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Verlinde bundles of families of hypersurfaces and their jumping lines

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Abstract
Verlinde bundles are vector bundles V_k arising as the direct image $\pi_*(\mathscr{L}^{\otimes k})$ of polarizations of a proper family of schemes $\pi : \mathfrak{X} \to S$. We study the splitting behavior of Verlinde bundles in the case where π is the universal family $\mathfrak{X} \to |\mathscr{O}(d)|$ of hypersurfaces of degree d in $|\mathscr{O}(d)|$ and calculate the cohomology class of the locus of jumping lines of the Verlinde bundles V_{d+1} in the cases $n = 2, 3$.

Keywords Verlinde bundles · Jumping lines · Cohomology class

Mathematics Subject Classification 14J60 · 14C15 · 14M15

1 Introduction

Let $\pi : \mathfrak{X} \to S$ be a proper family of schemes with a polarization $\mathscr{L}$. For $k \geq 1$, if the sheaf $\pi_*(\mathscr{L}^{\otimes k})$ is locally free, we call it the *k-th Verlinde bundle* of the family π.

For example (Iyer 2013), let $C \to T$ be a smooth projective family of curves of fixed genus. Consider the relative moduli space $\pi :\ \mathrm{SU}(r) \to T$ of semistable vector bundles of rank r and trivial determinant. This family is equipped with a polarization Θ, the determinant bundle. The Verlinde bundles $\pi_*(\Theta^k)$ of this family are projectively flat (Hitchin 1990; Axelrod et al. 1991), and their rank is given by the Verlinde formula.

In this article, we study the example of the universal family $\pi : \mathfrak{X} \to |\mathscr{O}_{\mathbb{P}^n}(d)|$ of hypersurfaces of degree d in the complex projective space $\mathbb{P}^n$, with $n > 1$. This family comes equipped with the polarization $\mathscr{L}$ given by the pullback of $\mathscr{O}(1)$ along the projection map $\mathfrak{X} \to \mathbb{P}^n$. For $k \geq 1$, the sheaf $\pi_*\mathscr{L}^{\otimes k}$ is locally free, as can be seen by considering the structure sequence of an arbitrary hypersurface of degree d in $\mathbb{P}^n$. For $k \geq 1$, we denote the k-th *Verlinde bundle* of the family π by V_k.

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To better understand V_k we study its splitting type when restricted to lines in $|\mathscr{O}(d)|$.

Let $T \subseteq |\mathscr{O}(d)|$ be a line. On $T = \mathbb{P}^1$, we define the vector bundle $V_{k,T} := V_k|_T$. The *splitting type* of $V_{k,T}$ is the unique non-increasing tuple $(b_1, \ldots, b_{r^{(k)}})$ of size $r^{(k)} := \operatorname{rk} V_k$ such that $V_{k,T} \simeq \bigoplus_i \mathscr{O}(b_i)$.

Sequence (2.1) puts constraints on the b_i: they are all non-negative and they sum up to $d^{(k)} := \deg(V_k)$. The set of such tuples (b_i) can be ordered by defining the expression $(b'_i) \geq (b_i)$ to mean

$$\sum_{i=1}^{s} b'_i \geq \sum_{i=1}^{s} b_i \quad \text{for all} \quad s = 1, \ldots, r.$$

With this definition, smaller types are more general: the vector bundle $\mathscr{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\mathscr{O}(b'_i)$ in the sense of Shatz (1976) if and only if $(b'_i) \geq (b_i)$.

If $d^{(k)} \leq r^{(k)}$, then the most generic possible type has thus the form $(1, \ldots, 1, 0, \ldots, 0)$. We call this the *generic splitting type*. A computation shows that $d^{(k)} \leq r^{(k)}$ if $k \leq 2d$.

We have the following result on the cohomology class of the *set of jumping lines*

$$Z := \left\{ T \in \mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|) \mid V_{d+1,T} \quad \text{has non-generic type} \right\}$$

in the Grassmannian of lines in $|\mathscr{O}(d)|$:

Theorem 1 *Let $n \leq 3$, let Z be set of jumping lines of V_{d+1}, and let $[Z]$ be the class of Z in the Chow ring $\mathrm{CH}(\mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|))$. We have*

$$\dim Z = n + 1 + \binom{d-1+n}{n}.$$

Furthermore, let b range over the integers with the property $0 \leq b < \frac{\dim Z}{2}$ and define $a = \dim Z - b$, $a' = a + \frac{\operatorname{codim} Z - \dim Z}{2}$, $b' = b + \frac{\operatorname{codim} Z - \dim Z}{2}$.

1. If $\dim Z$ is odd or $n = 2$, we have

$$[Z] = \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'}. \tag{1.1}$$

2. If $\dim Z$ is even and $n = 3$, we have

$$\begin{aligned}[Z] &= \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'} \\ &\quad + \binom{\frac{\dim Z}{2}+2}{n}\binom{\frac{\dim Z}{2}}{n} \sigma_{\frac{\dim Z}{2}, \frac{\dim Z}{2}}.\end{aligned}$$

The computation is carried out by the method of undetermined coefficients, leading into various calculations in the Chow ring of the Grassmannian. The assumption $n \leq 3$ is needed for a certain dimension estimation.

2 Attained splitting types

There exists a short exact sequence of vector bundles on $|\mathscr{O}(d)|$

$$0 \to \mathscr{O}(-1) \otimes H^0(\mathbb{P}^n, \mathscr{O}(k-d)) \xrightarrow{M} \mathscr{O} \otimes H^0(\mathbb{P}^n, \mathscr{O}(k)) \to V_k \to 0, \qquad (2.1)$$

as can be seen by taking the pushforward of a twist of the structure sequence of $\mathfrak{X}$ on $\mathbb{P}^n \times |\mathscr{O}(d)|$. The map M is given by multiplication by the section

$$\sum_I \alpha_I \otimes x^I \in H^0(|\mathscr{O}(d)|, \mathscr{O}(1)) \otimes H^0(\mathbb{P}^n, \mathscr{O}(d)).$$

In particular, we have $r^{(k)} = \binom{k+n}{n} - \binom{k+n-d}{n}$ and $d^{(k)} = \binom{k+n-d}{n}$.

Lemma 1 *Let $\mathscr{E}$ be a free $\mathscr{O}_{\mathbb{P}^1}$-module of finite rank, and let*

$$0 \to \mathscr{E}' \xrightarrow{\varphi} \mathscr{E} \xrightarrow{\psi} \mathscr{E}'' \to 0$$

be a short exact sequence of $\mathscr{O}_{\mathbb{P}^1}$-modules. Given a splitting $\mathscr{E}'' = \mathscr{E}_1'' \oplus \mathscr{O}$, we may construct a splitting $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{O}$ such that the image of φ is contained in $\mathscr{E}_1$.

Proof Define $\mathscr{E}_1 := \ker(\mathrm{pr}_2 \circ \psi)$, which is a locally free sheaf on $\mathbb{P}^1$. By comparing determinants in the short exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{O} \to 0$ we see that $\mathscr{E}_1$ is free, hence by an Ext^1 computation the sequence splits. The property $\mathrm{im}(\varphi) \subseteq \mathscr{E}_1$ follows from the definition. □

Proposition 1 *Let $f_1, f_2 \in |\mathscr{O}(d)|$ span the line $T \subseteq |\mathscr{O}(d)|$ and let p be the number of zero entries in the splitting type of $V_{k,T}$. We have*

$$p = \dim H^0(\mathbb{P}^n, \mathscr{O}(k)) - \dim(f_1U + f_2U).$$

Proof Note that the map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$. In particular, the image of $\mathscr{O}(-1) \otimes U$ is contained in $\mathscr{O} \otimes (f_1U + f_2U)$. It follows that $p \geq \dim H^0(\mathbb{P}^n, \mathscr{O}(k)) - \dim(f_1U + f_2U)$.

To prove the other inequality, consider the induced sequence

$$0 \to \mathscr{O}(-1) \otimes U \xrightarrow{M|_T} \mathscr{O} \otimes (f_1U + f_2U) \to \mathscr{E}'' \to 0$$

and assume for a contradiction that $\mathscr{E}'' \simeq \mathscr{E}_1'' \oplus \mathscr{O}$. By Lemma 1, we have a splitting $\mathscr{O} \otimes (f_1U + f_2U) \simeq \mathscr{E}_1 \oplus \mathscr{O}$ such that $\mathrm{im}(M|_T) \subseteq \mathscr{E}_1$.

Consider the map $\widetilde{M}|_T : (\mathscr{O} \otimes U) \oplus (\mathscr{O} \otimes U) \to \mathscr{O} \otimes (f_1U + f_2U)$ defined by

$$\widetilde{M}|_T(a \otimes \theta_1, b \otimes \theta_2) = a \otimes f_1\theta_1 + b \otimes f_2\theta_2.$$

We obtain the matrix description of $\widetilde{M}|_T$ from the matrix description of $M|_T$ as follows. If $M|_T$ is represented by the matrix A with coefficients $A_{i,j} = \lambda_{i,j}s + \mu_{i,j}t$, then $\widetilde{M}|_T$ is represented by a block matrix

$$B = \left(A' \middle| A'' \right)$$

with $A'_{i,j} = \lambda_{i,j}$ and $A''_{i,j} = \mu_{i,j}$.

The property $\operatorname{im}(M|_T) \subseteq \mathscr{E}_1$ implies that after some row operations, the matrix A has a zero row. By the construction of $\widetilde{M}|_T$, the same row operations lead to the matrix B having a zero row, but this is a contradiction, since the map $\widetilde{M}|_T$ is surjective. $\square$

Corollary 1 *Let $T \subseteq |\mathscr{O}(d)|$ be a line spanned by the polynomials f_1, f_2. Assume that $d^{(k)} \leq r^{(k)}$. Let θ range over a monomial basis of $H^0(\mathbb{P}^n, \mathscr{O}(k-d))$. The bundle $V_{k,T}$ has the generic splitting type if and only if $\langle f_1\theta, f_2\theta \mid \theta\rangle$ is a linearly independent set in $H^0(\mathbb{P}^n, \mathscr{O}(k))$.* $\square$

Corollary 2 *Let $T \subseteq |\mathscr{O}(d)|$ be a line spanned by the polynomials f_1, f_2, and let $d^{(k)} \leq r^{(k)}$. The bundle $V_{k,T}$ has not the generic type if and only if* $\deg(\gcd(f_1, f_2)) \geq 2d-k$. *In particular, if $d^{(k)} \leq r^{(k)}$ but $k > 2d$ then the generic type never occurs.*

Proof By Corollary 1, the bundle $V_{k,t}$ has non-generic type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathbb{P}^n, \mathscr{O}(k-d))$ such that $g_1 f_1 + g_2 f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d-k$ then $\deg(f_i/h) \leq k-d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h, respectively.

On the other hand, given such g_1 and g_2, we have $f_1 \mid g_2 f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. $\square$

Proposition 2 *Let $k = d+1$. No types of V_k other than* $(1, \ldots, 1, 0, \ldots, 0)$ *and* $(2, 1, \ldots, 1, 0, \ldots, 0)$ *occur.*

Proof Assume that the type of V_k at some line (f_1, f_2) is other than the two above. Then the type has at least two more zero entries than the general type. By Proposition 1, we have $\dim\langle f_1\theta, f_2\theta \mid \theta\rangle \leq 2d^{(k)} - 2$, so we find $g_1, g_2, g'_1, g'_2 \in H^0(\mathbb{P}^n, \mathscr{O}(1))$ and two linearly independent equations

$$\begin{aligned} g_1 f_1 + g_2 f_2 &= 0 \\ g'_1 f_1 + g'_2 f_2 &= 0, \end{aligned}$$

with both sets (g_1, g_2), (g'_1, g'_2) linearly independent. From the first equation it follows that $f_1 = g_2 h$ and $f_2 = -g_1 h$, for some common factor h. Applying this to the second equation, we find $g'_1 g_2 = g'_2 g_1$, hence $g'_1 = \alpha g_1$ and $g'_2 = \alpha g_2$ for some scalar α, a contradiction. $\square$

Corollary 3 *Let $k = d+1$, let $T \subset |\mathscr{O}(d)|$ be a line spanned by f_1, f_2. The type* $(2, 1, \ldots, 1, 0, \ldots, 0)$ *occurs if and only if* $\deg(\gcd(f_1, f_2) \geq d-1$. $\square$

3 The cohomology class of the set of jumping lines

Definition 1 Let $k \geq 1$ and (b_i) be a splitting type for V_k. We define the set $Z_{(b_i)}$ of all points $t \in \mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|)$ such that $V_{k,t}$ has splitting type (b_i). For the set of points t where $V_{k,t}$ has generic splitting type, we also write Z_{gen}, and define the *set of jumping lines* $Z \coloneqq \mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|) \backslash Z_{gen}$.

Now let $k = d + 1$. By Corollary 3, Z is the subvariety given as the image of the finite, generically injective multiplication map

$$\varphi \colon \mathbb{G}\mathrm{r}(1, |\mathscr{O}(1)|) \times |\mathscr{O}(d-1)| \to \mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|)$$

sending the tuple $((sg_1 + tg_2)_{(s:t)\in\mathbb{P}^1}, h)$ to the line $(shg_1 + thg_2)_{(s:t)\in\mathbb{P}^1}$.

To perform calculations in the Chow ring A of $\mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|)$, we follow the conventions found in Eisenbud and Harris (2016). We assume $\mathrm{char}(k) = 0$ for simplicity. Let $N \coloneqq \dim H^0(\mathscr{O}(d)) = \binom{n+d}{n}$. For $N - 2 \geq a \geq b$, we have the Schubert cycle

$$\Sigma_{a,b} \coloneqq \{T \in \mathbb{G}\mathrm{r}(1, |\mathscr{O}(d)|) : T \cap H \neq \varnothing, T \subseteq H'\},$$

where $(H \subset H')$ is a general flag of linear subspaces of dimension $N - a - 2$ resp. $N - b - 1$ in the projective space $|\mathscr{O}(d)|$. The ring A is generated by the Schubert classes $\sigma_{a,b}$ of the cycles $\Sigma_{a,b}$. The class $\Sigma_{a,b}$ has codimension $a + b$, and we use the convention $\sigma_a \coloneqq \sigma_{a,0}$.

Proof (*of Theorem* 1) We have $\dim Z = n + 1 + \binom{d-1+n}{n}$ since Z is the image of the generically injective map φ.

Let $Q \subset |\mathscr{O}(d)|$ be the image of the multiplication map

$$f\colon\ |\mathscr{O}(1)| \times |\mathscr{O}(d-1)| \to |\mathscr{O}(d)|\,.$$

The map f is birational on its image, since a general point of Q has the form gh with h irreducible. The Chow group $A^{\operatorname{codim} Z}$ is generated by the classes $\sigma_{a',b'}$ with $N-2 \geq a' \geq b' \geq \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$ and $a' + b' = \operatorname{codim} Z$, while the complementary group $A^{\dim Z}$ is generated by the classes $\sigma_{\dim Z - b, b}$ with $b \in 0, \ldots, \lfloor \frac{\dim Z}{2} \rfloor$. Write

$$[Z] = \sum_{a',b'} \alpha_{a',b'} \sigma_{a',b'}.$$

We have $\sigma_{a',b'}\sigma_{a,b} = 1$ if $b' - b = \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$ and 0 else. Hence, multiplying the above equation with the complementary classes $\sigma_{a,b}$ and taking degrees gives

$$\alpha_{a',b'} = \deg([Z] \cdot \sigma_{a,b}).$$

Using Giambelli's formula $\sigma_{a,b} = \sigma_a\sigma_b - \sigma_{a+1}\sigma_{b-1}$ (Eisenbud and Harris 2016, Prop. 4.16), we reduce to computing $\deg([Z] \cdot \sigma_a\sigma_b)$ for $0 \leq b \leq \lfloor \frac{\dim Z}{2} \rfloor$. By Kleiman transversality, we have

$$\deg([Z]\cdot\sigma_a\sigma_b) = |\{T \in Z : T \cap H \neq \varnothing, T \cap H' \neq \varnothing\}|,$$

where H and H' are general linear subspaces of $|\mathscr{O}(d)|$ of dimension $N-a-2$ and $N-b-2$, respectively.

To a point $p = g_p h_p \in Q$ with $g_p \in |\mathscr{O}(1)|$ and $h_p \in |\mathscr{O}(d-1)|$, associate a closed reduced subscheme $\Lambda_p \subset Q$ containing p as follows. If h_p is irreducible, let Λ_p be the image of the linear embedding $|\mathscr{O}(1)| \times \{h_p\} \to |\mathscr{O}(d)|$ given by $g \mapsto gh_p$.

If h_p is reducible, define the subscheme Λ_p as the union $\bigcup_h \operatorname{im}(|\mathscr{O}(1)| \times \{h\} \to |\mathscr{O}(d)|)$, where h ranges over the (up to multiplication by units) finitely many divisors of p of degree $d-1$.

Note that for all points p, the spaces $\operatorname{im}(|\mathscr{O}(1)| \times \{h\} \to |\mathscr{O}(d)|)$ meet exactly at p.

By the definition of Z, all lines $T \in Z$ lie in Q. Furthermore, if T meets the point p, then $T \subseteq \Lambda_p$. For $H \subseteq |\mathscr{O}(d)|$ a linear subspace of dimension $N-a-2$, define $Q' := H \cap Q$. For general H, the subscheme Q' is a smooth subvariety of dimension $b-n+1$ such that for a general point $p = gh$ of Q' with $h \in |\mathscr{O}(d)|$, the polynomial h is irreducible.

Next, we consider the case $n = 2$ or $\dim Z$ odd.

Claim For genereal H, for each point $p \in Q'$ we have $\Lambda_p \cap H = \{p\}$.

Proof (*of Claim*) Let $\mathscr{H}$ denote the Grassmannian $\operatorname{Gr}(\dim H + 1, N)$. Define the closed subset $X \subseteq Q \times \mathscr{H}$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

The fibers of the induced map $X \to \mathscr{H}$ have dimension at least one. Hence, to prove that the desired condition on H is an open condition, it suffices to prove $\dim(X) \leq \dim(\mathscr{H})$.

The fiber of the map $X \to Q$ over a point p consists of the union of finitely many closed subsets of the form $X'_p = \{H \in \mathscr{H} : \dim(H \cap \Lambda'_p) \geq 1\}$, where $\Lambda'_p \simeq \mathbb{P}^n \subseteq |\mathscr{O}(d)|$ is one of the components of Λ_p. The space X'_p is a Schubert cycle

$$\Sigma_{\dim Q-b, \dim Q-b} = \{H \in \operatorname{Gr}(\dim H + 1, N) : \dim(H \cap H_{n+1}) \geq 2\},$$

with H_{n+1} an $(n+1)$-dimensional subspace of $H^0(\mathscr{O}(d))$. The codimension of the cycle is $2(\dim Q - b)$, hence also $\operatorname{codim}(X_p) = 2(\dim Q - b)$. Finally, we have $\dim(\mathscr{H}) - \dim(X) = \operatorname{codim}(X_p) - \dim(Q) = \dim Q - 2b$.

If $\dim Z$ is odd, then $\dim Q - 2b \geq \dim Q - \dim Z + 1 = 3 - n \geq 0$. If $n = 2$, we instead estimate $\dim Q - 2b \geq \dim Q - \dim Z = 2 - n \geq 0$. □

Next, let

$$\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|\mathscr{O}(1)| \times \mathrm{pr}_2 f^{-1}(Q'))$$

and

$$\Lambda'' := |\mathscr{O}(1)| \times \mathrm{pr}_2 f^{-1}(Q').$$

By the choice of H, the map $f^{-1}(Q') \to Q'$ is birational and the map $f^{-1}(Q') \to \mathrm{pr}_2 f^{-1}(Q')$ is even bijective. It follows that Λ'' and hence Λ have dimension $b+1$.

The intersection of Λ with a general linear subspace H' of dimension $N-b-2$ is a finite set of points. For each point $p \in Q'$, the linear subspace H' intersects each component Λ'_p of Λ_p in at most one point. For each point $p' \in H' \cap \Lambda$ there exists a unique p such that $p' \in \Lambda_p$.

The only line $T \in Z$ meeting both p and H' is the one through p and p'. If the intersection $H' \cap \Lambda_p$ is empty, then there will be no line meeting p and H'. Hence, $\deg([Z] \cdot \sigma_a \sigma_b)$ is the number of intersection points of Λ with a general H'.

Finally, the pre-image $f^{-1}(Q') = f^{-1}(H)$ is smooth for a general H by Bertini's Theorem. If ζ is the class of a hyperplane section of $|\mathscr{O}(d)|$ we have $f^*(\zeta) = \alpha + \beta$, where α and β are classes of hyperplane sections of $|\mathscr{O}(1)|$ and $|\mathscr{O}(d)|$, respectively. Since pr_2 and f have degree one, we compute

$$[\Lambda''] = [\mathrm{pr}_2^{-1} \mathrm{pr}_2 f^{-1}(H)] = \mathrm{pr}_2^* \mathrm{pr}_{2,*} f^*[H] = \binom{\operatorname{codim} H}{n} \beta^{\operatorname{codim} H - n}.$$

Hence, by the push–pull formula:

$$\begin{aligned} \deg([\Lambda] \cdot H') &= \deg([\Lambda''] \cdot (\alpha+\beta)^{\operatorname{codim} H'}) \\ &= \binom{\operatorname{codim} H}{n} \binom{\operatorname{codim} H'}{n} = \binom{a+1}{n} \binom{b+1}{n}. \end{aligned}$$

We then use Giambelli's formula to obtain Eq. 1.1.

In case $n = 3$ and $\dim Z$ even, we show that for $b = \dim Z/2$ we have $\deg([Z] \cdot \sigma_{b,b}) = 0$. In this case, the hyperplanes H and H' have the same dimension $N-b-2$.

For $p \in Q$, the set Λ_p is defined as before.

Claim for general H of dimension $N-b-2$, we have $\dim(\Lambda_p \cap H) = 1$.

Proof (*of Claim*) Define as before the closed subset $X \subseteq Q \times \mathscr{H}$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

The generic fiber of the projection map $\varphi \colon X \to \mathscr{H}$ is one-dimensional, hence we have $\dim \varphi(X) = \dim(X) - 1 = \dim \mathscr{H}$. The last equation holds with $n = 3$ and $2b = \dim Z$. Hence for all $H \in \mathscr{H}$ we have $\dim(\Lambda_p \cap H) \geq 1$.

On the other hand, the equality $\dim(\Lambda_p \cap H) = 1$ is attained by some, and hence by a general, H. Indeed, Define the closed subset $X \subseteq Q \times \mathscr{H}$ by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

By a similar argument as before, one needs to show that $\dim(\mathscr{H}) - \dim(X) + 1 \geq 0$. The fiber X_p is a Schubert cycle of codimension $3(\dim Q - b + 1)$. Lastly, a computation shows $\dim(\mathscr{H}) - \dim(\widetilde{X}) + 1 = \operatorname{codim}(\widetilde{X}_p) - \dim(Q) + 1 = \frac{1}{2}(2 \dim Q + 18 - 5n) \geq 0$. $\square$

Now, define Λ'' as above. We have $\dim \Lambda'' = \dim |\mathscr{O}(1)| + \dim \mathrm{pr}_2 f^{-1}(Q') = b$. Since f is generically of degree one, we still have $\dim \Lambda'' = \Lambda$, hence $\dim \Lambda + \dim H' = N - 2 < \dim |\mathscr{O}(d)|$. It follows that a generic H' does not meet any of the lines $T \subset Z$, hence $\sigma_b \sigma_b \cdot [Z] = 0$. $\square$

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