Orlando Marigliano Fall Term 2022

Commutative Algebra and Algebraic Geometry
Challenging Exercises 1

These exercises are purely optional and give no extra credit. Hand in your solutions directly to me at `orlandom@kth.se`. There is no due date. The reward for completing a challenging exercise sheet is the next challenging exercise sheet.

Exercise 1. Let K be a field and $f \in K[t_1, \ldots, t_n]$ be a polynomial in n variables. Then we can see f as a function $K^n \to K$ by sending a point $x = (x_1, \ldots, x_n)$ to $f(x) \coloneqq f(x_1, \ldots, x_n)$. The *vanishing set* of f is $V(f) \coloneqq \{x \in K^n \mid f(x) = 0\}$.

(1) Let K be infinite. Show that $V(f) = K^n$ if and only if $f = 0$.
(2) Let K be algebraically closed. Show that $V(f) = \emptyset$ if and only if f is a unit.

In the following, k shall always denote an algebraically closed field. Recall that a *k-algebra* is a ring A together with a ring homomorphism $k \to A$, and that a *homomorphism* of k-algebras $A \to B$ is a ring homomorphism compatible with the structure homomorphisms $k \to A$ and $k \to B$. An example of a k-algebra is the ring $k[t_1, \ldots, t_n]$.

Exercise 2. An *affine algebraic variety* over k is a subset $X \subseteq k^n$ of the form

$$X = V(E) \coloneqq \{x \in k^n \mid f(x) = 0 \text{ for all } f \in E\}$$

for some $n \in \mathbb{N}$ and $E \subseteq k[t_1, \ldots, t_n]$. If X is an affine variety, we write

$$I(X) \coloneqq \{f \in k[t_1, \ldots, t_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

A *regular function* on X is a function $X \to k$ of the form $x \mapsto f(x)$ for some polynomial $f \in k[t_1, \ldots, t_n]$. We set

$$A(X) \coloneqq \{f : X \to k \mid f \text{ regular}\}.$$

(1) Briefly show that $A(X)$ is naturally a k-algebra. Show that $I(X)$ is an ideal and show that there is a k-algebra isomorphism
$$A(X) \simeq k[t_1, \ldots, t_n]/I(X).$$
(2) Show that for all affine varieties X and ideals $\mathfrak{a} \subseteq k[t_1, \ldots, t_n]$ we have
$$\operatorname{rad} \mathfrak{a} \subseteq I(V(\mathfrak{a})) \quad \text{and} \quad X = V(I(X)).$$
(3) Let $E \subseteq k[t_1, \ldots, t_n]$ be a subset and $\mathfrak{a}$ the ideal generated by E. Show that
$$V(E) = V(\operatorname{rad} \mathfrak{a}).$$

Exercise 3. A *morphism* between two affine varieties $X \subseteq k^n$ and $Y \subseteq k^m$ is a map $f : X \to Y$ of the form $x \mapsto (f_1(x), \ldots, f_m(x))$, for some polynomials $f_1, \ldots, f_m$ in $k[t_1, \ldots, t_n]$.

(1) Let $f : X \to Y$ be a map and define $f^\sharp : A(Y) \to A(X)$ by $\eta \mapsto f \circ \eta$. Show that f is a morphism if and only if $f^\sharp$ is a k-algebra homomorphism.
(2) Show that $f \mapsto f^\sharp$ defines a bijection between the set of morphisms $X \to Y$ and the set of k-algebra homomorphisms $A(Y) \to A(X)$. What is its inverse?