Cohomology

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Contents

1 Cohomology

Cohomology is widely used in physics. There are many ways to construct cohomology, here, we first construct it from cochain complex, then we construct it from differential forms.[1]

1.1 Singular Cohomology

Cohomology can be constructed by dualizing the construction of homology. Recall that we constructed homology from (singular) chain complex

$$... \mapsto S_{p+1}(X) \mapsto S_p(X) \mapsto S_{p-1}(X) \mapsto ... \mapsto S_0(X) \mapsto 0$$

where n-chain can be seen as "nD cells".
Now we define n-cochain, which is "functions on nD cells ":

$$S^p(X, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(S_p(X), \mathbb{Z})$$

$S^p(X, \mathbb{Z})$ is the collection of all the $\mathbb{Z}$-linear functions from $S_p(X)$ to $\mathbb{Z}$.
Or we can use $\mathbb{R}$ as coefficient:

$$S^p(X, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(S_p(X), \mathbb{R})$$

which is the collection of all the $\mathbb{R}$ -linear functions from $S_p(X)$ to $\mathbb{R}$.
In general, define

$$S^p(X, \Lambda) = \mathrm{Hom}(S_p(X), \Lambda)$$

where Λ can be $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_2....$

Since the elements in $S^p(X)$ are functions on $S_p(X)$, we can define "inner product" $\langle c, \gamma \rangle \in \Lambda$ for $c \in S^p(X)$ and $\gamma \in S_p(X)$.

We define the coboundary operator $\delta^p : S^p \longmapsto S^{p+1}$ by

$$\langle \delta c, \alpha \rangle = \langle c, \partial \alpha \rangle$$

for all $c \in S^p$ and $\alpha \in S_{p+1}$.
Thus we have the (sigular) cochain complex:

$$0 \longmapsto S^1(X, \Lambda) \xmapsto{\delta^1} S^2(X, \Lambda) \xmapsto{\delta^2} S^3(X, \Lambda) \xmapsto{\delta^3} ...... \longmapsto S^p(X, \Lambda) \xmapsto{\delta^p} S^{p+1}(X, \Lambda) \longmapsto ...$$

We can proof that δ is nilpotent, i.e. $\delta^2 = 0$.

$$\langle \delta^2 c, \beta \rangle = \langle \delta c, \partial \beta \rangle = \langle c, \partial^2 \beta \rangle = 0$$

So we can define the **(singular) cohomology group**

$$H^p(X, \Lambda) = \frac{Z^p(X, \Lambda)}{B^p(X, \Lambda)}$$
$$Z^p(X, \Lambda) = \mathbf{ker}\ \delta^p$$
$$B^p(X, \Lambda) = \mathbf{im}\ \delta^{p-1}$$

[1]To learn more about cohomology, one can refer to Chapter 12, *An Introduction to Algebraic Topology* by Rotman.

$Z^p(X,\Lambda)$ is called the **p-cocycle group**, $B^p(X,\Lambda)$ is called the **p-coboundary group**. One can derive that

$$Z^p = \{c \in S^p | c(B_p) = 0\}$$

which means the elements in Z^p vanishes on all boundaries of (p+1)-chain.

We also want to note that now $\mathrm{Hom}(\_\_,\Lambda)$ becomes a functor if we define $\mathrm{Hom}(\partial_{p+1},\Lambda) = \delta^p$. But it isn't a **covariant functor** which keeps the direction of arrows as S_p and H_p do, it is a **contravariant functor** which reverses the direction of arrows. Formally,

$$F(f \circ g) = F(f) \circ F(g) \quad \text{for covariant functor F}$$

$$F(f \circ g) = F(g) \circ F(f) \quad \text{for contrvariant functor F}$$

The nilpotent property of δ can be obtained formally by the property of functors:

$$\delta^2 = \mathrm{Hom}(\partial,\Lambda)\mathrm{Hom}(\partial,\Lambda) = \mathrm{Hom}(\partial^2,\Lambda) = \mathrm{Hom}(0,\Lambda) = 0$$

1.2 de Rham Cohomology

de Rham cohomology is constructed by the diffrential forms on manifold, so first we have to introduce the diffrential forms on manifold.

For a manifold M, we have the following sequence :

$$0 \longmapsto \Omega^0(M) \overset{d}{\longmapsto} \Omega^1(M) \overset{d}{\longmapsto} \Omega^2(M) \longmapsto ...$$

where $\Omega^p(M)$ is the collection of all the **differential p-forms** on M.

What is a manifold ? A manifold is a topological space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension n. [2]

Whitney embedding theorem states that a n-manifold can be embedded into $\mathbb{R}^{2n+1}$. So a manifold can be seen as a curved surface in a high dimension Euclidean space. Thus, we can define tangent space at each point of the manifold. We write the **tangent space** at x of M be T_xM. The elements in T_xM are called vectors.

We can set coordinate on the manifold. Then at every point of the manifold, we will have a set of basis $\{e_\mu\}$ for T_xM.[3] A vector $v \in T_xM$ can be written as $v^\mu e_\mu(x)$. If $v(x)$ can be defined continuously on M, we call $v(x)$ a vector field on M.

Define the dual vector space T_x^*M at $x \in M$ be the space of $\mathbb{R}$-linear functions from T_xM to $\mathbb{R}$. A dual vector $w \in T_x^*M$ can be written as $w_\nu \mathrm{d}x^\nu$, where

[2] For an introduction to manifold, one can refer to Chapter 5, *Geometry, Topology and physics* by M.Nakahara. Introduction to differential forms can also be found in Chapter 5 and introduction to de Rham cohomology can be found in Chapter 6 in this book.

[3] The details can be found in Nakahara's book.

$\{dx^\nu\}$is the basis of $T_x^* M$. We can do inner product $\langle w, v\rangle = w_\mu v^\mu$, where we have used the relation $\langle \mathrm{d}x^\nu, e_\mu\rangle = \delta_\mu^\nu$.

Now, let's define the p-form on M.

0-forms: functions (infinitely differentiable) on M

1-forms: dual vector field $w_\mu(x)dx^\mu$ on M

2-forms: $w_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \in T^*M \bigwedge T^*M$

3-forms: $w_{\mu\nu\lambda}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \in T^*M \bigwedge T^*M \bigwedge T^*M$

....

where $\wedge$ is wedge product which is antisymmetric.

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$$

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\lambda = \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

We denote the collection of p-forms on M by $\Omega^p(M)$.
Now, we define the exterior derivative operator $d : \Omega^p(M) \to \Omega^{p+1}(M)$

$$\mathrm{d}f(x) = \partial_\mu f(x)\mathrm{d}x^\mu$$

$$\mathrm{d}(w_\mu \mathrm{d}x^\mu) = \partial_\nu w_\mu \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$$

$$\mathrm{d}(w_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}) = \partial_\nu w_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$$

Since wedge product is antisymmetric, we can demand $w_{\mu_1\mu_2\ldots\mu_p}$ to be antisymmetric too, [4] because its symmetric part vanishes when encounters $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$. One can verify the exterior derivative operator d is well defined and $d^2 = 0$. Thus, we will have the following chain

$$0 \longmapsto \Omega^0(M) \overset{d}{\longmapsto} \Omega^1(M) \overset{d}{\longmapsto} \Omega^2(M) \longmapsto \ldots$$

Define the de Rham cohomology group

$$H^p_{dR}(M, \mathbb{R}) = \frac{Z^p(M)}{B^p(M)}$$

where

$$Z^p(M) = \{w \in \Omega^p(M) | dw = 0\}$$

$$B^p(M) = \{dw \in \Omega^p(M) | w \in \Omega^{p-1}(M)\}$$

Are de Rham cohomology groups isomorphic to singular cohomology groups ? The answer is yes. To see why, we have to define the integration of differential forms first.

[4] In the defination of p-forms, we also demand $w_{\mu_1\mu_2\ldots\mu_p}$ to be smooth on M

1.3 Stokes' Theorem

$w = w_{i_1...i_p} \mathrm{d}x^{i_1} \wedge ... \wedge \mathrm{d}x^{x_p}$ is a p-form on M, a singular p-simplex $\sigma : \Delta^p \to M$ determines n coordinate functions σ_i (if $y \in \Delta^p$, then $\sigma(y) = (\sigma_1(y), ..., \sigma_n(y)) \in M$). Define the integration of w on σ as

$$\int_\sigma w = \int_{\sigma(\Delta^p)} \sigma_{\#} w$$

where $\sigma_{\#} w = w_{i_1...i_p} \sigma J \mathrm{d}x_{i_1} ... \mathrm{d}x_{i_p}$ if σ is differentiable and J is the Jacobian $\det(\partial\sigma_{i_j}/\partial x_{i_k})$.

Stokes' theorem for chains

Let $w \in \Omega^{p-1}(M)$ and $\sigma \in S_p(M)$. Then

$$\int_\sigma \mathrm{d}w = \int_{\partial\sigma} w$$

Since we have defined the integration of differential forms on chains in manifold, then we can define the intergration of differential forms on the whole manifold, and obtain the following theorem:

Stokes' theorem

Let $w \in \Omega^{p-1}(M)$ and M is an orientable manifold, ∂M denotes the boundary of M. Then

$$\int_M \mathrm{d}w = \int_{\partial M} w$$

This theorem unifies Newton-Leibniz formula, Green theorem, Gauss theorem and classical Kelvin-Stokes theorem.

Now, we are able to see that de rham cohomology is isomorphic to singular cohomology.

Given $w \in \Omega^p(M)$ and $\sigma \in S_p(M)$, we can have a integration, so we can view w as a function on $S_p(M)$, inner product is just defined by

$$\langle w, \sigma \rangle = \int_\sigma \mathrm{d}w$$

We immeditaly notice that

$$\langle \mathrm{d}w, \alpha \rangle = \langle w, \partial\alpha \rangle$$

for $w \in \Omega^{p+1}(M)$ and $\alpha \in S_p(M)$ by Stokes' theorem. This is exactly consistent with the defination of the operator d in sigular cohomology. Finally, we can proof

$$H^*_{d\mathbb{R}}(M, \mathbb{R}) \cong H^*(M, \mathbb{R})$$

which is called the **de Rham theorem**.

1.4 Fundamental Homology Class

In Stokes' theorem, the manifold is orientable, but what is orientable ? We know that sphere is orientable while Mobius ring is not by intuition, but a more mathematical defination is needed. Here, we define orientable by fundamental homology class.

For a n-manifold M, $x \in M$, we have the relative homology groups

$$H_i(M, M-x, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

One can imagine this relative homology by keeping a small neighbourhood of x fixed and contract all other points to one point, so M becomes S^n, whose homology groups are $\mathbb{Z}$ when $i = n$ and 0 when $i \neq n$.

Defination: A local orientation μ_x for M at x is a choice of one of the two possible generators for $H_n(M, M-x, \mathbb{Z})$.

Defination: An orientation for M is a function which assigns to each $x \in M$ a local orientation μ_x and μ_x varies continously with x. i.e. For each x , there should exist a compact neighbuorhood N and a class $\mu_N \in H_n(M, M-N)$ so that $\rho_y(\mu_N) = \mu_y$ for each $y \in N$, where $\rho_y : H_n(M, M-N) \to H_n(M, M-y)$ is induced by $i : (M, M-N) \to (M, M-y)$. If M has such orientation, it is orientable.

If we take N to be M, then we get orientation $\mu_M \in H_n(M, \emptyset)$. For $\forall x \in M, \rho_x(\mu_M) = \mu_x$. Thus we have the following theorem:

For all orientable manifold M, $H_n(M) \cong \mathbb{Z}$, generated by μ_M, μ_M is called the **fundamental homology class**.

Now, with the fundamental homology class, we can define the integration as

$$\int_M w = \langle w, \mu_M \rangle$$

It's easy to verify this integration only depends on the cohomology class of w,

$$\langle w, \mu_M \rangle = \langle [w], \mu_M \rangle$$

2 Applications of Cohomology

2.1 Gauge Theory and Monopole

Maxwell's eletromagnetism theory can be written in the form of differential forms.

The four-potential $A_\mu = (\phi, -\vec{A})$, we write $A = A_\mu \mathrm{d}x^\mu$. The electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, can we write it as a differential form ? Yes, note

$$dA = \partial_\mu A_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

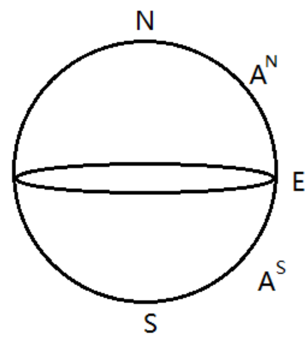
we write

$$F = dA = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

Here, we should note that A is not a 1-form in general, becuase it can not be continuously defined globally in general. For example, if there is a magnetic monopole[5] at the center, then we won't find a globally defined A on the sphere.

Fortunately, physically, we do not need A to be defined continuous, this is because A is a gauge field, A and $A + \mathrm{d}f$ represent the same physical field and we think they are equivalent to each other. So we just need the equivalent class $[A]$ to be defined continuously. For example, consider the sphere surrounding a magnetic nomopole, although we can not define a global A, we can define A^N on the north hemisphere and A^S on the south hemisphere. A^N and A^S conincide on the equator. Physically, they can different from each other by a gauge term on the equator

$$A^N|_E = A^S|_E + df|_E$$



On the other hand, F is certainly a 2-form since it is gauge invariant

$$F = dA = d(A + df)$$

Moreover, F is closed, that is $dF = 0$.[6] So $F \in Z^2(M, \mathbb{R})$ and can be classified by $H^2(M, \mathbb{R})$. The integration of F on the manifold gives a topological invarivant which called the **first Chern number**:

$$C = \frac{1}{2\pi} \int_M F \quad \in \mathbb{Z}$$

where M is an oriented compact 2-manifold. We can see why this happen by following calculation.

In the example of monopole above, the integration of F on S^2 is exactly the flux through the surface

$$\Phi = \int_{S^2} F$$

We define

$$F^N = dA^N$$

[5]The reader can read 2.6 of *Modern Quantum Mechanics* by Sakurai and 10.5 of *Geometry, Topology and Physics* by Nakahara to know the magnetic monopole.

[6]The reader can verify it.

$$F^S = dA^S$$

to be the electromagnetic field on north hemisphere D^N and southhemisphere D^S respectively. Then

$$\int_{S^2} F = \int_{D^N} F^N + \int_{D^S} F^S = \int_{D^S} dA^N + \int_{D^S} dA^S$$

By Stokes's theorem

$$\int_{S^2} F = \int_{S^1} A^N - \int_{S^1} A^S = \int_{S^1} df$$

where S^1 is the equator. If the gauge term df is arbitary, then

$$\int_{S^1} df \in \mathbb{R}$$

and we won't get a quantized value. However, A is a $U(1)$ gauge field, the gauge term should be of following form

$$df = ie^{if(\phi)}\partial_\phi e^{-if(\phi)}$$

then the integration becomes the formula of winding number

$$\int_{S^1} df = i\int_{S^1} e^{if(\phi)}\partial_\phi e^{-if(\phi)} = 2\pi v, \quad v \in \mathbb{Z}$$

So we have prooved

$$\frac{1}{2\pi}\int_{S^2} F = C \quad \in \mathbb{Z}$$

This result shows the flux and monopole are quantized.

There is a theorem: For oriented compact 2-manifold M, $H^2(M, \mathbb{Z})$ bijectives to isomorphism classes of $U(1)$ principle bundle over M.

Since we know $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ for oriented 2-manifold, we can see the $U(1)$ gauge fields on M are certainly characterized by $\mathbb{Z}$.

2.2 Chern-Simons Theory

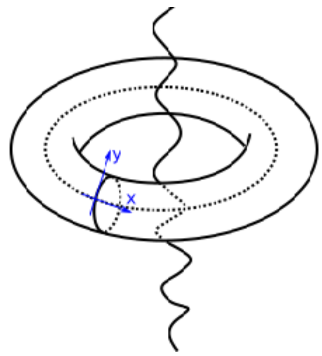
Consider a torus and a one-form field $a = a_\mu dx^\mu$ on the surface.

$$da = (\partial_x a_y - \partial_y a_x)dx \wedge dy = f_{xy} dx \wedge dy = f$$

Consider the integration of a along a circle on the torus

$$\oint_{C_i} a = \oint_{C_i} a_\mu dx^\mu = \int_{D_i} da = \int_{D_i} f = \int_{D_i} f_{xy} dx \wedge dy = \Phi_i \in \mathbb{R}$$

where Φ_i is the flux through the circle. There are two kinds of circles on T^2 and thus two kinds of fluxes. If $da = 0$, i.e. $a \in Z^1(T^2, \mathbb{R})$, then $f = 0$ at every point $x \in T^2$, and $\Phi_i = 0$, there is no flux. If a is also a gauge field, $a \sim a + d\alpha$, we find the guage equivalence is just the cohomology equivalence, the gauge eqivalence classes $[a] \in H^1(T^2, \mathbb{R})$.



Chern-Simons theory

The Lagrangian of Chern-Simons theory is

$$S = \frac{k}{4\pi}\int_M \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda d^3x = \frac{k}{4\pi}\int_M ada, \quad a = a_\mu dx^\mu$$

It is a new type gauge theory which compeletly different from Maxwell theory in 2+1 dimensions.
The classical solution of Chern-Simons theory can be obtained by $\delta S = 0$:

$$\delta S = \frac{k}{4\pi}\int (\delta a da + ad(\delta a))$$

by $ad(\delta a) = a\delta(da) = d(a\delta a) + da\delta a$,[7]

$$\delta S = \frac{k}{4\pi}\int (\delta a da + da\delta a + d(a\delta a)) = \frac{k}{2\pi}\int \delta a da + \frac{k}{4\pi}\int_S a\delta a$$

If the boundary term vanishes,

$$\delta S = \frac{k}{2\pi}\int \delta a da = 0 \Rightarrow da = 0$$

This equation of motion seems too trival, but there are many ways to make Chern-Simons theory interesting.

Since $da = 0$, the gauge equalence is just the cohomology equalence, and a can be classified by $H^1(M, \mathbb{R})$.

Quantization of Chern-Simons theory

$$[a_x(r), a_y(r')] = \frac{2\pi i}{k}\delta^2(r - r')$$

3 Universal Coefficients Theorems for Cohomology

What's the relation between $H^*(X, G)$ and $H_*(X)$? There is a theorem.

[7]The sign before $da\delta a$ is positive because for extorior derivative, $d(xy) = ydx - xdy$.

Theorem:(Dual Universal Coefficients)

$$H^p(X,G) \cong \mathrm{Hom}(H_p(X),G) \oplus \mathrm{Ext}(H_{p-1}(X),G)$$

Special cases: If G is divisible (Such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}, U(1)$)

$$H^p(X,G) \cong \mathrm{Hom}(H_p(X),G)$$

What's the meaning of $\mathrm{Ext}(A,G)$? There is a free resolution of over $\mathbb{Z}$

$$0 \longmapsto R \overset{i}{\longmapsto} F \overset{p}{\longmapsto} A \longmapsto 0, \qquad F/R = A$$

The functor $\mathrm{Hom}(\_\_,G)$ is applied to the resolution, we get

$$\begin{array}{ccccccc} 0 \longmapsto & \mathrm{Hom}(A,G) & \overset{p^\#}{\longmapsto} & \mathrm{Hom}(F,G) & \overset{i^\#}{\longmapsto} & \mathrm{Hom}(R,G) & \longmapsto 0 \\ & Exact & & Exact & & Not\ exact & \end{array}$$

Define

$$\mathrm{Ext}(A,G) = \textbf{coker}\ i^\# = \frac{\mathrm{Hom}(R,G)}{i^\#\mathrm{Hom}(F,G)}$$

There are some useful formulae

$$\begin{aligned} &\mathrm{Ext}(\sum A_j, G) \cong \prod \mathrm{Ext}(A_j, G) \\ &\mathrm{Ext}(A, \prod G_j) \cong \prod \mathrm{Ext}(A, G_j) \\ &\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG \\ &\mathrm{Ext}(F,D) = 0 \quad \text{F is free Abelian, D is divisible} \end{aligned}$$

Example:

$$\begin{aligned} &\mathrm{Ext}(\mathbb{Z}_N, \mathbb{Z}) \cong \mathbb{Z}_N \\ &H^2(RP^2,\mathbb{Z}) \cong \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) \cong \mathbb{Z}_2 \\ &H^1(RP^2,\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) \cong 0 \\ &H^0(RP^2,\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}(0,\mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$