p(C_*)$, we have

$$\begin{aligned} H_1(\Delta, \mathbb{R}) &= \frac{\mathbf{ker}\ \partial}{0} \cong \mathbb{R} \\ H_0(\Delta, \mathbb{R}) &= \frac{\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}}{\mathbb{R} \oplus \mathbb{R}} \cong \mathbb{R} \end{aligned}$$

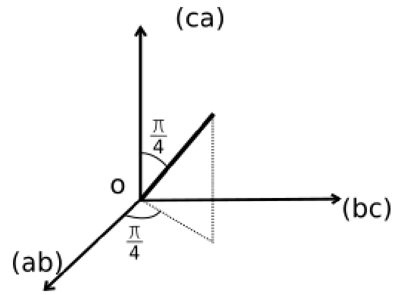


Figure 2: **ker** ∂ is a 1-D subspace of $\mathbb{R}^3$

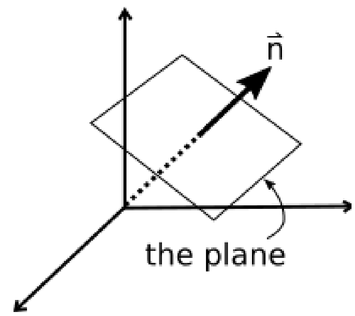


Figure 3: **im** ∂ is a plane in $\mathbb{R}^3$ with normal vector $\vec{n}$

2.2 n-simplex

In the above example, C_1 and C_0 is the free abelian group generated by line segments and points respectively. We call a point 0-simplex and a line segments 1-simplex. In general, a **n-simplex** is defined as a n-D body with n+1 vertices. For example, 2-simplex is a triangular and 3-simplex is a tetrahedron and so on. We can donate a n-simplex whose vertices are $0, 1, 2, ..., n$ as $(0, 1, 2, ..., n)$.

We define operator ∂ maps a simplex to its boundary,

$$\begin{aligned} \partial(0, 1, ..., p) &= (1, 2, ..., p) - (0, 2, 3, ..., p) + ... + (-1)^p(0, 1, 2, ..., p-1) \\ &= \sum_{k=0}^{p} (-1)^k (0, 1, 2, ...\hat{k}, ..., p) \end{aligned}$$

where the hat means it is erased in the tuple. The $(-1)^p$ in the definition is necessary for ∂_p to be a good map as we will illustrate later. Also it leads to $\partial^2 = 0$.

We like to note that (p_0, p_1) is distinguished from (p_1, p_0), because $\partial(p_0, p_1) = -\partial(p_1, p_0)$. That is to say, the complexes we defined above are oriented. To be in consistent with the definition of ∂, we demand

$$(p_0, p_1, ...p_n) = \text{sgn}(P)(0, 1, ..., n)$$

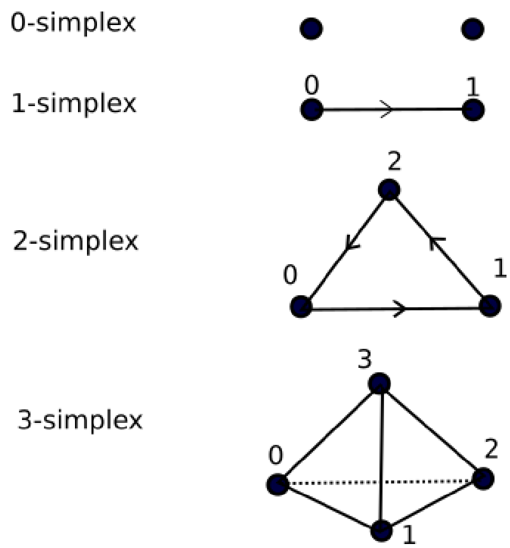


Figure 4: simplexes

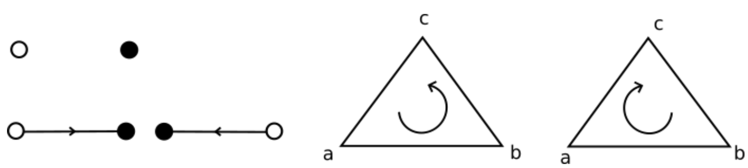


Figure 5: orientation

where

$$P = \begin{pmatrix} 0, & 1, & ..., & n \\ p_0, & p_1, & ..., & p_n \end{pmatrix}.$$

we call such simplex **oriented n-simplex**.

To see why above definitions work, we consider the following example. There is an oriented rhombus, we divided it into two oriented 2-simplexes $(a_0\ a_1\ a_3)$ and $(a_1\ a_2\ a_3)$, then the rhombus is $(a_0\ a_1\ a_3)+(a_1\ a_2\ a_3)$. Take the boundaries of the 2-simplexes,

$$\partial(a_0\ a_1\ a_3) = (a_1\ a_3) - (a_0\ a_3) + (a_0\ a_1)$$
$$\partial(a_1\ a_2\ a_3) = (a_2\ a_3) - (a_1\ a_3) + (a_1\ a_2)$$

then the boundary of the rhombus is

$$\partial[(a_0\ a_1\ a_3) + (a_1\ a_2\ a_3)] = (a_0\ a_1) + (a_1\ a_2) + (a_2\ a_3) + (a_3\ a_0)$$

This is exactly the ture boundary of the rhombus. This good result is due to the $(-1)^p$ in the definition of ∂.

The following example is similar

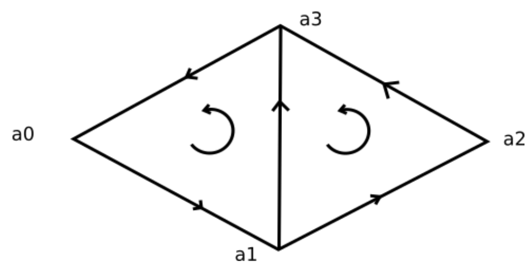
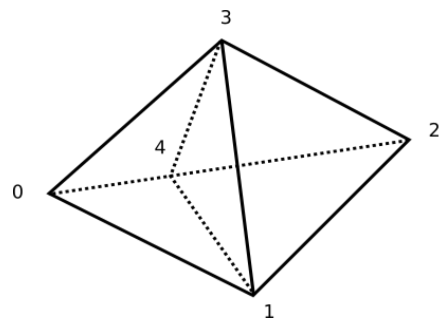


Figure 6: The oriented boundary of the rhombus is the sum of the oriented boundaries of the two 2-simplexes



$$\partial(a_0\ a_1\ a_3\ a_4) = (a_1\ a_3\ a_4) - (a_0\ a_3\ a_4) + (a_0\ a_1\ a_4) - (a_0\ a_1\ a_3)$$
$$\partial(a_1\ a_2\ a_3\ a_4) = (a_2\ a_3\ a_4) - (a_1\ a_3\ a_4) + (a_1\ a_2\ a_4) - (a_1\ a_2\ a_3)$$

2.3 Triangulation

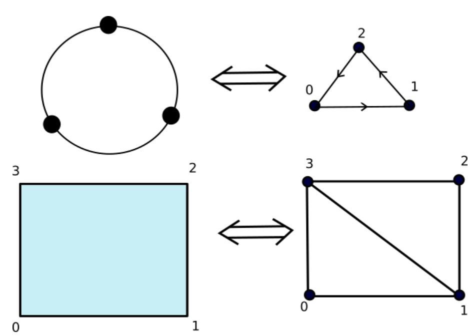


Figure 7: Triangulation

In section 2.1, we constructed the homology groups of a triangular without interior. That object can be seen as a set of well attached simplexes. In general, a set of well attached simplexes is called a **simplicial complex**[3]. For a simplical

[3]For strict definition, refer to Rotman's book.

complex, we can construct its homology groups as we do in 2.1. But can we construct the homology groups for a space that is not a simplical complex ? Notice in constrction of the homology groups, what we care is the relation betwwen simplexes, and we actually not care whether the faces of the simplexes are flat. So "curve simplexes" will do as good as "flat simplexes". Thus, if an object X can be deformed continuously[4] to a shape of simplical complex, we will think it has the same homotopy groups with the simplical complex. This is the thought of triangulation.

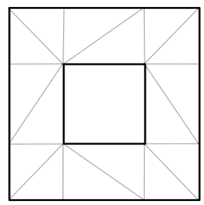


Figure 8: Triangulation of a rectangular ring

If a topological space X can be approximated by attaching simplexes, which we call triangulation, then we can calculate its homology groups. For example, a disk can be approxiamated by a 2-simplex.

Good traingulation: the intersection of any two simplexes is contracable.[5]

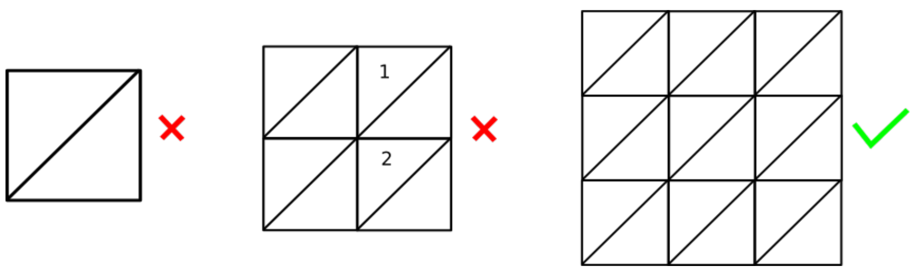


Figure 9: Good and not good triangulation of torus

For computation it is not necessary to use good triangulation. Now we consider an example of torus.

Here we have 2-simplexes Δ_1 and Δ_2

$$\partial\Delta_1 = l_2 + l_2 - l_1$$
$$\partial\Delta_2 = l_1 - l_3 - l_2$$

1-simplexes l_1, l_2, l_3

$$\partial l_i = 0$$

0-simplex a

[4]In mathematical language, it is homeomorphism.

[5]This is equivalent to using a simplicial complex to approximate the space.

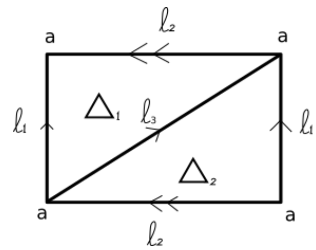


Figure 10: A triangulation of torus

We have the chain

$$0 \longmapsto \mathbb{R}^2 \overset{\partial_2}{\longmapsto} \mathbb{R}^3 \overset{\partial_1}{\longmapsto} \mathbb{R} \longmapsto 0$$

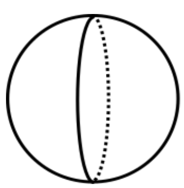
$$\begin{aligned}
&\partial_1(x\Delta_1 + y\Delta_2) = (y-x)l_1 + (x-y)l_2 + (x-y)l_3 \\
&\textbf{ker } \partial_2 = \{(x,x)^T\} \\
&\textbf{im } \partial_2 = \{(x,-x,-x)^T\} \\
&\textbf{ker } \partial_1 = \mathbb{R}^3 \\
&\textbf{im } \partial_1 = 0
\end{aligned}$$

Finally, we get the homology groups of torus:

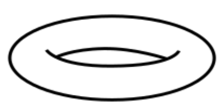
$$\begin{aligned}
&H_2(T_2^2) \cong \mathbb{R} \\
&H_1(T_2^2\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R} \\
&H_0(T_2^2\mathbb{R}) \cong \mathbb{R}
\end{aligned}$$

2.4 Homology groups of Riemann Surfaces

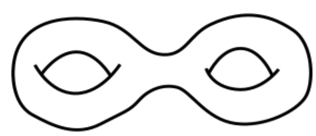
Here are some examples of Riemann surfaces.



$H_1(S^2, \mathbb{R}) = 0$

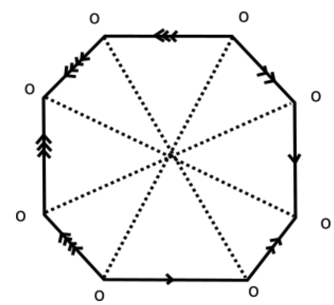


$H_1(T^2, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$

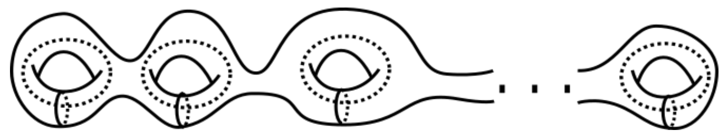


$H_1(\Re^2, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

where $\Re^n$ means torus liked object with n holes.



$$0 \longmapsto \mathbb{R}^{4g} \longmapsto \mathbb{R}^{2g+4g} \longmapsto \mathbb{R}^2 \longmapsto 0$$
$$\chi = 2 - 2g - 4g + 4g = 2(1-g)$$



There are 2g generators of $H_1(\Re^g, \mathbb{R})$, In general we have

$$H_1(\Re^g, \mathbb{R}) = \mathbb{R}^{2g}$$
$$H_2(\Re^g, \mathbb{R}) = \mathbb{R}$$
$$H_0(\Re^g, \mathbb{R}) = \mathbb{R}$$

Euler character

$$\chi_{\Re^g} = 1 - 2g + 1 = 2(1-g)$$

A theorem[6] about χ is:

$$\chi = \sum_{k=0}^{2} (-1)^k b_k = \frac{2}{4\pi} \int R_{\mu\nu} dx^\mu \wedge dx^\nu$$

Here we have some questions:
1: Is $H_*(X, \mathbb{Z})$ independent of the triangulation ?
2: Is it also independent of ordering of 0-simplices ?

The answers are yes, but we do not proof them here. Rather, we turn to a more formal and more useful definition: singular homology.

3 Singular Homology

3.1 Construction of Singular Homology

Let $S_p(X)$ be the free abelian group generated by all continious functions from Δ^p to X, i.e. all elements in $Hom(\Delta^p, X)$.
How to define the boundary operator:

$$S_p \overset{\partial}{\longmapsto} S_{p-1}$$

[6] It is the Gauss-Bonnet formula.

We note that if we can define the boundaries of the basis of S_p, i.e. of all $\varphi \in Hom(\Delta^p, X)$, then operator ∂ can be defined. In general, $\partial\varphi \in S_{p-1}$ is a linear combination of some $\varphi'_k \in Hom(\Delta^{p-1}, X)$, i.e. $\partial\varphi = \sum_k m_k \varphi'_k$. **im** φ is a "twist p-simplex"[7] in X while **im** φ'_k are a set of "twist (p-1)-simplexes" in X. It's natural to define $\partial\varphi$ to let $\cup_k$**im** φ'_k gives the "boundary" of **im** φ and every **im** φ'_k ought to be a "face" of **im** φ. How can we find these φ'_k? In fact, φ maps the faces of Δ^p to the "faces" of **im** φ, if we construct a map $\varepsilon_k : \Delta^{p-1} \longmapsto \Delta^p$ mapping Δ^{p-1} to a face of Δ^p, then $\varphi \circ \varepsilon_k : \Delta^{p-1} \longmapsto X$ will be a φ'_k we want. So we define ε_k which called the k-th face map as

$$\varepsilon_k : (b_0, b_1, ..., b_{p-1}) \longmapsto (a_0, a_1, ..., \hat{a_k}, ..., a_p)$$

where $k = 0, 1, 2...p$. Now we have

$$\Delta^{p-1} \overset{\varepsilon_k}{\longmapsto} \Delta^p \overset{\varphi}{\longmapsto} \chi$$

Combine all the faces and consider the orientation, we finally define ∂ as

$$\partial : \varphi \longmapsto \partial\varphi = \sum_{k=0}^{p} (-1)^k \varphi \circ \varepsilon_k$$

The reader can verify $\partial^2 = 0$.

Here is an examples.

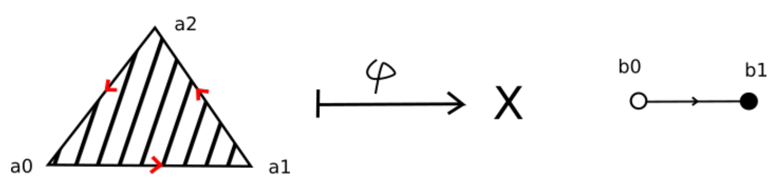


Figure 11: $\varphi : \Delta^2 \longmapsto X$ and $\varepsilon_k : \Delta^1 \longmapsto \Delta^2$

We have $\varphi : \Delta^2 \longmapsto X$, what is $\partial\varphi$? By definition, $\partial\varphi = \varphi\circ\varepsilon_0 - \varphi\circ\varepsilon_1 + \varphi\circ\varepsilon_2$, where $\varepsilon_0 : (b_0, b_1) \longmapsto (a_1, a_2), \varepsilon_1 : (b_0, b_1) \longmapsto (a_0, a_2), \varepsilon_2 : (b_0, b_1) \longmapsto (a_0, a_2)$.

Now let's consider the following chain:

$$... \mapsto S_{p+1} \mapsto S_p \mapsto S_{p-1} \mapsto ... \mapsto S_p \mapsto 0$$

It is very similar to the C_p chain. We can also define the homology group

$$H_p(S_*) = \textbf{ker}\ \partial_p / \textbf{im}\ \partial_{p+1}$$

For example, Let X be a point, then each $S_p(pt, \mathbb{Z})$ is $\mathbb{Z}$ and we have the chain

$$... \longmapsto \mathbb{Z} \longmapsto \overset{S_2}{\mathbb{Z}} \overset{\partial_2}{\longmapsto} \overset{S_1}{\mathbb{Z}} \overset{\partial_1}{\longmapsto} \overset{S_0}{\mathbb{Z}} \longmapsto 0$$

[7] This is just a heuristic think, not accurate.

Write the basis of $S_p(pt, \mathbb{Z})$ as α_i:

$$\alpha_0 : \circ \longmapsto pt$$

$$\alpha_1 : \Delta^1 \longmapsto pt$$

$$\alpha_2 : \Delta^2 \longmapsto pt$$

The boundary of α_i:

$$\begin{aligned}\partial\alpha_1 &= \alpha_1 \circ \varepsilon_0 - \alpha_1 \circ \varepsilon_1 = 0 \\ \partial\alpha_2 &= \alpha_2 \circ \varepsilon_0 - \alpha_2 \circ \varepsilon_1 + \alpha_2 \circ \varepsilon_2 = \alpha_1\end{aligned}$$

After a little think, we have

$$\begin{aligned}&H_0(pt, \mathbb{Z}) \cong \mathbb{Z} \\ &H_n(pt, \mathbb{Z}) = 0, n \geq 1\end{aligned}$$

Here we want to note that $S_p(X)$ is much larger than $C_p(X)$[8], but $H_p(S_*, \mathbb{Z}) \cong H_p(C_*, \mathbb{Z})$[9]. That is to say, the homology groups constructed by different methods are isomorphic. In fact, there are other methods to construct homology groups which give the same results. We will discuss this issue later.

3.2 S_p is a functor

Since S_* leads to the same homology groups, why we use the monster S_* ?
The answer is: S_p is a functor from $Tops$ to $FreeAbs$.

To make S_p a functor, we have to know what is $f_\# = S_p(f)$ for $f : X \mapsto Y$?

$$X \xmapsto{f} Y$$

$$S_p(X) \xmapsto{f_\#} S_p(Y)$$

For φ which is a generator of $S_p(X)$

$$\Delta^p \xmapsto{\varphi} X \xmapsto{f} Y$$

We define $f_\#\varphi = f \circ \varphi$, it is a generator of $S_p(Y)$. Then for $\forall\phi \in S_p(X)$, $f_\#\phi \in S_p(Y)$, it's easy to verify $f_\# : S_p(X) \mapsto S_p(Y)$ is a group homomorphism. Now we have made S_p a functor.

Furthermore, we have the following diagram:

[8] Recall that $C_p(X)$ is generated by the simplexes in the triangulation of X, whose number is much less than the number of the basis of $S_p(X)$.

[9] We do not proof it here.

$$\begin{tikzcd}
S_p(X) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "f_\#"'] & S_{p-1}(X) \arrow[d, mapsto, "f_\#"] \\
S_p(Y) \arrow[r, mapsto, "\partial"] & S_{p-1}(Y)
\end{tikzcd}$$

Here $f_\#$ and ∂ is commutative:

$$f_\#\partial\varphi = f_\#(\sum_k(-1)^k\varphi\circ\varepsilon_k) = \sum_k(-1)^k f\circ(\varphi\circ\varepsilon_k)$$

$$\partial(f_\#\varphi) = \partial(f\circ\varphi) = \sum_k(-1)^k(f\circ\varphi)\circ\varepsilon_k = f_\#\partial\varphi$$

This kind of diagrams are called **commutative diagrams**.

Furthermore, we can have the following commutative ladder:

$$\begin{tikzcd}
\cdots \arrow[r, mapsto] & S_{p+1}(X) \arrow[r, mapsto, "\partial^X_{p+1}"] \arrow[d, mapsto, "f_\#"'] \arrow[dr, phantom, "\mathrm{II}"] & S_p(X) \arrow[r, mapsto, "\partial^X_p"] \arrow[d, mapsto, "f_\#"'] \arrow[dr, phantom, "\mathrm{I}"] & S_{p-1}(X) \arrow[r, mapsto] \arrow[d, mapsto, "f_\#"'] & \cdots \\
\cdots \arrow[r, mapsto] & S_{p+1}(Y) \arrow[r, mapsto, "\partial^Y_{p+1}"] & S_p(Y) \arrow[r, mapsto, "\partial^Y_p"] & S_{p-1}(Y) \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

3.3 H_p is a functor

Since S_p is a functor, we expect H_p also be a functor.

In the ladder diagram above,

region I :

$\Rightarrow$ If $x \in \mathbf{ker}\ \partial^X_p$, then $f_\#(X) \in \mathbf{ker}\ \partial^Y_p \Rightarrow$ $\mathbf{ker}\ \partial^X_p \xmapsto{f_\#} \mathbf{ker}\ \partial^Y_p$

region II :

$\Rightarrow f_\#(\mathbf{im}\ \partial^X_{p+1}) \subset \mathbf{im}\ \partial^Y_{p+1}$

Thus $f_\#$ is actually a homomorphism from $H_p(X)$ to $H_p(Y)$[10],

$$(X \xmapsto{f} Y)$$

$$(H_p(X) \xmapsto{f_\#} H_p(Y))$$

[10]The proof of it leaves as an exercise for the reader.

An upshoot of above discuss is that H_p is a functor $Top \longmapsto Abs$.

An immediate consequence is that homeomorphic topological spaces have isomorphic H_p for all p.

Proof: If X is homeomorphic to Y, $X \cong Y$, then $\exists f : X \mapsto Y,\ g : Y \mapsto X$ satisfy $f \circ g = 1_Y,\ g \circ f = 1_X$. $H_p(f \circ g) = H_p(f) \circ H_p(g) = H_p(1_Y) = 1_{H_p(Y)}$, $H_p(g \circ f) = H_p(g) \circ H_p(f) = H_p(1_X) = 1_{H_p(X)}$, so $H_p(f)$ and $H_p(g)$ are bijective. So $H_p(X) \cong H_p(Y)$.

So the homology groups are topological invariant.

Recall that

$$\pi_1 : hTop \longmapsto Grps$$

we also want to to prove $H_n : hTops \mapsto Abs$, so we need to prove $H_n(f_0) = H_n(f_1)$ if there is $F : f_0 \simeq f_1$, where $f_0, f_1 : X \mapsto Y$.

Proof:

Since there is $F : f_0 \simeq f_1$, we can define map $\lambda_0^X,\ \lambda_1^X : X \mapsto I \times X$ satisfy $F \circ \lambda_0^X = f_0,\ F \circ \lambda_0^X = f_1$.

$$X \overset{\lambda_i^X}{\longmapsto} I \times X \overset{F}{\longmapsto} Y$$

$$H_n(f_i) = H_n(F \circ \lambda_i^X) = H_n(F) \circ H_n(\lambda_i^X)$$

Thus if $H_n(\lambda_0^X) = H_n(\lambda_1^X)$ then $H_n(f_0) = H_n(f_1)$.

Lemma 1. For Any convex subscape X of $\mathbb{R}^N$,there is $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \geq 1$.

Lemma 2. M_* is an exact sequence, P is a free abelian group,

$$\begin{tikzcd}
 & P \arrow[dl, mapsto, "\breve{\varphi}"'] \arrow[d, mapsto, "\varphi"] \arrow[dr, mapsto, "0"] & \\
M" \arrow[r, mapsto, "f"'] & M \arrow[r, mapsto, "f"] & M'
\end{tikzcd}$$

If $\mathbf{im}\ \varphi \subset \mathbf{ker}\ f \Rightarrow \exists \breve{\varphi}$ staisfies $f" \circ \breve{\varphi} = \varphi$.

Proof: Suppose P has basis $\{e_i\}$, then $\varphi(e_i) \in \mathbf{ker}\ f = \mathbf{im}\ f" \Rightarrow \exists m_i", s.t\ f"(m_i") = \varphi(e_i)$. Thus define $\breve{\varphi}$ by defining $\breve{\varphi}(e_i) = m_i"$, which satisfies $f" \circ \breve{\varphi} = \varphi$.

Lemma 3a.

$$
\begin{tikzcd}
\cdots \arrow[r, mapsto] & S_{p+1}(\Delta^n) \arrow[r, mapsto, "\partial_{p+1}"] \arrow[d, mapsto, "\lambda^1_\# - \lambda^0_\#"'] & S_p(\Delta^n) \arrow[r, mapsto, "\partial_p"] \arrow[d, mapsto, "\lambda^1_\# - \lambda^0_\#"'] \arrow[dl, mapsto, "P_p^{\Delta^n}"'] & S_{p-1}(\Delta^n) \arrow[r, mapsto, "\partial_{p-1}"] \arrow[d, mapsto, "\lambda^1_\# - \lambda^0_\#"'] \arrow[dl, mapsto, "P_{p-1}^{\Delta^n}"'] & S_{p-2}(\Delta^n) \arrow[r, mapsto] \arrow[d, mapsto, "\lambda^1_\# - \lambda^0_\#"'] \arrow[dl, mapsto, "P_{p-2}^{\Delta^n}"'] & \cdots \\
\cdots \arrow[r, mapsto] & S_{p+1}(\Delta^n) \arrow[r, mapsto, "\overline{\partial_{p+1}}"] & S_p(I \times \Delta^n) \arrow[r, mapsto, "\overline{\partial_p}"] & S_{p-1}(I \times \Delta^n) \arrow[r, mapsto, "\overline{\partial_{p-1}}"] & S_{p-2}(I \times \Delta^n) \arrow[r, mapsto] & \cdots
\end{tikzcd}
$$

$$\exists P_p^{\Delta^n} s.t.\ P_{p-1}^{\Delta^n} \circ \partial_p + \overline{\partial_{p+1}} \circ P_p^{\Delta^n} = \lambda^1_\# - \lambda^0_\# \ for\ p \geq 0$$

Proof: (i) We start from $p = 0$,

$$
\begin{tikzcd}
S_1(\Delta^n) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto] & S_0(\Delta^n) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto] \arrow[dl, mapsto, "P_0"'] & 0 \arrow[dl, mapsto, "P_{-1}"'] \\
S_1(I \times \Delta^n) \arrow[r, mapsto, "\bar{\partial}"] & S_0(I \times \Delta^n) \arrow[r, mapsto, "\bar{\partial}"] & 0
\end{tikzcd}
$$

For $x \in S_0(\Delta^n)$, $(\lambda^1_\# - \lambda^0_\#)(x) = (1, x) - (0, x)$. Let $P_0(x) :\ I \longmapsto I \times \Delta^n$ be $t \longmapsto (t, x)$, then we have $\bar{\partial} P_0(x) = (1, x) - (0, x)$.

(ii)Induction. Suppose for $k \leq p - 1$, we can find $P_k^{\Delta^n} s.t.\ P_{k-1}^{\Delta^n} \circ \partial_k + \overline{\partial_{k+1}} \circ P_k^{\Delta^n} = \lambda^1_\# - \lambda^0_\#$, then

$$
\begin{aligned}
&\overline{\partial_p} \circ (\lambda^1_\# - \lambda^0_\# - P_{p-1}^{\Delta^n} \circ \partial_p) \\
=&(\lambda^1_\# - \lambda^0_\#) \circ \partial_p - (\lambda^1_\# - \lambda^0_\# - P_{p-2}^{\Delta^n} \circ \partial_{p-1}) \circ \partial_p \\
=&0
\end{aligned}
$$

so **im** $(\lambda^1_\# - \lambda^0_\# - P_{p-1}^{\Delta^n} \circ \partial_p) \subset$ **ker** $\overline{\partial_p}$.

By Lemma 1, we know $H_p(I \times \Delta^p) \cong 0$ for $p \geq 1$, so it is exact at $S_p(I \times \Delta^p)$ for $p \geq 1$.

By Lemma 2,

$$\Rightarrow \exists P_p^{\Delta^p} s.t.(\overline{\partial_{p+1}} \circ P_p^{\Delta^n} = \lambda^1_\# - \lambda^0_\# - P_{p-1}^{\Delta^n} \circ \partial_p)$$

Lemma 3b.

$$\begin{tikzcd}
S_n(\Delta^n) \arrow[r, mapsto, "P_n^{\Delta}"] \arrow[d, mapsto, "\sigma_{\#}"'] & S_{n+1}(\Delta^n) \arrow[d, mapsto, "(1\times\sigma)_{\#}"] \\
S_n(X) \arrow[r, mapsto, "P_n^{X}"] & S_{n+1}(I\times X)
\end{tikzcd}$$

$\exists P_n^X, s.t. for\ \forall\sigma : \Delta^n \longmapsto X$ the diagrama is commutative.

Proof: let $\mathbb{1}_{\Delta^n} : \Delta^n \longmapsto \Delta^n$ to be the identity map. Define $P_n^X(\sigma) := (\mathbb{1}\times\sigma)_{\#}\circ P_n^{\Delta}(\mathbb{1}_{\Delta^n})$, then the diagram is commutative. Because

$$\begin{aligned}
P_n^X \circ \sigma_{\#}(\alpha) &= P_n^X(\sigma\circ\alpha) \\
&= (\mathbb{1}\times\sigma\alpha)_{\#}\circ P_n^{\Delta}(\mathbb{1}_{\Delta^n}) \\
(\mathbb{1}\times\sigma)_{\#}\circ P_n^{\Delta}(\alpha) &= (\mathbb{1}\times\sigma)_{\#}\circ(\mathbb{1}\times\alpha)_{\#}\circ P_n^{\Delta}(1_{\Delta^n}) \\
&= (\mathbb{1}\times\sigma\alpha)_{\#}\circ P_n^{\Delta}(\mathbb{1}_{\Delta^n})
\end{aligned}$$

Now we check our definition of P_n^X for the following diagram:

$$\begin{tikzcd}
S_n(\Delta^n) \arrow[r, mapsto, "P_n^{\Delta}"] \arrow[d, mapsto, "\alpha_{\#}"'] & S_{n+1}(I\times\Delta^n) \arrow[d, mapsto, "(1\times\alpha)_{\#}"] \\
S_n(\Delta^n) \arrow[r, mapsto, "\breve{P}_n^{\Delta}"] & S_{n+1}(I\times\Delta^n)
\end{tikzcd}$$

$$\begin{aligned}
\breve{P}_n^{\Delta^n}(\alpha) &:= (\mathbb{1}\times\alpha)_{\#}\circ P_n^{\Delta^n}(\mathbb{1}_{\Delta^n}) \\
P_n^{\Delta^n}(\alpha) &= P_n^{\Delta^n}(\alpha_{\#}(\mathbb{1}_{\Delta^n})) \\
&= (\mathbb{1}\times\alpha)_{\#}\circ P_n^{\Delta^n}(1_{\Delta^n})
\end{aligned}$$

Lemma 4.

$$\begin{tikzcd}[column sep=large, row sep=huge]
\cdots \arrow[r, mapsto] & S_{n+1}(X) \arrow[r, mapsto, "\partial_{n+1}"] \arrow[d, mapsto, "\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#}"'] & S_n(X) \arrow[r, mapsto, "\partial_n"] \arrow[dl, mapsto, "P_n^X"'] \arrow[d, mapsto, "\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#}"'] & S_{n-1}(X) \arrow[r, mapsto, "\partial_{n-1}"] \arrow[dl, mapsto, "P_{n-1}^X"'] \arrow[d, mapsto, "\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#}"'] & S_{n-2}(X) \arrow[r, mapsto] \arrow[dl, mapsto, "P_{n-2}^X"'] \arrow[d, mapsto, "\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#}"'] & \cdots \\
\cdots \arrow[r, mapsto] & S_{n+1}(I \times X) \arrow[r, mapsto, "\overline{\partial_{n+1}}"] & S_n(I \times X) \arrow[r, mapsto, "\overline{\partial_n}"] & S_{n-1}(I \times X) \arrow[r, mapsto, "\overline{\partial_{n-1}}"] & S_{n-2}(I \times X) \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

$$\overline{\partial} \circ P_n^X = \lambda^{1,x}_{\#} - \lambda^{0,x}_{\#} - P_n^X \circ \partial$$

where P_n^X is defined in the proof of Lemma 3b.

Proof:

$$\begin{aligned}
\overline{\partial}^X \circ P_n^X(\sigma) &= \overline{\partial}^X (\mathbb{1} \times \sigma)_{\#} \circ P_n^{\Delta}(\mathbb{1}_{\Delta}) \\
&= (\mathbb{1} \times \sigma)_{\#} \circ \overline{\partial}^{\Delta} \circ P_n^{\Delta}(\mathbb{1}_{\Delta}) \\
&= (\mathbb{1} \times \sigma)_{\#} \circ (\lambda^{1,\Delta}_{\#} - \lambda^{0,\Delta}_{\#} - P^{\Delta}_{n-1} \circ \partial_n)
\end{aligned}$$

$$\begin{aligned}
&(\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#} - P^X_{n-1} \circ \partial)(\sigma) \\
&\quad = (\lambda^{1,X}_{\#} - \lambda^{0,X}_{\#} - P^X_{n-1} \circ \partial) \circ \sigma_{\#}(\mathbb{1}_{\Delta}) \\
&\quad = (\lambda^{1,X} \circ \sigma)^{\mathbb{1}_{\Delta}}_{\#} - (\lambda^{0,X} \circ \sigma)_{\#}(\mathbb{1}_{\Delta}) - P^X_{n-1} \circ \sigma_{\#} \circ \partial(\mathbb{1}_{\Delta}) \\
&\quad = (\mathbb{1} \times \sigma)_{\#}(\lambda^{1,\Delta}_{\#}(\mathbb{1}_{\Delta}) - \lambda^{0,\Delta}_{\#}(\mathbb{1}_{\Delta})) - (\mathbb{1} \times \sigma)_{\#} \circ P^{\Delta}_n \circ \partial(\mathbb{1}_{\Delta}) \\
&\quad = (\mathbb{1} \times \sigma)_{\#} \circ (\lambda^{1,\Delta}_{\#} - \lambda^{0,\Delta}_{\#} - P^{\Delta}_{n-1} \circ \partial_n)
\end{aligned}$$

$$\begin{tikzcd}[row sep=huge]
S^n(\Delta^n) \arrow[r, mapsto, "\lambda^{i,\Delta}_{\#}"] \arrow[d, mapsto, "\sigma_{\#}"'] & S^n(I \times \Delta^n) \arrow[d, mapsto, "(I \times \sigma)_{\#}"] & & \Delta^n \arrow[r, mapsto] & \Delta^n \arrow[r, mapsto, "\lambda^i"] \arrow[d, mapsto, "\sigma_{\#}"'] & I \times \Delta^n \arrow[d, mapsto, "(I \times \sigma)_{\#}"] \\
S^n(X) \arrow[r, mapsto, "\lambda^{i,X}_{\#}"] & S^n(I \times X) \arrow[r, phantom, "\Rightarrow"] & & & X \arrow[r, mapsto, "\lambda^i"] & I \times X
\end{tikzcd}$$

Now, we have find P_n^X s.t. $\overline{\partial} \circ P_n^X = \lambda^{1,X}_{\#} - \lambda^{0,X}_{\#} - P_n^X \circ \partial$, then the proof of $H_n(\lambda_0) = H_n(\lambda_1)$ is at hand. For $\forall c_n \in \ker \partial_n$,

$$(\lambda^1_{\#} - \lambda^0_{\#})c_n = P_{n-1} \circ \partial c_n + \overline{\partial} \circ P_n c_n = \overline{\partial}(P_n c_n)$$

It means $\lambda^1_{\#} c_n$ and $\lambda^0_{\#} c_n$ only different from each other by a boundary. With a little think we will get $H_n(\lambda_0) = H_n(\lambda_1)$.

Here we summarize the main clue of the proof:

Step 1:

To proof $H_n(f_0) = H_n(f_1)$ for $f_0 \simeq f_1$, we construct $\lambda_0^X,\ \lambda_1^X : X \mapsto I \times X$

$$X \xmapsto{\lambda_i^X} I \times X \xmapsto{F} Y$$

then we just need to proof $H_n(\lambda_0) = H_n(\lambda_1)$.
Step 2:

Find the homotpoy from f to g.

$$\begin{tikzcd}
\cdots \arrow[r, mapsto] & C_{n+1} \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "f-g"'] & C_n \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "f-g"'] \arrow[dl, mapsto, "P_n"'] & C_{n-1} \arrow[d, mapsto, "f-g"'] \arrow[dl, mapsto, "P_{n-1}"'] \\
\cdots \arrow[r, mapsto] & \overline{C_{n+1}} \arrow[r, mapsto, "\overline{\partial}"] & \overline{C_n} \arrow[r, mapsto, "\overline{\partial}"] & \overline{C_{n-1}}
\end{tikzcd}$$

If $\exists P s.t.\ f_n - g_n = P_{n-1} \circ \partial + \overline{\partial} \circ P_n$ (this P is called a homotopy from g to f), Then $H_n(g) = H_n(f)$.
Step 3:

Construct a homotopy from $\lambda_{\#}^0 : S_*(X) \mapsto S_*(I \times X)$ to $\lambda_{\#}^1 : S_*(X) \mapsto S_*(I \times X)$ by first constructing a homotopy from $\lambda_{\#}^0 : S_*(\Delta^p) \mapsto S_*(I \times \Delta^p)$ to $\lambda_{\#}^1 : S_*(\Delta^p) \mapsto S_*(I \times \Delta^p)$.

3.4 Relative Homology

$A \subset X$, $i : A \mapsto X$ is a inclusion map, it is injective. Consider the diagram:

$$\begin{tikzcd}
\cdots & 0 \arrow[d, mapsto] & 0 \arrow[d, mapsto] & 0 \arrow[d, mapsto] & \cdots \\
\cdots \arrow[r, mapsto, "\partial"] & S_{p+1}(A) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i_{\#}"'] & S_p(A) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i_{\#}"'] & S_{p-1}(A) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i_{\#}"'] & \cdots \\
\cdots \arrow[r, mapsto, "\partial"] & S_{p+1}(X) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "p_{\#}"'] & S_p(X) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "p_{\#}"'] & S_{p-1}(X) \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "p_{\#}"'] & \cdots \\
\cdots \arrow[r, mapsto, "\bar{\partial}"] & S_{p+1}(X,A) \arrow[r, mapsto, "\bar{\partial}"] \arrow[d, mapsto] & S_p(X,A) \arrow[r, mapsto, "\bar{\partial}"] \arrow[d, mapsto] & S_{p-1}(X,A) \arrow[r, mapsto, "\bar{\partial}"] \arrow[d, mapsto] & \cdots \\
\cdots & 0 & 0 & 0 & \cdots
\end{tikzcd}$$

where $S_p(X,A) := \dfrac{S_p(X)}{S_p(A)}$ with generators $\varphi + S_p(A)$[11].
$\overline{\partial}$ is defined by

$$\bar{\partial}(\varphi + S_p(A)) = \partial\varphi + S_{p-1}(A)$$

Recall the form of the short exact sequence:

$$0 \longmapsto A \overset{i}{\longmapsto} \Gamma \overset{p}{\longmapsto} \Gamma/A \longmapsto 0$$

so every column is a short exact sequence.

It's obvious that $\overline{\partial}^2 = 0$, so we can define the homology group $H_n(X,A)$, which we call the **homology group of X relative to A**. To understand what this group means, we have a close look at the cycle groups and the boundary groups.

$Z_n(X) = \textbf{ker}\ \partial_n^X$ is the **n-cycle group** of X, and $B_n(X) = \textbf{im}\ \partial_{n+1}^X$ is the **n-boundary group** of X. But $Z_p(X,A)$ is not $Z_n(X)/Z_n(A)$. The reader can work out the following result by himself(or herself):

$$\begin{aligned}
Z_p(X,A) &= \{x \in S_p(X) \mid \partial x \in S_{p-1}(A)\} \\
B_p(X,A) &= \{x \in S_p(X) \mid x - x' \in B_p(X) for\ some\ x' \in S_p(A)\} \\
&= B_p(X) + S_p(A) \\
H_p(X,A) &\cong \frac{Z_n(X,A)}{B_n(X,A)}
\end{aligned}$$

These results can be intuitively understood by thinking A as one point in X.

Long Exact Sequence

For $A \subset X$ or pair (X,A), we can have the sequence:

$$...H_p(A) \overset{i_\#}{\longmapsto} H_p(X) \overset{p_\#}{\longmapsto} H_p(X,A) \overset{\partial}{\longmapsto} H_{p-1}(A) \longmapsto ...$$

If $f : (X,A) \longmapsto (Y,B)$ is continius with $f(A) \subset B$, then there is a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, mapsto] & H_p(A) \arrow[r, mapsto] \arrow[d, mapsto, "f_\#"'] & H_p(X) \arrow[r, mapsto] \arrow[d, mapsto, "f_\#"'] & H_p(X,A) \arrow[r, mapsto] \arrow[d, mapsto, "f_\#"'] & H_{p-1}(A) \arrow[r, mapsto] \arrow[d, mapsto, "f_\#"'] & \cdots \\
\cdots \arrow[r, mapsto] & H_p(B) \arrow[r, mapsto] & H_p(Y) \arrow[r, mapsto] & H_p(Y,B) \arrow[r, mapsto] & H_{p-1}(B) \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

[11] In order not to confuse the reader, we explain what we mean by $\varphi + S_p(A)$. $S_p(A) \subset S_p(X)$, thus for $a, b \in S_p(X)$, we can define $a \sim b$ if $a = b + \phi,\ \phi \in S_p(A)$, a, b is said to be of the same class. We know the factor group $S(X,A)$ is constructed by taking all classes as group elements. The class that $\varphi \in S_p(X)$ belongs to is given by $\{\varphi + \phi | \phi \in S_(A)\}$, we write it as $\varphi + S_p(A)$ for short.

4 Eilenberg-Steenrod Axioms

Homotopy group can be constructed in different ways, but usually we get the same thing. The **Eilenberg-Steenrod Axioms** tells us why this happens. In fact, if all axioms be satisfied, then the homology theory is fixed, no matter how we constructed it. So the Eilenberg-Steenrod Axioms are also called Homology theory's axioms.

Here are the axioms.

Homotopy Axiom:

If $f_0, f_1 : (X, A) \longmapsto (Y, B)$ are homotophic, then $H_n(f_0) = H_n(f_1) : H_n(X, A) \longmapsto H_n(X, B)$ for all $n \geq 0$.

Exactness Axiom:

For (X, A) , there is an exact sequence

$$... \longmapsto H_p(A) \longmapsto H_p(X) \longmapsto H_p(X, A) \longmapsto H_{p-1}(A) \longmapsto ...$$

Excision Axiom:

For every pair (X, A) and every open subset U of X with $\bar{U} \subset \mathring{A}$, the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces isomorphism

$$H_p(X - U, A - U) \cong H_p(X, A)$$

Dimension Axiom:

If X is a one-point space , then $H_p(X) = 0$ for all $p \geq 1$, and $H_0(X)$ is called the coefficient group.

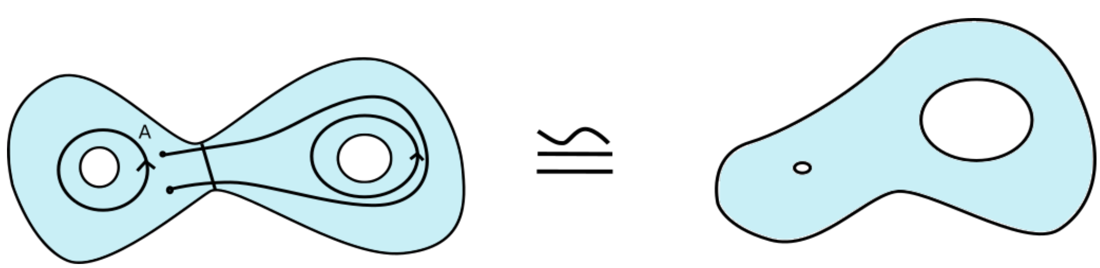
Here comes the important theorem.

Theorem(Eilenberg-Steenrod)

All homology theories with isomorphic coefficient groups on the category $\mathscr{C}$ of all compact polyhodral pairs are isomorphic.

One may think the dimension axiom is not necessary, however, it is. If we do not have the dimension axiom, we might get homology theory different from what we get as usual. In fact, homology theory without dimension axiom is called **general homology theory**. One example is K-theory.

Extradinary (co) homology (K -theory) (wait to be covered)



$$\begin{aligned}
H_1(X) &\cong \mathbb{Z} \oplus \mathbb{Z} \\
H_0(X) &\cong \mathbb{Z} \\
H_1(A) &\cong \mathbb{Z} \\
H_0(A) &\cong \mathbb{Z} \\
H_1(X, A) &\cong \mathbb{Z} \\
H_0(X, A) &\cong 0
\end{aligned}$$

Theorem $H_p(X, A) \cong H_p(X)$ for all $p \geq 1$

Theorem $H_0(X, A) \cong 0$

X path connected, A nonempty,

$$0 \longmapsto H_1(A) \longmapsto H_1(X) \longmapsto H_1(X, A) \overset{\partial}{\longmapsto} H_0(A) \longmapsto H_0(X) \cong \mathbb{Z}$$

which is isomorphic to

$$0 \longmapsto \mathbb{Z} \longmapsto \mathbb{Z} \oplus \mathbb{Z} \longmapsto \mathbb{Z} \overset{0}{\longmapsto} \mathbb{Z} \longmapsto \mathbb{Z} \longmapsto 0$$

$$0 \longmapsto \mathbb{Z} \longmapsto \mathbb{Z} \oplus \mathbb{Z} \longmapsto \mathbb{Z} \longmapsto 0 \qquad and \qquad 0 \longmapsto \mathbb{Z} \longmapsto \mathbb{Z} \longmapsto 0$$