Counting Covers of Elliptic Curves

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Contents

1. Quasimodular forms

This section introduces quasimodular forms as described in [2].

1.1. The space of modular forms

Let $\mathcal{H} = \{\tau \in \mathbb{C}\,;\ \Im(\tau) > 0\}$ denote the upper half-plane. For $\tau \in (H)$, define $q = \exp(2\pi\tau)$ and $Y = 4\pi\Im(\tau)$. Further, let $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{C})$ denote the full modular group. Then $\mathrm{SL}_2(\mathbb{Z})$ operates on $\mathcal{H}$ by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).^1$$

Definition. A *modular form (of weight k)* is a holomorphic function f on $\mathcal{H}$ satisfying the modular condition $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all τ in $\mathcal{H}$, which is holomorphic at infinity.

A function satisfying the modular condition is $\mathbb{Z}$-periodic, hence induces a map $f_\infty(\zeta)$, holomorphic for $\zeta \neq 0$, such that $f(\tau) = f_\infty(q)$. The condition that f should be holomorphic at infinity means that the function f_∞ should be holomorphic at zero.

Note that if k is odd, then any function satisfying the modular condition of k is zero.

The modular forms of weight k form a vector space, denoted by M_k. Multiplying two modular forms having the weights k and l yields a modular form of weight $k + l$, giving the space $\bigoplus_k \mathrm{M}_k$ the structure of a graded ring, denoted by M_*.

Examples. For an even integer $k \geq 2$, the *Eisenstein series of weight* k is the function

$$E_k(\tau) = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where b_k is the k-th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$. By definition, these functions are holomorphic at infinity. For $k \geq 4$, the Eisenstein series of weight k is a modular form of weight k. One proves this for example by showing that for $k \geq 4$, the series E_k is a multiple of the function $G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\smallsetminus(0,0)} (m\tau + n)^{-k}$, which is indeed modular of weight k.

The function $\Delta = 2^{-6}3^{-3}(E_4^3 - E_6^2)$ is a modular form of weight 12. By a theorem of Jacobi, one has

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q)^{24}.$$

[1] To see that $\gamma\tau \in \mathcal{H}$, note that $\Im(\gamma\tau) = \Im(\tau)/|c\tau + d|^2$.

The theory of modular forms, including the above equality, is developed in more detail in [5]. Therein, one also finds a proof of the following proposition, which characterizes the space of modular forms.

Proposition 1.1.1. *There is an isomorphism of graded rings* $\mathbb{C}[X_4, X_6] \to \mathrm{M}_*$ *mapping* X_i *to* E_i*, where the former ring is graded by assigning to* X_i *the degree* i*. In particular, there are no nonzero modular forms of negative weight.*

1.2. The space of quasimodular forms

Definition. An *almost holomorphic modular form (of weight k)* is a function F on $\mathcal{H}$ of the form

$$F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$$

satisfying the modular condition $F(\gamma\tau) = (c\tau + d)^k F(\tau)$, where the f_m are holomorphic functions, holomorphic at infinity.

Even though Y is $\mathbb{Z}$-periodic, it is not a priori clear whether the modular condition already implies that the f_m are $\mathbb{Z}$-periodic, which is required to justify the above definition. Nevertheless, this is a consequence of the following proposition, which allows comparing Y-coefficients.

Proposition 1.2.2. *Let* F *be a function of the form* $F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$*, for some holomorphic* f_m*. If* $F = 0$ *on* $\mathcal{H}$*, then all the coefficients* f_m *are zero on* $\mathcal{H}$*.*

Proof. For the differential operator $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}$ one has $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} Y^{-m} = -2\pi i m Y^{-m-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} f_m = 0$, hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} F(\tau) = -2\pi i \sum_{m=1}^{M} f_m(\tau) Y^{-m-1} = -2\pi i Y^{-2} \Big(\sum_{m=0}^{M-1} f_{m+1} \tau Y^{-m}\Big).$$

By induction this implies that the f_m are zero for $m \geq 1$, hence also $f_0 = 0$. □

Corollary 1.2.3. *Let* $F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$ *be an almost holomorphic modular form. Then the leading coefficient* f_M *is a modular form of weight* $k - 2M$*. In particular, if* $f_M \neq 0$*, then* $2M \leq k$*.*

Proof. This follows after comparing the coefficients of Y^{-M} in both sides of the modularity condition $F(\gamma\tau) = (c\tau + d)^k F(\tau)$, using the equality

$$Y^{-1}(\gamma\tau) = (c\tau + d)^2 Y(\tau)^{-1} + \frac{c(c\tau + d)}{2\pi i}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$. □

The almost holomorphic modular forms of weight k form a vector space, denoted by $\widehat{\mathrm{M}}_k$. Let $\widehat{\mathrm{M}}_*$ denote the associated graded ring.

Definition. An element in the image of the map $\widehat{\mathrm{M}}_k \to \mathcal{O}(\mathbb{C})$ taking an almost holomorphic modular form $F = \sum_{m=0}^{M} f_m Y^{-m}$ of weight k to f_0 is called a *quasi-modular form of weight k.* Hence a quasimodular form is a holomorphic function on the upper plane appearing as the constant term of an almost holomorphic modular form.

Again, denote the vector space of quasimodular forms of weight k by $\widetilde{\mathrm{M}}_k$ and the associated graded ring by $\widetilde{\mathrm{M}}_*$. The definition gives a surjective graded ring homomorphism $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ and one has $\widehat{\mathrm{M}}_k \cap \widetilde{\mathrm{M}}_k = \mathrm{M}_k$.

Example. Consider the second Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $\sigma_1(n) = \sum_{d|n} d$. For the weight 12 modular form $\Delta(\tau) = q \prod_{n=1}^{\infty}(1-q)^{24}$, one has the identity $2\pi i E_2(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} \log(\Delta(\tau))$, which is proven by a straightforward computation. Using the modularity of Δ, one then computes

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau + d)}{\pi i},$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$.

Now, since $Y^{-1}(\gamma\tau) = (c\tau + d)^2 Y(\tau)^{-1} + \frac{c(c\tau+d)}{2\pi i}$, it follows that $E_2^* = E_2 - 12/Y$ is an almost holomorphic modular form of weight 2. Hence, E_2 is a quasimodular form of weight 2.

Proposition 1.2.4. *The space $\widetilde{\mathrm{M}}_*$ of quasimodular forms satisfies the following properties.*

1. *The canonical graded homomorphism $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ is an isomorphism.*
2. *There is an isomorphism of graded rings $\mathrm{M}_* \otimes \mathbb{C}[X_2] \simeq \mathbb{C}[X_2, X_4, X_6] \to \widetilde{\mathrm{M}}_*$ mapping X_i to E_i, where the former ring is graded by assigning to X_i the degree i.*
3. *Quasimodular forms are closed under taking derivatives.*

Proof. 1. The map $\widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*$ is surjective by definition. Injectivity follows from Calculation 9.1.1. Given an almost holomorphic modular form $F(\tau) = \sum_{m=1}^{M} f_m(\tau) Y^{-m}$ with constant term zero, the strategy is to solve the modularity equation for the coefficients f_m. This way, one finds for a fixed argument τ a polynomial equation in the lower row components c, d of any transformation $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, involving the coefficients $f_m(\tau)$. By varying the transformation γ, one may force these coefficients to be zero.

2. Express the map $\mathbb{C}[X_2, X_4, X_6] \to \widetilde{\mathrm{M}}_*$ as the composition

$$\mathbb{C}[X_2^*, X_4, X_6] \to \widehat{\mathrm{M}}_* \to \widetilde{\mathrm{M}}_*,$$

where the first map takes X_2^* to E_2^* and X_i to E_i, and the second map is the canonical map, which is an isomorphism by the first point above.

To prove the surjectivity of the first map, let $F(\tau) = \sum_{m=0}^{M} f_m(\tau) Y^{-m}$ be an almost holomorphic modular form. Then $f_M(E_2^*/12)^M$ is an almost holomorphic modular form of weight k, since f_M is modular of weight $k-2M$, and the difference $F - f_M(E_2^*/12)^M$ has degree smaller than M. Now use induction on M.

To get injectivity, let $F = \sum_{\alpha=0}^{k/2} (E_2^*)^\alpha f_{k-2\alpha}$ be an almost holomorphic modular form of weight k, in the image of the first map, where the f_m are modular of weight m. If $F = 0$, then by comparing the coefficients of $Y^{-k/2}$ one obtains $0 = f_0$. Now it follows by induction on k that the other coefficients f_m are zero. Hence F was the image of the zero element in $\mathrm{M}_* \otimes \mathbb{C}[X_2^*]$.

3. To prove the last statement, one verifies that $(6/\pi i)E_2' - E_2^2$ is modular of weight 4, and that if f is modular of weight k, then $(6/\pi i)f' - kE_2 f$ is modular of degree $2 + k$. Now use the second point above.

$\square$

2. Basic facts and definitions

In this section we will fix some notation and recall the definitions and basic properties of the objects of this thesis.

2.1. Covering spaces

Definition. Let X be a topological space, F a set, G a group operating on both X and F. Define the fibred product $X \times_G F$ to be the topological space $(X \times F) / \sim$, where $(x, f) \sim (gx, gf)$ for all g in G.

Proposition 2.1.1. *Let X be a connected, locally pathwise connected, and semi-locally simply connected topological space. Let $p : \widetilde{X} \to X$ be a universal cover. Furthermore, choose a point $\widetilde{x}_0$ of $\widetilde{X}$, and let x_0 be the image of $\widetilde{x}_0$ in X. Then there is an eqivalence of categories*

$$\{\textit{Unbranched covers of } X\} \longrightarrow \{\pi_1(X, x_0)\textit{-sets}\},$$

defined by the pair of quasi-inverse functors

$$(p_Y : Y \to X) \mapsto p_Y^{-1}(x_0) \quad \textit{and} \quad F \mapsto \widetilde{X} \times_{\pi_1} F.$$

Proof. One verifies by hand that the given functors are mutually quasi-inverse, by using elementary covering theory. Nonetheless, the needed isomorphisms between objects are given below.

Let F be a π_1-set and $p_F : \widetilde{X} \times_{\pi_1} F \to X$ the associated covering. Define a map $\zeta_F : F \to p_F^{-1}(x_0)$ by sending an element f to the class of $(\widetilde{x}_0, f)$. This map is surjective by definition, and is injective since the π_1-action on $\widetilde{X}$ is free.

On the other hand, let $p_Y : Y \to X$ be a cover of X. Define a map

$$\eta_Y : \widetilde{X} \times_{\pi_1} p_Y^{-1} \to Y$$

as follows. For a given class $(\widetilde{x}, f)$, let $\beta : [0, 1] \to \widetilde{X}$ be a path starting in $\widetilde{x}_0$ and ending in $\widetilde{x}$. Consider the projection $p\beta$ of β to X and lift the path $p\beta$ to a path $\widetilde{\beta}_f$ in Y, with starting point f. Finally, set $\eta_Y(\widetilde{x}, f) = \widetilde{\beta}_f(1)$. Note that since $\widetilde{X}$ is simply connected, this is independent of the choice of the path β. Also, the map is well-defined, since $p\beta\widetilde{\gamma} = p\beta$ for any lift $\widetilde{\gamma}$ of a loop in X.

η_Y is surjective: for $y \in Y$, let β be a path in X with starting point $p_Y(y)$ and endpoint x_0. Let $f = \widetilde{\beta}_y(1)$, be the endpoint of the lift of β to Y with starting point y. Then y is the image of $(\widetilde{\beta}_{\widetilde{x}_0}(1), f)$ under η_Y, where $\widetilde{\beta}_{\widetilde{x}_0}$ is a lift of β to $\widetilde{X}$ with starting point $\widetilde{x}_0$. To see that the map is injective, given any two points $(\widetilde{x}_1, f), (\widetilde{x}_2, g)$ mapping to the same point in Y, define a path in $\widetilde{X}$ connecting $\widetilde{x}_1$ to $\widetilde{x}_2$, and use the paths given by the definition of η_Y to construct the loop in X that will take $(\widetilde{x}_1, f)$ to $(\widetilde{x}_2, g)$ $\square$

Remark. In the above proposition, if X has the structure of a Riemann surface, then the first category may be taken to be the category of unbranched covers of Riemann surfaces over X. Indeed, every cover inherits a complex structure from X such that the structure map becomes holomorphic, and morphisms of covers of X are automatically holomorphic: in general, if fg and f are holomorphic, then g is.

Furthermore, let X be a Riemann surface, let $S \subset X$ be a finite set. Then putting $(C,p) \mapsto (C \smallsetminus p^{-1}(S), p)$ defines an equivalence of categories between the category of finite covers of X with ramification locus S and the category of finite unbranched covers of $X \smallsetminus S$. The reason is roughly that the local data of an unbranched cover around a "missing" branch point uniquely characterizes that of any extention of that cover to a ramified one, e.g. the local degree of the cover map will correspond to the ramification index. The topic of extending unbranched covers to branched ones is discussed in detail in [4], 4.6.

2.2. Complex curves

Proposition 2.2.2. *The assignment $C \mapsto K(C)$ defines a contravariant equivalence of categories between the category of irreducible smooth curves over $\mathbb{C}$ and the category of finitely generated, transcendence degree one, field extensions of $\mathbb{C}$. By definition, degree d maps of curves correspond to degree d field extensions.*

Proof. See [7] pp. 20-22. □

Proposition 2.2.3 (Riemann-Hurwitz formula)**.** *Let $\varphi\colon C_1 \to C_2$ be a finite, degree d map of smooth curves of genera g_1 and g_2, respectively. Then*

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{x \in C_1} (\mathrm{e}_\varphi(x) - 1),$$

where $\mathrm{e}_\varphi(x)$ is the ramification index of φ at x.

2.3. Further definitions

Definition. Let X be a set. A *weighting* on X is a function $w\colon X \to \mathbb{R}$. For an element x of X, the value $w(x)$ is called the *weight* of x. The *weighted count of the elements of X* is defined as the sum $\sum_{x \in X} w(x)$.

3. Covers of an elliptic curve

3.1. Connected covers

In the following, let $\mathbb{C}$ be the ground field for all varieties considered.

Definition. Let E be an elliptic curve.

1. A *(degree d, genus g, connected) cover of E* is a finite, degree d morphism $p\colon C \to E$ of an irreducible smooth curve C of genus g onto E. Denote such a cover by (C, p), possibly omitting the structure map p.

2. If $S = b_1, \ldots, b_{2g-2}$ is a set of $2g-2$ distinct points of E, call a cover C *simply branched over S*, if it is simply branched over each point of S. This means that for all points b of S there is exacly one point x in $p^{-1}(b)$ with ramification index $\mathrm{e}_p(x) = 2$, the others having a ramification index of one.

   It follows from the Riemann-Hurwitz formula of Proposition 2.2.3 that every point not in the pre-image of S has a ramification index of one. This justifies the choice of the number of points in S.

3. Two covers C_1, C_2 are to be considered isomorphic, if there is an isomorphism $C_1 \to C_2$ commuting with the respective structure maps into E. Accordingly, define the automorphism group $\mathrm{Aut}_p(C) = \mathrm{Aut}(C)$ of the cover (C, p) to be the group of cover isomorphisms $C \to C$.

Proposition 3.1.1. *Let C be a connected cover of E. Then the automorphism group of C is finite.*

Proof. By Proposition 2.2.2, if C is a degree d connected cover, the elements of $\mathrm{Aut}(C)$ correspond to the automorphisms of the degree d field extension $K(C)/K(E)$, of which only finitely many exist. □

Remark. The degree d connected covers of an elliptic curve E form a set. Indeed, they correspond by Proposition 2.2.2 to elements of the power set of the algebraic closure of $K(E)$.

Definition. Let E be an elliptic curve, $S = b_1, \ldots, b_{2g-2}$ a set of $2g-2$ distinct points of E.

1. Denote the set of isomorphism classes of degree d, genus g, simply branched over S, connected covers of E by $\mathrm{Cov}(E, S)^{\circ}_{g,d}$.

2. Any isomorphism of two equivalent covers defines a bijection of their automorphism groups. This allows to define the *weight* of the class $[(C, p)]$ to be the number $1/|\mathrm{Aut}_p(C)|$.

3. Define $N_{g,d}$ to be the weighted count

$$\sum_{C \in \mathrm{Cov}(E,S)^\circ_{g,d}} \frac{1}{|\operatorname{Aut}(C)|}.$$

The elliptic curve E and the set of points S are omitted from the notation, a priori for brevity. It will turn out that $N_{g,d}$ is finite and does not depend on the choice of E and S.

Definition. For any $g \geq 1$, define F_g to be the generating series

$$F_g(q) = \sum_{d \geq 1} N_{g,d} q^d$$

counting covers of genus g.

Example. By the theory of elliptic curves, $N_{1,d} = \sum_{j|d} 1/j$. Indeed, Let E be defined by the lattice $\Omega = \langle 1, i \rangle$. The set of divisors of d classify the covers of E by assigning to some $j|d$ the elliptic curve C_j defined by $\Omega_j = \left\langle 1, di/j^2 \right\rangle$ and the cover map $C_j \to E\colon z \mapsto jz$. There are j automorphisms $C_j \to C_j$ of this cover, given by $z \mapsto z + k/j, k = 0, 1, \ldots, j-1$.

Further, by using the Mercator series expansion for the logarithm, one finds that $-\sum_{n \geq 1} \log(1 - q^n) = \sum_{d \geq 1} \sum_{j|d} \frac{1}{j} q^d$. Hence, the first generating function is given by

$$F_1(q) = -\sum_{n \geq 1} \log(1 - q^n).$$

This thesis shall prove the following result.

Theorem 3.1.2 (Dijkgraaf)**.** *Let $g \geq 2$, and for $\tau \in \mathbb{C}$ let $q(\tau) = \exp(2\pi i \tau)$. Then the function $F_g \circ q$ is a quasimodular form of weight $6g - 6$.*

The strategy to prove the theorem will involve considering a larger class of curves covering the fixed elliptic curve, also allowing “disconnected” covers. The covers in this more general sense will be easier to count.

3.2. Covers

Definition. Let E be an elliptic curve, $S = b_1, \ldots, b_{2g-2}$ a set of $2g - 2$ distinct points of E.

1. A *(degree d, genus g,) cover* of E is a finite, degree d morphism $p\colon C \to E$ of a disjoint union $C = \cup_i C_i$ of k irreducible smooth curves C_i of genus g onto E. Again, often a cover will be identified with its source C.

2. A cover C is *simply branched over S*, if it is simply branched over each point of S. Hence the cover C has $2g-2$ ramification points.

3. We define the notion of isomorphic covers and the automorphism group $\operatorname{Aut}_p(C)$ of a cover as before.

4. For a cover $(\cup_i C_i, p)$ we define the maps p_i to be the restrictions to the C_i of the structure map p. These are finite maps, whose degrees we denote by d_i.

Remark. By the Riemann-Hurwitz formula, the maps p_i have $2g_i - 2$ ramification points on C_i. Hence, the following relations hold:

$$\sum_i d_i = d, \text{ and } \sum_i (2g_i - 2) = 2g - 2.$$

Remark. The automorphism group of a cover $C = C_1 \cup \cdots \cup C_k$ is the semidirect product

$$\operatorname{Aut}_p(C) = \prod_i \operatorname{Aut}_{p_i}(C_i) \rtimes \Gamma,$$

where $\Gamma \subset \mathrm{S}_k$ is the subgroup of the permutations of the components such that each orbit is contained in an isomorphism class of connected covers over E.

Indeed, since cover isomorphisms must permute isomorphic components, there is a homomorphism of $\operatorname{Aut}(C)$ into Γ which is the identity on Γ, viewed as a subset of $\operatorname{Aut}(C)$, having as kernel the product $\prod_i \operatorname{Aut}_{p_i}(C_i)$.

If the cover C is simply branched over S, then no two components of genus greater than one are isomorphic as connected covers, since any isomorphism would have to preserve ramification indices (see for example [7], prop. 2.6 c), but no two components share a branched point over E. In particular, if there are no components of genus one, then $\Gamma = 1$.

On the other hand, each component of genus one is unramified over E, and could be isomorphic to other components of genus one, in which case Γ is nontrivial.

Definition. Let E be an elliptic curve, $S = b_1, \ldots, b_{2g-2}$ a set of $2g-2$ distinct points of E.

1. Denote the set of isomorphism classes of degree d, genus g, simply branched over S, covers of E by $\operatorname{Cov}(E,S)_{g,d}$.

2. Assign to an element $[(C,p)]$ of $\operatorname{Cov}(E,S)_{g,d}$ the *weight* $1/|\operatorname{Aut}_p(C)|$. This is again well-defined.

3. Define $\widehat{N}_{g,d}$ to be the weighted count of the elements of $\operatorname{Cov}(E,S)_{g,d}$ with the weighting defined above. As before, the data E and S are omitted from the notation, since $\widehat{N}_{g,d}$ will turn out not to depend on them.

Definition. The generating functions $Z(q,\lambda)$, respectively $\widehat{Z}(q,\lambda)$, for the quantities $N_{g,d}$, respectively $\widehat{N}_{g,d}$, are defined as follows:

$$Z(q,\lambda) = \sum_{g\geq 1}\sum_{d\geq 1} \frac{N_{g,d}}{(2g-2)!} q^d \lambda^{(2g-2)} = \sum_{g\geq 1} \frac{F_g(q)}{(2g-2)!} \lambda^{(2g-2)},$$

$$\widehat{Z}(q,\lambda) = \sum_{g\geq 1}\sum_{d\geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{(2g-2)}.$$

Lemma 3.2.3. *The above generating functions satisfy the relation*

$$\widehat{Z}(q,\lambda) = \exp(Z(q,\lambda)) - 1.$$

Proof. The proof is subdivided into three parts. First, some notation and terminology is introduced. Second, the coefficient of $q^d\lambda^{2g-2}$ in $\exp(Z(Q,\lambda)) - 1$ is expressed in terms of the new notation. Third, combinatorial arguments are used to prove that this coefficient is equal to $\widehat{N}_{g,d}/(2g-2)!$.

1. Let C be a degree d, genus g cover. The *combinatorial type* of C is the tuple $\kappa = (k_j, g_j, d_j)_{j=1}^r$ of natural numbers, such that for each j, the space C contains exactly k_j connected components C_j of genus g_j such that the cover map $C_j \to E$ is of degree d_j. For simplicity, denote the Euler characteristics $2g-2$ and $2g_j-2$ by χ and χ_j, respectively. Then

$$\sum_j d_j = d, \text{ and } \sum_j \chi_j = \chi.$$

   Further, define $\widehat{N}_\kappa$ to be the weighted count of the covers of combinatorial type κ. Then

$$\widehat{N}_{g,d} = \sum_{|\kappa|=(\chi,d)} \widehat{N}_\kappa,$$

   where $|\kappa|$ is defined as the tuple $(\sum_j k_j\chi_j,\ \sum_j k_jd_j)$, for $\kappa = (k_j, g_j, d_j)_j$. Finally, note that the relation

$$q^d\lambda^\chi = \prod_{j=1}^r q^{k_jd_j}\lambda^{k_j\chi_j}$$

   holds for each $\kappa = (k_j, g_j, d_j)_j$ such that $|\kappa| = (\chi, d)$.

2. The exponential of $Z(q,\lambda)$ is given by

$$\exp(Z(q,\lambda)) = \prod_{g\geq 1}\prod_{d\geq 1}\sum_{k\geq 0} \frac{N_{g,d}^k}{k!\chi!} q^{kd}\lambda^{k\chi}.$$

   Expanding, one finds that the expression for $\exp(Z(q,\lambda))$ is a sum over terms of the form

$$\prod_{j=1}^{<\infty} \left(\frac{N_{g_j,d_j}}{\chi_j!}\right)^{k_j} \frac{1}{k_j} q^{k_jd_j}\lambda^{k_j\chi_j},$$

for some choices of parameters g_j, d_j, k_j. Such choices may be collected to form combinatorial types $\kappa = (g_j, d_j, k_j)_j$. Now, by collecting the summands arising from choices that induce combinatorial types of the same absolute value $|\kappa|$, one obtains that the coefficient of $q^d \lambda^\chi$ in $\exp(Z(q,\lambda))$ is equal to the sum $\sum_{|\kappa|=(\chi,d)} a_\kappa$, where

$$a_\kappa = \prod_{j=1}^{r} \left(\frac{N_{g_j,d_j}}{\chi_j!} \right)^{k_j} \frac{1}{k_j} q^{k_j d_j} \lambda^{k_j \chi_j}.$$

3. It remains to prove that $a_\kappa = \widehat{N}_\kappa$ for each combinatorial type κ. ...

$\square$

4. Classifying covers via the fundamental group

Let E be an elliptic curve, $S = \{b_1, \ldots, b_{2g-2}\}$ a set of $2g-2$ distinct points of E. Fix a basis point $b_0 \in E \smallsetminus S$, and denote the fundamental group $\pi_1(E \smallsetminus S, b_0)$ by π_1. Recall the equivalence of categories from 2.1.:

$$\{\text{Finite ramified covers of } E \text{ with ramification locus } S\} \longrightarrow \{\pi_1\text{-sets}\}.$$

The goal of this section is to use this equivalence of categories to classify those π_1-sets giving rise to unbranched covers that, after adding the branched points, become the covers we are interested in, i.e. the over S simply branched, genus g, degree d covers. To obtain natural π_1-actions on the set of d fibre points of b_0, it is convenient to introduce markings on the set of fibres.

4.1. Marked covers and the monodromy map

Definition. A *marked (degree d, genus g, simply branched over S) cover* of E is a triple (C, p, m), where $(C, p) \in \mathrm{Cov}(E, S)_{g,d}$ and $m \colon p^{-1}(b_0) \to \{1, \ldots, d\}$ is a bijective map, the *marking* of (C, p, m).

Two marked covers (C_1, p_1, m_1) and (C_2, p_2, m_2) are considered equivalent, if there is an isomorphism of covers $\phi \colon C_1 \to C_2$ such that $m_1 = m_2\phi$. Let $\widetilde{\mathrm{Cov}}(E, S)_{g,d}$ denote the set of equivalence classes of marked covers with respect to this relation.

Definition. Let (C, p) be a cover of E. Denote the group operation of π_1 on the fibre of $p^{-1}(b_0)$ by $(\gamma, x) \mapsto \gamma \,.\, x$. Define the monodromy map

$$\mathrm{mon} \colon \widetilde{\mathrm{Cov}}(E, S)_{g,d} \to \mathrm{Hom}(\pi_1, \mathrm{S}_d)$$

by $\mathrm{mon}(C, p, m)(\gamma)(i) = m(\gamma \,.\, m^{-1}(i))$.

Let the symmetric group S_d operate on the first set by $\sigma \,.\, (C, p, m) = (C, p, \sigma m)$, and on the second by $\sigma \,.\, \psi = \mathrm{inn}(\sigma)\psi$, i.e. by inner automorphisms. Then mon becomes a morphism of S_d-sets. Furthermore, for an element $\psi = \mathrm{mon}(C, p, m)$ of the image of mon, the group action "forgetting the marking"

$$m^{-1}\psi(\_)m \colon \pi_1 \to \mathrm{Aut}(p^{-1}(b_0))$$

on the fiber of b_0 is the same as the one defined by the above equivalence of categories.

Definition. The S_d-set $\widehat{T}_{g,d}$ is defined by

$$\widehat{T}_{g,d} = \{(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2) \in \mathrm{S}_d^{2g} \,;\, \text{each } \tau_i \text{ is a simple transposition}, \\ \tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\},$$

where the S_d-action is defined by conjugation in each component, after noting that conjugates of transpositions are transpositions.

Proposition 4.1.1. *The image of* mon *is isomorphic as a* S_d*-set to* $\widehat{T}_{g,d}$.

Proof. The fundamental group π_1 of $E \smallsetminus S$ is described by the following generating set and relation:

$$\pi_1 = \left\langle \gamma_1, \dots, \gamma_{2g-2}, \alpha_1, \alpha_2 \,;\, \gamma_1 \cdots \gamma_{2g-2} = \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \right\rangle .$$

For over S simply branched covers, the image of each loop γ_i under the monodromy map is a simple transposition τ_i. Namely, there is over b_i exactly one branch point of index 2, and τ_i interchanges the two fiber points corresponding to the two sheets of the branching, leaving the other fiber points unchanged.

Combining these remarks, one finds that putting

$$\psi \mapsto (\psi(\gamma_1), \dots, \psi(\gamma_{2g-2}), \psi(\alpha_1), \psi(\alpha_2))$$

defines the required isomorphism, which is compatible with the S_d-action. □

Proposition 4.1.2. *The morphism of* S_d*-sets* $\rho\colon \widetilde{\mathrm{Cov}}(E,S)_{g,d} \to \widehat{T}_{g,d}$ *induces a bijection on the sets of orbits*

$$\mathrm{S}_d \backslash \widetilde{\mathrm{Cov}}(E,S)_{g,d} \to \mathrm{S}_d \backslash \widehat{T}_{g,d}.$$

Proof. To see that ρ is surjective, let $t \in \widehat{T}_{g,d}$, and let $\psi_t : \pi_1 \to \mathrm{S}_d$ be the corresponding group homomorphism. By the above equivalence of categories, the π_1-action on $\{1, \dots, d\}$ defined by ψ_t gives a finite, unbranched cover of Riemann surfaces $C' \to E \smallsetminus S$, which may be extended to a branched cover $C \to E$, see the remark in 2.1.. The π_1-action on $\{1, \dots, d\}$ gives the π_1-action on the fiber of the basis point b_0 associated to (C, p), showing that the extension C has the right branching.

For the injectivity on the sets of orbits, let $\rho(C_1, p_1, m_1) = t$ and $\rho(C_2, p_2, m_2) = \sigma \,.\, \psi_t$, for some $t \in \widehat{T}_{g,d}$ and $\sigma \in \mathrm{S}_d$. Then $\rho(C_2, p_2, \sigma^{-1} m_2) = t$. Let ψ_t define the associated group action on $\{1, \dots, d\}$, hence the group action on the fibers. From the equivalence of categories follows that the two marked covers differ only by the marking: $C_1 \simeq C_2$. Hence, the two marked covers are in the same orbit. □

Remark. The S_d-orbits of $\widetilde{\mathrm{Cov}}(E,S)_{g,d}$ are in one-to-one correspondence with the elements of $\mathrm{Cov}(E,S)_{g,d}$. The above proposition gives thus a bijection of $\mathrm{Cov}(E,S)_{g,d}$ with the set of S_d-orbits of $\widehat{T}_{g,d}$.

4.2. Counting covers

By the above discussion, we get an algebraic description of the weighted count $\widehat{N}_{g,d}$ of genus g, degree d, simply branched over S, covers of E.

Proposition 4.2.3. *Let (C, p, m) be a marked cover and t its image under ρ. Then there is a group isomorphism* $\mathrm{Aut}_p(C) \to \mathrm{Stab}(t)$.

Proof. Let ϕ_t be the group homomorphism $\pi_1 \to \mathrm{S}_d$ corresponding to t. By the equivalence of categories, $\mathrm{Aut}_p(C)$ is isomorphic to the group of automorphisms of the π_1-action on $\{1, \ldots, d\}$ defined by ψ_t, i.e. those elements σ in the symmetry group S_d commuting with ψ_t, i.e. such that $\psi_t = \mathrm{inn}(\sigma)\psi_t$. This condition translates under the isomorphism of S_d-sets in 4.1.1 $\square$

Lemma 4.2.4. *The following equality for the weighted count $\widehat{N}_{g,d}$ holds:*

$$\widehat{N}_{g,d} = |\widehat{T}_{g,d}|/d!.$$

Proof. By propositions 4.1.2 and 4.2.3, the weighted count $\widehat{N}_{g,d}$ is equal to the weighted count of the S_d-orbits of $\widehat{T}_{g,d}$, where each orbit is weighted by $1/|\mathrm{Stab}(t)|$, for any element t in the orbit (this is well-defined since elements of the same orbits have isomorphic stabilizer subgroups). Now, it follows from the formula $|\mathrm{Orb}(t)| = |\mathrm{S}_d|/|\mathrm{Stab}(t)|$ that this weighted count equals $|\widehat{T}_{g,d}|/d!$. $\square$

5. Conjugacy classes of the symmetric group

In this section, we further the computation of $\widehat{N}_{g,d}$ by using similar techniques to the one applied when counting cycles in a graph. The rough picture is one of a graph with vertices the conjugacy classes of S_d and edges representing the passage from one class to another by multiplication with a simple transposition. We seek to count not cycles, but cycles starting and ending with the same representative, in the sense specified in the section. To do this, we make use of an analogon of the adjacency matrix for a graph.

To abbreviate, we use the term "transposition" for simple transpositions.

5.1. Conjugacy cycles

Recall the definition

$$\widehat{T}_{g,d} = \{(\tau_1, \ldots, \tau_{2g-2}, \sigma_1, \sigma_2) \in \mathrm{S}_d^{2g}\,;\text{ each } \tau_i \text{ is a transposition,}\\ \tau_1 \cdots \tau_{2g-2} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\}.$$

Our aim is now to rewrite this definition using conjugacy classes. Note that the condition in the definition is equivalent to

$$(\tau_1 \cdots \tau_{2g-2})\sigma_2 = \sigma_1\sigma_2\sigma_1^{-1}. \tag{1}$$

Definition. For $\sigma_2 \in \mathrm{S}_d$, define

$$P_{g,d}(\sigma_2) = \{(\tau_1, \ldots, \tau_{2g-2}) \in \mathrm{S}_d^{2g-2}\,;\text{ each } \tau_i \text{ is a transposition,}\\ \tau_1 \cdots \tau_{2g-2}\sigma_2 \text{ is conjugate to } \sigma_2\}.$$

If $g = 1$, define $P_{g,d}$ to be the singleton set $\{\bullet\}$. Further, let $c(\sigma_2)$ denote the conjugacy class of σ_2.

Proposition 5.1.1. *Let $\mathcal{R} = (\sigma_2^{(1)}, \ldots, \sigma_2^{(r)})$ be a system of (distinct) representatives of the conjugacy classes of S_d. Then*

$$|\widehat{T}_{g,d}| = \sum_{\sigma_2 \in \mathcal{R}} d!|P_{g,d}(\sigma_2)|.$$

Proof. Let $\sigma_2 \in \mathrm{S}_d$, let $(\tau_1, \ldots, \tau_{2g-2}) \in P_{g,d}(\sigma_2)$ and let $\sigma_1^{;}$ be an element such that $(\tau_1 \cdots \tau_{2g-2})\sigma_2 = \sigma_1^{;}\sigma_2(\sigma_1^{;})^{-1}$. Then there is a bijection of the set of elements σ_1 satisfying (1) onto the set of elements commuting with σ_2, given by sending σ_1 to $(\sigma_1^{;})^{-1}\sigma_1$. The number of elements commuting with σ_2 is given by the cardinality of the stabilizer $|\operatorname{Stab}(\sigma_2)| = |\mathrm{S}_d|/|c(\sigma_2)| = d!/|c(\sigma_2)|$. Thus, one obtains

$$|\widehat{T}_{g,d}| = \sum_{\sigma \in \mathrm{S}_d} \frac{d!}{|c(\sigma)|}|P_{g,d}(\sigma)|.$$

Further, the function $|P_{g,d}| : \mathrm{S}_d \to \mathbb{C}$ is constant on conjugacy classes. Indeed, for $\sigma \in \mathrm{S}_d$ there is a bijection of $P_{g,d}(\sigma_2)$ onto $P_{g,d}(\sigma\sigma_2\sigma^{-1})$ given by conjugation with σ in each component. From this follows the required equality. $\square$

Corollary 5.1.2. *The above proposition, together with Lemma 4.2.4, give the equality*

$$\widehat{N}_{g,d} = \sum_{\sigma_2 \in \mathcal{R}} |P_{g,c}(\sigma_2)|.$$

From now on, let $\mathcal{R} = (\sigma_2^{(1)}, \dots, \sigma_2^{(r)})$ be a fixed system of representatives of the conjugacy classes of S_d. Then the cardinality $r = \mathrm{part}(d)$ of $\mathcal{R}$ is the number of (unordered) partitions of $\{1, \dots, d\}$. This follows essentially from the fact that conjugation with a permutation acts on cycles by applying the permutation to the entries of the cycle.

5.2. Adjacency matrices

Definition. Let $d \geq 1$ and $k \geq 0$.

1. For $1 \leq i, j \leq r$, define the sets $N^k_{d,i,j}$ by

$$N^k_{d,i,j} = \{(\tau_1, \dots, \tau_k) \in \mathrm{S}_d^k \,; \text{each } \tau_i \text{ is a transposition}, \\ \tau_1 \cdots \tau_k \sigma_2^{(i)} \in c(\sigma_2^{(j)})\}.$$

   For $k = 0$, define $N^0_{d,i,j} = \delta_{i,j}$ (Kronecker delta).

2. Define the size r square matrix M_d by

$$(M_d)_{i,j} = |N^1_{d,i,j}|.$$

   This does not depend on the choice of system of representatives $\mathcal{R}$.

Remark. If k is odd, applying the signum homomorphism to the defining condition shows that $N^k_{d,i,i}$ is empty. If $k = 2g - 2$ is even, then $N^{2g-2}_{d,i,i} = P_{g,d}(\sigma_2^{(i)})$.

Proposition 5.2.3. *The entries of M_d^k are given by $(M_d^k)_{i,j} = |N^k_{d,i,j}|$.*

Proof. The proof is by induction on k. For $k = 0, 1$, there is nothing to show. For the induction step, note that if i (resp. j) are fixed, the sets $N^k_{d,i,j}$ are pairwise disjoint for varying j (reps. i). Now define a function

$$\coprod_{l=1}^{r} N^k_{d,i,l} \times N^1_{d,l,j} \to N^{k+1}_{d,i,j}$$

as follows: for a given element $((\tau_1, \dots, \tau_k), \tau_0)$, let $\sigma \in \mathrm{S}_d$ be the unique element such that $\tau_1 \cdots \tau_k \sigma_2^{(i)} = \sigma\sigma_2^{(l)}\sigma^{-1}$, and define the image of $((\tau_1, \dots, \tau_k), \tau_0)$ to be

$(\sigma\tau_0\sigma^{-1}, \tau_1, \ldots, \tau_k)$. By the definition of matrix multiplication, it suffices to prove that this function is a bijection.

Injectivity is clear by the uniqueness of σ in the definition. For surjectivity, given an element $(\tau_0', \tau_1, \ldots, \tau_k)$ in the target, choose an l such that $\tau_1 \cdots \tau_k \sigma_2^{(i)}$ is conjugate to $\sigma_2^{(l)}$, say $\tau_1 \cdots \tau_k \sigma_2^{(i)} = \sigma\sigma_2^{(l)}\sigma^{-1}$. Then $(\sigma^{-1}\tau_0\sigma)\sigma_2^{(l)}$ is conjugate to $\sigma_2^{(j)}$. ☐

Lemma 5.2.4. *Let $d \geq 1$ and $r = \mathrm{part}(d)$. Let $\mu_{1,d}, \ldots, \mu_{r,d}$ be the eigenvalues of M_d, listed according to their algebraic multiciplicities. Then*

$$\widehat{Z}(q, \lambda) = \sum_{d\geq 1}\sum_{i=1}^{r} \exp(\mu_{i,d}\lambda)q^d.$$

Proof. Recall the definition of $\widehat{Z}$:

$$\widehat{Z}(q, \lambda) = \sum_{g\geq 1}\sum_{d\geq 1} \frac{\widehat{N}_{g,d}}{(2g-2)!} q^d \lambda^{2g-2}.$$

The above proposition and remark give $(M_d^{2g-2})_{i,i} = |P_{g,d}(\sigma_2^{(i)})|$ and $(M_d^k)_{i,i} = 0$ if k is odd, for all i. Hence, by 5.1.2 one has $\widehat{N}_{g,d} = \mathrm{Tr}(M_d^{2g-2}) = \sum_{i=1}^r \mu_{i,d}^{2g-2}$, and since the terms for k odd vanish,

$$\begin{aligned}
\widehat{Z}(q, \lambda) &= \sum_{g\geq 1}\sum_{d\geq 1} \frac{\mathrm{Tr}(M_d^{2g-2})}{(2g-2)!)} q^d \lambda^{2g-2} \\
&= \sum_{d\geq 1}\sum_{i=1}^{r}\sum_{g\geq 1} \frac{\mu_{i,d}^{2g-2}}{(2g-2)!} \lambda^{2g-2} q^d \\
&= \sum_{d\geq 1}\sum_{i=1}^{r} \exp(\mu_{i,d}\lambda)q^d.
\end{aligned}$$

☐

6. The group algebra of the symmetric group

Let $\mathbb{C}[\mathrm{S}_d]$ be the group algebra of the symmetric group, let $\mathcal{Z}_d$ be its centre. This is a commutative algebra, acting on itself linearly by multiplication. In this section, we relate this linear action to the matrix M_d of the previous section, and we use the representation and character theory of the symmetric group to compute its eigenvalues.

6.1. The centre of the group algebra

Definition. Let $\mathcal{Z}_d \subset \mathbb{C}[\mathrm{S}_d]$ be the centre of the group algebra. If c is a conjugacy class of S_d, define the element $z_c \in \mathcal{Z}_d$ by

$$z_c = \sum_{\sigma \in c} \sigma.$$

Remark. The elements z_c lie in the centre since $\alpha c = c\alpha$ for all conjugacy classes c and elements α of S_d. Further, the z_c form a basis of $\mathcal{Z}_d$. Indeed, linear independence follows from the linear independence of the distinct elements $\sigma \in \mathrm{S}_d \subset \mathbb{C}[\mathrm{S}_d]$. Further, if $z \in \mathcal{Z}_d$, then the equalities $\alpha z \alpha^{-1} = z$ show that the $\mathbb{C}$-coefficients of elements in the same conjugacy class are equal. Hence $\mathcal{Z}_d$ is r-dimensional, with $r = \operatorname{part}(d)$.

Recall the definition of M_d from the previous section. There, we fixed a system of representatives for the equivalence classes of S_d. However, since the definition does not depend from the chosen representatives, we may also define M_d to be a matrix indicised by the conjugacy classes of S_d, ordered in the same way as before. The new, equivalent definition is as follows.

Definition. Let c', c be conjugacy classes of S_d. Define the matrix M_d by

$$(M_d)_{c',c} = |\{\tau\,;\ \tau \text{ is a transposition such that } \tau\sigma_2 \in c'\}|,$$

where σ_2 is any representative of c.

From now on, we choose the ordering of the basis $\{z_c\}_c$ and the ordering of the columns of M_d to be compatible, i.e. coming from the same fixed ordering of the conjugacy classes $\{c\}$.

Proposition 6.1.1. *Let t be the conjugation class containing all transpositions, z_t the corresponding basis element of $\mathcal{Z}_d$. Let M_t be the size r square matrix matrix representing the $\mathbb{C}$-linear map $(z_t\,\cdot) : \mathcal{Z}_d \to \mathcal{Z}_d$ given by multiplication with z_t. Then $M_t = (M_d)^\top$.*

Proof. Let c, c' be conjugacy classes. Note that if $z = \sum_{\sigma \in \mathrm{S}_d} \lambda_\sigma \sigma = \sum_{c''} \lambda_{c''} z_{c''}$, then the coefficient $\lambda_{c''}$ is equal to the coefficient λ_{σ_2}, for any $\sigma_2 \in c''$. Now let $\sigma_2 \in c$, and consider the product

$$z_t z_{c'} = (\sum_{\tau \in t} \tau)(\sum_{\sigma' \in c'} \sigma') = \sum_{\sigma \in \mathrm{S}_d} (\sum_{\tau\sigma' = \sigma} 1)\sigma.$$

In this expansion, the coefficient λ_{σ_2} of any element $\sigma_2 \in c$ is the quantity $|\{\tau \in t\,;\ \tau^{-1}\sigma_2 \in c'\}|$. It follows that $(M_t)_{c', c} = \lambda_c = \lambda_{\sigma_2} = (M, d)_{c, c'}$ □

6.2. Irreducible characters of the symmetric group

We have reduced our problem of computing the eigenvalues of M_d to the computation of the eigenvalues of M_t. More generally, we find that $\mathcal{Z}_d$ actually has a basis $\{w_\chi\}$, indicised by the irreducible characters of S_d, such that each w_χ is an eigenvector for all linear maps defined by multiplication with any element of $\mathcal{Z}_d$, and such that the corresponding eigenvalues are easy to compute.

Definition. 1. Let ρ be an irreducible representation of $\mathbb{C}[\mathrm{S}_d]$, i.e. a group homomorphism $\rho\colon \mathrm{S}_d \to \mathrm{GL}(\mathbb{C}^n)$ such that for each $\sigma \in \mathrm{S}_d$ there are no $\rho(\sigma)$-invariant subspaces. The *irreducible character associated to* ρ is defined as the map $\chi_\rho\colon \mathrm{S}_d \to \mathbb{C}$, $\sigma \mapsto \mathrm{Tr}(\rho(\sigma))$

2. An *irreducible character* of S_d is a map $\chi\colon \mathrm{S}_d \to \mathbb{C}$ of the form $\chi = \chi_\rho$ for some irreducible representation ρ. Its *dimension* $\dim(\chi)$ is defined as the dimension of the associated representation $\dim \rho = \chi(1)$.

For brevity, we will refer to irreducible characters simply as characters.

Remark. Characters are constant on conjugacy classes. It is therefore justified to write $\chi(c) \in \mathbb{C}$ for a character χ and a conjugacy class c.

Remark. The number of irreducible representations of a finite group, up to isomorphism, is equal to the number of its conjugacy classes (see for example [6], p. 19, Thm. 7). In the case of the symmetric group, both the set of conjugacy classes and the set of irreducible representations are indicised by the set of Young diagrams, in a natural way. The irreducible representations are recovered from the Young diagrams via Specht modules.

Proposition 6.2.2. *Let χ, χ' be characters. Then*

$$\sum_{\sigma \in \mathrm{S}_d} \chi(\sigma)\chi'(\sigma^{-1}\sigma_1) = \begin{cases} \frac{d!}{\dim(\chi)}\chi(\sigma_1) & \text{if } \chi = \chi' \\ 0 & \text{else.} \end{cases}$$

Further, if c, c', then

$$\sum_{\chi} \chi(c)\chi(c') = \begin{cases} \frac{d!}{|c|} & \text{if } c = c' \\ 0 & \text{else,} \end{cases}$$

where the χ runs through the irreducible characters of S_d.

Proof. ... □

Definition. Let χ be a character of S_d. Define the element $w_\chi \in \mathcal{Z}_d$ by

$$w_\chi = \frac{\dim(\chi)}{d!} \sum_c \chi(c^{-1}) z_c = \frac{\dim(\chi)}{d!} \sum_{\sigma \in \mathrm{S}_d} \chi(\sigma^{-1})\sigma.$$

Proposition 6.2.3. *The w_χ form a basis of $\mathcal{Z}_d$. With respect to this basis, if $z = \sum_\chi a_\chi w_\chi$ is any element of $\mathcal{Z}_d$, then the linear map $(z\,\cdot)$ is represented by the matrix* $\mathrm{Diag}((a_\chi)_\chi)$. *With this notation, if $z = z_t$, then $a_\chi = \binom{d}{2}\chi(t)/\dim(\chi)$.*

Proof. The two formulae in the above proposition lead to the formulae

$$w_\chi w_\chi' = \begin{cases} w_\chi & \text{if } \chi = \chi' \\ 0 & \text{else} \end{cases} \tag{1}$$

and

$$z_c = \sum_\chi \left(\frac{|c^{-1}|\chi(c^{-1})}{\dim(\chi)}\right) w_\chi \tag{2}$$

respectively. By (1), the w_χ are linearly independent (multiply a linear relation with one of the w_χ), and by (2) they span $\mathcal{Z}_d$. The second statement follows directly from (1). The last statement follows with (2) from $t = t^{-1}$ and $|t| = \binom{d}{2}$. □

Lemma 6.2.4. *The eigenvalues of M_d are given by*

$$\mu_{i,d} = \frac{\binom{d}{2}\chi(t)}{\dim(\chi)},$$

where χ is the i-th character and t is the conjugation class of S_d containing all transpositions.

Proof. By proposition 6.1.1, the eigenvalues of M_d are the same as the eigenvalues of M_t. Now the statement follows from the second and third statements of the above proposition, since the matrix M_t represents multiplication with z_t. □

7. Subsets of the half integers

In this section, we use a formula of Frobenius to express the function $\widehat{Z}_{(q,\lambda)}$ as the constant term of a certain product of Laurent series. This is exactly what is needed in the next section to prove that Z_g is quasimodular for $g \geq 2$. The formula also exhibits a way to concretely compute the number of disconnected covers of given genus and degree.

Recall that the irreducible characters of S_d are parametrized by Young diagrams of size d. For example, ...

Definition. Define the *positive half integers* $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ by

$$\mathbb{Z}_{\geq 0} + \frac{1}{2} = \left\{ \frac{2k+1}{2} \; ; \; k \in \{0, 1, 2 \dots\} \right\}$$

Proposition 7.0.1. *There is a bijection between the set of Young diagrams of size d and the set of pairs (U, V) of finite subsets of $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ such that $|U| = |V|$ and $d = \sum_{u \in U} u + \sum_{v \in V} v$.*

Proof. Consider any a Young diagram of size d. Starting with the upper left corner, cut it diagonally in two pieces. This gives s “cut” columns in the lower piece and s “cut” rows in the upper piece. Let $u_i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ denote the number of squares in the i-th cut row and v_i the number of squares in the i-th cut column. Define $U = \{u_1, \dots, u_s\}$ and $V = \{v_1, \dots, v_s\}$. Then $|U| = |V|$ and $d = \sum_{u \in U} u + \sum_{v \in V} v$. Conversely, let two such U and V be given. The associated Young diagram is obtained by arranging both U and V in ascending order and then iteratively gluing the rows with u_i squares to the columns with v_i squares, for the appropriate elements $u_i \in U$ and $v_i \in V$ respectively. □

Proposition 7.0.2. *Let χ be the character associated to the Young diagram corresponding to the subsets $U, V \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ of equal cardinality s. Then*

$$\frac{\binom{d}{2} \chi(t)}{\dim(\chi)} = \frac{1}{2} \left(\sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} v_i^2 \right).$$

Proof. See [3], p. 52. □

Definition. Define the Laurent series $\theta(\zeta, q, \lambda)$ in ζ with coefficients formal power series in q and λ as follows:

$$\theta(\zeta, q, \lambda) = \prod_{u \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(1 + \zeta q^u e^{u^2 \lambda / 2} \right) \prod_{v \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(1 + \zeta^{-1} q^v e^{-v^2 \lambda / 2} \right).$$

Lemma 7.0.3. *The counting function $\widehat{Z}(q, \lambda)$ is the coefficient of ζ^0 in the series $\theta(\zeta, q, \lambda) - 1$.*

Proof. By expanding the product, one finds that $\theta(\zeta, q, \lambda) = \sum_{U,V \subset \mathbb{Z}_{\geq 0}+\frac{1}{2}} a_{U,V}$, where

$$a_{U,V} = \zeta^k q^d \exp(\mu_{U,V}\lambda).$$

Here,

1. $k = |U| - |V|$
2. $d = \sum_{u \in U} u + \sum_{v \in V} v$
3. $\mu_{U,V} = \frac{1}{2}\left(\sum_{i=1}^{s} u_i^2 - \sum_{i=1}^{s} v_i^2\right)$.

Using the bijection in proposition 7.0.1, let the eigenvalues of the matrix M_d be indicized by pairs (U, V) of subsets of $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ such that $|U| = |V|$ and $d = \sum_{u \in U} u + \sum_{v \in V} v$. By lemma 6.2.4 and proposition 7.0.2, the eigenvalue indicized by the pair (U, V) is equal to $\mu_{U,V}$.

Now consider the coefficient of ζ^0 in $\theta(\zeta, q, \lambda) - 1$. There, the coefficient of q^d is $\sum_{U,V} \exp \mu_{U,V}\lambda$, where the $\mu_{U,V}$ are the eigenvalues of M_d. By 5.2.4, this sum is equal to the coefficient of q^d in $\widehat{Z}(q, \lambda)$. This proves the lemma. $\square$

8. Quasimodularity of the generating function

In this section, we use the theorem of Kaneko and Zagier about the generalized Jacobi function found in [2] to prove that the generating function F_g counting connected covers of genus g is quasimodular for $g \geq 2$. Recall that F_g was defined as the series

$$Z(q,\lambda) = \sum_{g \geq 1} \frac{F_g(q)}{(2g-2)!} \lambda^{2g-2}.$$

For an element τ of the upper half plane, set $q(\tau) = exp(2\pi i \tau)$. For convenience, we sometimes write q instead of $q(\tau)$. Also, sometimes q will be viewed as a formal variable.

Proposition 8.0.1. *Let $a(x) = \sum_{k \geq 1} a_k x^k$ be a formal power series in x, with holomorphic functions a_k on the upper half plane as coefficients. Let $\exp(a(x)) = \sum_{k \geq 1} b_k x^k$ be its formal exponential. Assume that each of the coefficients b_k is quasimodular of weight kr, for some r. Then the a_k are also quasimodular of weight kr.*

Proof. This follows essentially by computing by hand the coefficients b_i. □

Definition. Define the Laurent series $\Theta(\zeta, q, \lambda)$ in ζ with coefficients formal power series in q and λ as follows:

$$\Theta(\zeta, q, \lambda) = (\prod_{n \geq 1} (1 - q^n)) \theta(\zeta, q, \lambda).$$

Further, let $\Theta_0(q, \lambda)$ denote the coefficient of ζ^0 in $\Theta(\zeta, q, \lambda)$.

The following theorem about the quasimodularity of the coefficients of Θ_0 is proved in [2].

Theorem 8.0.2 (Kaneko, Zagier)**.** *Let $\Theta_0(q, \lambda) = \sum_k A_k(q) \lambda^k$ be the constant ζ-coefficient of Θ. Then the coefficient $A_k(q)$ is a quasimodular form of weight $3k$.*

We may now prove the main result:

Theorem 8.0.3 (Dijgraaf)**.** *For $g \geq 2$, the function $F_g \circ q$ is a quasimodular form of weight $6g - 6$.*

Proof. Lemma 7.0.3 gives the equality

$$\Theta_0(q, \lambda) = (\prod_{n \geq 1} (1 - q^n))(\widehat{Z}(q, \lambda) + 1).$$

By the previous theorem, the coefficient of λ^{2g-2} in this product is quasimodular of weight $6g-6$. By Lemma 3.2.3 one obtains, after taking the logarithm of both sides of the above equality,

$$\log \Theta_0(q,\lambda) = \sum_{n\geq 1} \log(1+q^n) + Z(q,\lambda).$$

As seen in section 3., $F_1 = -\sum_{n\geq 1} \log(1+q^n)$. Hence, in $\log \Theta_0(q,\lambda)$ the coefficient of λ^0 is zero. Thus, we may apply proposition 8.0.1 and the previous theorem to find that the coefficient of λ^{2g-2} in $\log \Theta_0(q,\lambda)$, that is $F_g(q)/(2g-2)!$, is a quasimodular form of weight $6g-6$. This concludes the proof. $\square$

9. Appendix: Calculations

9.1. Quasimodular forms

Calculation 9.1.1. This calculation follows the one found in [1]. Let $F(\tau) = \sum_{m=1}^{M} f_m(\tau)Y^{-m}$ be an almost holomorphic modular form, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_n(\mathbb{Z})$, and $\tau \in \mathcal{H}$. Write $j = c\tau + d$, and $a = 6cj/2\pi i$. Then $Y^{-1}(\gamma\tau) = a + j^2 Y(\tau)^{-1}$. Hence,

$$\begin{aligned} F(\gamma\tau) &= \sum_{m=1}^{M} f_m(\gamma\tau)(a + j^2 Y^{-1})^m \\ &= \sum_{m=1}^{M}\sum_{l=0}^{m} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} Y^{-l} \\ &= \sum_{m=1}^{M} f_m(\gamma\tau)a^m + \sum_{l=1}^{M}\sum_{m=l}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l} y^{-l}. \end{aligned}$$

On the other hand,

$$F(\gamma\tau) = \sum_{l=1}^{M} f_l(\tau) j^k Y^{-l},$$

by the modularity condition. By comparing the coefficients of Y^{-l}, one obtains the equalities

$$\sum_{m=1}^{M} f_m(\gamma\tau)a^m = 0 \tag{1}$$

and

$$j^k f_l(\tau) = \sum_{m=l}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l} j^{2l}.$$

Rewriting the second equality yields

$$f_l(\gamma\tau) = f_l(\tau) j^{k-2l} - \sum_{m=l+1}^{M} \binom{m}{l} f_m(\gamma\tau) a^{m-l}. \tag{2}$$

The latter may be solved recursively, starting by f_M, to get equalities of the form

$$f_l(\gamma\tau) = (\text{a polynomial in the } f_{\geq l}(\tau)\text{ , } j \text{ and } c). \tag{3}$$

The first two equalities are

$$\begin{aligned} f_M(\gamma\tau) &= f_M(\tau) j^{k-2M} \\ f_{M-1}(\gamma\tau) &= f_{M-1}(\tau) j^{k-2M+2} - \text{const} \cdot f_M(\tau) j^{k-2M+1} c. \end{aligned}$$

In general, a straightforward inductive argument shows that in the summands of the expression (2) for $f_l(\gamma\tau)$, the variable j appears with a power lower than or equal to $k-2l$. Now let r be the greatest index such that $f_r \neq 0$. Equation (1) finally gives, after substituting back the expressions for j and a and using (2) for $l=r$, the relation

$$\begin{aligned}
0 &= \kappa_1 f_r(\gamma\tau)(c\tau+d)^r c^r + \sum_{l=r+1}^{M} \kappa_3 f_l(\gamma\tau)(c\tau+d)^l c^l \\
&= \kappa_1 f_r(\tau)(c\tau+d)^{k-r} c^r - \\
&- \sum_{m=r+1}^{M} \kappa_2 \binom{m}{r} f_m(\gamma\tau)(c\tau+d)^{m-r} c^{m-r} + \sum_{l=r+1}^{M} \kappa_3 f_l(\gamma\tau)(c\tau+d)^l c^l,
\end{aligned}$$

where the κ_i are some nonzero constants. To obtain a contradiction, choose a point τ in the upper half-plane and consider the last relation as a polynomial equation in c and d, letting $P(c,d)$ denote the right-hand side of the equation. First look for the possible coefficients of monomials of the form $c^r d^{\geq 1}$. This excludes the third summand from the picture, since there c will always appear with a power greater than r. Next look for the possible coefficients of the monomial $c^r d^{k-r}$. As seen when recursively solving the equations for $f_l(\gamma\tau)$, the second summand will include only terms where $(c\tau+d)$ appears with a power lower than $k-r$. Hence the coefficient of $c^r d^{k-r}$ in $P(c,d)$ is $\kappa_1 f_r(\tau)$.

Now, if $c \in \mathbb{Z}$, then there are infinitely many $d \in \mathbb{Z}$ such that $P(c,d)=0$. Indeed, there are infinitely many d with $\gcd(c,d)=1$. For these d, find $a,b \in \mathbb{Z}$ such that $ad-bc=1$. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, it follows that $P(c,d)=0$. Similarly, for all $d \in \mathbb{Z}$, there are infinitely many c such that $P(c,d)=0$. It this follows that $P(c,d)=0$ holds for all $c,d \in \mathbb{C}$. These remarks may be summarized by the statement that the set of all c,d belonging to the lower row of some matrix in $\mathrm{SL}_2(\mathbb{Z})$ is Zariski-dense in $\mathbb{C}^2$.

Concluding, since P is zero as a function on $\mathbb{C}^2$, it is also zero as a polynomial, hence the coefficient $\kappa_1 f_r(\tau)$ is zero. Since τ was arbitrary, one finds $f_r=0$, a contradiction.

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