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Commutative algebra and algebraic geometry – Exercises

There will be one exercise per week. Exercise n is *released* on Week n of the lecture and *due* on Week $n+1$. Week 1 of the lecture corresponds to calendar week 35. The deadline for handing in exercises is on Wednesdays at 11:59 PM. Exercises are to be handed in *online* via the course website. Please write your solutions in LaTeX if you can.

Grading: At the exam, you will get a percentile score. Regularly handing in solutions to these exercises will *improve* this score, adding an amount of percentage points on top of it. Your final grade will be determined by this combined score. The specifics of this system are not yet fixed, but will be communicated around the middle of the term.

TA session exercise. Let K be a field and $f \in K[t_1, \ldots, t_n]$ be a polynomial in n variables. Then we can see f as a function $K^n \to K$ by sending a point $x = (x_1, \ldots, x_n)$ to $f(x) \coloneqq f(x_1, \ldots, x_n)$. The *vanishing set* of f is $V(f) \coloneqq \{x \in K^n \mid f(x) = 0\}$.

(1) Let K be infinite. Show that $V(f) = K^n$ if and only if $f = 0$.
(2) Let K be algebraically closed. Show that $V(f) = \emptyset$ if and only if f is a unit.

In the following, k shall always denote an algebraically closed field.

Let $\mathfrak{a}$ be an ideal of a ring A. A *set of generators* for $\mathfrak{a}$ is a subset $E \subseteq \mathfrak{a}$ such that $\mathfrak{a} = (E) \coloneqq \{\sum_{i=1}^n x_i a_i \mid x_i \in A, a_i \in E\}$. For a subset $E \subseteq A$, it can be shown that the set (E) is the smallest ideal of A that contains E. The set (E) is called the *ideal generated by E*. If $E = \{a_1, \ldots, a_k\}$ is finite, we write $(E) = (a_1, \ldots, a_k)$.

Exercise 1. Let $A \coloneqq k[t_1, \ldots, t_n]$ and let $x \in k^n$ be a point.

(1) Show that the evaluation map $\varphi_x : A \to k$ defined by $\varphi_x(f) = f(x)$ is a surjective ring homomorpism.
(2) Let $\mathfrak{m}_x \coloneqq \ker(\varphi_x)$. Give a finite set of generators for $\mathfrak{m}_x$.
(3) Show that $\mathfrak{m}_x$ is a maximal ideal of A.
(4) Let $y \in k^n$ be another point. Show that if $x \neq y$ then $\mathfrak{m}_x \neq \mathfrak{m}_y$.
(5) Let

$$A_x \coloneqq \left\{ \frac{f}{g} \mid f, g \in A, g(x) \neq 0 \right\}.$$

This is a subring of the function field $k(t_1, \ldots, t_n)$. Show that A_x is a local ring.