Counting Covers of Elliptic Curves

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1. Introduction

Let E be an elliptical curve over $\mathbb{C}$ and fix a genus $g \geq 1$. The main interest of this thesis are the simply ramified morphisms $p\colon C \to E$, where C is an arbitrary smooth irreducible complex curve of genus g. This thesis seeks to count such morphisms, in the way described below.

For any $d \geq 1$, the isomorphism classes of morphisms p of degree d may be classified by applying the theory of covering spaces in topology to the ramified covers (i. e. the morphisms) of interest. One finds that there are (up to isomorphism) finitely many such covers, so that one may ask about their number. Since a cover may have nontrivial automorphisms, a natural way to count covers would be to first apply the weighting $p \mapsto 1/|\operatorname{Aut}(p)|$ to any cover p. The goal is to study the generating series $F_g(q)$ over the weighted counts $N_{g,d}$ of (connected) covers of genus g and degree d.

The main theorem of this thesis states that for $g \geq 2$, if q is viewed as the complex variable $\exp(2\pi i\tau)$ and the generating series $F_g(q)$ as a function of τ, then the function F_g is a *quasimodular form* of weight $6g-6$. This is one of the main theorems of [Dij95].

The concept of quasimodularity generalizes that of modularity; the definition here used comes from [KZ95]. The same article also contains a theorem that will provide the last step to the proof of the main theorem. As for getting to the last step, [Dij95] contains the sketch of an argument, expanded upon in [Rot09], for turning the generating series F_g into something more useful, i. e. to which one may apply the theorem in [KZ95]. The main part of this thesis follows the outline of [Rot09].

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Einleitung

Sei E eine elliptische Kurve über $\mathbb{C}$, sei $g > 1$ ein festes Geschlecht. Diese Arbeit befasst sich hauptsächlich mit den einfach verzweigten Morphismen $p\colon C \to E$, wobei C eine beliebige glatte irreduzible komplexe Kurve vom Geschlecht g ist. Ziel der Arbeit ist es, solche Morphismen auf die unten beschriebene Weise zu zählen.

Für $d > 1$ können die Isomorphieklassen von Morphismen p vom Grad d klassifiziert werden, indem man topologische Überlagerungstheorie auf die zu untersuchende verzweigte Überlagerungen (also auf die Morphismen) anwendet. Es

stellt sich heraus, dass es nur endlich viele solche Überlagerungen (bis auf Isomorphie) gibt, weshalb die Frage nach deren Anzahl aufkommt. Überlagerungen können nichttriviale Automorphismen haben; somit wäre eine natürliche Weise, die Überlagerungen p zu zählen, dadurch gegeben, dass man zunächst die Gewichtung $p \mapsto 1/|\operatorname{Aut}(p)|$ anwendet. Von Interesse ist die erzeugende Funktion $F_g(q)$ über die gewichteten Anzahlen $N_{g,d}$ der (zusammenhängenden) Überlagerungen vom Geschlecht g und Grad d.

Das Hauptresultat dieser Arbeit besagt, dass für $g \geq 2$, falls man q als die komplexe Variable $\exp(2\pi i\tau)$ und die erzeugende Reihe $F_g(q)$ als Funktion von τ auffasst, die Funktion F_g eine *quasimodulare Form* vom Gewicht $6g - 6$ ist. Dieses Ergebnis ist eine der Hauptaussagen in [Dij95].

Der Begriff der Quasimodularität verallgemeinert den der Modularität, und ist aus [KZ95] übernommen. Dieser Artikel liefert außerdem einen Satz, der den letzten Schritt des Beweises des Hauptresultats ermöglicht. Zum Erreichen dieses letztes Schrittes findet sich in [Dij95] die Skizze eines Arguments, welches in [Rot09] ausgeführt wird. Mit diesem kann die erzeugende Reihe F_g so umgeformt werden, dass sich der Satz aus [KZ95] anwenden lässt. Der Hauptteil dieser Arbeit orientiert sich an [Rot09].

2. Quasimodular forms

This section introduces quasimodular forms as described in [KZ95].

2.1. The space of modular forms

Let $\mathcal{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane. For $\tau \in \mathcal{H}$, define $q = \exp(2\pi i\tau)$ and $Y = 4\pi\,\mathrm{Im}(\tau)$. Further, let $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{C})$ denote the full modular group. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Definition 2.1. Let $f\colon \mathcal{H} \to \mathbb{C}$ be a function, let $k \in \mathbb{Z}$.

1. The function f is $\mathbb{Z}$-*periodic*, if it satisfies $f(\tau + 1) = f(\tau)$ for all $\tau \in \mathcal{H}$. In this case there exists a function $\tilde{f}\colon B \setminus \{0\} \to \mathbb{C}$, defined on the open unit ball $B \subset \mathbb{C}$ with the origin removed, such that $f(\tau) = \tilde{f}(q)$ for all τ. Now let f be holomorphic. Then so is $\tilde{f}$. We say that f is *holomorphic at infinity*, if $\tilde{f}$ has a holomorphic continuation to the whole of B.

2. The function f is said to satisfy the *modular condition of weight* k, if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for all τ in $\mathcal{H}$ and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Such a function is $\mathbb{Z}$-periodic, as can be seen by setting $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

3. The function f is a *modular form (of weight* k*)* if it is holomorphic, satisfies the modular condition and is holomorphic at infinity.

Note that if k is odd, then any function satisfying the modular condition of weight k is zero. This follows by using the modular condition with $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. There are several alternate conventions for handling the weights k. Some authors for instance replace k by $2k$ throughout, so that "modular forms of weight $2k$" are considered. This is the convention used by [Ser12a].

The modular forms of weight k form a vector space, denoted[1] by M_k. Multiplying two modular forms of weights k respectively l yields a modular form of weight $k + l$, giving the space $\bigoplus_k \mathrm{M}_k$ the structure of a graded ring, denoted by M_*.

Examples 2.2. For an even integer $k \geq 2$, the *Eisenstein series*[2] *of weight* k is the function

$$E_k(\tau) = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

[1] In [Ser12a], the space of modular forms of weight $2k$ is denoted by M_k.

where b_k is the k-th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$. By definition, these functions are holomorphic at infinity.

For $k \geq 4$, the Eisenstein series of weight k is a modular form of weight k. One proves this for example by showing that for $k \geq 4$, the series E_k is a multiple of the function $G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\smallsetminus(0,0)}(m\tau+n)^{-k}$, which is indeed modular of weight k, see [Ser12a, Ch. VII, Prop. 8] and [Ser12a, Ch. VII, 2.3].

The function $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$ is a modular form of weight 12. By a theorem of Jacobi [Ser12a, Ch. VII, Thm. 6], one has

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q)^{24}.$$

Proposition 2.3. *We have the following dimension formula:*

$$\dim(\mathrm{M}_k) = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \geq 0 \text{ and } k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \geq 0 \text{ and } k \not\equiv 2 \pmod{12} \\ 0 & \text{if } k < 0 \end{cases}.$$

Proof. An elementary approach is given in [Ser12a, Ch. VII, Corollary 1]. Alternatively, this formula arises as a corollary (8.39) to a more general formula proven in the Appendix, Theorem 8.38. □

Proposition 2.4. *There is an isomorphism of graded rings*

$$\mathbb{C}[X_4, X_6] \xrightarrow{\sim} \mathrm{M}_*$$

mapping X_i to E_i, where the former ring is graded by assigning to X_i the degree i.

Proof. See [Ser12a, Ch. VII, Corollary 2]. □

2.2. The space of quasimodular forms

Let $\mathcal{O}(\mathcal{H})$ denote the vector space of $\mathbb{C}$-valued holomorphic functions on $\mathcal{H}$. Recall the imaginary part function $Y(\tau) = 4\pi\,\mathrm{Im}(\tau)$. The following proposition shows that one may compare coefficients of elements of $\mathcal{O}(\mathcal{H})[Y^{-1}]$ as if Y was a formal variable.

Proposition 2.5. *Let $F = \sum_{m=0}^{M} f_m Y^{-m}$ be an element of $\mathcal{O}(\mathcal{H})[Y^{-1}]$. If $F = 0$, then $f_m = 0$ for all m.*

[2] In [Ser12a], the Eisenstein series of weight k as defined below is denoted by $E_{k/2}$. A similar remark applies to the function G_k below.

Proof. For the differential operator $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}$ one has $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}Y^{-m} = -2\pi i m Y^{-m-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}f_m = 0$, hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}F(\tau) = -2\pi i \sum_{m=1}^{M} f_m(\tau)Y^{-m-1} = -2\pi i Y^{-2}(\sum_{m=0}^{M-1} f_{m+1}\tau Y^{-m}).$$

By induction this implies that the f_m are zero for $m \geq 1$, hence also $f_0 = 0$. □

Corollary 2.6. *Let $F = \sum_{m=0}^{M} f_m Y^{-m}$ be an element of $\mathcal{O}(\mathcal{H})[Y^{-1}]$ satisfying the modular condition of weight k. Then the f_m are $\mathbb{Z}$-periodic.*

Definition 2.7. An *almost holomorphic modular form (of weight k)* is an element

$$F = \sum_{m=0}^{M} f_m Y^{-m}$$

of $\mathcal{O}(\mathcal{H})[Y^{-1}]$ such that F satisfies the modular condition and the $f_m \colon \mathcal{H} \to \mathbb{C}$ are holomorphic at infinity.

Proposition 2.8. *Let $F(\tau) = \sum_{m=0}^{M} f_m(\tau)Y^{-m}$ be an almost holomorphic modular form. Then the leading coefficient f_M is a modular form of weight $k - 2M$. In particular, if $f_M \neq 0$, then $2M \leq k$.*

Proof. This follows after comparing the coefficients of Y^{-M} in both sides of the modularity condition $F(\gamma\tau) = (c\tau + d)^k F(\tau)$, using the equality

$$Y^{-1}(\gamma\tau) = (c\tau + d)^2 Y(\tau)^{-1} + \frac{c(c\tau + d)}{2\pi i}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$. □

The almost holomorphic modular forms of weight k form a vector space, denoted by $\widehat{\mathrm{M}}_k$. Let $\widehat{\mathrm{M}}_*$ denote the associated graded ring.

Definition 2.9. An element in the image of the map $\widehat{\mathrm{M}}_k \to \mathcal{O}(\mathcal{H})$ taking an almost holomorphic modular form $F = \sum_{m=0}^{M} f_m Y^{-m}$ of weight k to f_0 is called a *quasimodular form of weight k*. Hence a quasimodular form is a holomorphic function on the upper plane appearing as the constant term of an almost holomorphic modular form.

Again, denote the vector space of quasimodular forms of weight k by $\widetilde{\mathrm{M}}_k$ and the associated graded ring by $\widetilde{\mathrm{M}}_*$. The definition gives a surjective graded ring homomorphism $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ and one has $\widehat{\mathrm{M}}_k \cap \widetilde{\mathrm{M}}_k = \mathrm{M}_k$, the intersection being taken in the set of functions $\mathcal{H} \to \mathbb{C}$.

Example 2.10. Consider the second Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $\sigma_1(n) = \sum_{d|n} d$. For the weight 12 modular form $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q)^{24}$, one has the identity $2\pi i E_2(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\log(\Delta(\tau))$, which is proven by a straightforward computation. Using the modularity of Δ, one then computes

$$E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) + \frac{6c(c\tau+d)}{\pi i},$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$.

Now, since $Y^{-1}(\gamma\tau) = (c\tau+d)^2 Y(\tau)^{-1} + \frac{c(c\tau+d)}{2\pi i}$, it follows that $E_2^* = E_2 - 12/Y$ is an almost holomorphic modular form of weight 2. Hence, E_2 is a quasimodular form of weight 2.

Proposition 2.11. *The space $\widetilde{\mathrm{M}}_*$ of quasimodular forms satisfies the following properties.*

1. *The canonical graded homomorphism $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ is an isomorphism.*

2. *There is an isomorphism of graded rings $\mathrm{M}_* \otimes \mathbb{C}[X_2] \simeq \mathbb{C}[X_2, X_4, X_6] \to \widetilde{\mathrm{M}}_*$ mapping X_i to E_i, where the former ring is graded by assigning to X_i the degree i.*

3. *Quasimodular forms are closed under taking derivatives. More precisely, the derivative of a quasimodular form of weight k is a quasimodular form of weight $k+2$.*

Proof.

1. The map $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ is surjective by definition. Injectivity follows from Calculation 2.12 below. Given an almost holomorphic modular form $F(\tau) = \sum_{m=1}^{M} f_m(\tau) Y^{-m}$ with constant term zero, the strategy is to solve the modularity equation for the coefficients f_m. This way, one finds for a fixed argument τ a polynomial equation in the lower row components c, d of any transformation $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, involving the coefficients $f_m(\tau)$. By varying the transformation γ, one may force these coefficients to be zero.

2. Express the map $\mathbb{C}[X_2, X_4, X_6] \to \widetilde{\mathrm{M}}_*$ as the composition

$$\mathbb{C}[X_2^*, X_4, X_6] \to \widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*,$$

where the first map takes X_2^* to E_2^* and X_i to E_i, and the second map is the canonical map, which is an isomorphism by the first point above.

To prove the surjectivity of the first map, let $F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$ be an almost holomorphic modular form of weight k. Then $f_M (E_2^*/12)^M$ is an almost holomorphic modular form of weight k, since f_M is modular of weight $k-2M$, and the difference $F - f_M(E_2^*/12)^M$ has degree smaller than M. Now use induction on M.

To get injectivity, let $F = \sum_{\alpha=0}^{k/2} (E_2^*)^\alpha f_{k-2\alpha}$ be an almost holomorphic modular form of weight k, in the image of the first map, where the f_m are modular of

weight m. If $F = 0$, then by comparing the coefficients of $Y^{-k/2}$ one obtains $0 = f_0$. Now it follows by induction on k that the other coefficients f_m are zero. Hence F was the image of the zero element in $\mathrm{M}_* \otimes \mathbb{C}[X_2^*]$.

3. To prove the last statement, one verifies that $(6/\pi i)E_2' - E_2^2$ is modular of weight 4, and that if f is modular of weight k, then $(6/\pi i)f' - kE_2 f$ is modular of weight $2 + k$. Now use the second point above. □

Calculation 2.12. This calculation follows the one found in [BO00]. Let $F(\tau) = \sum_{m=1}^{M} f_m(\tau)Y^{-m}$ be an almost holomorphic modular form, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$, and $\tau \in \mathcal{H}$. Write $j = c\tau + d$, and $a = 6cj/2\pi i$. Then $Y^{-1}(\gamma\tau) = a + j^2 Y(\tau)^{-1}$. Hence,

$$\begin{aligned} F(\gamma\tau) &= \sum_{m=1}^{M} f_m(\gamma\tau)(a + j^2 Y^{-1})^m \\ &= \sum_{m=1}^{M} \sum_{l=0}^{m} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} Y^{-l} \\ &= \sum_{m=1}^{M} f_m(\gamma\tau)a^m + \sum_{l=1}^{M} \sum_{m=l}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} y^{-l}. \end{aligned}$$

On the other hand,

$$F(\gamma\tau) = \sum_{l=1}^{M} f_l(\tau) j^k Y^{-l},$$

by the modularity condition. By comparing the coefficients of Y^{-l}, one obtains the equalities

$$\sum_{m=1}^{M} f_m(\gamma\tau)a^m = 0 \tag{1}$$

and

$$j^k f_l(\tau) = \sum_{m=l}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l}.$$

Rewriting the second equality yields

$$f_l(\gamma\tau) = f_l(\tau) j^{k-2l} - \sum_{m=l+1}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l}. \tag{2}$$

The latter may be solved recursively, starting by f_M, to get equalities of the form

$$f_l(\gamma\tau) = (\text{a polynomial in the } f_{\geq l}(\tau)\ ,\ j \text{ and } c). \tag{3}$$

The first two equalities are

$$\begin{aligned} f_M(\gamma\tau) &= f_M(\tau) j^{k-2M} \\ f_{M-1}(\gamma\tau) &= f_{M-1}(\tau) j^{k-2M+2} - \text{const} \cdot f_M(\tau) j^{k-2M+1} c. \end{aligned}$$

In general, a straightforward inductive argument shows that in the summands of the expression (2) for $f_l(\gamma\tau)$, the variable j appears with a power lower than or equal to $k-2l$. Now let r be the greatest index such that $f_r \neq 0$. Equation (1) finally gives, after substituting back the expressions for j and a and using (2) for $l=r$, the relation

$$\begin{aligned}
0 &= \kappa_1 f_r(\gamma\tau)(c\tau+d)^r c^r + \sum_{l=r+1}^{M} \kappa_3 f_l(\gamma\tau)(c\tau+d)^l c^l \\
&= \kappa_1 f_r(\tau)(c\tau+d)^{k-r} c^r - \\
&- \sum_{m=r+1}^{M} \kappa_2 \binom{m}{r} f_m(\gamma\tau)(c\tau+d)^{m-r} c^{m-r} + \sum_{l=r+1}^{M} \kappa_3 f_l(\gamma\tau)(c\tau+d)^l c^l,
\end{aligned}$$

where the κ_i are some nonzero constants. To obtain a contradiction, choose a point τ in the upper half-plane and consider the last relation as a polynomial equation in c and d, letting $P(c,d)$ denote the right-hand side of the equation. First look for the possible coefficients of monomials of the form $c^r d^{\geq 1}$. This excludes the third summand from the picture, since there c will always appear with a power greater than r. Next look for the possible coefficients of the monomial $c^r d^{k-r}$. As seen when recursively solving the equations for $f_l(\gamma\tau)$, the second summand will include only terms where $(c\tau+d)$ appears with a power lower than $k-r$. Hence the coefficient of $c^r d^{k-r}$ in $P(c,d)$ is $\kappa_1 f_r(\tau)$.

Now, if $c \in \mathbb{Z}$, then there are infinitely many $d \in \mathbb{Z}$ such that $P(c,d)=0$. Indeed, there are infinitely many d with $\gcd(c,d)=1$. For these d, find $a,b \in \mathbb{Z}$ such that $ad-bc=1$. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, it follows that $P(c,d)=0$. Similarly, for all $d \in \mathbb{Z}$, there are infinitely many c such that $P(c,d)=0$. It this follows that $P(c,d)=0$ holds for all $c,d \in \mathbb{C}$. These remarks may be summarized by the statement that the set of all c,d belonging to the lower row of some matrix in $\mathrm{SL}_2(\mathbb{Z})$ is Zariski-dense in $\mathbb{C}^2$.

Concluding, since P is zero as a function on $\mathbb{C}^2$, it is also zero as a polynomial, hence the coefficient $\kappa_1 f_r(\tau)$ is zero. Since τ was arbitrary, one finds $f_r=0$, a contradiction.

3. Covers of an elliptic curve

In this section we define the central notions and objects of interest, i. e. finite covers of an elliptic curve with simple ramification type, the weighted counts of isomorphism classes thereof, and the generating functions associated to such weighted counts.

In the following, let $\mathbb{C}$ be the ground field for all varieties considered. We begin by recalling some basic properties of complex curves.

Proposition 3.1. *The assignment $C \mapsto K(C)$ defines a contravariant equivalence of categories between the category of irreducible smooth curves over $\mathbb{C}$ and the category of finitely generated field extensions of $\mathbb{C}$ of trascendence degree one. By definition, degree d maps of curves correspond to degree d field extensions.*

Proof. See [Sil09, pp.20-22] □

Proposition 3.2 (Riemann–Hurwitz formula)**.** *Let $\varphi\colon C_1 \to C_2$ be a finite map of smooth curves of genera g_1 and g_2, respectively. Let d be the degree of φ. Then*

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{x \in C_1} (\mathrm{e}_\varphi(x) - 1),$$

where $\mathrm{e}_\varphi(x)$ is the ramification index of φ at x.

Proof. See [Sil09, Thm. 5.9] or [Lam09, 7.2.1]. □

Definition 3.3. Let E be an elliptic curve.

1. A *cover* of E is a finite morphism $p\colon C \to E$ of a disjoint union $C = \cup_{i=1}^{k} C_i$ of k irreducible smooth curves C_i. We shall denote the genus of C by g and the degree of p by d. Often a cover will be referred to by its source C.

2. Let $S = \{b_1, \ldots, b_{2g-2}\}$ be a set of $2g-2$ distinct points of E. A cover C of genus g is *simply branched over* S, if it is simply branched over each point of S. This means that for all points b of S there is exacly one point x in $p^{-1}(b)$ with ramification index $\mathrm{e}_p(x) = 2$, the others having ramification index one.

It follows from the Riemann-Hurwitz formula of Proposition 3.2 that for a simply branched cover $C \to E$, every point not in the pre-image of S has ramification index one. This justifies the choice of the number of points in S.

3. Two covers C_1, C_2 are to be considered isomorphic, if there is an isomorphism $C_1 \to C_2$ commuting with the respective structure maps into E. Accordingly, define the automorphism group $\mathrm{Aut}(C)$ of the cover C to be the group of cover isomorphisms $C \to C$.

4. A *connected cover* is a cover with connected source C, i. e. with only one irreducible component.

Remark 3.4. Let $C = C_1 \cup \cdots \cup C_k$ be a cover of genus g with structure map p of degree d. For all i, let p_i be the connected cover defined by the restriction $p|_{C_i}$. Denote the genus of C_i by g_i and the degree of p_i by d_i. By the Riemann-Hurwitz formula, the maps p_i have $2g_i - 2$ ramification points on C_i. Hence, the following relations hold:

$$\sum_i d_i = d, \text{ and } \sum_i (2g_i - 2) = 2g - 2.$$

Proposition 3.5. *Let C be a connected cover of E. Then the automorphism group of C is finite.*

Proof. By Proposition 3.1, if C is a connected cover of E, then the elements of the group $\mathrm{Aut}(C)$ correspond to the automorphisms of the finite field extension $K(C)/K(E)$, of which only finitely many exist. □

Proposition 3.6. *Let $C = C_1 \cup \cdots \cup C_k$ be a cover, and $p_i := p|_{C_i}$. Then the automorphism group of C is given by the semidirect product*

$$\mathrm{Aut}(C) = \prod_i \mathrm{Aut}(C_i) \rtimes \Gamma,$$

where $\Gamma \subset \mathrm{Sym}\{C_1, \ldots, C_k\}$ is the subgroup generated by the automorphisms that permute isomorphic components. In particular, $\mathrm{Aut}(C)$ is finite.

Proof. The map $\mathrm{Aut}(C) \to \Gamma$ given by looking at the action of an automorphism on the set $\{C_1, \ldots, C_k\}$ is part of a short exact sequence

$$1 \longrightarrow \textstyle\prod_i \mathrm{Aut}(C_i) \longrightarrow \mathrm{Aut}(C) \longrightarrow \Gamma \longrightarrow 1$$

which admits a splitting $\Gamma \to \mathrm{Aut}(C)$ given by the inclusion. □

Remark 3.7. If the cover C is simply branched over S, then no two components of genus greater than one are isomorphic as connected covers, since any isomorphism would have to preserve ramification indices (see for example [Sil09, II, Prop. 2.6]), but no two components share a branched point over E. In particular, if there are no components of genus one, then $\Gamma = \{1\}$.

On the other hand, each component of genus one is unramified over E, and could be isomorphic to other components of genus one, in which case Γ is nontrivial.

Furthermore, note that the C_k need not be connected for the statement of the previous proposition to hold.

Definition 3.8. Let E be an elliptic curve, $S = \{b_1, \ldots, b_{2g-2}\}$ a set of $2g - 2$ distinct points of E.

1. Let $\mathrm{Cov}(E, S)_{g,d}$ be the set of isomorphism classes of covers of E of genus g and degree d that are simply branched over S.

2. Any isomorphism of two equivalent covers defines a bijection of their automorphism groups. This allows one to define the *weight* of the class $[C]$ to be the number $1/|\operatorname{Aut}(C)|$.

3. Define $\widehat{N}_{g,d}$ to be the weighted count

$$\widehat{N}_{g,d} := \sum_{C \in \operatorname{Cov}(E,S)_{g,d}} \frac{1}{|\operatorname{Aut}(C)|}$$

of the (classes of) covers of E.

4. Let $\operatorname{Cov}(E,S)^{\circ}_{g,d} \subset \operatorname{Cov}(E,S)_{g,d}$ be the subset of classes $[C]$ such that C is connected.

5. Similarly, define $N_{g,d}$ to be the the weighted count

$$N_{g,d} := \sum_{C \in \operatorname{Cov}(E,S)^{\circ}_{g,d}} \frac{1}{|\operatorname{Aut}(C)|}$$

of the connected covers of E. To shorten the notation, the elliptic curve E and the set of points S are omitted from the notation. It will turn out that $\widehat{N}_{g,d}$ and $N_{g,d}$ are finite and do not depend on the choice of E and S.

Definition 3.9. For any $g \geq 1$, define F_g to be the generating series

$$F_g(q) := \sum_{d \geq 1} N_{g,d}\, q^d$$

counting connected covers of genus g.

Example 3.10. By the theory of elliptic curves we have $N_{1,d} = \sum_{j|d} 1/j$. To see this, we use the fact that the covers of degree d and genus 1 of an elliptic curve defined by a lattice Γ of $\mathbb{C}$ correspond to the subgroups of Γ of index d. If d is prime, then the number of such subgroups is $d+1$. Furthermore, any such cover has exactly d automorphisms.

Let $\mathcal{R}$ be the set of lattices of $\mathbb{C}$ and $\mathbb{Z}\mathcal{R}$ the free abelian group generated by the set $\mathcal{R}$. For $n \in \mathbb{N}$, define the endomorphism $T(n)\colon \mathbb{Z}\mathcal{R} \to \mathbb{Z}\mathcal{R}$ by

$$T(n)\Gamma = \sum_{[\Gamma' : \Gamma] = n} \Gamma'.$$

It is shown in [Ser12a, Ch. VII, Prop. 10] that the $T(n)$ satisfy

$$T(m)T(n) = T(mn) \text{ for } (m,n) = 1$$

and

$$T(p^n)T(p) = T(p^{n+1}) + pT(p^{n-1})R_p \text{ for } p \text{ prime}, n \geq 1,$$

where R_p is the homotety operator $\Gamma \mapsto p\Gamma$. The number of index n subgroups of Γ is $c_1(T(n)\Gamma)$, where c_1 is the linear extension to $\mathbb{Z}\mathcal{R}$ of the constant function

$c_1 \colon \mathcal{R} \to \mathbb{Z}$, $\Gamma' \mapsto 1$ on $\mathcal{R}$. Now define the function $c \colon \mathbb{N} \to \mathbb{N}$ by $c(n) = c_1(T(n)\Gamma)$ and define $\sigma_1 \colon \mathbb{N} \to \mathbb{N}$ by $\sigma_1(n) = \sum_{j|n} j$. It follows from the above equations that the function c satisfies

$$c(m)c(n) = c(mn) \text{ for } (m,n) = 1$$

and

$$c(p^n)c(p) = c(p^{n+1}) + pc(p^{n-1}) \text{ for } p \text{ prime}, n \geq 1.$$

Since the function σ_1 satisfies the same conditions and since $c(p) = \sigma_1(p)$ holds for p prime, the two functions are equal. Finally, since any cover of degree d and genus 1 has exactly d automorphisms, we have $N_{1,d} = \sigma_1(d)/d = \sum_{j|d} 1/j$.

By using the power series expansion for the logarithm

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

for $|z| < 1$, we find $-\sum_{n\geq 1} \log(1-q^n) = \sum_{d\geq 1} \sum_{j|d} \frac{1}{j} q^d$. Hence, the first generating function is given by

$$F_1(q) = -\sum_{n\geq 1} \log(1-q^n).$$

This thesis shall present a proof the following result.

Theorem 3.11 ([Dij95])**.** *Let $g \geq 2$, and for $\tau \in \mathbb{C}$ let $q(\tau) = \exp(2\pi i \tau)$. Then the function $F_g(q)$ is a quasimodular form of weight $6g-6$.*

The function F_1 cannot be quasimodular. Indeed, if F_1 was quasimodular of some weight $k \geq 0$, then by Proposition 2.11 the derivative F_1' would be quasimodular of weight $k+2$. We have $F_1'(q) = 2\pi i \sum_d \sigma_1(d) q^d$, so $1 + (24/2\pi i) F_1' = E_2$. This is a contradiction since the same proposition implies that the sum of two quasimodular forms of different weights cannot be quasimodular.

The strategy to prove the main theorem will involve considering a more general generating function counting all covers of genus g and degree d. This generating function will be easier to compute.

Definition 3.12. The generating functions $Z(q,\lambda)$ and $\widehat{Z}(q,\lambda)$ for $N_{g,d}$ and $\widehat{N}_{g,d}$ respectively, are defined as follows:

$$Z(q,\lambda) := \sum_{g\geq 1} \sum_{d\geq 1} \frac{N_{g,d}}{(2g-2)!} q^d \lambda^{2g-2} = \sum_{g\geq 1} \frac{F_g(q)}{(2g-2)!} \lambda^{2g-2},$$

$$\widehat{Z}(q,\lambda) := \sum_{g\geq 1} \sum_{d\geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{2g-2}.$$

Lemma 3.13. *The above generating functions satisfy the relation*

$$\widehat{Z}(q,\lambda) = \exp(Z(q,\lambda)) - 1.$$

Proof. The proof is subdivided into three parts. First, some notation and terminology is introduced. Second, the coefficient of $q^d\lambda^{2g-2}$ in $\exp(Z(q,\lambda))-1$ is expressed in terms of the new notation. Third, combinatorial arguments are used to prove that this coefficient is equal to $\widehat{N}_{g,d}/(2g-2)!$.

Let C be a degree d, genus g cover. The *combinatorial type* of C is the tuple $\kappa=(k_j,g_j,d_j)_{j=1}^r$ of natural numbers, such that for each j, the space C contains exactly k_j connected components C_j of genus g_j such that the cover map $C_j\to E$ is of degree d_j. For simplicity, denote the Euler characteristics $2g-2$ and $2g_j-2$ by χ and χ_j, respectively. Then

$$\sum_j d_j = d, \text{ and } \sum_j \chi_j = \chi.$$

Further, define $\widehat{N}_\kappa$ to be the weighted count of the covers of combinatorial type κ. Then

$$\widehat{N}_{g,d} = \sum_{|\kappa|=(\chi,d)} \widehat{N}_\kappa,$$

where $|\kappa|$ is defined as the tuple $(\sum_j k_j\chi_j,\ \sum_j k_j d_j)$, for $\kappa=(k_j,g_j,d_j)_j$. For any j, we also define the "unweighted" count $\widetilde{N}_{g_j,d_j}$ of connected covers of degree of genus g_j and degree d_j by

$$\widetilde{N}_{g_j,d_j} := |\operatorname{Cov}(E,S')^\circ_{g_j,d_j}|,$$

where $S'\subset S$ is any subset of cardinality χ_j. Finally, note that the relation

$$q^d\lambda^\chi = \prod_{j=1}^r q^{k_jd_j}\lambda^{k_j\chi_j}$$

holds for each $\kappa=(k_j,g_j,d_j)_j$ with $|\kappa|=(\chi,d)$.

The exponential of $Z(q,\lambda)$ is given by

$$\exp(Z(q,\lambda)) = \prod_{g\geq 1}\prod_{d\geq 1}\sum_{k\geq 0}\frac{N_{g,d}^k}{k!(\chi!)^k}q^{kd}\lambda^{k\chi}.$$

Expanding, one finds that the expression for $\exp(Z(q,\lambda))$ is a sum over terms of the form

$$\prod_{j=1}^{<\infty}\left(\frac{N_{g_j,d_j}}{\chi_j!}\right)^{k_j}\frac{1}{k_j!}q^{k_jd_j}\lambda^{k_j\chi_j},$$

for some choices of parameters g_j,d_j,k_j. Such choices may be collected to form combinatorial types $\kappa=(g_j,d_j,k_j)_j$. Now, by collecting the summands arising from choices that induce combinatorial types of the same absolute value $|\kappa|$, one obtains that the coefficient of $q^d\lambda^\chi$ in $\exp(Z(q,\lambda))$ is equal to the sum $\sum_{|\kappa|=(\chi,d)}a_\kappa$, where

$$a_\kappa = \prod_{j=1}^r\left(\frac{N_{g_j,d_j}}{\chi_j!}\right)^{k_j}\frac{1}{k_j!}.$$

It remains to prove that $a_\kappa = \widehat{N}_\kappa/(2g-2)!$ for each combinatorial type κ. Here, we shall often make implicit use of Proposition 3.6 and the remark after it.

There are $\binom{\chi}{\chi_1,\chi_1,\ldots,\chi_r} = \chi! \prod_{j=1}^r 1/(\chi_j!)^{k_j}$ ways to subdivide the ramification locus S into subsets that serve as the ramification loci of the connected components. Here, each χ_j appears in the binomial coefficient k_j times. Fix one such decomposition, say $S = S_1 \cup \cdots \cup S_r$. Subdivide the decomposition further into subpartions such as $(S_{k_{j-1}+1}, \ldots, S_{k_{j-1}+k_j})$, where all subsets belong to one type (g_j, d_j, k_j) and have cardinality χ_j.

For $g_j \geq 2$, the number of covers of type (g_j, d_j, k_j) which ramify according to the subpartition $(S_{k_{j-1}+1}, \ldots, S_{k_{j-1}+k_j})$ is exactly $(1/k_j!)\widetilde{N}^{k_j}_{g_j,d_j}$. By tracking the automorphism group of the components involved in each choice, we obtain a weighted count of $(1/k_j!)N^{k_j}_{g_j,d_j}$.

For $g_j \geq 1$, there is no ramification locus. Instead, divide the k_j components of genus 1 into isomorphism classes, so that there are, say, ℓ isomorphism classes with cardinalities $t_1, \ldots, t_\ell$. There are $\binom{k_j}{t_1,\ldots,t_\ell} = k_j! \prod_{i=1}^\ell 1/t_j!$ ways to perform such a subdivision, so that the number of of covers of type (g_j, d_j, k_j) is $(1/k_j!)(\prod_{i=1}^\ell t_j!)\widetilde{N}^{k_j}_{g_j,d_j}$. Applying the weighting to an isomorphism class with t_i elements gives an additional factor of $1/t_i!$ in the weighting, since the cardinality of its automorphism group as a cover has an additional factor of $t_i!$. Hence, the weighted count of covers of type (g_j, d_j, k_j) is $(1/k_j!)(\prod_{i=1}^\ell t_j!)(\prod_{i=1}^\ell 1/t_j!)N^{k_j}_{g_j,d_j}$, which is equal to $(1/k_j!)N^{k_j}_{g_j,d_j}$.

Since covers of different types (g_j, d_j, k_j) are not isomorphic, we deduce that the weighted count of covers with ramification type determined by the partition $(S_1, \ldots, S_r)$ of S is $\prod_{j=1}^r (N^{k_j}_{g_j,d_j}/k_j!)$. Finally, since covers belonging to different partitions are not isomorphic and since the number of such partitions is $\chi! \prod_{j=1}^r 1/(\chi_j!)^{k_j}$, we deduce that $\widehat{N}_\kappa = \chi a_\kappa$, as required. □

4. Classifying covers via the fundamental group

The goal of this section is to use the theory of covering spaces to classify the covers we are interested in, i. e. the covers of genus g and degree d that are simply branched over S. This will get us to the first step to understanding their weighted count $\widehat{N}_{g,d}$.

4.1. Covering spaces

Definition 4.1. Let X be a topological space, F a set endowed with the discrete topology, and G a group acting on both X and F. Define the fibred product $X \times_G F$ to be the topological space $(X \times F) \,/\sim$, where $(x, f) \sim (gx, gf)$ for all g in G.

Proposition 4.2. *Let X be a connected, locally pathwise connected, and semi-locally simply connected topological space. Let $p\colon \widetilde{X} \to X$ be a universal cover. Furthermore, choose a point $\widetilde{x}_0$ of $\widetilde{X}$, and let x_0 be the image of $\widetilde{x}_0$ in X. Denote the fundamental group $\pi_1(X, x_0)$ by π_1. Then there is an eqivalence of categories*

$$\{\textit{Unbranched covers of } X\} \longrightarrow \{\pi_1\textit{-sets}\},$$

defined by the pair of quasi-inverse functors

$$(p_Y\colon Y \to X) \mapsto p_Y^{-1}(x_0) \quad \textit{and} \quad F \mapsto \widetilde{X} \times_{\pi_1} F.$$

Proof. One verifies by hand that the given functors are mutually quasi-inverse, by using elementary covering theory. Nonetheless, the needed isomorphisms between objects are given below.

Let F be a π_1-set and $p_F\colon \widetilde{X} \times_{\pi_1} F \to X$ the associated covering. Define a map $\zeta_F\colon F \to p_F^{-1}(x_0)$ by sending an element f to the class of $(\widetilde{x}_0, f)$.

On the other hand, let $p_Y\colon Y \to X$ be a cover of X. Define a map

$$\eta_Y\colon \widetilde{X} \times_{\pi_1} p_Y^{-1} \to Y$$

as follows. For a given class $(\widetilde{x}, f)$, let $\beta\colon [0,1] \to \widetilde{X}$ be a path starting in $\widetilde{x}_0$ and ending in $\widetilde{x}$. Consider the projection $p\beta$ of β to X and lift the path $p\beta$ to a path $\widetilde{\beta}_f$ in Y, with starting point f. Finally, set $\eta_Y(\widetilde{x}, f) = \widetilde{\beta}_f(1)$. Note that since $\widetilde{X}$ is simply connected, this is independent of the choice of the path β. Also, the map is well-defined, since $p\beta\widetilde{\gamma} = p\beta$ for any lift $\widetilde{\gamma}$ of a loop in X. □

Remark 4.3. In the above proposition, if X has the structure of a Riemann surface, then the first category may be taken to be the category of unbranched covers of Riemann surfaces over X. Indeed, every cover inherits a complex structure from X such that the structure map becomes holomorphic, and morphisms

of covers of X are automatically holomorphic. Indeed, if $g\colon C' \to C$ is a continuous map and $f\colon C \to X$ is an open and holomorphic map such that $f \circ g$ is holomorphic, then g is holomorphic; see [Lam09, 1.3.7].

Furthermore, let X be a Riemann surface, let $S \subset X$ be a finite set. Then putting $(C, p) \mapsto (C \setminus p^{-1}(S), p)$ defines an equivalence of categories between the category of finite covers of X with ramification locus contained in S and the category of finite unbranched covers of $X \setminus S$. The reason is roughly that the local data of an unbranched cover around a "missing" branch point uniquely characterizes that of any extention of that cover to a ramified one, e. g. the local degree of the cover map will correspond to the ramification index. The topic of extending unbranched covers to branched ones is discussed in detail in [Lam09, 4.6].

Corollary 4.4. *Let X be a connected Riemann surface, $S \subset X$ a finite set, $x_0 \in X \setminus S$ a point. Denote the fundamental group $\pi_1(X \setminus S, x_0)$ by π_1. There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{Finite ramified covers of } X \\ \textit{with ramification locus contained in } S \end{array} \right\} \longleftrightarrow \{\pi_1\textit{-sets}\}.$$

Remark 4.5. Let $p\colon C \to X$ be a ramified cover with ramification locus contained in S. Let $x_0 \in X \setminus S$ and $s \in S$. Let $\gamma \in \pi_1$ be a loop of degree one about s and $\theta\colon p^{-1}(x_0) \to p^{-1}(x_0)$ the permutation induced by γ. If θ decomposes in ℓ cycles, say $\theta = \theta_1 \cdots \theta_\ell$, having lengths $k_1, \ldots, k_\ell$ respectively, then the preimage $p^{-1}(s)$ consists of ℓ points $s_1, \ldots, s_\ell$ with ramification indices $k_1, \ldots, k_\ell$ respectively.

4.2. Marked covers and the monodromy map

Let E be an elliptic curve, $S = \{b_1, \ldots, b_{2g-2}\}$ a set of $2g-2$ distinct points of E. Fix a basis point $b_0 \in E \setminus S$, and denote the fundamental group $\pi_1(E \setminus S, b_0)$ by π_1. The equivalence of categories from the previous corollary is the equivalence

$$\left\{ \begin{array}{c} \text{Finite ramified covers of } E \\ \text{with ramification locus contained in } S \end{array} \right\} \longleftrightarrow \{\pi_1\text{-sets}\}.$$

Note that a simply branched cover of genus g is ramified over exactly $2g-2$ points of E. It follows that if its ramification locus S_0 is contained in S, then $S_0 = S$.

Definition 4.6. A *marking* m on a cover $(C, p) \in \mathrm{Cov}(E, S)_{g,d}$ is a bijective map $m\colon p^{-1}(b_0) \to \{1, \ldots, d\}$. A *marked cover* is a triple (C, p, m), where (C, p) is a cover and m is a marking on it.

Two marked covers (C_1, p_1, m_1) and (C_2, p_2, m_2) are considered equivalent, if there is an isomorphism of covers $\psi\colon C_1 \to C_2$ such that $m_1 = m_2\psi$. Let $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ denote the set of equivalence classes of marked covers with respect to this relation.

Definition 4.7. Let (C,p) be a cover of E. Denote the group action of π_1 on the fibre of $p^{-1}(b_0)$ by $(\gamma, x) \mapsto \gamma \,.\, x$. Define the monodromy map

$$\mathrm{mon}\colon \widetilde{\mathrm{Cov}}(E,S)_{g,d} \to \mathrm{Hom}(\pi_1, \mathfrak{S}_d)$$

by $\mathrm{mon}(C,p,m)(\gamma)(i) = m(\gamma \,.\, m^{-1}(i))$.

Let the symmetric group $\mathfrak{S}_d$ act on the first set by $\sigma \,.\, (C,p,m) = (C,p,\sigma m)$, and on the second by $\sigma \,.\, \varphi = \mathrm{inn}(\sigma)\varphi$, i.e. by inner automorphisms. Then mon becomes a morphism of $\mathfrak{S}_d$-sets. Furthermore, for $\varphi = \mathrm{mon}(C,p,m)$ in the image of mon, the group action "forgetting the marking"

$$m^{-1}\varphi(\_)m\colon \pi_1 \to \mathrm{Aut}(p^{-1}(b_0))$$

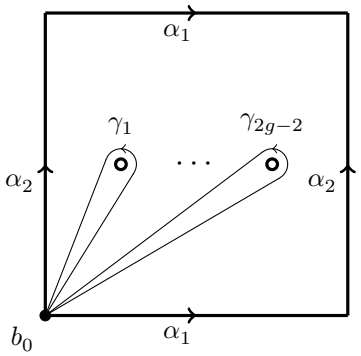
on the fiber of b_0 is the same as the one defined by the above equivalence of categories.

Definition 4.8. Let t denote the set of simple transpositions in $\mathfrak{S}_d$. Define the set

$$\widehat{T}_{g,d} = \{(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2) \in \mathfrak{S}_d^{2g}\,;\ \tau_i \in t \text{ for all } i,\ \tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\}.$$

We define an $\mathfrak{S}_d$-action on $\widehat{T}_{g,d}$ by conjugation in each component. This action is well-defined, since conjugates of transpositions are transpositions, and makes $\widehat{T}_{g,d}$ into an $\mathfrak{S}_d$-set.

Remark 4.9. We can describe the fundamental group π_1 of $E \setminus S$ using generators and relations. Choose a generating set $\{\alpha_1, \alpha_2\}$ of $\pi_1(E, b_0)$ such that the images of the α_i in E do not intersect the ramification locus S. Furthermore, for $i \in \{1, \ldots, 2g-2\}$ let γ_i' be a simple loop about the point b_i in S. For each i, there is a homotopy with image in $E \setminus S$ transforming γ_i' into a loop γ_i about b_i that contains the point b_0. The picture below illustrates the situation.



The loops we defined satisfy the relation $\gamma_1 \cdots \gamma_{2g-2} = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$. From the Seifert–Van Kampen theorem, it follows that the fundamental group π_1 is described by the following generating set and relation:

$$\pi_1 = \left\langle \gamma_1, \ldots, \gamma_{2g-2}, \alpha_1, \alpha_2 \,;\; \gamma_1 \cdots \gamma_{2g-2} = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1} \right\rangle.$$

In particular, any homomorphism of the group π_1 into any group H is uniquely determined by the images of the elements $\gamma_1, \ldots, \gamma_{2g-2}, \alpha_1$, and α_2. On the other hand, any tuple $(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2)$ of elements of H satisfying the relation $\tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$ defines a homomorphism of π_1 into H.

Proposition 4.10. *The image of* mon*, as a $\mathfrak{S}_d$-set, is isomorphic to $\widehat{T}_{g,d}$.*

Proof. Define the set $\widehat{T}'_{g,d}$ by

$$\widehat{T}'_{g,d} = \{(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2) \in \mathfrak{S}_d^{2g} \,;\; \tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\}.$$

By the previous remark there is a bijection $\psi\colon \operatorname{Hom}(\pi_1, \mathfrak{S}_d) \to \widehat{T}'_{g,d}$ given by $\varphi \mapsto (\varphi(\gamma_1), \ldots, \varphi(\gamma_{2g-2}), \varphi(\alpha_1), \varphi(\alpha_2))$. The map ψ is also a morphism of $\mathfrak{S}_d$-sets, where the $\mathfrak{S}_d$-action on $\widehat{T}_{g,d}$ is defined by compenent-wise conjugation. The preimage of $\widehat{T}_{g,d}$ under ψ consists of the homomorphisms $\varphi\colon \pi_1 \to \mathfrak{S}_d$ such that for all i, the permutation $\varphi(\gamma_i)$ is a simple transposition.

For covers that are simply branched over S, the image τ_i of each loop γ_i under the monodromy map is a simple transposition of the points in the fiber. Namely, there is over b_i exactly one branch point of index 2, and τ_i interchanges the two fiber points corresponding to the two sheets of the branching, leaving the other fiber points unchanged. Hence, the image of mon is contained in $\psi^{-1}(\widehat{T}_{g,d})$.

It is left to show that $\operatorname{Im}(\text{mon})$ contains $\psi^{-1}(\widehat{T}_{g,d})$. Let $\varphi\colon \pi_1 \to \mathfrak{S}_d$ be a homomorphism such that $\varphi(\gamma_i)$ is a simple transposition for all i. By the equivalence of categories at the beginning of the section, the π_1-action on $\{1, \ldots, d\}$ defined by φ gives a finite, branched cover of Riemann surfaces $p\colon C \to E$. The cover (C, p) also comes with a natural marking $m\colon p^{-1}(b_0) \to \{1, \ldots, d\}$, namely the inverse of the map $\zeta_{\{1,\ldots,d\}}$ defined in the proof of Proposition 4.2. By the same proposition, for all $\theta \in \pi_1$ and all $k, k' \in \{1, \ldots, d\}$, if $\theta \,.\, m^{-1}(k) = m^{-1}(k')$ then $k' = \varphi(\theta)(k)$. This shows that the π_1-action defined by (C, p, m) on the set $\{1, \ldots, d\}$ is the same as the action defined by φ. Finally, the condition that $\varphi(\gamma_i)$ be a simple transposition for all i implies by Remark 4.5 that the cover (C, p) is simply branched over S. Therefore, we have $(C, p, m) \in \widetilde{\operatorname{Cov}}(E, S)_{g,d}$ and $\text{mon}(C, p, m) = \varphi$. □

Define the morphism of $\mathfrak{S}_d$-sets $\rho\colon \widetilde{\operatorname{Cov}}(E, S)_{g,d} \to \widehat{T}_{g,d}$ to be the composition of mon with the isomorphism $\operatorname{Im}(\text{mon}) \xrightarrow{\sim} \widehat{T}_{g,d}$ of the previous proposition.

Proposition 4.11. *Let (C, p, m) be a marked cover and θ its image under ρ. Then there is a group isomorphism $\operatorname{Aut}_p(C) \xrightarrow{\sim} \operatorname{Stab}_{\mathfrak{S}_d}(\theta)$.*

Proof. Let $\varphi\colon \pi_1 \to \mathfrak{S}_d$ be the preimage of θ in the isomorphism of Proposition 4.10. By the equivalence of categories, the group $\mathrm{Aut}_p(C)$ is isomorphic to the group H of automorphisms of the π_1-action on $\{1,\dots,d\}$ defined by φ. The group H consists of all elements $\sigma \in \mathfrak{S}_d$ commuting with φ, i. e. such that $\varphi = \mathrm{inn}(\sigma)\varphi$. Since the isomorphism of Proposition 4.10 is $\mathfrak{S}_d$-equivariant, we have $H = \mathrm{Stab}_{\mathfrak{S}_d}(\theta)$. □

Proposition 4.12. *The morphism ρ induces a bijection on the sets of orbits*

$$_{\mathfrak{S}_d}\backslash\widetilde{\mathrm{Cov}}(E,S)_{g,d} \xrightarrow{\sim} {}_{\mathfrak{S}_d}\backslash\widehat{T}_{g,d}.$$

Proof. The map ρ is surjective as the composition of surjective maps, hence the induced map on the sets of orbits is surjective too.

For the injectivity on the sets of orbits, let $\rho(C_1,p_1,m_1) = \theta$ and $\rho(C_2,p_2,m_2) = \sigma\,.\,\theta$, for some $\theta \in \widehat{T}_{g,d}$ and $\sigma \in \mathfrak{S}_d$. Then $\rho(C_2,p_2,\sigma^{-1}m_2) = \theta$. It follows from the equivalence of categories that $(C_1,p_1) \simeq (C_2,p_2)$, and hence the that the two marked covers only differ by the marking, so that they are in the same $\mathfrak{S}_d$-orbit. □

Corollary 4.13. *The morphism ρ induces a bijection between $\mathrm{Cov}(E,S)_{g,d}$ and the set of $\mathfrak{S}_d$-orbits of $\widehat{T}_{g,d}$. In particular, the set $\mathrm{Cov}(E,S)_{g,d}$ is finite.*

Proof. The statement follows from the last proposition, since the $\mathfrak{S}_d$-orbits of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ are in one-to-one correspondence with the elements of $\mathrm{Cov}(E,S)_{g,d}$. □

Lemma 4.14. *The following equality for the weighted count $\widehat{N}_{g,d}$ holds:*

$$\widehat{N}_{g,d} = |\widehat{T}_{g,d}|/d!.$$

Proof. By Corollary 4.13 and Proposition 4.11, the weighted count $\widehat{N}_{g,d}$ is equal to the weighted count of the $\mathfrak{S}_d$-orbits of $\widehat{T}_{g,d}$, where each orbit is weighted by $1/|\,\mathrm{Stab}_{\mathfrak{S}_d}(\theta)|$, for any element θ in the orbit. Now, it follows from the formula $|\,\mathfrak{S}_d\,.\,\theta| = |\,\mathfrak{S}_d\,|/|\,\mathrm{Stab}(\theta)|$ that this weighted count equals $|\widehat{T}_{g,d}|/d!$. □

5. Conjugacy classes of the symmetric group

In this section, we further the computation of $\widehat{N}_{g,d}$ by using techniques inspired by graph theory. The rough picture is one of a graph with vertices the conjugacy classes of $\mathfrak{S}_d$ and edges representing the passage from one class to another by multiplication with a simple transposition. We seek to count not cycles, but cycles starting and ending with the same representative, in the sense specified below. To do this, we make use of an analogue of the adjacency matrix for a graph.

To abbreviate, we use the term "transposition" for simple transpositions. Again, we denote by t the set of simple transpositions in $\mathfrak{S}_d$.

5.1. Conjugacy cycles

Recall the definition

$$\widehat{T}_{g,d} = \{(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2) \in \mathfrak{S}_d^{2g} \,;\; \tau_i \in t \text{ for all } i,\, \tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\}.$$

Our aim is now to rewrite this definition using conjugacy classes. Note that the condition in the definition is equivalent to

$$(\tau_1 \cdots \tau_{2g-2})\sigma_2 = \sigma_1\sigma_2\sigma_1^{-1}.$$

In particular, it is necessary for the product $(\tau_1 \cdots \tau_{2g-2})\sigma_2$ to be conjugated to σ_2.

For $\sigma \in \mathfrak{S}_d$, denote the conjugacy class of σ in $\mathfrak{S}_d$ by $c(\sigma)$.

Definition 5.1. For $\sigma \in \mathfrak{S}_d$, define

$$P_{g,d}(\sigma) = \{(\tau_1, \ldots, \tau_{2g-2}) \in \mathfrak{S}_d^{2g-2} \,;\; \tau_i \in t \text{ for all } i,\, c(\tau_1 \cdots \tau_{2g-2}\sigma) = c(\sigma)\}.$$

If $g = 1$, define $P_{g,d}(\sigma)$ to be the singleton set $\{\bullet\}$.

Proposition 5.2. *Let $\mathcal{R} = \{\sigma^{(1)}, \ldots, \sigma^{(r)}\}$ be a system of (distinct) representatives of the conjugacy classes of $\mathfrak{S}_d$. Then*

$$|\widehat{T}_{g,d}| = \sum_{\sigma \in \mathcal{R}} d!|P_{g,d}(\sigma)|.$$

Proof. We begin by noting that if $(\tau_1, \ldots, \tau_{2g-2}) \in P_{g,d}$ and $\sigma \in \mathfrak{S}_d$, then there is a bijection

$$\{\sigma' \in \mathfrak{S}_d \,;\; (\tau_1 \cdots \tau_{2g-2})\sigma = \sigma'\sigma(\sigma')^{-1}\} \to \mathrm{Stab}_{\mathfrak{S}_d}(\sigma).$$

Indeed, choose an arbitrary element σ'_0 of the first set and define the bijection by sending σ' to $(\sigma'_0)^{-1}\sigma'$.

To count the elements in $\widehat{T}_{g,d}$, we fix σ and we ask first first how many tuples $(\tau_1, \ldots, \tau_{2g-2})$ satisfy the necessary condition of $(\tau_1, \ldots, \tau_{2g-2})\sigma$ being conjugate to σ, next we count the number of ways to realize such a conjugacy relation, which is exactly $|\operatorname{Stab}_{\mathfrak{S}_d}(\sigma)|$. We obtain

$$|\widehat{T}_{g,d}| = \sum_{\sigma \in \mathfrak{S}_d} |\operatorname{Stab}_{\mathfrak{S}_d}||P_{g,d}(\sigma)| = \sum_{\sigma \in \mathfrak{S}_d} \frac{d!}{|c(\sigma)|}|P_{g,d}(\sigma)|.$$

The function $|P_{g,d}|\colon \mathfrak{S}_d \to \mathbb{Z}$ is constant on conjugacy classes. Indeed, for $\sigma' \in \mathfrak{S}_d$ there is a bijection of $P_{g,d}(\sigma)$ onto $P_{g,d}(\sigma'\sigma(\sigma')^{-1})$ given by conjugation with σ' in each component. From this follows the required equality. □

The above proposition, together with Lemma 4.14, give the following reduction step:

Corollary 5.3. *We have the equality*

$$\widehat{N}_{g,d} = \sum_{\sigma \in \mathcal{R}} |P_{g,c}(\sigma)|.$$

From now on, let $\mathcal{R} = \{\sigma^{(1)}, \ldots, \sigma^{(r)}\}$ be a fixed system of representatives of the conjugacy classes of $\mathfrak{S}_d$. Then the cardinality $r = \operatorname{part}(d)$ of $\mathcal{R}$ is the number of (unordered) partitions of $\{1, \ldots, d\}$. This follows essentially from the fact that the conjugacy class of a permutation is uniquely determined by its *cycle shape*, that is the multiset $\{a_1, \ldots, a_s\}$ whose elements are the respective sizes of the s cycles making up the (reduced) cycle decomposition of the permutation. Such sets are in a one-to-one correspondence with the partitions of d, since each cycle entry $1, \ldots, d$ must appear exactly once in the cycle decomposition.

5.2. Adjacency matrices

Definition 5.4. Let $d \geq 1$ and $k \geq 0$.

1. For $1 \leq i, j \leq r$, define the sets $E^k_{d,i,j}$ by

$$E^k_{d,i,j} = \{(\tau_1, \ldots, \tau_k) \in \mathfrak{S}_d^k;\ \tau_i \in t \text{ for all } i,\ c(\tau_1 \cdots \tau_k \sigma^{(i)}) = c(\sigma^{(j)})\}.$$

For $k = 0$, define $E^0_{d,i,j}$ to be the empty set if $i \neq j$ and the singleton set if $i = j$.

2. Define the square matrix M_d by

$$(M_d)_{i,j} = |E^1_{d,i,j}|.$$

The matrix M_d is a square matrix of size r, its definition does not depend on the choice of system of representatives $\mathcal{R}$.

Remark 5.5. If k is odd, applying the signum to the defining condition shows that $E^k_{d,i,i}$ is empty. If $k = 2g - 2$ is even, then $E^{2g-2}_{d,i,i} = P_{g,d}(\sigma^{(i)})$.

Proposition 5.6. *The entries of M_d^k are given by $(M_d^k)_{i,j} = |E_{d,i,j}^k|$.*

Proof. The proof is by induction on k. For $k = 0, 1$, there is nothing to show. For the induction step, note that if i (resp. j) are fixed, the sets $E_{d,i,j}^k$ are pairwise disjoint for varying j (reps. i). We seek to define a bijection

$$\phi\colon \coprod_{l=1}^{r} E_{d,i,l}^k \times E_{d,l,j}^1 \to E_{d,i,j}^{k+1}.$$

First, for each element $(\tau_1, \ldots, \tau_k)$ of $E_{d,i,l}^k$, choose an element $\sigma \in \mathfrak{S}_d$ such that $\tau_1 \cdots \tau_k \sigma^{(i)} = \sigma \sigma^{(l)} \sigma^{-1}$. Second, define the image of $((\tau_1, \ldots, \tau_k), \tau_0)$ under ϕ to be $(\sigma\tau_0\sigma^{-1}, \tau_1, \ldots, \tau_k)$. By the definition of matrix multiplication, it suffices to prove that ϕ is a bijection.

Injectivity is clear. For surjectivity, given an element $(\tau_0', \tau_1, \ldots, \tau_k)$ in the target, choose an l such that $\tau_1 \cdots \tau_k \sigma^{(i)}$ is conjugate to $\sigma^{(l)}$, say $\tau_1 \cdots \tau_k \sigma^{(i)} = \sigma \sigma^{(l)} \sigma^{-1}$. Then $(\sigma^{-1}\tau_0\sigma)\sigma^{(l)}$ is conjugate to $\sigma^{(j)}$. □

Lemma 5.7. *Let $d \geq 1$ and $r = \mathrm{part}(d)$. Let $\mu_{1,d}, \ldots, \mu_{r,d}$ be the eigenvalues of M_d, listed according to their algebraic multiciplicities. Then*

$$\widehat{Z}(q, \lambda) = \sum_{d \geq 1} \sum_{i=1}^{r} \exp(\mu_{i,d}\lambda) q^d.$$

Proof. Recall the definition of $\widehat{Z}$:

$$\widehat{Z}(q, \lambda) = \sum_{g \geq 1} \sum_{d \geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{2g-2}.$$

The above proposition and remark give $(M_d^{2g-2})_{i,i} = |P_{g,d}(\sigma^{(i)})|$ and $(M_d^k)_{i,i} = 0$ if k is odd, for all i. Hence, by 5.3 one has $\widehat{N}_{g,d} = \mathrm{Tr}(M_d^{2g-2}) = \sum_{i=1}^{r} \mu_{i,d}^{2g-2}$, and since the terms for k odd vanish,

$$\begin{aligned}
\widehat{Z}(q, \lambda) &= \sum_{g \geq 1} \sum_{d \geq 1} \frac{\mathrm{Tr}(M_d^{2g-2})}{(2g-2)!)} q^d \lambda^{2g-2} \\
&= \sum_{d \geq 1} \sum_{i=1}^{r} \sum_{g \geq 1} \frac{\mu_{i,d}^{2g-2}}{(2g-2)!} \lambda^{2g-2} q^d \\
&= \sum_{d \geq 1} \sum_{i=1}^{r} \exp(\mu_{i,d}\lambda) q^d.
\end{aligned}$$

□

6. The group algebra of the symmetric group

Let $\mathbb{C}[\mathfrak{S}_d]$ be the group algebra of the symmetric group, let $\mathcal{Z}_d$ be its centre. This is a commutative algebra, acting on itself linearly by multiplication. In this section, we relate this linear action to the matrix M_d of the previous section, and we use the representation and character theory of the symmetric group to compute its eigenvalues.

6.1. The centre of the group algebra

Definition 6.1. Let $\mathcal{Z}_d \subset \mathbb{C}[\mathfrak{S}_d]$ be the centre of the group algebra. If c is a conjugacy class of $\mathfrak{S}_d$, define the element $z_c \in \mathcal{Z}_d$ by

$$z_c = \sum_{\sigma \in c} \sigma.$$

Remark 6.2. The elements z_c lie in the centre since $\alpha c = c\alpha$ for all conjugacy classes c and elements α of $\mathfrak{S}_d$. Further, the z_c form a basis of $\mathcal{Z}_d$. Indeed, linear independence follows from the linear independence of the distinct elements $\sigma \in \mathfrak{S}_d \subset \mathbb{C}[\mathfrak{S}_d]$. Furthermore, if $z \in \mathcal{Z}_d$, then the equalities $\alpha z \alpha^{-1} = z$ show that the $\mathbb{C}$-coefficients of elements in the same conjugacy class are equal. Hence $\mathcal{Z}_d$ is r-dimensional, with $r = \text{part}\,(d)$.

Recall the definition of M_d from the previous section. There, we fixed a system of representatives for the conjugacy classes of $\mathfrak{S}_d$. However, since the definition does not depend on the chosen representatives, we may also define M_d to be a matrix indexed by the conjugacy classes of $\mathfrak{S}_d$, ordered in the same way as before. The new, equivalent definition is as follows.

Definition 6.3. Let c', c be conjugacy classes of $\mathfrak{S}_d$. Define the matrix M_d by

$$(M_d)_{c',c} = |\{\tau\,;\ \tau \text{ is a transposition such that } \tau\sigma \in c'\}|,$$

where σ is any representative of c.

From now on, we choose the ordering of the basis $\{z_c\}_c$ and the ordering of the columns of M_d to be compatible, i. e. coming from the same fixed ordering of the conjugacy classes $\{c\}$.

Proposition 6.4. *Let t be the conjugacy class containing all transpositions. Let z_t be the corresponding basis element of $\mathcal{Z}_d$. Let M_t be the size r square matrix representing the $\mathbb{C}$-linear map $(z_t\,\cdot) : \mathcal{Z}_d \to \mathcal{Z}_d$ given by multiplication with z_t. Then the matrix M_t is the transpose of M_d.*

Proof. Let c, c' be conjugacy classes. Note that if $z = \sum_{\sigma \in \mathfrak{S}_d} \lambda_\sigma \sigma = \sum_{\tilde{c}} \lambda_{\tilde{c}} z_{\tilde{c}}$, then the coefficient $\lambda_{\tilde{c}}$ is equal to the coefficient λ_σ, for any $\sigma \in \tilde{c}$. Now let $\sigma \in c$,

and consider the product

$$z_t z_{c'} = \Big(\sum_{\tau\in t}\tau\Big)\Big(\sum_{\sigma'\in c'}\sigma'\Big) = \sum_{\sigma\in\mathfrak{S}_d}\sum_{\tau\sigma\in c'}\sigma.$$

In this expansion, the coefficient λ_σ of any element $\sigma \in c$ is the quantity

$$|\{\tau \in t\,;\ \tau\sigma \in c'\}|.$$

It follows that $(M_t)_{c',c} = \lambda_c = \lambda_\sigma = (M_d)_{c,c'}$. □

6.2. Irreducible characters of the symmetric group

We have reduced our problem of computing the eigenvalues of M_d to the computation of the eigenvalues of M_t. More generally, we find that $\mathcal{Z}_d$ actually has a basis $\{w_\chi\}$, indexed by the irreducible characters of $\mathfrak{S}_d$, such that each w_χ is an eigenvector for all linear maps defined by multiplication with any element of $\mathcal{Z}_d$, and such that the corresponding eigenvalues are easy to compute.

Definition 6.5.

1. Let ρ be an irreducible representation of $\mathbb{C}[\mathfrak{S}_d]$, i. e. a group homomorphism $\rho\colon \mathfrak{S}_d \to \mathrm{GL}(\mathbb{C}^n)$ such that for each $\sigma \in \mathfrak{S}_d$ there are no $\rho(\sigma)$-invariant subspaces. The *irreducible character associated to* ρ is defined as the map $\chi_\rho\colon \mathfrak{S}_d \to \mathbb{C}$, $\sigma \mapsto \mathrm{Tr}(\rho(\sigma))$.

2. An *irreducible character* of $\mathfrak{S}_d$ is a map $\chi\colon \mathfrak{S}_d \to \mathbb{C}$ of the form $\chi = \chi_\rho$ for some irreducible representation ρ. Its *dimension* $\dim(\chi)$ is defined as the dimension of the associated representation $\dim\rho = \chi(1)$.

For brevity, we will refer to irreducible characters simply as characters.

Remark 6.6. Characters are constant on conjugacy classes. It is therefore justified to write $\chi(c) \in \mathbb{C}$ for a character χ and a conjugacy class c.

Remark 6.7. The number of irreducible representations of a finite group, up to isomorphism, is equal to the number of its conjugacy classes (see for example [Ser12b, p. 19, Thm. 7]). In the case of the symmetric group, both the set of conjugacy classes and the set of irreducible representations are indexed by the set of Young diagrams, in a natural way. The irreducible representations are recovered from the Young diagrams via Specht modules.

Proposition 6.8. *Let χ, χ' be characters, let $\sigma_1 \in \mathfrak{S}_d$. Then*

$$\sum_{\sigma\in\mathfrak{S}_d}\chi(\sigma)\chi'(\sigma^{-1}\sigma_1) = \begin{cases}\frac{d!}{\dim(\chi)}\chi(\sigma_1) & \text{if } \chi = \chi'\\ 0 & \text{else.}\end{cases}$$

Further for conjugacy classes c and c_1 we have

$$\sum_{\chi}\chi(c)\chi(c_1^{-1}) = \begin{cases}\frac{d!}{|c|} & \text{if } c = c'\\ 0 & \text{else,}\end{cases}$$

where the χ runs through the irreducible characters of $\mathfrak{S}_d$.

Proof. See [Isa13, Thm. 2.13 and Thm. 2.18]. □

Definition 6.9. Let χ be a character of $\mathfrak{S}_d$. Define the element $w_\chi \in \mathcal{Z}_d$ by

$$w_\chi = \frac{\dim(\chi)}{d!} \sum_c \chi(c^{-1}) z_c = \frac{\dim(\chi)}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi(\sigma^{-1}) \sigma.$$

Proposition 6.10. *The w_χ form a basis of $\mathcal{Z}_d$. With respect to this basis, if $z = \sum_\chi a_\chi w_\chi$ is any element of $\mathcal{Z}_d$, then the linear map $(z\,\cdot)$ is represented by the matrix $\operatorname{Diag}((a_\chi)_\chi)$. Moreover, the matrix representing the map $(z_t\,\cdot)$ has the diagonal entries $a_\chi = \binom{d}{2}\chi(t)/\dim(\chi)$.*

Proof. The two formulae in Proposition 6.8 lead to the formulae

$$w_\chi w_{\chi'} = \begin{cases} w_\chi & \text{if } \chi = \chi' \\ 0 & \text{else} \end{cases} \tag{1}$$

and

$$z_c = \sum_\chi \left(\frac{|c|\chi(c)}{\dim(\chi)} \right) w_\chi \tag{2}$$

respectively. By (1), the w_χ are linearly independent (multiply a linear relation with one of the w_χ), and by (2) they span $\mathcal{Z}_d$. The second statement follows directly from (1). The last statement follows with (2) from $t = t^{-1}$ and $|t| = \binom{d}{2}$. □

Lemma 6.11. *With the notation from Lemma 5.7, the eigenvalues of M_d are given by*

$$\mu_{i,d} = \frac{\binom{d}{2}\chi(t)}{\dim(\chi)},$$

where χ is the i-th character and t is the conjugation class of $\mathfrak{S}_d$ containing all transpositions.

Proof. By Proposition 6.4, the eigenvalues of M_d are the same as the eigenvalues of M_t. By Proposition 6.10, the w_χ are eigenvectors for the map $(z_t\,\cdot)$, represenented by the matrix M_t. The eigenvalues are given by the a_χ from the same proposition. □

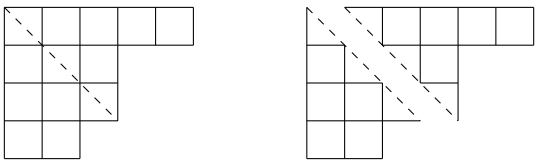
7. Quasimodularity of the generating function

In this section, we use a formula of Frobenius to express the function $\widehat{Z}(q,\lambda)$ as the constant term of a certain product of Laurent series. This is exactly what is needed to prove that the generating function F_g is quasimodular for $g \geq 2$, using the theorem about the generalized Jacobi function found in [KZ95]. The formula also exhibits a way to concretely compute the number of disconnected covers of given genus and degree.

7.1. Subsets of the half integers

Recall that the irreducible characters of $\mathfrak{S}_d$ are parametrized by Young diagrams of size d. We will make use of the set of *positive half integers* $\mathbb{Z}_{\geq 0} + \frac{1}{2}$.

Proposition 7.1. *There is a bijection between the set of Young diagrams of size d and the set of pairs (U,V) of finite subsets of $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ such that $|U| = |V|$ and $d = \sum_{u\in U} u + \sum_{v\in V} v$.*



Proof. Consider any a Young diagram of size d. Starting with the upper left corner, cut it diagonally in two pieces (see picture). This gives s "cut" columns in the lower piece and s "cut" rows in the upper piece. Let $u_i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ denote the number of squares in the i-th cut row and v_i the number of squares in the i-th cut column. Define $U = \{u_1, \ldots, u_s\}$ and $V = \{v_1, \ldots, v_s\}$. Then $|U| = |V|$ and $d = \sum_{u\in U} u + \sum_{v\in V} v$. Conversely, let two such U and V be given. The associated Young diagram is obtained by arranging both U and V in ascending order and then iteratively gluing the rows with u_i squares to the columns with v_i squares, for the appropriate elements $u_i \in U$ and $v_i \in V$ respectively. □

Proposition 7.2. *Let χ be the character associated to the Young diagram corresponding to the subsets $U, V \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ of equal cardinality s. Then*

$$\frac{\binom{d}{2}\chi(t)}{\dim(\chi)} = \frac{1}{2}\left(\sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} v_i^2\right).$$

Proof. See [FH91], p. 52. □

Definition 7.3. Define the Laurent series $\theta(\zeta, q, \lambda)$ in ζ with coefficients formal power series in q and λ as follows:

$$\theta(\zeta,q,\lambda) = \prod_{u\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} \left(1 + \zeta q^u e^{u^2\lambda/2}\right) \prod_{v\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} \left(1 + \zeta^{-1} q^v e^{-v^2\lambda/2}\right).$$

Lemma 7.4. *The counting function $\widehat{Z}(q,\lambda)$ is the coefficient of ζ^0 in the series $\theta(\zeta,q,\lambda)-1$.*

Proof. By expanding the product, one finds that $\theta(\zeta,q,\lambda) = \sum_{U,V\subset\mathbb{Z}_{\geq 0}+\frac{1}{2}} a_{U,V}$, where

$$a_{U,V} = \zeta^k q^d \exp(\mu_{U,V}\lambda).$$

Here,

1. $k = |U| - |V|$
2. $d = \sum_{u\in U} u + \sum_{v\in V} v$
3. $\mu_{U,V} = \frac{1}{2}\left(\sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} v_i^2\right)$.

Using the bijection in proposition 7.1, let the eigenvalues of the matrix M_d be indexed by pairs (U,V) of subsets of $\mathbb{Z}_{\geq 0}+\frac{1}{2}$ such that $|U| = |V|$ and $d = \sum_{u\in U} u + \sum_{v\in V} v$. By lemma 6.11 and proposition 7.2, the eigenvalue indexed by the pair (U,V) is equal to $\mu_{U,V}$.

Now consider the coefficient of ζ^0 in $\theta(\zeta,q,\lambda)-1$. There, the coefficient of q^d is $\sum_{U,V}\exp(\mu_{U,V}\lambda)$, where the $\mu_{U,V}$ are the eigenvalues of M_d. By 5.7, this sum is equal to the coefficient of q^d in $\widehat{Z}(q,\lambda)$. This proves the lemma. □

7.2. The coefficients of the theta function

Recall that F_g was defined as the series

$$Z(q,\lambda) = \sum_{g\geq 1} \frac{F_g(q)}{(2g-2)!}\lambda^{2g-2}.$$

For an element τ of the upper half plane, set $q = \exp(2\pi i\tau)$. Sometimes q will be viewed as a formal variable.

Proposition 7.5. *Let $a(x) = \sum_{k\geq 1} a_k x^k$ be a formal power series in x, with holomorphic functions a_k on the upper half plane as coefficients. Let $\exp(a(x)) = \sum_{k\geq 1} b_k x^k$ be its formal exponential. Assume that each of the coefficients b_k is quasimodular of weight kr, for some r. Then the a_k are also quasimodular of weight kr.*

Proof. This follows essentially by computing by hand the coefficients b_i. □

Definition 7.6. Define the Laurent series $\Theta(\zeta,q,\lambda)$ in ζ with formal power series in q and λ as coefficients by

$$\Theta(\zeta,q,\lambda) = (\prod_{n\geq 1}(1-q^n))\theta(\zeta,q,\lambda).$$

Further, let $\Theta_0(q,\lambda)$ denote the coefficient of ζ^0 in $\Theta(\zeta,q,\lambda)$.

The following theorem about the quasimodularity of the coefficients of Θ_0 is proved in [KZ95].

Theorem 7.7. *Let $\Theta_0(q,\lambda) = \sum_k A_k(q)\lambda^k$ be the constant ζ-coefficient of Θ. Then the coefficient $A_k(q)$ is a quasimodular form of weight $3k$.*

We may now prove the main result:

Theorem 7.8 ([Dij95])**.** *For $g \geq 2$, the function $F_g(q)$ is a quasimodular form of weight $6g-6$.*

Proof. Lemma 7.4 gives the equality

$$\Theta_0(q,\lambda) = (\prod_{n\geq 1}(1-q^n))(\widehat{Z}(q,\lambda)+1). \tag{1}$$

By the previous theorem, the coefficient of λ^{2g-2} in this product is quasimodular of weight $6g-6$. By Lemma 3.13 one obtains, after taking the logarithm of both sides of (1),

$$\log\Theta_0(q,\lambda) = \sum_{n\geq 1}\log(1-q^n) + Z(q,\lambda).$$

As seen in Example 3.10, we have $F_1 = -\sum_{n\geq 1}\log(1-q^n)$. Hence, in $\log\Theta_0(q,\lambda)$ the coefficient of λ^0 is zero. Thus, we may apply proposition 7.5 and the previous theorem to find that the coefficient of λ^{2g-2} in $\log\Theta_0(q,\lambda)$, that is $F_g(q)/(2g-2)!$, is a quasimodular form of weight $6g-6$. This concludes the proof. □

8. Appendix – Modular curves

One may weaken the definition of a modular form by requiring that the modular condition be met only for transformations lying in certain subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$. In this section we will calculate the dimension of the space $\mathrm{M}_k(\Gamma)$ of modular forms of even weight k associated to Γ, using the fact that modular forms can be seen as sections of a certain line bundle on a particular Riemann surface: the modular curve associated to the subgroup Γ. We will roughly follow [DS06, Ch. 1-3].

Taking $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ will provide a proof to the dimension formula in Proposition 2.3 for the space M_k of modular forms defined in Section 2.

8.1. Congruence subgroups and modular curves

Definition 8.1. Let $N \in \mathbb{Z}$.

1. The *principal congruence subgroup* of *level* N is the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,;\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

2. A *congruence subgroup* is a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{Z}$. In that case, the group Γ is said to have *level* N.

Remark 8.2. The subgroup $\Gamma(N)$ is the kernel of the component-wise congruence map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. It is hence normal in $\mathrm{SL}_2(\mathbb{Z})$ and of finite index. Consequently, each congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$, while not necessarily being normal.

Remark 8.3. Since $\Gamma(N) \subseteq \Gamma$ for some N, each congruence subgroup Γ contains an element of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.

For the rest of the section, fix a congruence subgroup Γ.

Definition 8.4.

1. For $M > 0$, define the set $\mathcal{N}_M \subset \mathbb{C} \cup \{\infty\}$ by

$$\mathcal{N}_M = \{\tau \in \mathcal{H} \,;\, \mathrm{Im}(\tau) > M\} \cup \{\infty\}.$$

2. Define the *compact upper half-plane* $\mathcal{H}^*$ to be the set

$$\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},$$

endowed with the topology generated by the union of the topology of $\mathcal{H}$ with the set

$$\{\alpha(\mathcal{N}_{\mathcal{M}}) \,;\, \alpha \in \mathrm{SL}_2(\mathbb{Q}), M \in \mathbb{R}_{>0}\}.$$

With this definition, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}^*$ by continuous maps. This group action is transitive on the subset $\mathbb{Q} \cup \{\infty\}$. Moreover, the space $\mathcal{H}$ is connected.

3. The *modular curve* $X(\Gamma)$ is defined as the quotient space ${}_\Gamma\backslash\mathcal{H}^*$. Denote the canonical projection map $\mathcal{H}^* \to X(\Gamma)$ by π.

For $\tau \in \mathcal{H}^*$, denote the stabilizer of τ under the action of Γ by Γ_τ.

Remark 8.5.

1. Define the function $s\colon \mathcal{H} \to \mathrm{SL}_2(\mathbb{R})$ by $s(x+iy) = \frac{1}{\sqrt{y}} \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. If $\tau \in \mathcal{H}$, then $s(\tau)\,.\,i = \tau$. Hence, the $\mathrm{SL}_2(\mathbb{R})$-operation on $\mathcal{H}$ is transitive.

2. The stabilizer of i with respect to the transitive group action of the topological group $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ is the compact subgroup $\mathrm{SO}_2(\mathbb{R})$.

3. Let $e_1, e_2 \in \mathcal{H}$. We have $\gamma(e_1) = e_2$ if and only if $\gamma \in s(e_2)\,\mathrm{SO}_2(\mathbb{R}) s(e_1)^{-1}$.

Proposition 8.6. *If $\tau_1, \tau_2 \in \mathcal{H}$, then there are open neighborhoods U_1' of τ_1 and U_2' of τ_2, such that for all but finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the sets $\gamma(U_1')$ and U_2' do not meet.*

Proof. Choose U_1' and U_2' to be any open neighborhoods belonging to the topology of $\mathcal{H}$, and with compact closure. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The previous remark implies that $\gamma(\overline{U_1'}) \cap \overline{U_2'} \neq \emptyset$ is equivalent to

$$\gamma \in \mathrm{SL}_2(\mathbb{Z}) \cap \bigcap_{\substack{e_1 \in \overline{U_1'} \\ e_2 \in \overline{U_2'}}} s(e_2)\,\mathrm{SO}_2(\mathbb{R}) s(e_1)^{-1}.$$

But this subgroup is compact and discrete, hence finite. □

Proposition 8.7. *Let τ_1 and τ_2 be elements of $\mathcal{H}^*$. Then there exist open neighborhoods U_1 of τ_1 and U_2 of τ_2 such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, if $\gamma(U_1)$ meets U_2 then $\gamma\tau_1 = \tau_2$.*

Proof. We consider three cases separately.

1. Let $\tau_1, \tau_2 \in \mathcal{H}$. By Proposition 8.6, there are open neighborhoods U_1' and U_2' of τ_1 and τ_2 respectively, such that the set

$$\{\gamma \in \mathrm{SL}_2(\mathbb{Z})\,;\ \gamma(U_1') \cap U_2' \neq \emptyset, \gamma(\tau_1) \neq \tau_2\}$$

is finite. We denote this set by F. For each $\gamma \in F$, choose disjoint open neighborhoods $U_{1,\gamma}$ and $U_{2,\gamma}$ of $\gamma\tau_1$ and τ_2, respectively, and put

$$U_1 = U_1' \cap \left(\bigcap_{\gamma \in F} \gamma^{-1}(U_{1,\gamma}) \right) \text{ and}$$

$$U_2 = U_2' \cap \left(\bigcap_{\gamma \in F} U_{2,\gamma} \right).$$

The open neighborhoods U_1 and U_2 satisfy the required propertieties.

2. Let $\tau_1 \in \mathbb{Q} \cup \{\infty\}$ and $\tau_2 \in \mathcal{H}$. Choose U_2 to be any open neighborhood in $\mathcal{H}$ with compact closure. Now, there is some $M \geq 0$ such that $\mathrm{SL}_2(\mathbb{Z})\bar{U}_2 \cup \mathcal{N}_M = \emptyset$. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha(\infty) = \tau_1$. Choose $U_1 = \alpha(\mathcal{N}_M)$. Then for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the sets $\gamma(U_2)$ and U_1 are disjoint.

3. Let $\tau_1, \tau_2 \in \mathbb{Q} \cup \{\infty\}$. Let $\alpha_1, \alpha_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha_1(\infty) = \tau_1$ and $\alpha_2(\infty) = \tau_2$. Choose $U_1 = \alpha_1(\mathcal{N}_2)$ and $U_2 = \alpha_2(\mathcal{N}_2)$. The open sets U_1 and U_2 satisfy the required properties. □

Corollary 8.8. *Let $\tau \in \mathcal{H}^*$. There is an open neighborhood U of τ such that for all $\gamma \in \mathcal{H}$, if $\gamma(U)$ meets U then $\gamma \in \Gamma_\tau$.*

Proposition 8.9. *The modular curve $X(\Gamma)$ is connected, compact, and Hausdorff.*

Proof. The connectedness of $X(\Gamma)$ follows from the connectedness of $\mathcal{H}^*$. For compactness, define the subsets

$$\mathcal{D} = \{\tau \in \mathcal{H}\,;\ |\tau| \geq 1, \mathrm{Re}(\tau) \leq 1/2\}$$

and $\mathcal{D}^* = \mathcal{D} \cup \{\infty\}$. The subset $\mathcal{D}^* \subset \mathcal{H}^*$ is compact and a fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathcal{H}^*$. Since Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, it follows that one possible fundamental domain for the Γ-action is given by the union of finitely many images of $\mathcal{D}^*$ under elements of $\mathrm{SL}_2(\mathbb{Z})$. Therefore, the modular curve $X(\Gamma)$ is compact. Finally, the Hausdorff property follows from the previous proposition. □

8.2. Modular curves as Riemann surfaces

The modular curve $X(\Gamma)$ may be given the structure of a Riemann surface. The needed local data is summarized below. We use the following convention: a subgroup G of $\mathrm{SL}_2(\mathbb{Z})$ need not contain the matrix $-\mathrm{id}$; we denote the subroup generated by G and $\{-\mathrm{id}\}$ by $\pm G$.

Proposition 8.10.

1. *For $\tau \in \mathcal{H}$, the isotropy group Γ_τ is finite cyclic.*

2. *For $s \in \mathbb{Q} \cup \{\infty\}$, the isotropy group Γ_s has finite index in the isotropy group $\mathrm{SL}_2(\mathbb{Z})_s$.*

Proof. 1. By Remark 8.5, the isotropy group Γ_τ is conjugated via an element of $\mathrm{SL}_2(\mathbb{R})$ to some discrete subroup of $\mathrm{SO}_2(\mathbb{R})$, but every such discrete subgroup is finite cyclic.

2. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha(\infty) = s$. We have $\Gamma_s = \alpha\Gamma_\infty\alpha^{-1}$ and $\mathrm{SL}_2(\mathbb{Z})_s = \alpha\,\mathrm{SL}_2(\mathbb{Z})_\infty\alpha^{-1}$. Now use that $\mathrm{SL}_2(\mathbb{Z})_\infty$ is generated by $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. □

Definition 8.11.

1. Let $\tau \in \mathcal{H}$. The *period* of τ is the number

$$h_\tau = |\pm\Gamma_\tau/\{\pm\mathrm{id}\}|$$

of maps in the isotropy group of τ. The period of τ only depends on the class $\Gamma\tau$. If $h_\tau > 1$, then we call the point τ, or interchangeably the point $\pi(\tau)$, an *elliptic point* for Γ. We further define the map $\delta_\tau\colon \mathcal{H}^* \to \widehat{\mathbb{C}}$ to be the map represented by the matrix

$$\delta_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

and the map $\rho_\tau\colon \mathbb{C} \to \mathbb{C}$ by $\rho_\tau(z) = z^{h_\tau}$.

2. Let $s \in \mathbb{Q} \cup \{\infty\}$. The *width* of s is the number

$$h_s = [\mathrm{SL}_2(\mathbb{Z})_s : \pm\Gamma_s].$$

This also only depends on the class Γs. We call the point s, or interchangeably the point $\pi(s)$, a *cusp* for Γ. Furthermore, we define $\delta_s \in \mathrm{SL}_2(\mathbb{Z})$ to be any map taking s to ∞. Finally, define the map $\rho_s\colon \mathcal{H}^* \to \mathbb{C}$ by $\rho_s(z) = \exp(2\pi i z/h_s)$. Note that this is well-defined at ∞ since we restrict to the upper half-plane.

Remark 8.12. There are only two elliptic points on $X(\mathrm{SL}_2(\mathbb{Z}))$, namely i and μ_3, where $\mu_3 = \exp(2\pi i/3)$; see [DS06, 2.3]. Their periods are 2 and 3, respectively. Since $\Gamma_\tau \subseteq \mathrm{SL}_2(\mathbb{Z})_\tau$, each elliptic point τ for Γ has period 2 or 3, and lies in one of the classes Γi or $\Gamma\mu_3$, according to wether its period is 2 or 3. Since Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, it follows that there are only finitely many elliptic points on $X(\Gamma)$.

Remark 8.13. The only cusp of $\mathrm{SL}_2(\mathbb{Z})$ is ∞, whose isotropy subgroup is the group generated by $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Hence the width of ∞ with respect to Γ is generally the smallest $h > 0$ such that $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$, cf. Remark 8.3. The only exception is when h is minimal such that $-\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$ but $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right) \notin \Gamma$, in which case h is the width, but $2h$ is minimal such that $\left(\begin{smallmatrix}1 & 2h\\ 0 & 1\end{smallmatrix}\right) \in \Gamma$. As before, there are only finitely many cusps of Γ.

For $\tau \in \mathcal{H}^*$, let U_τ be an open neighborhood of τ such that for all $\gamma \in \mathcal{H}$, if $\gamma(U_\tau)$ meets U_τ then $\gamma \in \Gamma_\tau$, as per Corollary 8.8. If τ is an elliptic point or a cusp for Γ, we may assume that U_τ contains no further elliptic points or cusps. If τ is neither, we may assume that U_τ contains no ellliptic points or cusps altogether. These assumptions are justified, since there are only finitely many elliptic points or cups.

Define the map $\varphi_\tau = \rho_\tau \circ \delta_\tau|_{U_\tau}$. This is a map with open image in $\mathbb{C}$. For $\tau_1, \tau_2 \in U_\tau$, we have $\varphi_\tau(\tau_1) = \varphi_\tau(\tau_2)$ if and only if $\pi(\tau_1) = \pi(\tau_2)$; see [DS06, p. 50, 60-61]. Hence, the map φ_τ induces an injective continuous map $\psi_\tau\colon U_\tau \to \mathbb{C}$. We take $(\pi(U_\tau), \psi_\tau)$ to be our chosen chart around $\pi(\tau)$.

Proposition 8.14. *The charts* $(\pi(U_\tau), \psi_\tau)_{\tau \in \mathcal{H}^*}$ *form a holomorphic atlas for the modular curve* $X(\Gamma)$.

Proof. See [DS06, 2.2, 2.4]. □

Proposition 8.15. *Let* Γ_1 *and* Γ_2 *be two conjugation subgroups with* $\Gamma_1 \subseteq \Gamma_2$. *If* $-\mathrm{id} \in \Gamma_2 \setminus \Gamma_1$, *then the induced morphism* $X(\Gamma_1) \to X(\Gamma_2)$ *has degree* $[\Gamma_2 : \Gamma_1]/2$, *else it has degree* $[\Gamma_2 : \Gamma_1]$. *The ramification index of a point* $\pi_1(\tau) \in X(\Gamma_1)$ *is* $[\pm\Gamma_{2,\tau} : \pm\Gamma_{1,\tau}]$.

Proof. See [DS06, p. 66-67]. □

Corollary 8.16. *Let* $f \colon X(\Gamma) \to X(\mathrm{SL}_2(\mathbb{Z}))$ *be the morphism induced by the inclusion* $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$. *If* τ *is a cusp for* Γ *or if* τ *is an elliptic point for* $\mathrm{SL}_2(\mathbb{Z})$ *but not for* Γ, *then the ramification index of* $\pi(\tau)$ *is* h_τ, *Else, the ramification index is* 1.

Let y_2, y_3, and y_∞ be the images of i, μ_3, and ∞, respectively, under the projection $\mathcal{H}^* \to X(\mathrm{SL}_2(\mathbb{Z}))$. For $h \in \{2, 3\}$, let ε_h be the number of elliptic points in $X(\Gamma)$ of period h. Let ε_∞ be the number of cusps in $X(\Gamma)$.

By Corollary 8.16, the ramification locus of f is contained in the set $\{y_1, y_2, y_\infty\}$. Now let d be the degree of f. By applying the previous corollary and the formula

$$d = \sum_{x \in f^{-1}(y_h)} \mathrm{e}_f(x),$$

we obtain the formulae

$$\sum_{x \in f^{-1}(y_h)} (\mathrm{e}_f(x) - 1) = \frac{h-1}{h}(d - \varepsilon_h)$$

and

$$\sum_{x \in f^{-1}(\infty)} (\mathrm{e}_f(x) - 1) = d - \varepsilon_\infty.$$

Proposition 8.17. *The genus* g *of the modular curve* $X(\Gamma)$ *is given by*

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}.$$

Proof. The statement follows from the Riemann–Hurwitz formula, the previous discussion, and the fact that $X(\mathrm{SL}_2(\mathbb{Z}))$ has genus 0, which will be proved later in Example 8.23. □

Corollary 8.18. *We have the inequality*

$$2g - 2 + \frac{1}{2}\varepsilon_2 + \frac{2}{3}\varepsilon_3 + \varepsilon_\infty \geq 0.$$

8.3. Automorphic forms and modular forms

Definition 8.19. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\mathrm{GL}_2(\mathbb{C})$ and let $f\colon \mathcal{H} \to \widehat{\mathbb{C}}$ be a holomorphic function. For $\tau \in \mathcal{H}$ define the *factor of automorphy*

$$\mathrm{j}(\gamma, \tau) := c\tau + d$$

and for $k \in \mathbb{Z}$ the function $f[\gamma]_k\colon \mathcal{H} \to \mathbb{C}$ by

$$f[\gamma]_k(\tau) := \det(\gamma)^{k/2}\mathrm{j}(\gamma, \tau)^{-k} f(\gamma\tau).$$

Remark 8.20. Let $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

1. The factor of automorphy satisfies $\mathrm{j}(\gamma\gamma', \tau) = \mathrm{j}(\gamma, \gamma'(\tau))\mathrm{j}(\gamma', \tau)$.

2. For all holomorphic functions $f : \mathcal{H} \to \mathbb{C}$, we have $f[\gamma\gamma']_k = (f[\gamma]_k)[\gamma']_k$.

Definition 8.21. Let $f\colon \mathcal{H} \to \mathbb{C}$ be a meromorphic function. Let $h \in \mathbb{N}$ be minimal with the property that $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. For $\tau \in \mathcal{H}$, set $q_h = \exp(2\pi i\tau/h)$.

1. The function f is *h-periodic*, if it satisfies $f(\tau + h) = f(\tau)$ for all $\tau \in \mathcal{H}$.

2. Assume that f is h-periodic. If f has no poles in the subset $\mathcal{N}_C$ for some $C > 0$, then there exists a meromorphic function $\tilde{f}\colon B \setminus \{0\} \to \mathbb{C}$ such that $f(\tau) = \tilde{f}(q_h)$ for all τ. We say that f is *meromorphic at infinity,* if it has no poles in some $\mathcal{N}_C$ and if the associated function $\tilde{f}$ has a meromorphic continuation to the whole of B. In this case, we may write $f(\tau) = \sum_{n=m}^{\infty} a_n q_h^n$, with $m \in \mathbb{Z}$ and $a_m \neq 0$. We call m the *order of f at infinity*, and we denote it by $\nu_\infty(f)$. The function f is *holomorphic at infinity* if $\nu_\infty(f) \geq 0$. We denote the order of f at a point τ of $\mathcal{H}$ by $\nu_\tau(f)$.

3. The function f is said to satisfy the *modular condition of weight k with respect to* Γ, if $f[\gamma]_k = f$ for all $\gamma \in \Gamma$. Such a function is h-periodic, since $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.

For all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the group $\alpha^{-1}\Gamma\alpha$ is a conjugation subgroup. Now assume that the function f satisfies the modular condition of weight k with respect to Γ. For all α, the function $f[\alpha]_k$ satisfies the modular condition of the same weight with respect to the subgroup $\alpha^{-1}\Gamma\alpha$, and is hence h_α-periodic for some h_α.

4. Let $s \in \mathbb{Q} \cup \{\infty\}$ be a cusp for Γ. We define f to be *meromorphic at s*, if for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ with $\alpha(\infty) = s$, the function $f[\alpha]_k$ is meromorphic at infinity. The *order* $\nu_s(f)$ of f at s is the order of the function $f[\alpha]_k$ at infinity. The function f is *holomorphic at s*, if $\nu_s(f) \geq 0$; it *vanishes at s*, if $\nu_s(f) > 0$. These notions do not depend on the choice of α.

5. The function f is an *automorphic form* with respect to Γ, if it satisfies the modular condition of some weight k with respect to Γ and is meromorphic at all cusps of Γ. We call k the *weight* of f.

6. An automorphic form f is a *modular form*, if it is holomorphic and is holomorphic at all cusps of Γ.

7. A modular form f is a *cusp form*, if it vanishes at all cusps of Γ.

We denote the space of automorphic forms of weight k by $\mathrm{A}_k(\Gamma)$, the space of modular forms of weight k by $\mathrm{M}_k(\Gamma)$, and the space of cusp forms of weight k by $\mathcal{S}_k(\Gamma)$.

Remark 8.22. The $\mathbb{C}$-algebra $\mathrm{A}_0(\Gamma)$ of automorphic forms of weight 0 is isomorphic to the algebra $\mathbb{C}(X(\Gamma))$ of meromorphic functions on the modular curve $X(\Gamma)$. To see this, note that the two notions of meromorphy at a cusp are equivalent. Similarly the space $\mathrm{M}_0(\Gamma)$ is the space of holomorphic functions on $X(\Gamma)$, so $\mathrm{M}_0(\Gamma) \simeq \mathbb{C}$. Accordingly, $\mathcal{S}_0(\Gamma) = \{0\}$.

Example 8.23. Let Δ and G_4 be the cusp form of weight 12, respectively the modular form of weight 4 of Example 2.2. The *j-invariant* is the automorphic form of weight 0 defined as $j = 1728G_4^3/\Delta$. The function Δ has no zero on $\mathcal{H}$, but it is a cusp form with a nontrivial Fourier coefficient at place 1 in its expansion at infinity (see [DS06, p. 73, eq. 3.1]). Hence the automorphic form j has only one pole, which is simple. Therefore the j-invariant may be seen as a holomorphic function $j\colon X(\mathrm{SL}_2(\mathbb{Z})) \to \widehat{\mathbb{C}}$ of degree 1. It follows that the modular curve $X(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to the sphere $\widehat{\mathbb{C}}$, and has hence genus 0.

Remark 8.24. The derivative j' of the j-invariant is a nonzero element of $\mathrm{A}_2(\Gamma)$. Hence for k even with $k \geq 2$, the algebra $\mathrm{A}_k(\Gamma)$ contains nonzero elements. Furthermore, if f is a nonzero element of $\mathrm{A}_k(\Gamma)$, then division by f yields an isomorphism

$$\mathrm{A}_k(\Gamma) \simeq \mathrm{A}_0(\Gamma) f.$$

Since the factor of holomorphy $\mathrm{j}(\gamma, \tau)$ has no zeroes or poles at non-cusp points, the order of an automorphic form at a point τ of $\mathcal{H}^*$ does not depend on the Γ-class of τ. Hence we may define the order of an automorphic form at points of a modular curve, while taking the local coordinates into account.

Definition 8.25. Let f be an automorphic form of weight k, let $\tau \in \mathcal{H}^*$. Let U_τ be a neighborhood of τ such that $\pi(U_\tau)$ is a chart neighborhood of the point $\pi(\tau)$ of $X(\Gamma)$, as per Proposition 8.14. Write $f|U_\tau = f_{\mathrm{local}} \circ \rho \circ \delta$, with $f_{\mathrm{local}}\colon \phi(U_\tau) \to V$. The *order* $\nu_{\pi(\tau)}(f)$ of f at the point $\pi(\tau)$ is defined as the order of f_{local} at 0. The order at $\pi(\tau)$ does not depend on the choice of the representative τ.

Remark 8.26. We have

$$\nu_{\pi(\tau)}(f) = \begin{cases} \dfrac{\nu_\tau(f)}{h_\tau} & \text{if } \tau \in \mathcal{H}, \\ \dfrac{\nu_\tau(f) h_\tau}{p_\alpha} & \text{if } \tau \text{ is a cusp,} \end{cases}$$

where in the second case α is some element of $\mathrm{SL}_2(\mathbb{Z})$ with $\alpha(\infty) = \tau$, and p_α is the smallest period of the function $f[\alpha]_k$. Following Remark 8.13, the number p_α is in most cases equal to the width h_τ, the exception takes place when k is odd and $(\alpha\Gamma\alpha^{-1})_\infty = \left\langle -\left(\begin{smallmatrix} 1 & h_\tau \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$, in which case $p_\alpha = 2h_\tau$. Note that if $(\alpha\Gamma\alpha^{-1})_\infty = \left\langle -\left(\begin{smallmatrix} 1 & h_\tau \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$ but k is even, then Definition 8.21 treats $f[\alpha]_k$ as a $2h_\tau$-periodic function, when f is in fact even h_τ-periodic. For more detail, see [DS06, p. 74-75].

8.4. The dimension formula

Definition 8.27. Let Y be a Riemann surface. Let $\mathrm{T}^*(Y)$ denote the cotangent bundle of Y. For $n \in \mathbb{N}$, define the sheaf of *n-fold meromorphic differentials* $\Omega(Y)^{\otimes n}$ to be the sheaf of sections into the bundle $\mathrm{T}^*(Y)^{\otimes n}$.

If U is the domain of some chart for some Riemann surface, then the n-fold meromorphic differentials are naturally identified with the meromorphic functions on U. In the following, we shall make implicit use of this fact.

Definition 8.28. Any morphism $\phi\colon Y \to Y'$ of Riemann surfaces gives rise to a *pullback map* $\phi^*\colon \Omega(Y')^{\otimes n} \to \Omega(Y)^{\otimes n}$, given locally by $f \mapsto (\phi')^n(f \circ \phi)$.

Examples 8.29. The natural morphism $\pi\colon \mathcal{H} \to X(\Gamma)$ gives rise to the pullback map

$$\pi^*\colon \Omega(X(\Gamma))^{\otimes n} \to \Omega(\mathcal{H})^{\otimes n}.$$

For $\delta \in \mathrm{GL}_2(\mathbb{C})$ we have $\delta' = \det(\delta)\mathrm{j}(\delta,\tau)^{-2}$. Hence for $f \in \Omega(\mathcal{H})^{\otimes n}$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ we have the relation $f[\alpha]_{2n} = \alpha^* f$.

Remark 8.30. Let $\phi\colon Y_1 \to Y_2$ be a morphism of Riemann surfaces with dense image. Then the pullback map $\phi^*\colon \Omega(Y_2)^{\otimes n} \to \Omega(Y_1)^{\otimes n}$ is injective by the Morse-Sard theorem.

Proposition 8.31. *Let $\phi\colon Y_1 \to Y_2$ be a morphism of Riemann surfaces. Let f be a meromorphic function on Y_1. For each chart domain V in Y_2, let $\omega_V\colon V \to \widehat{\mathbb{C}}$ be a holomorphic function such that $\phi^*\omega_V = f|_{\phi^{-1}(V)}$. Then the collection $\{\omega_V\}_V$ forms an n-fold meromorphic differential on Y_2.*

Proof. Let V' and V be two chart domains in Y_2, let U be their intersection. Set $W = \phi^{-1}(U)$. Since the map $\phi|_W\colon W \to U$ is surjective, by Remark 8.30 it suffices to show that $(\phi|_W)^*\omega_{V'} = (\phi|_W)^*(g_{V',V}\ \omega_V)$, where $g_{V',V}$ is the corresponding cocycle in the cotangent bundle. But a calculation shows that $(\phi|_W)^*(g_{V',V}\omega_V) = (\phi|_W)^*\omega_V$, and by assumption $(\phi|_W)^*\omega_{V'} = (\phi|_W)^*\omega_V$. □

Proposition 8.32. *Let $\omega \in \Omega(X(\Gamma))^{\otimes n}$, write $\pi^*(\omega) = f$ for some meromorphic function $f\colon \mathcal{H} \to \widehat{\mathbb{C}}$. Then f is an automorphic form of weight $2n$.*

Proof. Let $\gamma \in \Gamma$. Then $\gamma^*\pi^*\omega = \pi^*\omega$, since $\pi\gamma = \gamma$. On the other hand, we have $\gamma^*\pi^*\omega = (\gamma')^n(f \circ \gamma)$ by the definition of the pullback map. Hence we have $f = f[\gamma]_{2n}$, so that the f satisfies the modular condition. Furthermore, f is meromorphic at each cusp because ω is. □

By this proposition we get a map of $\mathbb{C}$-vector spaces

$$\pi^*\colon \Omega(X(\Gamma))^{\otimes n} \to \mathrm{A}_{2n}(\Gamma).$$

Proposition 8.33. *The map π^* is bijective. Furthermore, if $\omega \in \Omega(X(\Gamma))^{\otimes n}$ and $f = \pi^*(\omega)$, then the orders of f and ω at a point $\pi(\tau) \in X(\Gamma)$ are related by the formula*

$$\nu_{\pi(\tau)}(\omega) = \begin{cases} \nu_{\pi(\tau)}(f) - n\left(1 - \dfrac{1}{h}\right) & \text{if } \tau \in \mathcal{H} \text{ and } \tau \text{ has period } h, \\ \nu_{\pi(\tau)}(f) - n & \text{if } \tau \in \mathbb{Q} \cup \{\infty\} \end{cases}.$$

Proof. The map π^* is injective by Remark 8.30, since the map π has dense image.

For surjectivity, let f be an automorphic form of weight $2n$, viewed as an element of $\Omega(\mathcal{H})^{\otimes n}$ when convenient. Let $\tau_0 \in \mathcal{H}^*$, and let $U := U_{\tau_0}$ be the corresponding open neighborhood as in 8.14, so that there is a chart of $X(\Gamma)$ of the form $\psi\colon \pi(U) \xrightarrow{\sim} V$. For convenience, we are omitting the index τ_0. Hence we have a diagram

$$\begin{array}{ccc} U & \xrightarrow{\pi} & \pi(U) \\ \wr\downarrow{\scriptstyle \delta} & & \wr\downarrow{\scriptstyle \psi} \\ \delta(U) & \xrightarrow{\rho} & V, \end{array}$$

with $\delta := \delta_{\tau_0}$ and $\rho := \rho_{\tau_0}$ as in Definition 8.11. Define the n-fold differential $\lambda = (\delta^{-1})^*(f|_{U\cap\mathcal{H}})$ on $\delta(U)$, which we also view as a meromorphic function. We distinguish between two cases, depending on whether τ_0 is a cusp or not.

Suppose τ_0 is not a cusp and has period h. The extended isotropy group $\pm(\delta\Gamma\delta^{-1})_0/\{\pm\mathrm{id}\}$ is generated by the rotation r_h represented by the matrix

$$\begin{pmatrix} \mu_{2h} & 0 \\ 0 & \mu_{2h}^{h-1} \end{pmatrix},$$

where $\mu_{2h} = \exp(2\pi i/h)$. Since the function f is Γ-invariant (under pullbacks), the function λ is $\delta\Gamma\delta^{-1}$-invariant. In particular, for $z \in \delta(U)$ we have $\lambda(r_h z) = \mathrm{j}(r_h, z)^{2n}\lambda(z)$, from which the relation $(\mu_h z)^n\lambda(\mu_h z) = z^n\lambda(z)$ follows. Hence the function $z \mapsto z^n\lambda(z)$ is invariant under the transformation $z \mapsto \mu_h z$, so that there is a meromorphic function g on $\delta(U)$ such that $g(z^h) = z^n\lambda(z)$ for all z. We have $h\nu_0(g) = n + \nu_{\tau_0}(f)$, hence $\nu_0(g) = \nu_{\pi(\tau_0)}(f) + n/h$.

We define the n-fold differential θ on V by putting $\theta(z) = g(z)/(hz)^n$, and set $\omega_{\pi(U)} = \psi^*\theta$. A calculation shows that $\rho^*\theta = \lambda$, therefore we have $\pi^*\omega_{\pi(U)} = f|_U$.

The case where τ_0 is a cusp is treated analogously: here, V is the open unit ball, now the extended isotropy group $\pm(\delta\Gamma\delta^{-1})_\infty/\{\pm\mathrm{id}\}$ is generated by the matrix $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$, where h is the width of τ_0, and the function λ is $h\mathbb{Z}$-periodic. Hence there is a meromorphic function g on $V\setminus\{0\}$ such that $g(e^{2\pi iz/h}) = \lambda(z)$ for all z. Since f is holomorphic at the cusp τ_0, the function g has a meromorphic continuation on all of V. Its order at 0 is given by $\nu_0(g) = \nu_{\pi(\tau_0)}(f)$.

Define the n-fold differential θ on V by putting $\theta(z) = h^n g(z)/(2\pi iz)^n$, and set again $\omega_{\pi(U)} = \psi^*\theta$, an n-fold differential on $\pi(U)$. Another calculation shows that $\rho^*\theta = \lambda$, hence $\pi^*\omega_{\pi(U)} = f|_{U\cap\mathcal{H}}$.

Now we put everything together: by Proposition 8.31, the collection $\{\omega_{\pi(U)}\}_U$ forms an n-fold differential on $X(\Gamma)$, as required.

By construction, we obtain the orders of the differential as in the statement. □

We now come to the Riemann–Roch theorem, stated here without proof, and its application to computing the dimension of $\mathrm{M}_k(\Gamma)$ for k even. Let X be a compact Riemann surface. To a divisor $D \in \mathrm{Div}(X)$ we associate its *linear space* $L(D)$, a finite-dimensional vector space over $\mathbb{C}$ whose dimension we denote by $\ell(D)$, defined as

$$L(D) = \{f \in \mathbb{C}(X)\,;\ \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

Theorem 8.34 (Riemann–Roch)**.** *Let X be a compact Riemann surface, g its genus. Let K be a canonical divisor on X. If $D \in \mathrm{Div}^0(X)$, then*

$$\ell(D) = \deg(D) - g + 1 + \ell(K - D).$$

Remark 8.35. From the Riemann–Roch theorem follow:

1. We have $\ell(K) = g$ and $\deg(K) = 2g - 2$.

2. If $\deg(D) < 0$, then $\ell(D) = 0$.

3. If $\deg(D) > 2g - 2$, then $\ell(D) = \deg(D) - g + 1$.

Definition 8.36. Let X be a Riemann surface, k an even integer.

1. Define the group $\mathrm{Div}_{\mathbb{Q}}(X)$ of *rational divisors* on X by $\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes \mathbb{Q}$. This group is equipped with a relation $\leq$, which extends the relation on $\mathrm{Div}(X)$.

2. Let $f \in \mathrm{A}_k(\Gamma)$ be a nonzero automorphic form. The *rational divisor* $\mathrm{div}(f)$ of f is the rational divisor $\mathrm{div}(f) = \sum_{x \in X(\Gamma)} \nu_x(f)x$.

3. The floor function $\lfloor\ \rfloor : \mathrm{Div}_{\mathbb{Q}}(X) \to \mathrm{Div}(X)$ is defined by applying the floor function coefficient-wise.

Remark 8.37. Any differential in $\Omega(X(\Gamma))^{\otimes n}$ has degree $2n(g-1)$. To see this, choose a nonzero element f of $\mathrm{A}_2(\Gamma)$, and let $\lambda \in \Omega(X(\Gamma))^{\otimes 1}$ be the preimage of f under π^*. The divisor $\mathrm{div}(\lambda)$ is a canonical divisor and has hence degree $2(g-1)$ by the previous remark. The element λ^n of $\Omega(X(\Gamma))^{\otimes n}$ has degree $2n(g-1)$. Since we may write $\Omega(X(\Gamma))^{\otimes n} = \mathbb{C}(X(\Gamma))\lambda^n$, it follows that all elements of $\Omega(X(\Gamma))^{\otimes n}$ have the same degree as λ^n.

Theorem 8.38. *Let k be an even integer, set $n = k/2$. Let g be the genus of the modular curve $X(\Gamma)$; for $h \in \{2, 3\}$, let ε_h denote the number of elliptic points in $X(\Gamma)$ of period h. Let ε_∞ be the number of cusps. The following formula for the dimension of the space $\mathrm{M}_k(\Gamma)$ of modular forms with respect to the congruence subgroup Γ holds:*

$$\dim(\mathrm{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\,\varepsilon_\infty & \text{if } k \geq 2\ , \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k \leq 0 \end{cases}.$$

Proof. First let $k \leq 0$. We have $\mathrm{M}_0(\Gamma) \simeq \mathbb{C}$ and $\mathcal{S}_0(\Gamma) = \{0\}$. If $k < 0$ and $f \in \mathrm{M}_k(\Gamma)$, then $f^{12}\Delta^{-k}$ lies in $\mathcal{S}_0(\Gamma)$, where Δ is the usual weight 12 cusp form. Hence $f = 0$.

Now let $k \geq 2$. By Remark 8.24, there is a nonzero element f of $\mathrm{A}_k(\Gamma)$. Let $\omega \in \Omega(X(\Gamma))^{\otimes n}$ be the preimage of f under π^*. Using the isomorphism $\mathrm{A}_k \simeq \mathbb{C}(X(\Gamma))f$, we obtain an isomorphism

$$\mathrm{M}_k(\Gamma) \simeq \{f_0 \in \mathbb{C}(X(\Gamma))\,;\ \mathrm{div}(f_0) + \mathrm{div}(f) \geq 0\} \cup \{0\}.$$

Since $\mathrm{div}(f_0)$ is integral, the above condition is equivalent to the condition $\mathrm{div}(f_0) + \lfloor \mathrm{div}(f) \rfloor \geq 0$, hence there is an isomorphism $\mathrm{M}_k(\Gamma) \simeq L(\lfloor \mathrm{div}(f) \rfloor)$, and we have $\dim(\mathrm{M}_k(\Gamma)) = \ell(\lfloor \mathrm{div}(f) \rfloor)$.

Using the formula for the orders found in Proposition 8.33, we obtain

$$\mathrm{div}(\omega) = \lfloor \mathrm{div}(f) \rfloor - \left(\sum_{x_2} \left\lfloor \frac{k}{4} \right\rfloor x_2 + \sum_{x_3} \left\lfloor \frac{k}{3} \right\rfloor x_3 + \sum_{x_\infty} \frac{k}{2} x_\infty \right).$$

Here the variables x_2, x_3 and x_∞ run through the elliptic points of period 2, the elliptic points of period 3, and the cusps, respectively; all seen as points of $X(\Gamma)$. It follows with Remark 8.37 that

$$\deg(\lfloor \mathrm{div}(f) \rfloor) = k(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty.$$

Since $k \geq 2$, we have $\lfloor k/4 \rfloor \geq (k-2)/4$ and $\lfloor k/3 \rfloor \geq (k-2)/3$. Together with Corollary 8.18, this implies

$$\deg(\lfloor \mathrm{div}(f) \rfloor) > 2g - 2.$$

Therefore, by Corollary 8.35,

$$\ell(\lfloor \mathrm{div}(f) \rfloor) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2}\varepsilon_\infty.$$

□

Corollary 8.39. *Let k be an even integer. The following dimension formula holds:*

$$\dim(\mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))) = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \geq 0 \text{ and } k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \geq 0 \text{ and } k \not\equiv 2 \pmod{12} \\ 0 & \text{if } k < 0 \end{cases}.$$

Proof. For $k \geq 2$, Theorem 8.38 with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ gives

$$\dim(\mathrm{M}_k(\Gamma)) = (1-k) + \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \frac{k}{2}.$$

Now write $k = 12q + r$ with $0 \leq r \leq 11$ even, so that the formula becomes

$$\dim(\mathrm{M}_k(\Gamma)) = 1 + q + \lfloor r/4 \rfloor + \lfloor r/3 \rfloor - r/2,$$

and verify that the sum of the last three terms is 0 if $r \neq 2$ and -1 else. □

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