Homology Algebra and Applications

2019-11-13

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Contents

1 A little comment on theoritical physics

We all hear that experiment is the motive force of physics. However, theoritical physics is not derived by experiments.[1] I think the motive force of theoritical physics is "writing the theory in more elegent and more profound mathematical language". For example, classical mechanics had been established by Newton in 17th century and there was no new great physics theory until Maxwell in 19th century. What did theoritical physicists do in that peroid ? In fact, we all know that Newton's mechanics was not the end of mechanics. Lagrangian mechanics and Homiltonian mechanics were developed after it. It seems that analytical mechanics is just a rewriting of Newton's mechanics, which add nothing new. But now, we all konw analytical mechanics is improtant for quantum mechanics.[2] Lagrange and Hamilton were seen as mathematicians rather than physicists in their ages, and their work was not well-known by physicists until early 20th century. [3] But they are surely physicists in our view and their work has become the basic vocabulary of physics. Another example is group theory, which was called "groupest" by Pauli in the beginning, but applied everywhere in physics sooner. So, something seems to be "too mathematic" today may become important physics tomorrow. It is necessary for you to learn some new mathematical language beyond what you learn in "Methods of Mathematical Physics".

2 Long Exact Sequence for calculation

We all know what means "equal" from our primary school. When you solving a equation, the right hand is equal to the left hand. This kind of equal means two numbers are equal. The old concept of equalence is too strict to reveal the structure, so wider concept of equalence has been adopted in modern math.

We can think two groups are "equal" if they are isomorphic to each other. Thus, an exact sequence is an "equation" of groups. If we konw the groups on some sites, then we can derive the unknown gropus on other sites. For example, if we have a short exact sequence

$$0 \longmapsto A \longmapsto \Gamma \longmapsto B \longmapsto 0$$

and we know A and Γ, then we can derive $B \cong \Gamma/A$. In last lecture, we have shown that we can construct long exaxt sequences of homology groups, which will enable us to calculate the unknown homology groups. Thus, Long exact sequence is an important calculation technique.

In general, if we have the following diagram:

$$0 \longmapsto A_* \overset{i}{\longmapsto} C_* \overset{p}{\longmapsto} \frac{C_*}{A_*} \longmapsto 0$$

[1] We do not deny that experiment results in the needs of new theory, however, the development of theoritical physics has its own way.

[2] According to Feynman, it is worth to write a theory in different forms because you don't know which form will be useful when tackling a new problem.

[3] We recall that Dirac was not familiar with Possion bracket and had to go to library.

which is actually:

$$
\begin{tikzcd}
 & \cdots \arrow[d, mapsto] & \cdots \arrow[d, mapsto] & \cdots \arrow[d, mapsto] & \\
0 \arrow[r, mapsto] & A_n \arrow[r, mapsto] \arrow[d, mapsto] & C_n \arrow[r, mapsto] \arrow[d, mapsto] & \dfrac{C_n}{A_n} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
0 \arrow[r, mapsto] & A_m \arrow[r, mapsto] \arrow[d, mapsto] & C_m \arrow[r, mapsto] \arrow[d, mapsto] & \dfrac{C_m}{A_m} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}
$$

i is the inclusion map, p is the projection map. Every raw is an exact sequence of groups and the diagram is commutative. Then we can have a long exact sequence:

$$... \longmapsto H_n(A_*) \longmapsto H_n(C_*) \longmapsto H_n(\frac{C_*}{A_*}) \overset{\partial}{\longmapsto} H_{n-1}(A_*) \longmapsto ...$$

i) We have applied this to

$$0 \longmapsto S_*(Y) \longmapsto S_*(X) \longmapsto S_*(X, Y) \longmapsto 0$$

with $Y \subset X$, giving us the axiom of long exact sequence

$$... \longmapsto H_n(Y) \longmapsto H_n(X) \longmapsto H_n(X, Y) \longmapsto H_{n-1}(Y) \longmapsto ...$$

ii) Consider $\overset{\circ}{X} \bigcup \overset{\circ}{Y} = Z$, and $X \bigcap Y = A$
we can have

$$0 \longmapsto S_*(A) \longmapsto S_*(X) \oplus S_*(Y) \longmapsto S_*(Z) \longmapsto 0$$

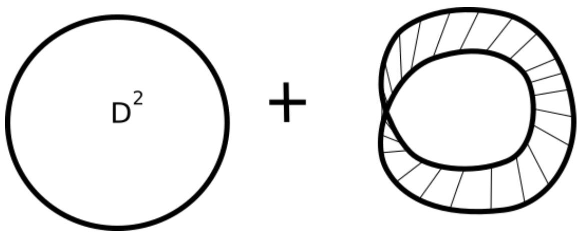
Then we can have a long exact sequence called Mayer-Veitoris exact sequence[4]:

$$... \longmapsto H_n(A) \overset{(i_{1\#}, i_{2\#})}{\longmapsto} H_n(X) \oplus H_n(Y) \overset{g_\# - j_\#}{\longmapsto} H_n(Z) \longmapsto H_{n-1}(A) \longmapsto ...$$

where $i_1 : A \mapsto X,\ i_2 : A \mapsto Y,\ g : X \mapsto Z,\ j : Y \mapsto Z$ are inclusions. And the map $(i_{1\#}, i_{2\#})$ is defined by $a_n \mapsto (i_{1\#}a_n, i_{2\#}a_n)$, $g_\# - j_\#$ is defined by $(x_n, y_n) \mapsto g_\# x_n - j_\# y_n$.

Example :RP^2 can be made by attaching a Mobius ring to a disk, we write $RP^2 = D^2 \cup M$ and $D^2 \cap M = S^1$.

[4]For serious reader who want to know this sequence clearly, we recommand to read Chapter 5 and Chapter 6 of *An Introduction to Algebraic Topology* by Rotman.



From $M \subset RP^2$, we have the long exact sequence:

$$H_2(M) \longmapsto H_2(RP^2) \longmapsto H_2(RP^2, M) \longmapsto H_1(M) \longmapsto H_1(RP^2) \longmapsto H_1(RP^2, M)$$

Note $H_n(RP^2, M) \cong H_n(S^2, *)$, it's easy to know that

$$\begin{aligned} &H_2(M) = 0 \\ &H_2(RP^2, M) = \mathbb{Z} \\ &H_1(M) = \mathbb{Z} \\ &H_1(RP^2, M) = 0 \end{aligned}$$

then the exact sequence turns into:

$$0 \longmapsto H_2(RP^2) \overset{f}{\longmapsto} \mathbb{Z} \overset{\times 2}{\longmapsto} \mathbb{Z} \overset{g}{\longmapsto} H_1(RP^2) \longmapsto 0$$

where the $\times 2$ is important. Because $\mathbf{ker}(\times 2) = 0 = \mathbf{im} f$ and f is injective, so $H_2(RP^2) = 0$. Then $H_1(RP^2) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$.

From $D^2 \cap M = S^1$ and $D^2 \cup M = RP^2$, we have another long exact sequence:

$$H_2(S^1) \longmapsto H^2(M) \oplus H_2(D^2) \longmapsto H_2(RP^2) \longmapsto H_1(S^1) \overset{\times 2}{\longmapsto} H_1(M) \oplus H_1(D^2) \longmapsto H_1(RP^2) \longmapsto \mathbb{Z}$$

From

$$H_2(S^1) \cong H_2(M) \cong H_2(D^2) = 0$$

$$H_1(S^1) \cong \mathbb{Z}$$

$$H_1(M) \oplus H_1(D^2) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$$

we can also derive $H_2(RP^2) = 0$. Finish writing the above sequence, we have

$$0 \longmapsto 0 \longmapsto 0 \longmapsto \mathbb{Z} \overset{\times 2}{\longmapsto} \mathbb{Z} \oplus 0 \longmapsto H_1(RP^2) \overset{f}{\longmapsto} \mathbb{Z} \overset{i}{\longmapsto} \mathbb{Z} \oplus \mathbb{Z} \overset{p}{\longmapsto} \mathbb{Z} \longmapsto 0$$

Because $\mathbf{im} f = \mathbf{ker} i = 0$, f maps $H_1(RP^2)$ to 0, we can have the following sequence:

$$0 \longmapsto \mathbb{Z} \overset{\times 2}{\longmapsto} \mathbb{Z} \longmapsto H_1(RP^2) \longmapsto 0$$

So we get $H_1(RP^2) \cong \mathbb{Z}_2$. The Euler character

$$\chi(RP^2) = 1 - 0 + 0 = 1$$

3 Homotopy of chain complex morphism

$$\begin{tikzcd}
\cdots \arrow[r, mapsto] & C_{n+2} \arrow[r, mapsto] \arrow[d, mapsto, "f_1 - f_0"'] & C_{n+1} \arrow[r, mapsto, "\partial^C"] \arrow[d, mapsto, "f_1 - f_0"'] \arrow[dl, mapsto, "P_{n+1}"'] & C_n \arrow[r, mapsto, "\partial^C"] \arrow[d, mapsto, "f_1 - f_0"'] \arrow[dl, mapsto, "P_n"'] & C_{n-1} \arrow[r, mapsto] \arrow[d, mapsto, "f_1 - f_0"'] \arrow[dl, mapsto, "P_{n-1}"'] & C_{n-2} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dl, mapsto, "P_{n-2}"'] & \cdots \\
\cdots \arrow[r, mapsto] & D_{n+2} \arrow[r, mapsto] & D_{n+1} \arrow[r, mapsto, "\partial^D"] & D_n \arrow[r, mapsto, "\partial^D"] & D_{n-1} \arrow[r, mapsto] & D_{n-2} \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

$f_0, f_1 : C_* \mapsto D_*$, if $\exists P$ satisfy

$$f_1 - f_0 = \partial^D \circ P_n + P_{n-1} \circ \partial^C$$

then P is called a homotopy from f_0 to f_1, $P : f_0 \simeq f_1$.

In last lecture, we have proved if $\exists P : f_0 \simeq f_1$, then $H_n(f_0) = H_n(f_1)$.

Applications:

i) Consider an exact sequence of free abelian groups (a free resolution of $\mathbb{Z}$)

$$... \longmapsto F_{n+1} \longmapsto F_n \longmapsto F_{n-1} \longmapsto ... \longmapsto F_0 \longmapsto \mathbb{Z} \longmapsto 0$$

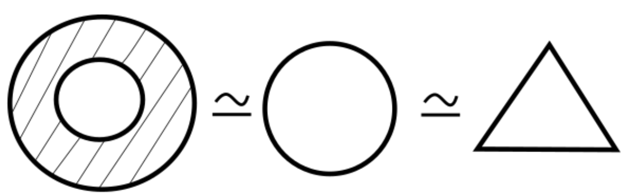
all such exact sequences are homotopy equivalent. We omit proof here, the result will be used in group (co)homology.

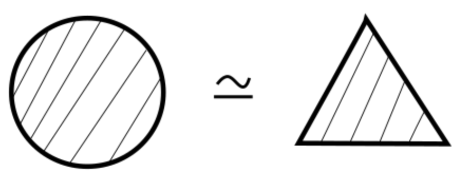
ii) We have prooved in last lecture that

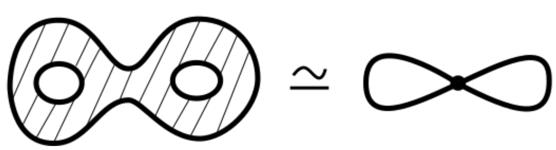
$$F : f_0 \simeq f_1 \Longrightarrow \exists P : f_{0\#} \simeq f_{1\#} \Longrightarrow H_*(f_0) = H_*(f_1)$$

This is also what the homotopy axiom says. An immediate consequence is : two spaces of the same homotopy type have isomorphic homology group.[5]

For example :

i) 

ii) 

iii) 

[5]The reader can proof it by copying the corresponding proof of the functor π_1.

4 Hurewicz Theorem

the Hurewicz Theorem states that : if X is path connected then

$$H_1(X) \cong \textit{the abelianization of } \pi_1(X)$$

For example, we have the following shape:

Then

$$\pi_1(\infty) \cong \mathbb{Z} * \mathbb{Z}$$
$$H_1(\infty) \cong Ab\pi_1(\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$\mathbb{Z} * \mathbb{Z}$ means $\pi_1(\infty)$ is generated by two generators a, b and $ab \neq ba$, $\pi_1(\infty)$ is not an abelian group. $Ab\pi_1(\infty)$ means the abelianization of $\pi_1(\infty)$, that is, we add a relation $ab = ba$, so $\mathbb{Z} * \mathbb{Z}$ becomes $\mathbb{Z} \oplus \mathbb{Z}$.

A group G can be defined by a set of generators $X = \{x_k : k \in K\}$ and relations : $\Delta = \{r_j = 1 : j \in J\}$. We define $G \cong F/R$, F is free[6] on X and R is the normal subgroup of F generated by $\{r_j : j \in J\}$, and G is presented as $(X|\Delta)$.

For example :

$$\pi(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} = (\{a, b\}|aba^{-1}b^{-1})$$

The abelianization of a group means we demand any two generators to commute by adding new relations.

5 Seifert-van Kampen Theorem

Seifert-van Kampen Theorem:

$\mathring{X}_1 \cup \mathring{X}_2 = X$ and $X_1 \cap X_2$ is path connected, if

$$\pi(X_1, x_0) = (K_1|\Delta_1)$$
$$\pi(X_2, x_0) = (K_2|\Delta_2)$$

Then:

$$\pi_1(X, x_0) = (K_1 \cup K_2|\Delta_1 \cup \Delta_2 \cup \{i^1_{\#}(g)i^2_{\#}(g^{-1})|g \in \pi(X_1 \cap X_2, x_0)\})$$

where $i^1 : X_1 \cap X_2 \mapsto X_1$ and $i^2 : X_1 \cap X_2 \mapsto X_2$ are inclusions, and

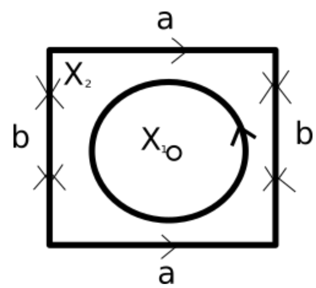
$$\pi_1(X_1 \cap X_2) \xmapsto{i^1_{\#}} \pi_1(X_1, x_0)$$

$$\pi_1(X_1 \cap X_2) \xmapsto{i^2_{\#}} \pi_1(X_2, x_0)$$

[6]F is free on X means F is generated by $\{x_k\}$ and there is no relation between the generators.

The theorem is really easy to understand although not easy to proof. It says, when you consider a big space which is the sum of two subspaces, the fundamental group is generated by the generators of the two subspaces and the old relations are held, except some new relations are added to identifying "the same things" in the intersectional area.

This theorem can help us to find the foundamental groups of some spaces. For example, consider T^2



We let X_2 be the squre substract the center point, and X_1 is a disk in the squre. Then we have

$X_2 \cong$  $\Rightarrow \pi_1(X_2) \cong \mathbb{Z} * \mathbb{Z}$

$X_1 \cong$  $\cong * \Rightarrow \pi_1(X_1) \cong 1$

$X_1 \bigcap X_2 \cong$  $\Rightarrow \pi_1(X_1 \cap X_2) \cong \mathbb{Z}$

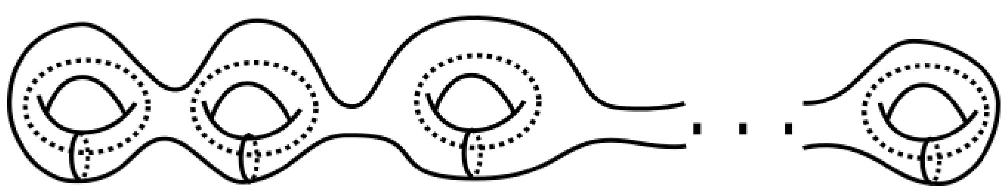
We write the above groups by presentations:

$$\pi_1(X_2) = (\{a, b\}|\emptyset)$$
$$\pi_1(X_1) = (\emptyset|\emptyset)$$
$$\pi_1(X_1 \cap X_2) = (\{c\}|\emptyset)$$

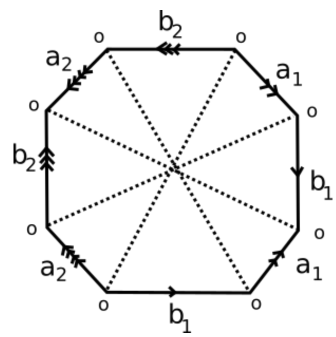
It's easy to see $i^1_{\#}c = 1,\ i^2_{\#}c = aba^{-1}b^{-1}$, so

$$\pi_1(T^2) = (\{a.b\}|aba^{-1}b^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

We can also find the foundamental group of $\mathfrak{R}^g$ by the Seifert-van Kampen Theorem



For example, we consider $\mathfrak{R}^2$



Define X_1 and X_2 like in the T^2 case, then we have

$$\pi_1(X_2) = (\{a_1, b_1, a_2, b_2\}|\emptyset)$$
$$\pi_1(X_1) = (\emptyset|\emptyset)$$
$$\pi_1(X_1 \cap X_2) = (\{c\}|\emptyset)$$

$$i^2{}_{\#}c = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$$
$$\Rightarrow \pi_1(\mathfrak{R}^2) \cong (\{a_1, b_1, a_2, b_2\}|a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1})$$

In general,

$$\pi_1(\mathfrak{R}_g) \cong (\{a_1, b_1, ..., a_2, b_2\}|a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} ... a_g b_g a_g^{-1} b_g^{-1})$$

if we define

$$[a, b] = aba^{-1}b^{-1}$$

then we can write

$$\pi_1(\mathfrak{R}^g) = (\{a_i, b_i\}|\prod_{i=1}^{g}[a_i, b_i])$$

Abelianization of $\pi_1(\mathfrak{R}^g)$ will give

$$H_1(\mathfrak{R}^g) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}$$

there are $2g$ $\mathbb{Z}$.

6 Applications in Physcis

6.1 Braid Groups

Consider n indentical particles in a space X, let them be hardcore particles, namely any two of them don't occupy the same position. We set Δ to be the space of $(x_1, ..., x_n)$ with $x_i = x_j$ for some pairs i, j. Then the configuration space is :

$$C_n(X) := (X^n - \Delta)/\sim = (X^n - \Delta)/S_n$$

where we think $(x_1, ..., x_n) \sim (y_1, ..., y_n)$ if there is a permutation $\sigma \in S_n$ $s.t.$ $x_i = y_{\sigma(i)}$, because the particles are indentical.

Define the braid group

$$B_n(X) = \pi_1(C_n(X), [\vec{x_0}])$$

On the two dimentional plane $\mathbb{R}^2$

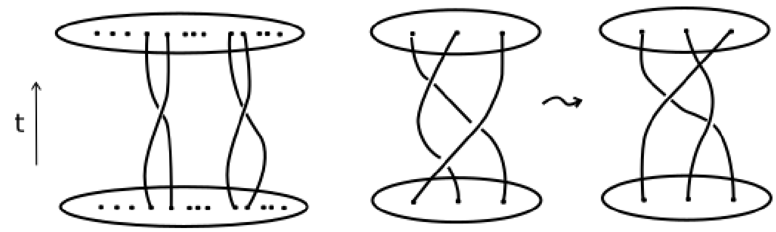
$$B_n := \pi_1(C_n(\mathbb{R}^2), [\vec{x_0}])$$

This group can be generated by a set of generators $\{\sigma_i\}$ with following relations[7]:

i) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$

ii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

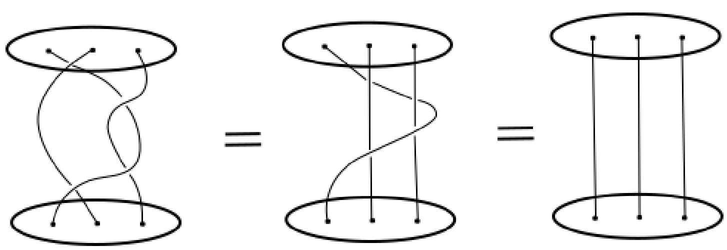
σ_i exchanges i_{th} and $(i+1)_{th}$ particles, we can represent the relations in pictures[8]



How about $\pi_1(C_n(S^2))$? There is 2 new relations :

$$\sigma_1 ... \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} .. \sigma_1 = 1$$
$$(\sigma_1 ... \sigma_{n+1})^n = 1$$

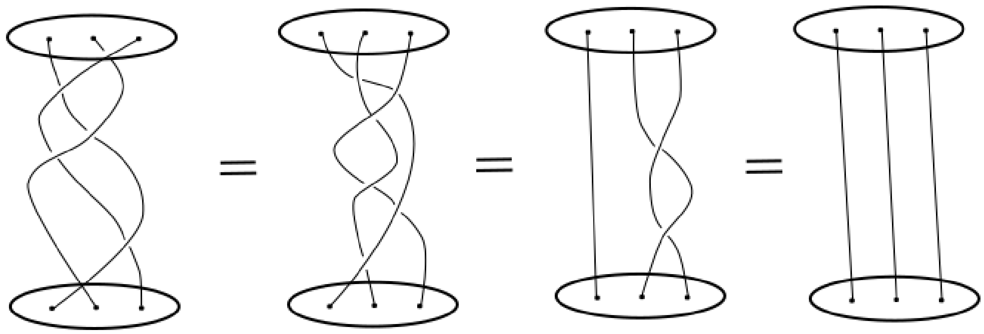
For example, when $n = 2$ we have $\sigma_1^2 = 1$ and $\pi_1(C_2(S^2)) \cong \mathbb{Z}_2$. When $n = 3$, the first new relation $\sigma_1 \sigma_2 \sigma_2 \sigma_1 = 1$ can be shown by



and $(\sigma_1 \sigma_2)^3 = 1$ is shown by

[7] We omit the proof.

[8] We can proof the braid group in $\mathbb{R}^2$ can be represented by this kind of pictures.

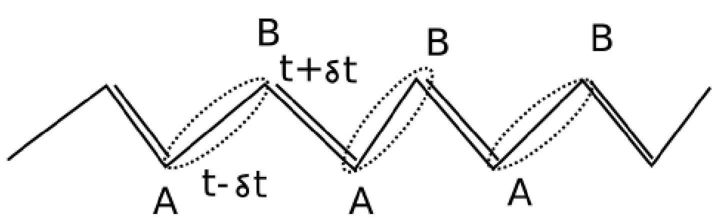


In 3D, $\pi_1(C_n(\mathbb{R}^3))$ is much simpler than in 2D.[9] Its generators are commutative, and one can proof $\pi_1(C_n(\mathbb{R}^3))$ is just the permutation group S_n.

Why braid group so improtant? Because it is related to the statistics of identical particles. In 3D, there are only two one-dimensional representations of S_n, which correspond to Bosons and Fermions respectively. But in 2D, the non-Abelian group B_n leads to exotic statistics and anyons.[10]

6.2 SSH Model and 1 D Topological Insulator

There is a bipartite lattice, all transitions happen between A and B, and there are two different transition amplitudes. This model is called **SSH model**, which is the simplest topological insulator(TI).



The Hamiltonian in second quantization form of SSH model is :

$$\hat{H} = \sum_i (t-\delta t) C^\dagger_{B_i} C_{A_i} + (t+\delta t) C^\dagger_{A_{i+1}} C_{B_i} + h.c.$$

where h.c. means the Hermitian conjugation of the fomer term.

Do Fourier transformation on the coefficients

$$C^\dagger_{A_j} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(kj)} C^\dagger_{A_k}$$

[9] For some problems, higer dimension cases are simpler than low dimension cases, another famous example is the proof of Poincaré conjucture.

[10] This lecture is too short to discuss this topic, a serious reader needs to read other materials. For example, the original paper (PhysRevLett.52.2103) by Yong-Shi Wu.

$$C^\dagger_{B_j} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(kj)} C^\dagger_{B_k}$$

Then

$$\hat{H} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} (C^\dagger_{A_k}, C^\dagger_{B_k}) \hat{H}_{SSH}(k) \begin{pmatrix} C_{A_k} \\ C_{B_k} \end{pmatrix}$$

where

$$\hat{H}_{SSH}(k) = \begin{pmatrix} 0 & (t-\delta t) + (t+\delta t)e^{ik} \\ (t-\delta t) + (t+\delta t)e^{-ik} & 0 \end{pmatrix} = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$$

Because $q(k) \neq 0$ for all the k, $\hat{H}_{SSH}(k)$ has an energy gap.

Now let us consider more generally.

For a Hamiltonian $\hat{H}(k)$, if $\{\hat{H}(k), \sigma_3\} = 0$, where

$$\sigma_3 = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$$

then it's easy to proof

$$\hat{H}(k) = \begin{pmatrix} 0 & Q(k) \\ Q^\dagger(k) & 0 \end{pmatrix}$$

If $\mathrm{Det}(Q(k)) \neq 0 \Rightarrow \hat{H}(k)$ has an energy gap. We call such $\hat{H}(k)$ has **chiral symmetry**.[11]

Note that

$Q(k) \in GL(N, \mathbb{C}) \simeq U(N)$[12] , $k \in BZ = S^1$ [13]

So Q can be seen as a map from S^1 to $GL(N,\mathbb{C})$, $Q : S^1 \longmapsto GL(N,\mathbb{C})$, topological distinct maps of this type are classified by the fundamental group $\pi_1(GL(N,\mathbb{C}))$, so 1D systems with chiral symmetry can also be classified by the foundamental group, this kind of topological insulators are in **class AIII**.

We have the following results:

$$\pi_1(GL(N,\mathbb{C})) \cong \pi_1(U(N)) \cong \mathbb{Z}$$

It can be obtained by the long exact sequence calculation technique. First, we construct a short sequence:

$$1 \longmapsto SU(N) \overset{i}{\longmapsto} U(N) \overset{\mathrm{Det}}{\longmapsto} U(1) \longmapsto 1$$ [14]

Then we can get a long exact sequence:

$$\ldots \longmapsto \pi_n(SU(N)) \longmapsto \pi_n(U(N)) \longmapsto \pi_n(U(1)) \longmapsto \pi_{n-1}(SU(N)) \longmapsto \ldots$$

[11] This symmetry has the similar form with the chiral symmery of the Dirac operator $D = i\gamma^\mu \partial_\mu$, which satisfies $\{D, \gamma^5\} = 0$, but it's not a good name. It's better to call it CT symmetry.

[12] $GL(N,\mathbb{C})$ and $U(N)$ are of the same homotopy type. We note that $GL(N,\mathbb{C})$ and $U(N)$ are Lie groups which are also topological spaces.

[13] BZ means the first Brillouin zone.

[14] $\mathrm{Det}(MN) = \mathrm{Det}(M) * \mathrm{Det}(N)$ and $\mathrm{Det}(U(N)) = U(1)$, so Det: $U(N) \mapsto U(1)$ is a group homeomorphism form $U(N)$ to $U(1)$. One can proof $U(1) \cong U(N)/SU(N)$.

For example, when $N=2$, $SU(2) \cong S^3$, [15] we have

$$\pi_1(S^3) \longmapsto \pi_1(U(2)) \longmapsto \pi_1(U(1)) \longmapsto \pi_0(S^3)$$

By

$$\begin{aligned} &\pi_1(S^3) = 0 \\ &\pi_0(S^3) = 0 \\ &\pi_1(U(1)) = \mathbb{Z} \end{aligned}$$

we get $\pi_1(U(2)) = \mathbb{Z}$. In general, we can get $\pi_1(U(N)) = \mathbb{Z}$.

Recall the winding number characterizes topological distinct loops in $U(1)$, we can calculate the winding number of $Q(k)$ by the formula:

$$Q \longmapsto q = \frac{Det(Q)}{|Det(Q)|}$$

$$v = \int_{-\pi}^{\pi} \frac{dk}{2\pi} q(k) \frac{dq^\dagger(k)}{dk} \quad \in \mathbb{Z}$$ [16]

Class AIII is just one class of TIs, now we introduce another two classes: **class A** and **class D**[17]

class A has a gapped Hermitain matrix like

$$\begin{aligned} &\hat{H} = \hat{U}^\dagger \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix} \hat{U} \qquad \sim \qquad \hat{U}^\dagger \begin{pmatrix} 1_N & 0 \\ 0 & -1_M \end{pmatrix} \hat{U} \\ &C_0 \simeq \frac{U(N+M)}{U(N) \times U(M)} \end{aligned}$$

class D has particle-hole symmetry

$$\{\hat{H}, \hat{C}\} = 0 \qquad \Rightarrow \hat{H} = -\hat{H}^*$$

the charge conjugate operator $\hat{C}$ satisfies $C^2 = 1,\ \{C, i\} = 0$. So we get $i\hat{H}$ is real. Class D's Hamiltonian looks like:

[15] If you don't know, try to proof it.

[16] What's the physical meaning of v ? First, it is a topological invariant of class AIII TIs. Physically speaking, if two gapped quantum phases can be transformed into each other through an adiabatic/a continuous path in the phase diagram without closing the gap (i.e., without encountering a quantum phase transition), then they are said to be topologically equivalent. If two AIII type systems are topological equivalent, they have the same winding number. Second, it corresponds to the number of zero modes when the system has a boundary, this close connection between nontrivial bulk topological properties and gapless boundary modes is known as the bulk-boundary correspondence.

[17] To learn the classification of topological insulators, one can refer to Rev. Mod. Phys. 88, 035005. This review covers nearly all we discussed in this section.

$$i\hat{H} = \hat{O}^T \begin{pmatrix} 0 & -E_1 & 0 & 0 & \dots \\ E_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -E_2 & \dots \\ 0 & 0 & E_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \hat{O} \quad \sim \quad \hat{O}^T \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \hat{O}$$

$$i\hat{H} \sim \hat{O}^T (i\tau_2 \otimes 1_N)\hat{O}$$

$$\Rightarrow \; i\hat{H} \in R_2 = \frac{O(2N)}{U(N)} \subset \frac{U(2N)}{U(N) \times U(N)}$$

We have

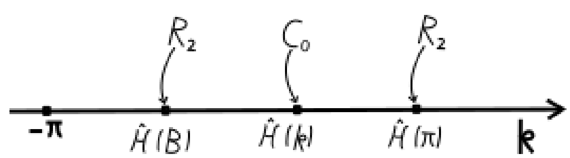
$$\pi_1(\frac{U(2N)}{U(N) \times U(N)}) \longmapsto \pi_1(C_0, R_2) \longmapsto \pi_0(R_2) \longmapsto \pi_0(C_0)$$

note the fact that $\pi_1(\frac{U(2N)}{U(N) \times U(N)}) = 0$ and $\pi_0(R_2) = \mathbb{Z}$, we get $\pi_1(C_0, R_2) \cong \mathbb{Z}$.

Superconductor wire in one dimension, (waiting to be covered)

Kitaev's majorana chain:

$$H_0(k) \qquad [H_0] \in \pi_1(C_0, R_2) \cong \mathbb{Z}_2$$



what is the topological invariant?

$$i\hat{H}(0) = \hat{O}^T(0) iT_2 \otimes 1_N \hat{O}(0)$$

$$i\hat{H}(\pi) = \hat{O}^T(\pi) iT_2 \otimes 1_N \hat{O}(\pi)$$

$$v = Det(O(0))Det(\pi) \in \{+1, -1\}$$

unpair Majorana modes: