# COMPLETE QUADRICS: SCHUBERT CALCULUS FOR GAUSSIAN MODELS AND SEMIDEFINITE PROGRAMMING

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Abstract. We establish connections among: the maximum likelihood degree for linear concentrations models, the degree of semidefinite programming and Schubert calculus for complete quadrics. We prove a conjecture by Sturmfels and Uhler about polynomiality of ML-degree. We also prove a conjecture by Nie, Ranestad and Sturmfels providing a formula for the degree of SDP. The interactions among the three fields shed new light on asymptotic behaviour of enumerative invariants for the varieties of complete quadrics.

## 1. Introduction

**Maximum likelihood degree and quadrics.** Although this paper is mainly about enumerative geometry and symmetric functions, the main motivations come from algebraic statistics and multivariate Gaussian models. These are generalizations of the well-known Gaussian distributions to higher dimensions. In the one dimensional case, in order to determine a Gaussian distribution on $\mathbb{R}$, one needs to specify its mean $\mu \in \mathbb{R}$ and its variance $\Sigma \in \mathbb{R}_{>0}$. In the $n$-dimensional case, the mean is a vector $\mu \in \mathbb{R}^n$, and the second parameter is a positive-definite $n \times n$ covariance matrix $\Sigma$. The corresponding Gaussian distribution on $\mathbb{R}^n$ is given by

$$f_{\mu,\Sigma}(x) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)},$$

where $^T$ denotes the transpose. Equivalently to determining it by $\mu$ and $\Sigma$, one may represent the distribution by $\mu$ and the *concentration matrix* $K := \Sigma^{-1}$, which is also positive definite. Our primary interest lies in *linear concentration models*, i.e. statistical models which assume that $K$ belongs to a fixed $d$-dimensional space $\mathcal{L}$ of $n \times n$ symmetric matrices. These were introduced by Anderson half a century ago [1]. In particular, this means that $\Sigma$ should belong to the set $\mathcal{L}^{-1}$ of inverses of matrices from $\mathcal{L}$.

In statistics, typically one gathers data as sample vectors $x_1, \dots, x_s \in \mathbb{R}^n$. This allows to estimate the mean $\mu$ as the mean of the $x_i$'s. Furthermore, each $x_i$ provides a matrix $\Sigma_i := (x_i - \mu)(x_i - \mu)^T$. Next one considers the *sample covariance matrix* $S$, that is the mean of the $\Sigma_i$'s. Note that in most situations, it is not true that $S \in \mathcal{L}^{-1}$. The aim is then to find $\Sigma$ that best explains the observations. From the point of view of statistics, it is natural to maximize the likelihood function

$$f_{\mu,\Sigma}(x_1) \cdots f_{\mu,\Sigma}(x_s),$$

that is, to find a positive definite matrix $\Sigma \in \mathcal{L}^{-1}$ for which the above value is maximal. Classical theorems in statistics assert that the solution to this optimization problem is essentially geometric [4, Theorem 3.6, Theorem 5.5], [19, Theorem 4.4]. Namely, under mild genericity assumptions, the optimal $\Sigma$ is the unique positive definite matrix in $\mathcal{L}^{-1}$ that maps to the same point as $S$ under projection from $\mathcal{L}^{\perp}$.

This is one of the main reasons why the variety that is the Zariski closure of $\mathcal{L}^{-1}$ (which abusing notation we also denote by $\mathcal{L}^{-1}$) and the rational map $\pi$ defined as the projection

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from $\mathcal{L}^\perp$ are intensively studied in algebraic statistics. Note that for generic $\mathcal{L}$, and after projectivization, $\pi$ becomes a finite map. The following is the central definition of the article.

**Definition 1.1** (ML-degree)**.** The *ML-degree* of a linear concentration model represented by a space $\mathcal{L}$ is the degree of the projection from the space $\mathcal{L}^\perp$ restricted to the variety $\mathcal{L}^{-1}$.

The ML-degree is the basic measure of the complexity of the model. When $\mathcal{L}$ is a generic space, the ML-degree only depends on the size $n$ of the symmetric matrices and on the (affine) dimension $d$ of $\mathcal{L}$. By a theorem of Teissier [30, 31] (cf. [17, Corollary 2.6]) or Sturmfels and Uhler [29, Theorem 1], the ML-degree equals the degree of the variety $\mathcal{L}^{-1}$. Following Sturmfels and Uhler [29] we denote it by $\phi(n,d)$. We refer algebraists interested in statistics to [9] for more information about the subject.

**Definition 1.2.** For $n \in \mathbb{Z}_{>0}$ and $1 \leq d \leq \binom{n+1}{2}$, we define $\phi(n,d)$ to be the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L}$ is a general $d$-dimensional linear subspace of $S^2\mathbb{C}^n$.

Thus, our main result concerns a very basic algebro-geometric object: the degree of the variety obtained by inverting all symmetric matrices in a general linear space. In Section 4, we confirm the following conjecture of Sturmfels and Uhler [29, p. 611], [17, Conjecture 2.8]:

**Theorem 1.3.** *For any fixed positive integer $d$, the ML-degree $\phi(n,d)$ is polynomial in $n$.*

Astonishingly, it appears that the numbers $\phi(n,d)$ were studied for the last 150 years! In 1879 Schubert presented his fundamental results on quadrics satisfying various tangency conditions [23]. His contributions shaped the field of enumerative geometry, inspiring many mathematicians for centuries to come. A nondegenerate quadric being given, the set of its tangent hyperplanes (its projective dual, in modern language) is nothing else than the inverse quadric. This implies that $\phi(n,d)$ is also the solution to the following enumerative problem:

> *What is the number of nondegenerate quadrics in $n$ variables, passing through $\binom{n+1}{2} - d$ general points and tangent to $d-1$ general hyperplanes?*

In modern language, such problems can be solved by performing computations in the cohomology ring of the *variety of complete quadrics*. This is now a classical topic with many beautiful results [24, 25, 33, 34, 7, 8, 13, 6, 32, 15]. In particular, the cohomology ring has been described by generators and relations, and algorithms have been devised that allow to compute any given intersection number. But this only applies for $n$ fixed. Algebraic statistics suggested to change the perspective and to fix $d$ instead of $n$. This explains, in a way, why the polynomiality property of $\phi(n,d)$ is only proved now.

**Semidefinite programming and projective duality.** The second domain of mathematics that inspired our research is semidefinite programming (SDP), a very important and effective subject in optimization theory. The goal is to study linear optimization problems over spectrahedra. This subject is a direct generalization of linear programming, that is optimization of linear functions over polyhedra. For a short introduction to the topic we refer to [18, Chapter 12].

The coordinates of the optimal solution for an SDP problem, defined over rational numbers, are algebraic numbers. Their algebraic degree is governed by *the algebraic degree of semidefinite programming*. For more information we refer to the fundamental article [20]. To stress the importance of this degree let us just quote this paper:

> *"The algebraic degree of semidefinite programming addresses the computational complexity at a fundamental level. To solve the semidefinite programming exactly essentially reduces to solve a class of univariate polynomial equations whose degrees are the algebraic degree."*

Let us provide a precise definition of the algebraic degree of SDP, in the language of algebraic geometry, without referring to optimization. (However, the fact that this definition is correct is actually a nontrivial result [20, Theorem 13].)

**Definition 1.4.** For $0 < m < \binom{n+1}{2}$ and $0 < r < n$, let $\mathcal{L} \subset S^2\mathbb{C}^n$ be a general linear space of symmetric matrices, of (affine) dimension $m+1$, and let $D^r_{\mathcal{L}} \subset \mathbb{P}(S^2\mathbb{C}^n)$ denote the projectivization of the cone of matrices of rank at most $r$ in $\mathcal{L}$. The *algebraic degree of semidefinite programming* $\delta(m,n,r)$ is the degree of the projective dual $(D^r_{\mathcal{L}})^*$ of $D^r_{\mathcal{L}}$ if this dual is a hypersurface (and zero otherwise).

Projective duality is a very classical topic, to which a huge literature has been devoted. Computing the degree of a dual variety is well-known to be very hard, especially when the variety in question is singular, which is almost always the case of our $D^r_{\mathcal{L}}$. Nevertheless, Ranestad and Bothmer [11] suggested to use conormal varieties, and managed to obtain an algebraic expression of $\delta(m,n,r)$ in terms of what we call the *Lascoux coefficients*. These are integer coefficients that govern the Segre classes of the symmetric square of a given vector bundle; algebraically, they are defined by the formal identity

$$\prod_{1\le i\le j\le s} \frac{1}{1-(x_i+x_j)} = \sum_I \psi_I s_{\lambda(I)}(x_1,\dots,x_s),$$

where the sum is over the increasing sets $I = (i_1 < i_2 < \cdots < i_s)$ of nonnegative integers, $\lambda(I) = (i_s - s + 1, \dots, i_2 - 1, i_1)$ is the associated partition, and $s_{\lambda(I)}(x_1,\dots,x_s)$ the corresponding Schur function in the variables $x_1,\dots,x_s$. These coefficients were introduced and studied in [13], whose influence on our work cannot be underestimated. Diving into the combinatorics of those coefficients, in Section 5 we confirm [20, Conjecture 21], providing an explicit formula for $\delta(m,n,r)$.

**Theorem 1.5.** *(NRS, Conjecture 21) Let $m,n,s$ be positive integers. Then*

$$\delta(m,n,n-s) = \sum_{\sum I \le m-s} (-1)^{m-s-\sum I} \psi_I b_I(n) \binom{m-1}{m-s-\sum I}$$

*where the sum goes trough all sets of nonnegative integers of cardinality $s$.*

In this formula, $\Sigma I = i_1 + \cdots + i_s$, and $b_I(n)$ is a polynomial function of $n$ defined inductively in Section 5. Actually, $b_I(n)$ is obtained by evaluating a Q-Schur polynomial on $n$ identical variables; by the work of Stembridge [27], it counts certain shifted tableaux of shape determined by $I$, numbered by integers not greater than $n$.

This also implies an explicit polynomial formula for the ML-degree, since elementary relations in the cohomology ring of the variety of complete quadrics imply the fundamental identity

$$\phi(n,d) = \sum_s s\delta(m,n,n-s).$$

So far the exact formula for $\phi(\cdot,d)$ was only known for $d \le 5$ [5, 28, 29, 17]. We compute it explicitely for $d \le 50$, confirming in particular [17, Conjecture 5.1].

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## 2. Notation and preliminaries

**Definition 2.1.** For a set of nonnegative integers $I = \{i_1,\dots,i_r\}$, we assume $i_r > i_{r-1} > \cdots > i_1$ and we define the corresponding partition

$$\lambda(I) := (i_r - (r-1), i_{r-1} - (r-2), \dots, i_2 - 1, i_1).$$

Analogously, for a partition $\lambda = (\lambda_1,\dots,\lambda_r)$, which also can end with zeros, we define the corresponding set

$$I(\lambda) := \{\lambda_r, \lambda_{r-1}+1, \dots, \lambda_2 + r - 2, \lambda_1 + r - 1\}.$$

We will abbreviate $\{0,\ldots,n-1\}$ to $[n]$. Let $\sum I := i_1+\cdots+i_r$ denote the sum of elements of $I$ and $|I|=r$ its cardinality. For two sets $I=\{i_1,\ldots,i_r\}$ and $J=\{j_1,\ldots,j_r\}$ we say that $I\le J$ if $i_k\le j_k$ for all $1\le k\le r$.

**Definition 2.2.** For a partition $\lambda$ we denote by $s_\lambda$ the corresponding Schur polynomial.

**Definition 2.3.** Let $I,J$ be two sets of non-negative integers of cardinality $r$. We define numbers $s_{I,J}$ to be the unique integers which satisfy the polynomial equation

$$s_{\lambda(I)}(x_1+1,\ldots,x_r+1)=\sum_{J\le I} s_{I,J}s_{\lambda(J)}(x_1,\ldots,x_r)$$

**Definition 2.4.** We define the *Lascoux coefficients* $\psi_I$ by the following formula:

$$H_d(\{x_i+x_j \mid 1\le i\le j\le k\})=\sum_{\substack{\lambda(I)\,\vdash\, d\\ |I|\,=\,k}}\psi_I s_{\lambda(I)}(x_1,\ldots,x_k),$$

Here $H_d$ is a complete symmetric polynomial of degree $d$, in the $\binom{n+1}{2}$ variables $x_i+x_j$. Hence, the coefficients $\psi_I$ appear in the expansion of the complete symmetric polynomial evaluated at sums of variables in the Schur basis.

Equivalently, the Lascoux coefficients appear in the expansion of the $d$-th Segre class of the second symmetric power of the universal bundle $\mathcal{U}$ over a Grassmannian $G(k,n)$ for $n\ge k+d$ *:

$$Seg_d(S^2\mathcal{U})=\sum_{\substack{\lambda(I)\,\vdash\, d\\ |I|\,=\,k}}\psi_I\sigma_{\lambda(I)},$$

where $\sigma_\lambda$ denote the Schubert classes in the Chow ring of the Grassmannian.

*Example* 2.5. Let us consider $k=2$ and $n=4$, i.e. the Grassmannian $G(2,4)$. The rank two universal vector bundle $\mathcal{U}$ has two Chern roots $x_1,x_2$. Recall that the cohomology ring of $G(2,4)$ is six-dimensional with basis corresponding to Young diagrams contained in the $2\times 2$ square. We have formal equalities:

$$x_1+x_2=-\ydiagram{1},\quad x_1\cdot x_2=\ydiagram{1,1}.$$

The Chern roots of $S^2\mathcal{U}$ are $2x_1,x_1+x_2,2x_2$. Computing the elementary symmetric polynomials in those we obtain the three respective Chern classes:

$$-3\ydiagram{1},\quad 2\ydiagram{2}+6\ydiagram{1,1},\quad -4\ydiagram{2,1}.$$

By inverting the Chern polynomial we obtain the Segre classes:

$$3\ydiagram{1},\quad \mathbf{7}\ydiagram{2}+\mathit{3}\ydiagram{1,1},\quad 10\ydiagram{2,1},\quad 3\ydiagram{2,2}.$$

Their coefficients are the Lascoux coefficients, precisely:

$$\psi_{0,2}=3,\psi_{0,3}=\mathbf{7},\psi_{1,2}=\mathit{3},\psi_{2,3}=10,\psi_{3,4}=3.$$

We use boldface and emphasis above and below to indicate the same numbers. We may also compute them by expanding complete symmetric polynomials, where now $x_1,x_2$ are simply formal variables.

$$H_2(2x_1,x_1+x_2,2x_2)=7x_1^2+7x_2^2+10x_1x_2=$$
$$=7(x_1^2+x_1x_2+x_2^2)+3x_1x_2=\mathbf{7}s_{2,0}(x_1,x_2)+\mathit{3}s_{1,1}(x_1,x_2).$$

*if $k\le n<k+d$ the identity is still true, but some of the Schubert classes $\sigma_{\lambda(I)}$ will be zero.

We note that Lascoux coefficients appear in many publications with different notation. For example one needs to be careful with the shift: $\psi_{\{j_1,\dots,j_s\}}$ as defined above equals $\psi_{\{j_1+1,\dots,j_s+1\}}$ in [11]. On the other hand our notation is consistent with [20].

For later reference, we recall the description of $\delta(m,n,r)$ in terms of the bidegree of a conormal variety:

**Theorem 2.6** ([20, Theorem 10])**.** *Let $Z_r \subseteq \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ be the conormal variety to the variety $D^r \subseteq \mathbb{P}(S^2V)$ of matrices of rank at most $r$. Explicitely, $Z_r$ consisit of pairs of symmetric matrices $(X,Y)$ with $\operatorname{rk} X \leq r$, $\operatorname{rk} Y \leq n-r$, and $X \cdot Y = 0$. Then the multidegree of $Z_r$ is given by*

$$[Z_r] = \sum_m \delta(m,n,r) H_1^m H_2^{\binom{n+1}{2}-m}$$

*Remark* 2.7. For our polynomiality results in Section 4, it will be useful to extend the definitions of $\phi$ and $\delta$:

- For $d > \binom{n+1}{2}$, we put $\phi(n,d) = 0$.
- For $m \geq \binom{n+1}{2}$ or $s \geq n$, we put $\delta(m,n,n-s) = 0$, with one exception: in the case $m = \binom{n+1}{2}$ and $s = n$, we define $\delta(m,n,n-s) = 1$. See also Remark 3.10.

Now $\phi(n,d)$ is defined for all $n, d > 0$, and $\delta(m,n,n-s)$ is defined for all $m,n,s > 0$.

## 3. Formulas for ML-degree via complete quadrics

3.A. **Space of complete quadrics.** Let $V$ be a vector space over $\mathbb{C}$. The space of complete quadrics $\Phi(V)$ is a particular compactification of the space of smooth quadrics in $\mathbb{P}(V)$, or equivalently, of the space of invertible symmetric matrices $\mathbb{P}(S^2(V))^\circ \subset \mathbb{P}(S^2(V))$. The space of complete quadrics $\Phi(V)$ has several equivalent descriptions, below we will describe some of them. For more information we refer the reader to [13, 32, 15].

For $A \in S^2(V)$ let $\bigwedge^k A \in S^2(\bigwedge^k V)$ be the corresponding operator on $\bigwedge^k V$. If we view $A$ as a symmetric matrix, then $\bigwedge^k A$ is the matrix of $k \times k$ minors of $A$. In particular, $\bigwedge^{n-1} A$ is the inverse of $A$ up to scaling.

**Definition 3.1.** The space of complete quadrics $\Phi(V)$ is the closure of $\phi(\mathbb{P}(S^2(V))^\circ)$, where

$$\phi : \mathbb{P}(S^2(V))^\circ \to \mathbb{P}(S^2(V)) \times \mathbb{P}\left(S^2\left(V \wedge V\right)\right) \times \ldots \times \mathbb{P}\left(S^2\left(\bigwedge^{n-1} V\right)\right)$$

is given by

$$A \mapsto \left(A, \bigwedge^2 A, \ldots, \bigwedge^{n-1} A\right).$$

The natural projection $\pi_j : \prod_{i=1}^{n-1} \mathbb{P}\left(S^2\left(\bigwedge^i V\right)\right) \to \mathbb{P}\left(S^2\left(\bigwedge^j V\right)\right)$ induces a map $\pi_j : \Phi(V) \to \mathbb{P}\left(S^2\left(\bigwedge^j V\right)\right)$. The map $\pi_1 : \Phi(V) \to \mathbb{P}(S^2(V))$ is an isomorphism on $\phi(\mathbb{P}(S^2(V))^\circ)$ and is a sequence of blow-downs. This provides the second description of the space of complete quadrics.

**Definition 3.2.** The space of complete quadrics $\Phi(V)$ is the successive blow-up of $\mathbb{P}(S^2(V))$:

$$\Phi(V) = Bl_{\widetilde{D}^{n-1}} Bl_{\widetilde{D}^{n-2}} \ldots Bl_{D^1} \mathbb{P}(S^2(V)),$$

where $\widetilde{D}^i$ is the proper transform of the space of rank $\leq i$ symmetric matrices under previous blow-ups.

The space of invertible symmetric matrices is a spherical homogeneous space:

$$\mathbb{P}(S^2(V))^\circ \simeq \mathrm{SL}_n / N(\mathrm{SO}_n),$$

where $N(\mathrm{SO}_n)$ is the normalizer of $\mathrm{SO}_n$. Moreover, the space of complete quadrics $\Phi(V)$ is an equivariant compactification of $\mathbb{P}(S^2(V))^\circ$. Similar to the case of toric varieties, equivariant partial compactifications (or embeddings) of spherical homogeneous spaces are described in terms of combinatorial objects; colored fans (see [21] for introduction to spherical geometry). This leads to the third definition of $\Phi(V)$.

**Definition 3.3.** The space of complete quadrics $\Phi(V)$ is the toroidal spherical embedding of $\mathrm{SL}_n / N(\mathrm{SO}_n)$ given by the colored fan which consists only of the valuation cone $\mathcal{V}$.

The last description of $\Phi(V)$ is based on [32, 17]. Let $V^*$ be the vector space dual to $V$. The space $V \oplus V^*$ has a natural symplectic form $\omega$. Let us define a $\mathbb{C}^*$ action on $V \oplus V^*$ via

$$t \cdot (v, w) = (tv, t^{-1}w).$$

This action preserves $\omega$ and hence descends to an action on the Lagrangian Grassmannian $LG(V \oplus V^*)$. Since the multiplication by $-1$ does not act on $LG$ the above action factors through $\mathbb{C}^* \to \mathbb{C}^*$ given by $t \mapsto t^2$, we will consider the corresponding effective action.

A $\mathbb{C}^*$ action on $LG$ lifts to an action on the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(LG, \beta)$ of stable maps.

**Definition 3.4.** Let $\beta \in H^2(LG, \mathbb{Z})$ be the class of the closure of a generic $\mathbb{C}^*$ orbit. Then the space of complete quadrics $\Phi(V)$ is the connected component of the $\mathbb{C}^*$ fixed locus of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(LG, \beta)$ which contains a generic orbit.

**Theorem 3.5.** *Definitions 3.1, 3.2, 3.3, and 3.4 of the space of complete quadrics are equivalent.*

*Proof.* One can check that the spaces defined in Definitions 3.1, 3.2, 3.3, and 3.4 are equivariant compactifications of the space of smooth quadrics which has $n-1$ simple normal crossings boundary divisors with simplex as the dual complex. The space of complete quadrics is uniquely determined by these properties. ∎

The space of complete quadrics has two series of special classes of divisors $S_1, \ldots, S_{n-1}$ and $L_1, \ldots, L_{n-1}$. Below we give several descriptions of $S_i$'s and $L_i$'s.

Divisors $S_i$ consist of tuples $(A_1, \ldots, A_{n-1}) \in \Phi(V) \subset \prod_{i=1}^{n-1} \mathbb{P}\left(S^2\left(\bigwedge^i V\right)\right)$, where $A_i$ has rank 1. This is equal to the closure in $\Phi(V)$ of the tuples $(A_1, \ldots, A_{n-1})$ where $A_1$ has rank $i$. Equivalently $S_i$ is the the $i$-th exceptional divisor of $Bl_{\widetilde{D}^{n-1}} Bl_{\widetilde{D}^{n-2}} \ldots Bl_{D^1} \mathbb{P}(S^2(V))$. Also, $S_1, \ldots, S_{n-1}$ are precisely the $\mathrm{SL}_n$-invariant prime divisors on $\Phi(V)$.

Divisors $L_i$ can be obtained as pullbacks of a hyperplane under $\pi_i : \Phi(V) \to \mathbb{P}\left(S^2\left(\bigwedge^i V\right)\right)$. The classes $S_1, \ldots, S_{n-1}$ and $L_1, \ldots, L_{n-1}$ are not independent in $\mathrm{Pic}(\Phi(V))$.

**Proposition 3.6.** *Classes $L_1, \ldots, L_{n-1}$ are independent and generate* $\mathrm{Pic}(\Phi(V))$*, the classes $S_1, \ldots, S_{n-1}$ generate an index $n$ sublattice of* $\mathrm{Pic}(\Phi(V))$*. Moreover there are the following relations between $L_i$'s and $S_i$'s:*

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

*where $L_0 = L_n = 0$.*

*Proof.* See for example [15, Proposition 3.6 and Theorem 3.13]. ∎

The inverse to the relations in Proposition 3.6 are given by the $(n-1) \times (n-1)$ matrix:

$$(M)_{i,j} = \min(i, j) - \frac{ij}{n},$$

in particular we have:

$$nL_1 = (n-1)S_1 + (n-2)S_2 + \ldots + S_{n-1},$$

$$nL_{n-1} = S_1 + 2S_2 + \ldots + (n-1)S_{n-1}. \tag{3.1}$$

Now we are ready to relate the computation of *ML*-degree and the algebraic degree of semidefinite programming to the intersection theory of $\Phi(V)$.

**Proposition 3.7.**

$$\phi(n,d) = L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1}$$

$$\delta(m,n,r) = S_r L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}$$

*Proof.* For $\phi$: since the morphisms $\pi_1$ and $\pi_{n-1}$ resolve the inversion map $\mathbb{P}(S^2V) \dashrightarrow \mathbb{P}(S^2V^*)$, we can compute the degree of $\mathcal{L}^{-1}$, for $\mathcal{L} \subseteq \mathbb{P}(S^2V^*)$ a general $d-1$-dimensional linear subspace, as $\pi_1^*(H_1^{\binom{n+1}{2}-d})\pi_{n-1}^*(H_{n-1}^{d-1})$, where $H_1$ and $H_{n-1}$ are hyperplane classes in $\mathbb{P}(S^2V)$ and $\mathbb{P}(S^2V^*)$ respectively.

For $\delta$: this follow from Theorem 2.6. ■

The bounds on $m$ when either $S_r L_1^{\binom{n+1}{2}-m-1} = 0$ or $S_r L_{n-1}^{m-1} = 0$ are known as *Pataki inequalities* [20, Proposition 5]. From our perspective they can be proved by looking when a general $m$ dimensional space $\mathcal{L}$, (resp. $\mathcal{L}^{\perp}$) intersects the locus of rank (resp. corank) $r$ matrices. The following inequalities, known as the *Pataki inequalities* are necessary and sufficient [20, Proposition 5 and Theorem 7] for $\delta(m,n,r) \neq 0$:

$$\binom{n-r+1}{2} \leq m \leq \binom{n+1}{2} - \binom{r+1}{2}. \tag{3.2}$$

Note that we can use (3.1) to write the ML-degree in terms of the SDP-degree, when $1 \leq d < \binom{n+1}{2}$:

$$\begin{aligned} \phi(n,d) &= L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1} \\ &= \frac{1}{n} L_1^{\binom{n+1}{2}-d-1} L_{n-1}^{d-1} \sum_{r=1}^{n-1} r S_{n-r} \\ &= \frac{1}{n} \sum_{r=1}^{n-1} r\delta(d,n,n-r). \end{aligned} \tag{3.3}$$

3.B. **Intersection theory.** In this subsection we will relate the computation of *ML*-degree and the algebraic degree of semidefinite programming to the intersection theory of the Grassmannian.

**Theorem 3.8** ([11][Theorem 1.1])**.** *For $0 < m < \binom{n+1}{2}$ and $0 < r < n$,*

$$\delta(m,n,r) = \sum_{\substack{I \subset [n] \\ |I| = n-r \\ \sum I = m-n+r}} \psi_I \psi_{[n]\setminus I}$$

*Proof idea.* Computing the intersection product $S_r L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}$ on $\Phi(V)$ is equivalent to computing the intersection product $L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}$ on $S_r$. Via the natural map $S_r \to Gr(r,V)$ this computation can be pushed forward to the Grassmannian $Gr(r,V)$, leading to the formula

$$\delta_{m,n,r} = \operatorname{Seg}_{(\binom{n+1}{2}-m-\binom{r+1}{2})}(S^2\mathcal{U}) \operatorname{Seg}_{(m-\binom{n-r+1}{2})}(S^2\mathcal{Q}^*).$$

We then obtain the theorem by expanding these Segre classes. ■

*Remark* 3.9. Recall that our definition of $\psi_I$ is shifted w.r.t. [11], which explains why our formula looks slightly different.

*Remark* 3.10. One can easily verify that the above formula is still true for the extended definition of $\delta$ from 2.7. The only nontrivial case is $\delta(\binom{n+1}{2}, n, 0) = \psi_{[n]}\psi_{[n]\setminus[n]} = 1$.

3.C. **Representation theory.** In this subsection we will deduce a formula which expresses the ML-degree as a linear combination of dimensions of irreducible representations of $\mathrm{SL}_n$. Our construction is based on the following lemma.

**Lemma 3.11.** *Let $X$ be a smooth complete algebraic variety with $\dim X = d$, and let $D_1, D_2$ be two divisors on $X$. Then the following identity holds:*

$$D_1^i D_2^{d-i} = \chi\left((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{d-i}\right).$$

*Proof.* By the additivity of Chern character we have:

$$\mathrm{ch}(1-\mathcal{O}(-D_i)) = \sum_{k\geq 1}(-1)^{k+1}\frac{D_i^k}{k!},$$

so we have

$$\mathrm{ch}\big((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{d-i}\big) = \left(\sum_{k\geq 1}(-1)^{k+1}\frac{D_1^k}{k!}\right)^i\left(\sum_{k\geq 1}(-1)^{k+1}\frac{D_2^k}{k!}\right)^{d-i} = D_1^i D_2^{d-i}.$$

Finally by Riemann-Roch theorem we get

$$\chi\left((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{d-i}\right) = \int_X D_1^i D_2^{d-i}\,\mathrm{td}(X) = D_1^i D_2^{d-i}.$$

∎

We are going to apply Lemma 3.11 for the computation of ML-degree $\phi(n, a+1) = L_1^{d-a}L_{n-1}^a$. For the rest of this subsection, let us denote the dimension of the space of complete quadrics by $d$, i.e. $d = \binom{n+1}{2} - 1 = \dim(\Phi(V))$. We will need the following theorem of Brion.

**Theorem 3.12** ([3])**.** *Let $L$ be a globally generated line bundle on a complete spherical variety $X$. Then $H^i(X, K_X \otimes L) = 0$ for any $i \neq \dim X - \kappa(L)$. Equivalently, by Serre duality, $H^j(X, L^*) = 0$ for $j \neq \kappa(L)$.*

Let us denote by $\mathcal{L}_i$ the line bundle on $\Phi(V)$ corresponding to the divisor $L_i$.

**Corollary 3.13.** *The following identity holds:*

$$\phi(n, a+1) = \sum_{\substack{0\leq i\leq d-a\\ 0\leq j\leq a}} (-1)^{i+j+d}\binom{d-a}{i}\binom{a}{j} h^0\left(\mathcal{L}_1^{i+1}\otimes\mathcal{L}_{n-1}^{j+1}\bigotimes_{i=1}^{n-1}\mathcal{L}_i\right),$$

*where for a line bundle $\mathcal{L}$ on $\Phi(V)$, $h^0(\mathcal{L}) = \dim H^0(\Phi(V), \mathcal{L})$ is the dimension of the space of sections of $\mathcal{L}$.*

*Proof.* First by Lemma 3.11 we have:

$$\phi(n, a+1) = \chi((1-\mathcal{L}_1^{-1})^i(1-\mathcal{L}_{n-1}^{-1})^{d-i}) = \sum_{\substack{0\leq i\leq d-a\\ 0\leq j\leq a}} (-1)^{i+j}\binom{d-a}{i}\binom{a}{j}\chi(\mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}).$$

Now, since both $\mathcal{L}_1$ and $\mathcal{L}_{n-1}$ are globally generated and $\kappa(\mathcal{L}_1) = \kappa(\mathcal{L}_{n-1}) = d$, by Theorem 3.12 we have

$$\chi(\mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}) = (-1)^d h^d(\Phi(V), \mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}) = (-1)^d h^0(\Phi(V), K_{\Phi(V)}\otimes\mathcal{L}_1^{i}\otimes\mathcal{L}_{n-1}^{j}).$$

Finally, the canonical divisor $K_\Phi$ of the space of complete quadrics is given by $K_{\Phi(V)} = \sum_{i=1}^{n-1}(L_i + S_i) = L_1 + L_{n-1} + \sum_{i=1}^{n-1} L_i$, so the corollary holds. ∎

Line bundles $\mathcal{L}_1^{i+1}\otimes\mathcal{L}_{n-1}^{j+1}\bigotimes_{i=1}^{n-1}\mathcal{L}_i$ from Corollary 3.13 are $\mathrm{SL}_n$-equivariant, hence the space of sections $H^0(\mathcal{L}_1^{i+1}\otimes\mathcal{L}_{n-1}^{j+1}\bigotimes_{i=1}^{n-1}\mathcal{L}_i)$ is a representation of $\mathrm{SL}_n$ for any $i, j$. The decomposition of the space of sections of equivariant line bundles into irreducible representations was obtained by De Concini and Procesi.

**Theorem 3.14.** *Let $\lambda \in \Gamma$, then $H^0(\Phi, L_\lambda) \neq 0$ if and only if $\lambda = \gamma + \sum 2t_i(\alpha_i)$ for some dominant $\gamma$ and $t_i \in \mathbb{Z}_+$. In this case*

$$H^0(\Phi, L_\lambda) = \bigoplus_{\gamma = \lambda - \sum 2t_i\alpha_i} V_\gamma^*,$$

*where the sum is taken over dominant $\gamma$ and $t_i \in \mathbb{Z}_+$.*

3.D. **Pfaffian formulas.**

**Lemma 3.15** ([12, (A.15.4)])**.** *Let $I = \{i_1, \dots, i_r\}$ be a set of nonnegative integers. Then*

$$\psi_I = \mathrm{Pf}(\psi_{\{i_k,i_l\}})_{0<k<l\leq n} \text{ for even } |I|,$$
$$\psi_I = \mathrm{Pf}(\psi_{\{i_k,i_l\}})_{0\leq k<l\leq n} \text{ for odd } |I|,$$

*where $\psi_{\{i_0,i_k\}} := \psi_{\{i_k\}}$.*

Let us recall two statements from linear algebra which will allow us to prove Pfaffian formula also for the set complements.

For an $n \times n$ matrix $A$ and sets $I, J \subset \{0, 1, \dots, n-1\}$ we denote by $A_{I,J}$ the $|I| \times |J|$ matrix which is obtained from $A$ by taking rows indexed by $I$ and columns indexed by $J$. Here we index rows and columns from 0. In the case $I = J$ we write simply $A_{I,I} = A_I$.

**Lemma 3.16.** *(Jacobi's Theorem) Let $A$ be an $n \times n$ matrix. Then*

$$\det(A_{[n]\setminus I,[n]\setminus J}) = \det(A^C_{I,J}) \det(A)^{|I|-1}$$

*for all sets $I, J \subset \{0, 1, \dots, n-1\}$ with $|I| = |J|$.*

**Corollary 3.17.** *Let $A$ be an $n\times n$ skew-symmetric matrix and let $A^C$ be its cofactor matrix. Then $a^C_{ij} = \mathrm{Pf}(A_{[n]\setminus\{i,j\}}) \mathrm{Pf}(A)$.*

**Lemma 3.18.** *Let $I = \{i_1, \dots, i_r\}$ be a set of nonnegative integers. Then*

$$\psi_{[n]\setminus I} = \mathrm{Pf}(\psi_{[n]\setminus\{i_k,i_l\}})_{0<k<l\leq r} \text{ for even } |I|,$$
$$\psi_{[n]\setminus I} = \mathrm{Pf}(\psi_{[n]\setminus\{i_k,i_l\}})_{0\leq k<l\leq r} \text{ for odd } |I|,$$

*where $\psi_{[n]\setminus\{i_0,i_k\}} := \psi_{[n]\setminus\{i_k\}}$.*

*Proof.* Let us consider the case when both $n$ and $|I|$ are even. Consider a skew-symmetric matrix $A = (\psi_{\{k,l\}})_{0\leq k<l<n}$. Then using Lemmas 3.16 and 3.15 we get

$$\psi_{[n]\setminus I} = \mathrm{Pf}(A_{[n]\setminus I}) = \mathrm{Pf}(A^C_I) \mathrm{Pf}(A)^{|I|-1} = \mathrm{Pf}(A^C_I),$$

since $\det(A) = \psi_{\{0,1,\dots,n-1\}} = 1$.

Moreover, by Lemma 3.17, the entries of the cofactor matrix are $\mathrm{Pf}(A_{[n]\setminus\{k,l\}}) \mathrm{Pf}(A) = \psi_{[n]\setminus\{i_k,i_l\}}$ which proves the lemma in this case.

The proof in other cases is analogous. The only difference is that we consider different matrix $A$. If $n$ is odd we take $A = (\psi_{\{k,l\}})_{-1\leq k<l<n}$ and if $n$ is even and $|I|$ is odd we take $A = (\psi_{\{k,l\}})_{-2\leq k<l<n}$. We interpret $\psi_{\{-1,k\}}$ and $\psi_{\{-2,k\}}$ as $\psi_{\{k\}}$ and we put $\psi_{\{-1,-2\}} = 1$. Then we conclude in the same way. ∎

**Corollary 3.19.**

$$|I|\psi_{[n]\setminus I} = \begin{cases} 2\sum_{1\leq k<l<n} (-1)^{k+l+1} \psi_{[n]\setminus\{i_k,i_l\}} \psi_{[n]\setminus(I\setminus\{i_k,i_l\})} & \text{if } |I| \text{ is even} \\ 2\sum_{0\leq k<l\leq n} (-1)^{k+l+1} \psi_{[n]\setminus\{i_k,i_l\}} \psi_{[n]\setminus(I\setminus\{i_k,i_l\})} & \text{if } |I| \text{ is odd.} \end{cases}$$

*Proof.* For every skew-symmetric $r \times r$ matrix $A$ (with $r$ even) and every $k = 1, \dots, r$, we have the following recursive formula for the Pfaffian:

$$\mathrm{Pf}(A) = \sum_{l=1}^{k-1} (-1)^{k+l} a_{k,l} \mathrm{Pf}(A_{\hat{k}\hat{l}}) - \sum_{l=k+1}^{r} (-1)^{k+l} a_{k,l} \mathrm{Pf}(A_{\hat{k}\hat{l}}),$$

where $A_{\hat{k}\hat{l}}$ is the submatrix obtained by removing the $k$-th and $l$-th rows and columns. Summing over all $k$ gives the desired equality. ∎

*Remark* 3.20. If we define $\psi_{[n]\setminus I} = 0$ for $I = \{i_1, \ldots, i_r\}$ a multiset/partition with at least one repeated entry, the recursion from Corollary 3.19 still holds.

## 4. Polynomiality of ML-degree

In this section and the next we present three proofs of the following polynomiality result for the algebraic degree of semidefinite programming:

**Theorem 4.1.** *For any fixed $m, s > 0$, the function $\delta(m, n, n-s)$ is a polynomial in $n$ that vanishes for $n = 0$.*

As an immediate corollary, we obtain the main theorem of this paper: the polynomiality of the ML-degree for linear concentration models. It was first conjectured by Sturmfels and Uhler [29] and confirmed in small, special cases in [5, 28, 17].

**Theorem 4.2.** *For any fixed $d > 0$, the function $\phi(n, d)$ is a polynomial for $n > 0$.*

*Proof.* We claim that for all $n, d > 0$,

$$\phi(n,d) = \frac{1}{n} \sum_{1 \le \binom{s+1}{2} \le d} s\delta(d, n, n-s). \tag{4.1}$$

Indeed: in the case $d < \binom{n+1}{2}$, by eq. (3.3) we have that $\phi(n,d) = \frac{1}{n}\sum_{s=1}^{n-1} s\delta(d,n,n-s)$. Moreover, the Pataki inequality eq. (3.2) implies that the terms on the right are 0 whenever $\binom{s+1}{2} > d$, hence we obtain (4.1). In the case $d \ge \binom{n+1}{2}$ the formula follows from our conventions in Remark 2.7 and the Pataki inequalities: if $d = \binom{n+1}{2}$ both sides of (4.1) are equal to 1, and if $d > \binom{n+1}{2}$ both sides are 0.

Now by Theorem 4.1 every term in the right hand side of (4.1) is a polynomial divisible by $n$, hence the theorem follows. ■

Our first two proofs of Theorem 4.1 are based on the following theorem.

**Theorem 4.3.** *Let $I = \{i_1, \ldots, i_r\}$ be a set of strictly increasing nonnegative integers. For $n \ge 0$ the function:*

$$P_I(n) := \begin{cases} \psi_{[n]\setminus I} & \text{if } I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

*is a polynomial of degree $\sum_{j=1}^{r}(i_j + 1)$. We call $P_I(n)$ the* Lascoux polynomials.

Before we prove Theorem 4.3 let us note that it immediately implies Theorem 4.1. Indeed, by Theorem 3.8, we have

$$\delta(m,n,n-s) = \sum_{\substack{I \subset [n] \\ |I|=s \\ \sum I = m-s}} \psi_I \psi_{[n]\setminus I} = \sum_{\substack{|I|=s \\ \sum I = m-s}} \psi_I P_I(n)$$

By Theorem 4.3, each of the summands is a polynomial in $n$ that vanishes for $n = 0$. Thus $\delta(m, n, n-s)$ is also a polynomial in $n$, which proves Theorem 4.1, and hence Theorem 4.2.

In the remainder of this section, we will present two proofs of Theorem 4.3.

### 4.A. First proof.

The following recursive relations are central for our first proof.

**Lemma 4.4.** *(1) For $j_1 > 0$ we have:*

$$\psi_{\{j_1,\ldots,j_s\}} = (s+1)\psi_{\{0,j_1,\ldots,j_s\}} - 2\sum_{\ell=1}^{s} \psi_{\{0,j_1,\ldots,j_\ell - 1,\ldots,j_s\}}, \tag{4.2}$$

*where the summation is over all $\ell$ for which $j_\ell - 1 > j_{\ell-1}$ and we set $j_0 := 0$.*

*(2) For $j_1 = 0$ we have:*

$$\psi_{\{j_1,j_2,\ldots,j_s\}} = \sum_{j_\ell \le j'_\ell < j_{\ell+1}} \psi_{\{j'_1,\ldots,j'_{s-1}\}}. \tag{4.3}$$

*Proof.* The first formula is [12, (A.15.7)].

To prove the second formula let $H_d$ be a complete homogeneous polynomial of degree $d$. We have:

$$H_d(\{x_i + x_j \mid 1 \le i \le j \le s\}) = \sum_{\substack{\lambda(I) \vdash d \\ |I| = s}} \psi_I s_{\lambda(I)}(x_1, \dots, x_s).$$

Substituting $x_s = 0$ we obtain:

$$\sum_{i=0}^{d} H_i(\{x_i + x_j \mid 1 \le i \le j \le s-1\}) H_{d-i}(x_1, \dots, x_{s-1}) =$$

$$= H_d(\{x_i + x_j \mid 1 \le i \le j \le s-1\}, x_1, \dots, x_{s-1}) = \sum_{\substack{\lambda(I) \vdash d \\ \mathrm{length}(\lambda(I)) \le s-1}} \psi_I s_{\lambda(I)}(x_1, \dots, x_{s-1}).$$

We note that $\mathrm{length}(\lambda(I)) \le s-1$ if and only if $0 \in I$. On the other hand we may apply Pieri rule to

$$\sum_{i=0}^{d} H_i(\{x_i + x_j \mid 1 \le i \le j \le s-1\}) H_{d-i}(x_1, \dots, x_{s-1}) =$$

$$\sum_{i=0}^{d} \left( \sum_{\substack{\lambda(I) \vdash d \\ |I| = s-1}} \psi_I s_{\lambda(I)}(x_1, \dots, x_{s-1}) \right) s_{(d-i)}(x_1, \dots, x_{s-1}).$$

Comparing the coefficients of Schur polynomials in both expressions gives the formula. ■

*First proof of Theorem 4.3.* We proceed by induction first on $|I|$, then on $\sum I := \sum_{i_j \in I} i_j$. The base case is $I = \emptyset$, when $\psi_{\{0,\dots,n-1\}} = 1$.

For the induction step, fix $I$, and assume the theorem has been shown for all $I'$ with $|I'| < |I|$, and for all $I'$ with $|I'| = |I|$ and $\sum I' < \sum I$. We consider two cases:

**Case 1.** $i_1 = 0$. We claim that for every $n \ge 0$,

$$P_I(n) = (n - r + 1) P_{I \setminus \{0\}}(n) - 2 \sum_{\ell : i_{\ell+1} > i_\ell + 1} P_{I \setminus \{0, i_\ell\} \sqcup \{i_\ell + 1\}}(n),$$

where for summation we formally assume $i_{r+1} = +\infty$. Indeed: if $n \le i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 4.4 (1).

**Case 2.** $i_1 > 0$. We claim that for every $n \ge 0$,

$$P_I(n) - P_I(n-1) = \sum_J P_J(n-1),$$

where the sum is over all $J \ne I$ of the form $\{i_1 - \epsilon_1, \dots, i_r - \epsilon_r\}$ with $\epsilon_\ell \in \{0, 1\}$. Again, if $n \le i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 4.4 (2).

In both cases, it follows that $P_I$ is a polynomial of the correct degree. ■

4.B. **Second proof.** Our second proof is based on explicit interpretation of $\psi_I$ as a sum of minors in Pascal triangle. We denote $E$ the Pascal triangle matrix, i.e. $E_{ij} = \binom{i}{j}$. We will always consider only finite submatrices of $E$ so despite the fact that it is an infinite matrix there will be no computations with infinite matrices.

For sets $K, C$ with $|K| = |C|$ we denote $V(K, C)$ the Vandermonde matrix with entries $V(K, C)_{ij} = k_{i+1}^{c_{j+1}}$. We also set $V(K) := V(K, [|K|])$, i.e. $V(K)_{ij} = k_{i+1}^{j}$.

For two sets $A, B$ we denote by $\varepsilon^{A,B}$ the sign of the permutation of $A \cup B$ determined by $A, B$ if they are disjoint. If they are not, we define $\varepsilon^{A,B} = 0$.

We begin with characterization of $\psi_I$ as a sum of the minors of the matrix $E$ which follows from [12, Proposition 2.8].

**Proposition 4.5.** *The following equality holds:*

$$\psi_I = \sum_{J \leq I} \det(E_{I,J}).$$

In what follows we will need the following lemma that may be easily proved by induction.

**Lemma 4.6.** *Let $a, b$ be nonnegative integers.*

*a) if $a > b$ then $\sum_{i=0}^{a}(-1)^i \binom{a}{i} i^b = 0$,*

*b) if $a = b$ then $\sum_{i=0}^{a}(-1)^{a-i} \binom{a}{i} i^b = a!$.*

To compute special minors of the matrix $E$ we use the following lemma.

**Lemma 4.7.** *Let $I = \{i_1, \dots, i_r\} \subset [n]$ be a set of nonnegative integers. Then*

$$\det E_{[n]\setminus[r],[n]\setminus I} = \frac{\prod_{1 \leq j < k \leq n-r}(i_k - i_j)}{(r-1)!(r-2)!\dots 2!1!} = \frac{\det(V(I))}{(r-1)!(r-2)!\dots 2!1!}$$

*Proof.* We fix $r$ and proceed by induction on $n$. The case $i_r < n-1$ is trivial. In the case $i_r = n-1$ we express the determinant via Laplace expansion on the $n$-th row, use the induction hypothesis and Lemma 4.6 to conclude. ∎

Now we are able to present our second proof of Theorem 4.3.

*Second proof of Theorem 4.3.* Let $|I| = r$ and $m := i_r + 1$. First, assume $n \geq m$. We use the formula from Proposition 4.5. We express the determinants $E_{[n]\setminus I,J}$ using the Laplace expansion along the first $m-r$ rows, we choose the columns indexed by set $L$. For the rest we use the Lemma 4.7. To simplify notation we let $K := [n] \setminus J$.

$$\begin{aligned}
\psi_{[n]\setminus I} &= \sum_{J \leq [n]\setminus I} \det(E_{[n]\setminus I,J}) \\
&= \sum_{J \leq [n]\setminus I} \sum_{\substack{L \subseteq J \\ |L|=m-r}} \varepsilon^{L,J\setminus L} \det(E_{[m]\setminus I,L}) \det(E_{[n]\setminus[m],J\setminus L}) \\
&= \sum_{\substack{|L|=m-r \\ L \leq [m]\setminus I}} \det(E_{[m]\setminus I,L}) \sum_{\substack{|K|=r \\ K\cap L=\emptyset \\ K\subset[n]}} \varepsilon^{L,[n]\setminus(K\cup L)} \det(E_{[n]\setminus[m],[n]\setminus(K\cup L)}) \\
&= \sum_{\substack{|L|=m-r \\ L \leq [m]\setminus I}} \det(E_{[m]\setminus I,L}) \sum_{\substack{|K|=r \\ K\subset[n]}} \varepsilon^{L,K} \varepsilon^{L,[n]\setminus L} \frac{\det(V(L\cup K))}{(m-1)!(m-2)!\dots 2!1!} \\
&= \sum_{\substack{|L|=m-r \\ L \leq [m]\setminus I}} \frac{\varepsilon^{L,[n]\setminus L} \det(E_{[m]\setminus I,L})}{(m-1)!(m-2)!\dots 2!1!} \sum_{\substack{|K|=r \\ K\subset[n]}} \det(V^*(L\cup K))
\end{aligned}$$

where $V^*(L \cup K)$ is the matrix $V(L \cup K)$ where we first put the rows indexed by $L$. Note that we may drop the assumption $L \leq [m] \setminus I$, since otherwise $\det(E_{[m]\setminus I,L}) = 0$. Similarly, we can extend our sum and drop the condition $L \cap K = \emptyset$ since we add only zero terms. If we fix $L$ and denote the elements of $K$ by $k_1 < \dots < k_r$, then $\det(V^*(L \cup K))$ is clearly a polynomial in $k_1, \dots, k_r$. Then

$$\sum_{\substack{|K|=r \\ K\subset[n]}} \det(V^*(L\cup K)) = \sum_{0 \leq k_1 < \dots < k_r < n} \det(V^*(L\cup K))$$

is a polynomial in $n$ for the fixed $L$. Moreover, the sum trough $L$ does not depend on $n$ and therefore also $\psi_{[n]\setminus I}$ is a polynomial in $n$. Our computations are correct only for $n \geq m$. However, the last expression makes sense and is a polynomial for all $n \geq 0$. Clearly, it is equal 0 for $n < m$. This proves the theorem. ∎

With this approach we can even compute the leading coefficient of the Lascoux polynomial $P_I$. For this we will need two technical lemmas. The proof of first one is straightforward, e.g. by induction.

**Lemma 4.8.** *Let $a_1, \dots, a_m$ be nonegative integers. Then*

$$\sum_{0 \le k_1 < \dots < k_m < n} k_1^{a_1} k_2^{a_2} \dots k_m^{a^m}$$

*is a polynomial in $n$ of degree $\sum_{i=1}^m a_i + m$.*

*Its leading coefficient is $\frac{1}{(a_1+1)(a_1+a_2+2)\dots(a_1+\dots+a_m+m)}$.*

**Lemma 4.9.**

$$\sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(n)})} = \frac{\prod_{i>j}(x_i - x_j)}{\prod_i x_i \prod_{i>j}(x_i + x_j)}$$

*Proof.* We prove it by induction on $n$. It is easy to check that for $n = 1, 2$ the statement holds. For $n > 2$, we split the sum depending on $\sigma(n)$ and apply induction hypothesis to the partial sums.

$$\begin{aligned}
&\sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(n)})} = \\
&= \frac{1}{x_1 + \dots + x_n} \sum_{k=1}^n \sum_{\substack{\sigma \in \mathbb{S}_n \\ \sigma(n)=k}} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(n-1)})} \\
&= \frac{1}{x_1 + \dots + x_n} \sum_{k=1}^n (-1)^{n-k} \frac{\prod_{i>j; i,j \ne k}(x_i - x_j)}{\prod_{i \ne k} x_i \prod_{i>j; i,j, \ne k}(x_i + x_j)} \\
&= \frac{1}{(x_1 + \dots + x_n) \prod_i x_i \prod_{i>j}(x_i + x_j)} \sum_{k=1}^n (-1)^{n-k} x_k \prod_{i>j; i,j \ne k} (x_i - x_j) \prod_{i \ne k}(x_i + x_k) \\
&= \frac{1}{(x_1 + \dots + x_n) \prod_i x_i \prod_{i>j}(x_i + x_j)} Q(x_1, \dots, x_n),
\end{aligned}$$

where $Q$ is a homogeneous polynomial of degree $\binom{n}{2} + 1$. Moreover, $Q$ is skewsymmetric, that if we exchange values of $x_i$ and $x_j$ we just change the sign. Therefore

$$Q(x_1, \dots, x_n) = \prod_{i>j}(x_i - x_j) R(x_1, \dots, x_n)$$

where $R$ is a symmetric polynomial of degree one. This means that $R$ is a multiple of $(x_1 + \dots + x_n)$. It is easy to check that the coefficient of $x_n^n x_{n-1}^{n-2} x_{n-2}^{n-3} \dots x_2$ in $Q$ is 1. Therefore $R = x_1 + \dots + x_n$. This proves the lemma. ∎

**Theorem 4.10.** *The polynomial $P_I$ is of degree $\sum I + |I|$. Its leading coefficient is equal to*

$$\frac{\prod_{j>k}(i_j - i_k)}{(i_1)! \dots (i_r)! \prod_j (i_j + 1) \prod_{j>k}(i_j + i_k + 2)}$$

*Proof.* We continue with the calculation from the second proof of Theorem 4.3. We do Laplace expansion of Vandermonde by first $m-r$ rows. We get

$$\sum_{\substack{|L|=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\sum_{\substack{|K|=r\\ K\subset[n]}}\det(V^*(L\cup K)) =$$
$$= \sum_{\substack{|L|=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\sum_{\substack{|C|=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\det(V(L,C))\sum_{\substack{|K|=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))$$
$$= \sum_{\substack{|C|=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\sum_{\substack{|L|=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\det(V(L,C)))\sum_{\substack{|K|=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))$$
$$= \sum_{\substack{|C|=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\det(\operatorname{diag}(1,-1,\dots,-1^{m-1})E_{[m]\setminus I,[m]}V([m],C))\sum_{\substack{|K|=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))$$

Consider the matrix $A := \big(\operatorname{diag}(1,-1,\dots,-1^{m-1})E_{[m]\setminus I,[m]}V([m],C)\big)$. Let $[m]\setminus I = \{b_1,\dots,b_{m-r}\}$, $C=\{c_1,\dots,c_{m-r}\}$, where, as always, we assume that the elements of sets are ordered. Notice that $c_{m-r}<b_{m-r}$ implies that the last row of the matrix $A$ is 0 by Lemma 4.6 and so is $\det(A)$. In general, if $c_i<b_i$, then $A_{[m-r]\setminus[i-1],[i]}=0$ and we also get $\det A = 0$.

The necessary condition for $\det A\neq 0$ is $c_i\geq b_i$ for all $1\leq i\leq m-r$. Therefore, we will sum only trough such sets $C$. In the border case when $C=[m]\setminus I$ we get that the matrix $A$ is upper triangular and by Lemma 4.6 we have $\varepsilon^{C,[m]\setminus C}\det A = (b_1)!\dots(b_{m-r})!$.

If we consider the sum $\sum_{\substack{|K|=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))$ it is clearly a polynomial in $n$ of degree at most $\sum([m]\setminus C)+r=\binom{m}{2}+r-\sum C$. Since we are summing only trough $C$ with $\sum C\geq\sum([m]\setminus I)$ we immediately get that the degree of the polynomial $P$ is at most $\sum I + r$.

Furthermore, the only summand which contributes to the term of degree $\sum I+r$ is the one with $C=[m]\setminus I$. We finish the proof of the theorem by computing this coefficient. In this case we have:

$$\tilde{P}_I(n) := \sum_{\substack{|K|=r\\ K\subset[n]}}\det(V(K,[m]\setminus C)) = \sum_{\sigma\in\mathbb{S}_r}\sum_{0\leq k_1<\dots<k_r<n}(-1)^\sigma k_1^{i_{\sigma(1)}}\dots k_r^{i_{\sigma(r)}}.$$

By Lemma 4.8 the leading coefficient of $\tilde{P}_I$ is

$$\sum_{\sigma\in\mathbb{S}_r}(-1)^\sigma\frac{1}{(i_{\sigma(1)}+1)(i_{\sigma(1)}+i_{\sigma(2)}+2)\dots(i_{\sigma(1)}+\dots+i_{\sigma(r)}+r)}$$

Now we apply Lemma 4.9 for $x_j=i_j+1$ to conclude that the leading coefficient of $\tilde{P}_I$ is equal to

$$\frac{\prod_{j>k}(i_j-i_k)}{\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)}$$

which is obviously non-zero. This shows that the degree of the polynomial $P_I$ is $\sum I+r$ and its leading coeffient is

$$\frac{1}{(m-1)!(m-2)!\dots1!}\cdot(b_1!)\dots(b_{m-r})!\cdot\frac{\prod_{j>k}(i_j-i_k)}{\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)} =$$
$$= \frac{\prod_{j>k}(i_j-i_k)}{(i_1)!\dots(i_r)!\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)}$$

■

## 5. Nie-Ranestad-Sturmfels conjecture

In this section we present a proof of the formula for the degree of semidefinite programming which was conjectured by Nie, Ranestad and Sturmfels [20]. The formula was known so far only for special values of the parameters. To state it we introduce the following coefficients.

**Definition 5.1** (Coefficients $b_I$)**.** Let $I$ be a set of $k$ nonnegative integers. We define $b_I(n)$ by the formula:

$$Q_{I+\mathbf{1}_k}(h/2,\dots,h/2) = b_I(n)\cdot h^{\sum I+k},$$

where $I+\mathbf{1}_k$ is the set obtained from $I$ by adding one to each of its elements. The function $Q_{I+\mathbf{1}_k}$ is the Schur $Q$-function [14, Section III.8] and its argument $h/2$ appears $n$ times.

These coefficients may be computed recursively as described in [20, Section 6]. We note that in this reference the authors use a convention that $I$ is a subset of the set $\{1,\dots,n\}$ while in this article $I\subset[n]=\{0,\dots,n-1\}$. This results in the difference in notation for the coefficient $b_I$ exchanging $I$ and $I+\mathbf{1}_k$.

The main theorem of this section, confirming Nie, Ranestad and Sturmfels conjecture, is stated below.

**Theorem 5.2.** *(NRS, Conjecture 21)*
*Let $m,n,s$ be positive integers. Then*

$$\delta(m,n,n-s) = \sum_{\sum I\le m-s} (-1)^{m-s-\sum I}\psi_I b_I(n)\binom{m-1}{m-s-\sum I}$$

*where the sum goes trough all sets of nonnegative integers of cardinality $s$.*

Note that Theorem 4.1 is an immediate corollary of Theorem 5.2, since the coefficients $b_I(n)$ are known to be polynomials. Hence, as soon as we have proven Theorem 5.2, we have a third proof of Theorem 4.1.

*Remark* 5.3. We note that if the Pataki inequality (3.2) $m\ge\binom{s+1}{2}$ is not satisfied, then both sides of the equality above are trivially zero.

For the rest of the section we fix the numbers $m,n,s$ as in the statement of the theorem.

Our proof is algebraic. Theorem 5.2 presents a relation between numbers $b_I(n)$ and $\psi_I$. The coefficients $s_{I,J}$ from Definition 2.3 will play a prominent role.

**Lemma 5.4.** *Let $I=\{i_1,\dots,i_r\}$ and $J=\{j_1,\dots,j_r\}$ be two sets of nonnegative integers. Let $M_{I,J}=(m_{kl})$ be the $r\times r$ matrix with $m_{kl}=\binom{i_k}{j_l}$. Then*

*a)* $s_{I,J}=\det(M_{I,J})$
*b)* $H_d(x_1+1,\dots,x_r+1)=\sum_{i=0}^{d}\binom{d+r-1}{d-i}H_i(x_1,\dots,x_r)$

*Proof.* Part a) is proved in [14, Section I.3, example 10]. In particular, it implies

$$s_{[r+d],[r+i]} = \binom{d+r-1}{r+i-1} = \binom{d+r-1}{d-i}.$$

From this, the equation in part b) becomes the defining equation for $s_{I,J}$. ∎

The following lemma describes the relation between $b_I(n)$ and $s_{I,J}$:

**Lemma 5.5.** *Let $I$ be a set of nonnegative integers. Then*

$$b_I(n) = \sum_{J\le I}\left(\frac{1}{2}\right)^{\sum I-\sum J} s_{I,J}\psi_{[n]\setminus J}$$

We present two proofs of this lemma: one based on simple algebra, one on methods from algebraic geometry.

*First proof.* We will use induction on the length of $I$, which we will denote by $k$. The base of induction, i.e. the cases $k = 1, 2$ are left for the reader.

We proceed with the general case $k > 2$. We will assue $k$ is even; the odd case is analoguous. Since $b_I = \mathrm{Pf}(b_{i_p,i_q})_{1\le p<q\le k}$, we have (as in Corollary 3.19) the following recursive relations between the $b_I$:

$$kb_I = 2 \sum_{1\le p<q\le n} (-1)^{p+q+1} b_{\{i_p,i_q\}} b_{I\setminus\{i_p,i_q\}}.$$

By induction, we need to show that

$$k \sum_{J\le I} 2^{\sum J} s_{I,J}\psi_{[n]\setminus J} =$$

$$2 \sum_{1\le p<q\le n} (-1)^{p+q+1} \left( \sum_{J\le\{i_p,i_q\}} 2^{\sum J} s_{\{i_p,i_q\},J}\psi_{[n]\setminus J} \right) \left( \sum_{J\le I\setminus\{i_p,i_q\}} 2^{\sum J} s_{I\setminus\{i_p,i_q\},J}\psi_{[n]\setminus J} \right).$$

This follows immediately from the following claim:

**Claim 5.6.** *For every $J \le I$, where $J$ can have repeated elements,*

$$ks_{I,J}\psi_{[n]\setminus J} = 2 \sum_{1\le p<q\le n} (-1)^{p+q+1} \left( \sum_{1\le s<t<n} s_{\{i_p,i_q\},\{j_s,j_t\}}\psi_{[n]\setminus\{j_s,j_t\}} s_{I\setminus\{i_p,i_q\},J\setminus\{j_s,j_t\}}\psi_{[n]\setminus(J\setminus\{j_s,j_t\})} \right).$$

*Proof.* Using Laplace expansion, for any $s, t$, we can write:

$$s_{I,J} = \sum_{p<q} (-1)^{p+q+s+t} s_{\{i_p,i_q\},\{j_s,j_t\}} s_{I\setminus\{i_p,i_q\},J\setminus\{j_s,j_t\}}.$$

Hence, the right hand side can be rewritten as

$$2s_{I,J} \sum_{1\le s<t<n} (-1)^{s+t+1}\psi_{[n]\setminus\{j_s,j_t\}}\psi_{[n]\setminus(J\setminus\{j_s,j_t\})}.$$

It remains to show that

$$k\psi_{[n]\setminus J} = 2 \sum_{1\le s<t<n} (-1)^{s+t+1}\psi_{[n]\setminus\{j_s,j_t\}}\psi_{[n]\setminus(J\setminus\{j_s,j_t\})}$$

which is precisely Corollary 3.19. ■

This finishes the first proof of the formula. ■

The ideas of the second proof were suggested to us by Andrzej Weber.

*Second proof.* We start with a projection formula, which is a special case of [10, (4.7)]. Note that this formula is stated in terms of Schur $P$-polynomials, while we work with Schur $Q$-polynomials which accounts for an additional factor of a power of two.

For a vector bundle $\mathcal{E}$ of rank $n$ over some base $X$, we consider the relative Grassmannian $G^k(\mathcal{E})$ of rank $k$ quotients of $\mathcal{E}$, with its projection $\pi$ to $X$. We denote by $\mathcal{K}$ and $\mathcal{Q}$ the relative tautological subbundle and quotient bundle of $\pi^*\mathcal{E}$, of respective ranks $r = n - k$ and $k$. Then

$$Q_{I+\mathbf{1}_k}(\mathcal{E}) = \pi_*(c_{top}(\mathcal{K}\otimes\mathcal{Q})Q_{I+\mathbf{1}_k}(\mathcal{Q})), \tag{5.1}$$

where by $+\mathbf{1}_k$ we mean adding 1 to all $k$ elements of $I$ (cf. [10, Example 2, p. 50]). Moreover, [10, (4.5)] can be written as

$$Q_{I+\mathbf{1}_k}(\mathcal{Q}) = 2^k c_{top}(\wedge^2\mathcal{Q})s_{\lambda(I)+\mathbf{1}_k}(\mathcal{Q}) = c_{top}(S^2\mathcal{Q})s_{\lambda(I)}(\mathcal{Q}).$$

Since $\pi^*\mathcal{E}$ is an extension of $\mathcal{Q}$ by $\mathcal{K}$, the bundle $\pi^*S^2\mathcal{E}$ admits a filtration whose successive quotients are $S^2\mathcal{Q}$, $\mathcal{K}\otimes\mathcal{Q}$ and $S^2\mathcal{K}$. Hence the identity

$$c(\mathcal{K}\otimes\mathcal{Q})c(S^2\mathcal{Q}) = s(S^2\mathcal{K}^*)\pi^*c(S^2\mathcal{E}).$$

Equation (5.1) can thus be rewritten as

$$Q_{I+\mathbf{1}_k}(\mathcal{E}) = c(S^2\mathcal{E})\pi_*(s(S^2\mathcal{K}^*)s_{\lambda(I)}(\mathcal{Q}))_{|deg=\Sigma I+k},$$

where the last symbols mean we only keep the component of degree $\sum I + k$.

Now suppose that $\mathcal{E} = \mathcal{E}_0 \otimes L$ for some line bundle $L$ and a trivial vector bundle $\mathcal{E}_0$. Then $G^k(\mathcal{E})$ is a trivial bundle over $X$, while $\mathcal{K} = \mathcal{K}_0 \otimes L$ and $\mathcal{Q} = \mathcal{Q}_0 \otimes L$ are obtained by pull-back of the tautological and quotient bundles $\mathcal{K}_0$, $\mathcal{Q}_0$ over a fixed Grassmannian $G^k(\mathbf{C}^n)$ (we omit the pull-backs for simplicity). By Definition 2.3 (where formally we need homogenize by replacing 1 by $c_1(L)$ and $x_i$ are the Chern roots of $\mathcal{Q}$) we have:

$$s_{\lambda(I)}(\mathcal{Q}) = \sum_{J\leq I} s_{I,J}s_{\lambda(J)}(\mathcal{Q}_0)\delta^{\Sigma I-\Sigma J},$$

where $\delta = c_1(L)$. Moreover, the Segre classes of $S^2\mathcal{K}_0^*$ and $S^2\mathcal{K}^*$are related by the formula

$$s(S^2\mathcal{K}^*) = \sum_{\ell\geq 0}(1+2\delta)^{-\binom{r+1}{2}-\ell}s_{(\ell)}(S^2\mathcal{K}_0^*).$$

Plugging these two formulas into the previous one, we get $Q_{I+\mathbf{1}_k}(\mathcal{E})$ as

$$\sum_{J\leq I}\sum_{L}(1+2\delta)^{\binom{n+1}{2}-\binom{r+1}{2}-|\lambda(L)|}\delta^{\Sigma I-\Sigma J}s_{I,J}\psi_L\pi_*(s_{\lambda(L)}(\mathcal{K}_0^*)s_{\lambda(J)}(\mathcal{Q}_0))_{|deg=\Sigma I+k}.$$

Now recall that the Schur classes $s_\alpha(\mathcal{K}_0^*)$ and $s_\beta(\mathcal{Q}_0)$, for partitions $\alpha \subset (k^r)$ and $\beta \subset (r^k)$, that are non zero, give dual bases of Schubert cycles on the Grassmannian $G^k(\mathbf{C}^n)$. This can be expressed as

$$\pi_*(s_{\lambda(L)}(\mathcal{K}_0^*)s_{\lambda(J)}(\mathcal{Q}_0)) = \delta_{L,[n]/J},$$

where $\delta_{L,[n]/J}$ is the Kronecker delta. Note that $L = [n]/J$ implies that $|\lambda(L)|+|\lambda(J)| = kr$. We thus get the formula

$$Q_{I+\mathbf{1}_k}(\mathcal{E}) = \Big(\sum_{J\leq I}(1+2\delta)^{k+\Sigma J}\delta^{\Sigma I-\Sigma J}s_{I,J}\psi_{[n]/J}\Big)_{|deg=\Sigma I+k}.$$

But since the degree of the polynomial into brackets is exactly $\Sigma I + k$, we just need to keep its top degree component, that is

$$Q_{I+\mathbf{1}_k}(\mathcal{E}) = \sum_{J\leq I} 2^{\Sigma J+k}s_{I,J}\psi_{[n]/J}\delta^{\Sigma I+k}.$$

We conclude by applying formally this formula to the bundle $E = \mathcal{O}(1/2)^{\oplus n}$ over the projective space. ■

**Lemma 5.7.** *Let $J$ be a set of nonnegative integers of length $s$ with $\sum J \leq m - s$. Then*

$$\sum_{\substack{I\geq J\\ \sum I\leq m-s}} \psi_I\left(-\frac{1}{2}\right)^{\sum I-\sum J} s_{I,J}\binom{m-1}{m-s-\sum I} = \begin{cases}0 & \text{if } \sum J < m-s\\ \psi_J & \text{if } \sum J = m-s\end{cases}$$

*Proof.* We prove the lemma at the same time for all $J$ by multiplying each equation for $J$ by the Schur polynomial $s_{\lambda(J)}(x_1,\ldots,x_s)$ and summing them up. Since Schur polynomials form a basis of the space of symmetric polynomials, the statement of the lemma is equivalent to the following polynomial identity:

$$\sum_{\sum J\leq m-s}\ \sum_{\substack{I\geq J\\ \sum I\leq m-s}} \psi_I\left(-\frac{1}{2}\right)^{\sum I-\sum J} s_{I,J}\binom{m-1}{m-s-\sum I}s_{\lambda(J)}(x_1,\ldots,x_s) =$$

$$= \sum_{\sum J=m-s} \psi_J s_{\lambda(J)}(x_1,\ldots,x_s)$$

By 2.4, the right hand side is equal to $H_{m-s-\binom{s}{2}}(x_i + x_j | 1 \le i \le j \le s)$. For the left hand side we can use Definition 2.3 of the coefficients $s_{I,J}$:

$$\sum_{\sum J \le m-s} \sum_{\substack{I \ge J \\ \sum I \le m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} s_{\lambda(J)}(x_1, \dots, x_s) =$$

$$\sum_{\sum I \le m-s} \psi_I \binom{m-1}{m-s-\sum I} \sum_{J \le I} \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} s_{\lambda(J)}(x_1, \dots, x_s) =$$

$$\sum_{\sum I \le m-s} \psi_I \binom{m-1}{m-s-\sum I} s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s} \sum_{\sum I = i} \binom{m-1}{m-s-i} \psi_I s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} \sum_{\sum I = i} \psi_I s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} H_{i-\binom{s}{2}}(x_i + x_j - 1 | 1 \le i \le j \le s) =$$

$$H_{m-s-\binom{s}{2}}(x_i + x_j | 1 \le i \le j \le s)$$

In the last equality we apply Lemma 5.4 for variables $x_i + x_j - 1$. ■

Now we are able to present the proof of Theorem 5.2:

*Proof of Theorem 5.2.* We replace $b_I(n)$ by the expression from Lemma 5.5, change the order of summation and use Lemma 5.7 in the last step:

$$\sum_{\sum I \le m-s} (-1)^{m-s-\sum I} \psi_I b_I(n) \binom{m-1}{m-s-\sum I} =$$

$$\sum_{\sum I \le m-s} \sum_{J \le I} s_{I,J} \psi_{[n]\setminus J} \left(\frac{1}{2}\right)^{\sum I - \sum J} (-1)^{m-s-\sum I} \psi_I \binom{m-1}{m-s-\sum I} =$$

$$\sum_{\sum J \le m-s} (-1)^{m-s-\sum J} \psi_{[n]\setminus J} \sum_{\substack{I \ge J \\ \sum I \le m-s}} s_{I,J} \left(-\frac{1}{2}\right)^{\sum I - \sum J} \psi_I \binom{m-1}{m-s-\sum I} =$$

$$\sum_{\sum J = m-s} (-1)^{m-s-\sum J} \psi_{[n]\setminus J} \psi_J = \delta(m, n, n-s).$$

■

*Remark* 5.8. While deriving the polynomiality result from the Nie-Ranestad-Sturmfels conjecture is not difficult, the reverse implication does not seem easy.

## 6. Skew-symmetric matrices and general matrices

The results from the previous sections have natural analogues if we replace the space of symmetric matrices with the space of skew-symmetric matrices, or with the space of all matrices.

6.A. **Type $D$: Skew-symmetric matrices.** In this section, we will be working with skew-symmetric matrices of even size $2n \times 2n$. Both $\delta(m,n,r)$ and $\phi(n,d)$ have natural analogues for skew-symmetric matrices:

**Definition 6.1.** Let $\delta_D(m,n,r)$ be the degree of the variety $(D_{\mathcal{L}}^{2r})^*$, where $\mathcal{L} \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ is a general linear space of skew-symmetric matrices of (projective) dimension $m$, and $^*$ denotes the dual variety. Equivalently, if we let $Z_r \subset \mathbb{P}(\wedge^2 V^*) \times \mathbb{P}(\wedge^2 V)$ be the variety of pairs $(X,Y)$ with $X \cdot Y = 0$, $\operatorname{rk} X \leq 2r$, $\operatorname{rk} Y \leq n - 2r$, then the multidegree

$$[Z_r] = \sum_m \delta_D(m,n,r) H_1^{\binom{n}{2}-m} H_2^m.$$

**Definition 6.2.** The number $\phi_D(n,d)$ is the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L} \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^{2n})$ is a general linear subspace of dimension $d-1$.

We can study these numbers using the *space of complete skew-forms*. Just as with complete quadrics, there are many ways of constructing this space. Here we give just one, referring the reader to the literature [2, 32, 16] for other equivalent definitions.

**Definition 6.3.** Let $V$ be a $2n$-dimensional vector space. The space of complete skew-forms $\Psi(V)$ is defined as the closure of $\phi(\mathbb{P}(\bigwedge^2(V))^\circ)$, where

$$\phi : \mathbb{P}(\bigwedge^2 V)^\circ \to \mathbb{P}(\bigwedge^2 V) \times \mathbb{P}\left(\bigwedge^4 V\right) \times \ldots \times \mathbb{P}\left(\bigwedge^{2n-2} V\right),$$

given by

$$A \mapsto (A, \wedge^2 A, \ldots, \wedge^{n-1} A).$$

We note that here $\wedge^i A$ is viewed as an element of $\bigwedge^{2i} V$, not of $\bigwedge^i \bigwedge^2 V$; see also [2, Section 3]. In coordinates, the map $\bigwedge^2 V \to \bigwedge^{2i} V$ sends the entries of a skew-symmetric matrix to the Pfaffians of its principal $2i \times 2i$ submatrices.

As with complete quadrics, the space of complete skew-forms has two series of special classes of divisors $S_1, \ldots, S_{n-1}$ and $L_1, \ldots, L_{n-1}$. Divisors $S_i$ consist of tuples $(A_1, \ldots, A_{n-1})$, where $A_{2i}$ is a pure wedge. This is equal to the closure in $\Psi(V)$ of the tuples $(A_1, \ldots, A_{n-1})$ where $A_1$ has rank $2i$. Divisors $L_i$ can be obtained as pullbacks of a hyperplane under $\pi_i : \Psi(V) \to \mathbb{P}(\bigwedge^{2i} V)$.

The analogue of Proposition 3.6 holds:

**Proposition 6.4.** *Classes $L_1, \ldots, L_{n-1}$ are independent and generate* $\operatorname{Pic}(\Psi(V))$*, the classes $S_1, \ldots, S_{n-1}$ generate an index $n$ sublattice of* $\operatorname{Pic}(\Psi(V))$*. Moreover there are the following relations between $L_i$'s and $S_i$'s:*

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

*where $L_0 = L_n = 0$.*

*Proof.* Follows from [16, Proposition 3.6, Theorem 3.9]. ■

As with symmetric matrices, the numbers $\phi_D$ and $\delta_D$ can be expressed as intersection products in the Chow ring of $\Psi(V)$:

**Proposition 6.5.**

$$\phi_D(n,d) = L_1^{\binom{2n}{2}-d} L_{\ell-1}^{d-1}$$

$$\delta_D(m,n,r) = S_r L_1^{\binom{2n}{2}-m-1} L_{\ell-1}^{m-1}$$

*Proof.* Analogous to the proof of Proposition 3.7. ■

From the two propositions above, we deduce that

$$\phi_D(n,d) = \frac{1}{n}\sum_{r=1}^{n-1} r\delta_D(d,n,n-r), \tag{6.1}$$

the analogue of eq. (3.3).

We can express $\delta_D(m,n,r)$ in terms of the *type D Lascoux coefficients*:

**Definition 6.6.** The type $D$ Lascoux coefficients are defined via

$$H_d(\{x_i + x_j \mid 1 \le i < j \le k\}) = \sum_{\substack{\lambda(I) \vdash d \\ |I| = k}} \alpha_I s_{\lambda(I)}(x_1,\dots,x_k),$$

(the difference with Definition 2.4 is $i < j$ as opposed to $i \le j$). Equivalently, for the universal bundle $\mathcal{U}$ over a Grassmannian $G(k,n)$:

$$Seg_d(\bigwedge^2 \mathcal{U}) = \sum_{\substack{\lambda(I) \vdash d \\ |I| = k}} \alpha_I \sigma_{\lambda(I)}.$$

For more about these coeffients, see [12, Proposition A.16], where they are denoted $\alpha_I$.

**Theorem 6.7.**

$$\delta_D(m,n,r) = \sum_{\substack{I \subset [2n] \\ |I| = 2n-2r \\ \sum I = m-2n+2r}} \alpha_I \alpha_{[2n]\setminus I}$$

*Proof.* Analogous to the proof of Theorem 3.8. ∎

We will now prove polynomiality (or more precisely: quasipolynomiality) of $\alpha_{[k]\setminus I}$. The following recursive relations will be central to our proof:

**Lemma 6.8.** *(1) For $j_1 > 0$ we have:*

$$\alpha_{\{j_1,\dots,j_s\}} = \begin{cases} \alpha_{\{0,j_1,\dots,j_s\}} & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd} \end{cases} \tag{6.2}$$

*(2) For $j_1 = 0$ we have:*

$$\alpha_{\{j_1,j_2,\dots,j_s\}} = \sum_{j_\ell \le j'_\ell < j_{\ell+1}} \alpha_{\{j'_1,\dots,j'_{s-1}\}}. \tag{6.3}$$

*Proof.* First formula is [12, (A.16.3)], the proof of the second formula is analogous to the proof of eq. (4.3) in Lemma 4.4. ∎

**Theorem 6.9.** *Let $I = \{i_1,\dots,i_s\}$ be a set of strictly increasing nonnegative integers. For $k \ge 0$ the function:*

$$P_I^D(k) := \begin{cases} \alpha_{[k]\setminus I} & \text{if } I \subset [k], \\ 0 & \text{otherwise.} \end{cases}$$

*is a quasi-polynomial in $k$ with period 2, i.e. for both even $k$ and odd $k$ it is a polynomial.*

*Proof.* We proceed as in the first proof of Theorem 4.3 by induction on $|I|$ and then on $\sum I$ using relations from Lemma 6.8. The difference is that in the case $i_0 = 0$ we have

$$P_I^D(n) = \begin{cases} P_{I\setminus 0}^D & \text{if } n - |I| \text{ is even} \\ 0 & \text{if } n - |I| \text{ is odd} \end{cases}$$

which is clearly by induction hypothesis a quasipolynomial in $n$ with period 2. The rest is analogous as in the proof of Theorem 4.3. ∎

From Theorem 6.7 and Theorem 6.9 we obtain polynomiality of $\delta_D$:

**Theorem 6.10.** *For every fixed $m, s$, the function $\delta_D(m, n, n-s)$ is a polynomial in $n$.*

Using eq. (6.1),we also get polynomiality of $\phi_D$:

**Theorem 6.11.** *For any fixed $d$, the function $\phi_D(n, d)$ is a polynomial for $n > 0$.*

6.B. **Type $A$: Arbitrary matrices.** We now turn our attention to the space of all $n \times n$ square matrices.

**Definition 6.12.** Let $\delta_A(m, n, r)$ be the degree of the variety $(D^r_{\mathcal{L}})^*$, where $\mathcal{L} \subset (\mathbb{C}^n \otimes \mathbb{C}^n)$ is a general linear space of matrices of (projective) dimension $m$, and $^*$ denotes the dual variety. Equivalently, if we let $Z_r \subset \mathbb{P}(V^* \otimes V) \times \mathbb{P}(V^* \otimes V)$ be the variety of pairs $(X, Y)$ with $X \cdot Y = Y \cdot X = 0$, $\operatorname{rk} X \leq r$, $\operatorname{rk} Y \leq n - r$, then the multidegree

$$[Z_r] = \sum_m \delta_A(m, n, r) H_1^{n^2-m} H_2^m.$$

**Definition 6.13.** The number $\phi_A(n, d)$ is the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L} \subseteq \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is a general linear subspace of dimension $d - 1$.

Now, the correct space to work it is the *space of complete collineations* [26, 33, 35, 13, 32, 15]. It can actually be defined for rectangular matrices, but for sake of simplicity we will restrict ourselves to square matrices.

**Definition 6.14.** Let $V$ and $W$ be two vector spaces of equal dimension $n$. The space $\mathbb{P}(V^* \otimes W)$ represents linear maps from $V$ to $W$; the open subset of rank $n$ linear maps is denoted by $\mathbb{P}(V^* \otimes W)^\circ$. Then the space of complete collineations $\Omega(V, W)$ is defined as the closure of the image of the map

$$\phi : \mathbb{P}(V^* \otimes W)^\circ \to \mathbb{P}(V^* \otimes W) \times \mathbb{P}\left(\bigwedge^2 V^* \otimes \bigwedge^2 W\right) \times \ldots \times \mathbb{P}\left(\bigwedge^{n-1} V^* \otimes \bigwedge^{n-1} W\right),$$

given by

$$A \mapsto (A, \wedge^2 A, \ldots, \wedge^{n-1} A).$$

As before, in coordinates this map sends a matrix to its minors.

As in the previous cases, the space of complete collineations has two series of special classes of divisors $S_1, \ldots, S_{n-1}$ and $L_1, \ldots, L_{n-1}$.

Divisors $S_i$ consist of tuples $(A_1, \ldots, A_{n-1})$, where $A_i$ is a rank one matrix. This is equal to the closure in $\Omega(V, W)$ of the tuples $(A_1, \ldots, A_{n-1})$ where $A_1$ has rank $i$. Divisors $L_i$ can be obtained as pullbacks of a hyperplane under $\pi_i : \Omega(V, W) \to \mathbb{P}\left(\bigwedge^i V^* \otimes \bigwedge^i W\right)$.

The analogue of Proposition 3.6 holds:

**Proposition 6.15.** *$L_1, \ldots, L_{n-1}$ form a basis of* $\operatorname{Pic}(\Omega(V, W))$*. Moreover there are the following relations between $L_i$'s and $S_i$'s:*

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

*where $L_0 = L_n = 0$.*

*Proof.* Follows from [15, Proposition 3.6, Theorem 3.13]. ∎

As before (Proposition 3.7), $\phi_A$ and $\delta_A$ are intersection products in the Chow ring of $\Omega(V, W)$:

**Proposition 6.16.**

$$\phi_A(n, d) = L_1^{n^2-d} L_{n-1}^{d-1}$$
$$\delta_A(m, n, r) = S_r L_1^{n^2-m-1} L_{n-1}^{m-1}$$

We again conclude the analogue of eq. (3.3):

$$\phi_A(n, d) = \frac{1}{n} \sum_{r=1}^{n-1} r \delta_A(d, n, n - r). \tag{6.4}$$

**Definition 6.17.** We define type $A$ Lascoux coefficients $d_{I,J}$ by

$$H_d(\{x_i+y_j \mid 1 \le i \le k, 1 \le j \le l\}) = \sum_{c=0}^{d} \sum_{\substack{\lambda(I)\vdash c \\ |I|=k}} \sum_{\substack{\lambda(J)\vdash d-c \\ |J|=l}} d_{I,J} s_{\lambda(I)}(x_1,\dots,x_k) s_{\lambda(J)}(y_1,\dots,y_l),$$

The main difference is that $d_{I,J}$ depends on two sets of nonnegative integers $I, J$.

Equivalently, for the product of universal bundles $\mathcal{U}_1 \otimes \mathcal{U}_2$ over a product of Grassmannians $G_1(k,n) \times G_2(l,n)$:

$$Seg_d(\mathcal{U}_1 \otimes \mathcal{U}_2) = \sum_{c=0}^{d} \sum_{\substack{\lambda(I)\vdash c \\ |I|=k}} \sum_{\substack{\lambda(J)\vdash d-c \\ |J|=l}} d_{I,J} \sigma_{\lambda(I),\lambda(J)}.$$

Then analogously to Theorem 3.8, we have the following formula for $\delta_A$:

**Theorem 6.18.**

$$\delta_A(m,n,r) = \sum_{\substack{I,J\subset[n] \\ |I|=|J|=n-r \\ \sum I + \sum J = m-n+r}} d_{I,J} d_{[n]\setminus I, [n]\setminus J}$$

We denote $D(t)$ the infinite matrix with entries $D(t)_{ij} = \binom{t+i+j}{i}$. This matrix gives us a formula for $d_{I,J}$ [12, Proposition 2.8].

**Proposition 6.19.** *Let $I = \{i_1,\dots,i_r\}$, $J = \{j_1,\dots,j_s\}$ be two sets of nonnegative integers with $r \le s$. Then*

$$d_{I,J} = \begin{cases} \det D(s-r)_{I,\{j_{s-r+1}-(s-r),\dots,j_s-(s-r)\}} & \text{if } j_i = i-1 \text{ for all } 1 \le i \le s-r \\ 0 & \text{otherwise .} \end{cases}$$

*In particular, if $|I| = |J|$ then $d_{I,J} = \det D(0)_{I,J}$.*

**Lemma 6.20.** *(1) Let $I = \{i_1,\dots,i_s\}$, $J = \{j_1,\dots,j_s\}$ with $i_1, j_1 > 1$. Write $I_0 = \{0\} \cup J$ and $J_0 = \{0\} \cup J$. Then*

$$d_{I,J} = (s+1) d_{I_0,J_0} - \sum_{p=1}^{s} d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} - \sum_{q=1}^{s} d_{I_0,J_0\setminus\{j_q\}\cup\{j_q-1\}}.$$

*(Here, if $I_0 \setminus \{i_p\} \cup \{i_p - 1\}$ is a multiset, then $d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} = 0$.)*

*(2) For $i_1 = 0$ or $j_1 = 0$ we have:*

$$d_{\{i_1,\dots,i_s\},\{j_1,j_2,\dots,j_s\}} = \sum_{\substack{i_\ell \le i'_\ell < i_{\ell+1} \\ j_\ell \le j'_\ell < j_{\ell+1}}} d_{\{i'_1,\dots,i'_{s-1}\},\{j'_1,\dots,j'_{s-1}\}}.$$

*Proof.* (1) We expand the determinant $\det D(0)_{I_0,J_0}$ in each row, and sum up:

$$
\begin{aligned}
(s+1)d_{I_0,J_0} = &\sum_{p,q=0}^{s} (-1)^{p+q} \binom{i_p+j_q}{i_p} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=& d_{I,J} + \sum_{p=1}^{s} (-1)^{p} d_{I_0\setminus\{i_p\},J} + \sum_{q=1}^{s} (-1)^{q} d_{I,J_0\setminus\{j_q\}} \\
&+ \sum_{p,q=1}^{s} (-1)^{p+q} \left( \binom{i_p+j_q-1}{i_p} + \binom{i_p+j_q-1}{i_p-1} \right) d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=& d_{I,J} + \sum_{p=1}^{s}\sum_{q=0}^{s} (-1)^{p+q} \binom{i_p+j_q-1}{i_p-1} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
&+ \sum_{q=1}^{s}\sum_{p=0}^{s} (-1)^{p+q} \binom{i_p+j_q-1}{i_p} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=& d_{I,J} + \sum_{p=1}^{s} d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} + \sum_{q=1}^{s} d_{I_0,J_0\setminus\{j_q\}\cup\{j_q-1\}}
\end{aligned}
$$

(2) Proof of the second formula is similar to the proof of formula 4.3 in Lemma 4.4. We consider only the case $i_1 = 0$ and in $H_d(\{x_i + y_j \mid 1 \le i, j \le s\})$ we subsitute $x_s = 0$. This yields

$$
d_{\{i_1,\dots,i_s\},\{j_1,j_2,\dots,j_s\}} = \sum_{j_{\ell-1} < j'_\ell \le j_\ell} d_{\{i_2-1,\dots,i_s-1\},\{j'_1,\dots,j'_s\}}.
$$

Then by Proposition 6.19 all summands with $j'_1 > 0$ are zero. This allows us to substitute $y_s = 0$ in $H_d(\{x_i + y_j \mid 1 \le i \le s-1, 1 \le j \le s\})$ and conclude the lemma analogously as the formula 4.3 in Lemma 4.4.

■

**Theorem 6.21.** *Let $I = \{i_1, \dots, i_r\}, J = \{j_1, \dots, j_r\}$ be two sets of strictly increasing nonnegative integers. For $n \ge 0$ the function:*

$$
Q_{I,J}(n) := \begin{cases} d_{[n]\setminus I,[n]\setminus J} & \textit{if } I, J \subset [n], \\ 0 & \textit{otherwise.} \end{cases}
$$

*is a polynomial in $n$.*

*Proof.* From Lemma 6.20 it follows that

$$
\begin{aligned}
Q_{I,J}(n) = &(n-r+1)Q_{I\setminus\{0\},J\setminus\{0\}}(n) \\
&- \sum_{\ell:i_{\ell+1}>i_\ell+1} Q_{I\setminus\{0,i_\ell\}\sqcup\{i_\ell+1\},J\setminus\{0\}}(n) - \sum_{\ell:j_{\ell+1}>j_\ell+1} Q_{I\setminus\{0\},J\setminus\{0,j_\ell\}\sqcup\{j_\ell+1\}}(n)
\end{aligned}
$$

if $i_0 = j_0 = 0$, and otherwise

$$
Q_{I,J}(n) = \sum_{I',J'} Q_{I',J'}(n-1),
$$

where the sum is over all pairs $(I', J')$ of the form $(\{i_1-\epsilon_1, \dots, i_r-\epsilon_r\}, \{j_1-\mu_1, \dots, j_r-\mu_r\})$ with $\epsilon_\ell, \mu_\ell \in \{0, 1\}$. As in the first proof of Theorem 4.3, it follows by induction that $Q_{I,J}$ is a polynomial. ■

**Theorem 6.22.** *For every fixed $m, s$, the function $\delta_A(m, n, n-s)$ is a polynomial in $n$.*

*Proof.* Follows from Theorems 6.18 and 6.21. ■

**Theorem 6.23.** *For any fixed $d$, the function $\phi_A(n, d)$ is a polynomial for $n > 0$.*

*Proof.* Follows from eq. (6.4) and Theorem 6.22. ■

6.C. **NRS in type A.** Let $M_n$ denote the space of complex matrices of size $n$, and $D_n^{n-r} \subset \mathbb{P}(M_n)$ the locus of matrices of rank at most $r$. Denote by $D_{n,m}^{n-r}$ its intersection with a general $m$-dimensional projective space. Its dimension is $d = m - r^2$ when this is non negative, otherwise it is empty. The analogs of the Pataki's inequalities are given by:

**Proposition 6.24.** *The dual variety of $D_{n,m}^{n-r}$ is a hypersurface if and only if*

$$r^2 \leq m \leq n^2 - (n-r)^2.$$

The degree of this dual variety can be computed by classical means when $D_{n,m}^r$ is smooth, which is equivalent to $r^2 \leq m \leq r^2 + 2r$. The class formula gives, in terms of topological Euler characteristics,

$$\deg(D_{n,m}^{n-r})^* = (-1)^d \Big(\chi(D_{n,m}^{n-r}) - 2\chi(D_{n,m-1}^{n-r}) + \chi(D_{n,m-2}^{n-r}\Big).$$

Euler characterics of smooth degeneracy loci have been computed by Pragacz [22]. For $\varphi : F \to E$ a morphism of vector bundles of ranks $f, e$ over a variety $X$, the formula is

$$\chi(D_r(\varphi)) = \int_X P_r(E,F)c(X),$$

where $c(X)$ denotes the total Chern class, while $P_r(E,F)$ is a universal polynomial in the Chern classes of $E$ and $F$:

$$P_r(E,F) = \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} D_{\lambda,\mu}^{n-r,m-r} s_{(m-r)^{n-r}+\lambda,\tilde{\mu}}(E-F),$$

where the sum is over partitions $\lambda$ and $\mu$ of length $n-r$ and $m-r$ respectively, and $\tilde{\mu}$ is the dual partition of $\mu$.

We want to apply this formula to $D_{n,m}^{n-r}$, which we consider as the degeneracy locus $D_{n-r}(\varphi)$ of the tautological morphism $\phi : F = \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n}$ over $X = \mathbb{P}^m$. Since $c(\mathbb{P}^m) - 2hc(\mathbb{P}^{m-1}) + h^2 c(\mathbb{P}^{m-2}) = (1+h)^{m-1}$, if $h$ denotes the hyperplane class, we get the formula

$$\deg(D_{n,m}^{n-r})^* = \sum_{\lambda,\mu} \binom{m-1}{r^2 + |\lambda| + |\mu|} D_{\lambda,\mu}^{r,r} s_{r^r+\lambda,\tilde{\mu}}(\underbrace{1,\ldots,1}_{n \text{ times}}),$$

the sum being taken over partitions $\lambda$ and $\mu$ of length $r$. Note that the dependence on $n$ for $r$ and $m$ fixed is only in the last term, more precisely in the number of one's on which the Schur funbctions are evaluated. This dependence is well known to be polynomial in $n$; very explicitely, for any partition $\nu$,

$$s_\nu(\underbrace{1,\ldots,1}_{n \text{ times}}) = \dim S_\nu \mathbb{C}^n = c_\nu(n)/h(\nu),$$

where $c_\nu$ is the content polynomial and $h(\nu)$ is the product of the hooklengths of $\nu$. A priori this formula is ony valid in the range $r^2 \leq m \leq r^2 + 2r$, when $D_{n,m}^{n-r}$ is smooth. Could it be true in general? That would be similar to the NRS conjecture.

## 7. Future directions and Conjectures

For symmetric matrices, the NRS conjecture asserts that when $D_{n,m}^{n-r}$ is not dual defective, that is, in the range

$$\binom{r+1}{2} \leq m \leq \binom{r+1}{2} + r(n-r),$$

its codegree depends polynomially on $n$ when $r$ and $m$ are fixed. What does happen in the defective cases? For example one has the formula

$$\operatorname{codegree}(D^r_{n,\binom{n+1}{2}}) = \operatorname{degree}(D^{n-r}_{n,\binom{n+1}{2}}) = \prod_{j=1}^{r} \frac{\binom{n+j-1}{r-j+1}}{\binom{2j-1}{j-1}},$$

which is clearly polynomial in $n$. Note that in this case the dual defect is $\binom{r+1}{2} - 1$, independently of $n$. Could it happen that the following holds?

**Conjecture.** *For any fixed $k \geq 0$,*

*(1) the dual defect of $D^r_{n,\binom{n+1}{2}-k}$ is equal to $\max(0, \binom{r+1}{2} - 1 - k)$, independently of $n$,*

*(2) the codegree of $D^r_{n,\binom{n+1}{2}-k}$ depends polynomially on $n$.*

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