

VERLINDE BUNDLES

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1. Universal Families of Extensions

Let X and S be Noetherian schemes over a field k . Let $f: X \rightarrow S$ be a flat, projective morphism, and let \mathcal{F} and \mathcal{G} be coherent \mathcal{O}_X -modules, flat over \mathcal{O}_X .

Recall that an element $\xi \in \operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences, or *extensions*, of the form

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on \mathcal{F} and \mathcal{G} . The set of these equivalence classes can be given the structure of an $H^0(S, \mathcal{O}_S)$ -module, see for example [Wei95, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$ -module structure.

Explicitly, the sum of two elements of Ext^1 corresponds to the Baer sum of the associated extensions, while the multiplication of an extension as above by a scalar $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

The next proposition shows there exists a k -scheme V that parametrizes the points of $\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})$.

Proposition 1.1. *Let $V := \mathbb{V}(\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$. There exists an extension*

$$\xi_{\mathrm{univ}}: 0 \rightarrow \mathrm{pr}_1^* \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathrm{pr}_1^* \mathcal{F} \rightarrow 0$$

over $X \times V$ such that for all Noetherian affine k -schemes Y , the map $\mathrm{Mor}_k(Y, V) \rightarrow \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$ defined by $\alpha \mapsto (\mathrm{id}_X \times \alpha)^ \xi_{\mathrm{univ}}$ is a bijection, functorial in Y . In particular, pulling back ξ_{univ} gives a bijection $\mathrm{Mor}_k(\mathrm{Spec}(k), V) \xrightarrow{\sim} \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})$.*

Proof. Write $Y = \mathrm{Spec}(A)$. We aim to construct a functorial isomorphism

$$\mathrm{Mor}_k(Y, V) \simeq \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y).$$

Given such an isomorphism for all Y , the required universal extension is the image of $\mathrm{id} \in \mathrm{Mor}_k(V, V)$.

Note that there exist functorial isomorphisms

$$\begin{aligned} \mathrm{Mor}_k(Y, V) &\simeq \mathrm{Hom}_{k\text{-alg}}(\mathrm{Sym} \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee, A) \simeq \mathrm{Hom}_{k\text{-mod}}(\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee, A) \\ &\simeq A \otimes_k \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}). \end{aligned}$$

For the final isomorphism $A \otimes_k \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$, it suffices to prove that the δ -functors¹ $A \otimes_k \mathrm{Hom}_X(\mathcal{F}, -)$ and $\mathrm{Hom}_{X_Y}(\mathcal{F}_Y, -_Y)$ are canonically isomorphic.

In fact, there exists a canonical homomorphism $A \otimes_k \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Hom}_{X \times Y}(\mathcal{F}_Y, \mathcal{G}_Y)$, functorial in \mathcal{G} , that sends an elementary tensor $a \otimes u$ to the homomorphism $a \otimes u$. This is an isomorphism on stalks by [Bou72, Ch. I, §2.10]. There, we need the assumptions that \mathcal{F} is coherent and X Noetherian. \square

The scheme V of Proposition 1.1 is a special case of the solution to the more general moduli problem of classifying relative families of sheaves. The rest of this section sketches the more general situation. The material is taken from the article [Lan83].

Definition 1.2. (i) The i -th relative Ext module $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ is the image of \mathcal{G} under the right-derived functor $R^i(f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})) : \mathrm{Mod}_{\mathcal{O}_X} \rightarrow \mathrm{Mod}_{\mathcal{O}_S}$.

(ii) For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s, \mathcal{F}, \mathcal{G}} : \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s)$$

by restricting extensions of \mathcal{F} by \mathcal{G} to the fiber X_s . This is well-defined, since \mathcal{F} is flat over S .

¹Recall that $A \otimes_k -$ is exact.

(iii) A family of extensions of \mathcal{F} by \mathcal{G} over S is a family

$$\xi_s \in \text{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s), \quad s \in S$$

such that there exists an open covering \mathfrak{U} of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \text{Ext}_{f^{-1}(U)}^1(\mathcal{F}_U, \mathcal{G}_U)$ with $\Phi_{s, \mathcal{F}_U, \mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

Remark 1.3. If S is affine, then we have $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G}) = \text{Ext}_X^i(\mathcal{F}, \mathcal{G})^\sim$.

Proposition 1.4. Let $g: Y \rightarrow S$ be a morphism of Noetherian schemes. There exists a number $N \geq 0$ depending on \mathcal{G} such that for all quasi-coherent \mathcal{O}_Y -modules \mathcal{M} , all $i \geq 1$ and $n \geq N$ we have

$$\mathcal{E}xt_{f_Y}^i(\mathcal{O}_{X_Y}(-n), \mathcal{G} \boxtimes \mathcal{M}) = 0$$

Proposition 1.5. Let $g: Y \rightarrow S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism

$$\tau_g^i: g^* \mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{E}xt_{f_Y}^i(g_X^* \mathcal{F}, g_X^* \mathcal{G}).$$

Furthermore, if g is flat, then τ_g^i is an isomorphism for all $i \geq 0$.

Definition 1.6. We say that $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change if for all morphisms of Noetherian schemes $g: Y \rightarrow S$, the base change homomorphism τ_g^i is an isomorphism.

Proposition 1.7. Let $s \in S$ be a point such that τ_s^i is surjective. Then there exists an open neighborhood U of s such that $\tau_{s'}^i$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ_s^{i-1} is surjective if and only if $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ is locally free on an open neighborhood of s .

Remark 1.8. (i) If τ_s^i is an isomorphism for all $s \in S$, then $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change.

(ii) From Proposition 1.7 we conclude that if $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ is locally free.

(iii) In case S is reduced, if $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ is locally free then $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 1.9. Let $u: Y' \rightarrow Y$ be a morphism of Noetherian S -schemes.

(i) We define a functoriality map $H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) \rightarrow H^0(Y', \mathcal{E}xt_{f_{Y'}}^1(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1 \otimes \text{id}} H^0(Y', u^* \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau_u^1)} H^0(Y', \mathcal{E}xt_{f_{Y'}}^1(u_{X_Y}^* \mathcal{F}_Y, u_{X_Y}^* \mathcal{G}_Y)). \end{aligned}$$

(ii) Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of \mathcal{F}_Y by \mathcal{G}_Y over Y , we set $(u^* \xi)_{y'} := u^* \xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^* \xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y' .

(iii) We define the functors

$$\begin{aligned} E, E' &: (\text{NoethSch}/S) \rightarrow (\text{Sets}); \\ E(Y) &:= H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)), \\ E'(Y) &:= \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

Remark 1.10. The spectral sequence $E_2^{p,q} = H^p(S, \mathcal{E}xt_f^q(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$ gives an exact sequence

$$\begin{aligned} 0 \rightarrow H^1(S, f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})) &\xrightarrow{\varepsilon} \text{Ext}_X^1(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})) \\ &\xrightarrow{d_2} H^2(S, f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})). \end{aligned}$$

Proposition 1.11. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S -schemes, the functors E and E' are isomorphic.*

Proposition 1.12. *Suppose that $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the \mathcal{O}_S -module $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.13. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S -schemes, the functor E' is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.14. *Suppose that S is affine and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. The functor*

$$(\text{Aff}/S) \rightarrow (\text{Sets}): Y \mapsto \text{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$$

is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.

Remark 1.15. As a special case of the above, we recover Proposition 1.1.

Remark 1.16. The article [Lan83] continues on to define a (projectivized) version of the problem, so that over $\text{Spec}(k)$, the scheme $\mathbb{P}(\text{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of \mathcal{F} by \mathcal{G} , modulo the action of k^\times . See also [HL10, Example 2.1.12].

2. Verlinde Bundles on Pencils of Quartics

The thesis [Hem15] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in \mathbb{P}^3 , after

summarizing some of its properties. We work over a field k , but omit it in most notation², e.g. we write \mathbb{P}^3 for \mathbb{P}_k^3 .

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in \mathbb{P}^3 . The quartics $\mathfrak{X}_t \subseteq \mathbb{P}^3$ parametrized by the $t \in |\mathcal{O}(4)|$ form a universal family $\pi: \mathfrak{X} \rightarrow |\mathcal{O}(4)|$ with fibers \mathfrak{X}_t . The family \mathfrak{X} is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$. The morphism π is projective and flat.

Throughout, the homogeneous coordinates of \mathbb{P}^3 will be denoted by x_i , $i = 0, \dots, 4$.

We define the line bundle \mathcal{L} on \mathfrak{X} as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to \mathfrak{X} . In other words³, the bundle \mathcal{L} is the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \rightarrow \mathbb{P}^3$.

Proposition 2.1. *Let $k \geq 1$. The following statements hold:*

- (i) *If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this number is independent of the point q .*
- (ii) *The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.*
- (iii) *For all cartesian diagrams of the form*

$$\begin{array}{ccc} \mathfrak{X}_Z & \longrightarrow & \mathfrak{X} \\ \pi_Z \downarrow & \times & \downarrow \pi \\ Z & \xrightarrow{\rho} & |\mathcal{O}(4)| \end{array}$$

we have $\rho^ \pi_* \mathcal{L}^{\otimes k} \simeq (\pi_Z)_* \mathcal{L}_Z^{\otimes k}$.*

Proof. For the first statement, see the proof of [Hem15, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. \square

Let $T \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}_K^1 \rightarrow |\mathcal{O}(4)|$, with K an extension field of k . We call T a *pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}_K^1}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}_K^1}$. The situation is summarized in the picture below:

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{P}_K^1} & \longrightarrow & \mathfrak{X} \\ \downarrow & \times & \downarrow \pi \\ \mathbb{P}_K^1 & \longrightarrow & |\mathcal{O}(4)| \end{array}$$

For $k \geq 1$, we define the k -th Verlinde bundles $V_k := \pi_* \mathcal{L}^{\otimes k}$ and $V_{k,T} := (\pi_{\mathbb{P}^1})_* \mathcal{L}_{\mathbb{P}^1}^{\otimes k}$. These bundles are related by $V_k|_T = V_{k,T}$ using Proposition 2.1.

²Most instances of the letter k will be used to denote a natural number instead.

³For a fiber product $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ and sheaves \mathcal{F} and \mathcal{G} on X resp. Y , write $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$.

Proposition 2.2. *There exists a short exact sequence of coherent $\mathcal{O}_{|\mathcal{O}(4)|}$ -modules*

$$0 \rightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \rightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \rightarrow V_k \rightarrow 0.$$

Let I_d range over the tuples of the form (i_0, \dots, i_3) with $\sum i_j = d$. The first map is then given by $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.

Proof. See [Hem15, Proposition 4.2]. □

Remark 2.3. Let T be a pencil of quartics.

(i) The sequence from Proposition 2.2 restricts to a sequence

$$0 \rightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \rightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \rightarrow V_{k,T} \rightarrow 0$$

over \mathbb{P}^1 .

(ii) The vector bundle $V_{k,T}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

Definition 2.4. Let $k \geq 1$.

(i) A *type candidate* for V_k is a non-increasing tuple $(d_1, \dots, d_{r^{(k)}})$ of non-negative integers with $r^{(k)} = \binom{k+3}{3} - \binom{k-1}{3}$ and $\sum d_i = \binom{k-1}{3}$.

(ii) The *general type candidate* for V_k is the unique type candidate for V_k of the form $(b^{(k)} + 1, \dots, b^{(k)} + 1, b^{(k)}, \dots, b^{(k)})$. The integer $b^{(k)}$ is determined by the equation $\binom{k-1}{3} = b^{(k)} r^{(k)} + a$, with $a < r^{(k)}$ becoming the number of occurrences of $b^{(k)} + 1$.

(iii) Let T be a pencil of quartics. The *type* of $V_{k,T}$ is the unique type candidate (d_i) such that $V_{k,T} \simeq \bigoplus_i \mathcal{O}(d_i)$.

(iv) We say that $V_{k,T}$ has *general type* if its type is a general type candidate.

The points of $\text{Gr}(2, 35)$ correspond to the pencils of quartics $T \subseteq |\mathcal{O}(4)|$ in the following way. Let P the universal \mathbb{P}^1 -bundle over $\text{Gr}(2, 35)$. It comes equipped with a projection map $P \rightarrow |\mathcal{O}(4)|$ such that for all pencils of quartics T there exists a unique point $t \in \text{Gr}(2, 35)$ such that the image of the fiber P_t in $|\mathcal{O}(4)|$ is T .

$$\begin{array}{ccccc} P_t & \xrightarrow{\quad} & P & \xrightarrow{p} & |\mathcal{O}(4)| \\ \downarrow & & \downarrow \varphi & & \\ \text{Spec}(\kappa(t)) & \xrightarrow{\quad} & \text{Gr}(2, 35) & & \end{array} \quad \times$$

For $t \in \text{Gr}(2, 35)$ corresponding to the pencil T , we write $V_{k,t} := V_{k,T}$.

Definition 2.5. Let $k \geq 1$ and (d_i) be a type candidate for V_k . We define the set $Z_{(d_i)}$ of all points $t \in \text{Gr}(2, 35)$ such that $V_{k,t}$ has type (d_i) . For the set of points t where $V_{k,t}$ has general type, we also write Z_{gen} .

Proposition 2.6. *The set Z_{gen} is Zariski open. Its complement is the union*

$$\text{Supp}(R^1\varphi_*p^*V_k(-b^{(k)} - 1)) \cup \text{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee)),$$

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \text{Gr}(2, 35)$, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and $b := b^{(k)}$. We have $t \in Z_{\text{gen}}$ if and only if $b \leq d_i \leq b+1$ for all i , which holds if and only if $H^1(P_t, V_{k,t}(-b-1)) = H^1(P_t, V_{k,t}(-b)^\vee) = 0$.

Next, we want to apply the Cohomology and Base Change Theorem [Vak17, 28.1.6] to the map $\varphi: P \rightarrow \text{Gr}(2, 25)$, which is a \mathbb{P}^1 -bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\text{Gr}(2, 35)$.

For all $t \in \text{Gr}(2, 35)$ we have $h^2(P_t, p^*V_{k,t}(-b-1)) = 0$ and $h^2(P_t, p^*V_{k,t}(-b)^\vee) = 0$. Since the sheaves $p^*V_{k,t}(-b-1)$ and $p^*V_{k,t}(-b)^\vee$ are locally free and coherent, we have

$$(R^1\varphi_*p^*V_k(-b-1))_t = H^1(P_t, V_{k,t}(-b-1))$$

and

$$(R^1\varphi_*(p^*V_k(-b)^\vee))_t = H^1(P_t, V_{k,t}(-b)^\vee).$$

By the previous characterization, we have

$$\text{Gr}(2, 35) \setminus Z_{\text{gen}} = \text{Supp}(R^1\varphi_*p^*V_k(-b-1)) \cup \text{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)),$$

which is a Zariski closed set. \square

Proposition 2.7. *The sets $\text{Supp}(R^1\varphi_*p^*V_k(-b^{(k)} - 1))$ and $\text{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee))$ are determinantal varieties.*

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k))$, $r_2 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4))$ and $b := b^{(k)}$, and rewrite the exact sequence from Proposition 2.2 as

$$0 \rightarrow \mathcal{O}(-1)^{r_2} \rightarrow \mathcal{O}^{r_1} \rightarrow V_k \rightarrow 0. \quad (\star)$$

Twisting the sequence (\star) with $\mathcal{O}(-b-1)$ and pulling back to P gives an exact sequence

$$0 \rightarrow p^*\mathcal{O}(-b-2)^{r_2} \rightarrow p^*\mathcal{O}(-b-1)^{r_1} \rightarrow p^*V_k(-b-1) \rightarrow 0.$$

For all $t \in \text{Gr}(2, 35)$ we have $h^2(P_t, \mathcal{O}(-b-2)^{r_2}) = 0$, hence $R^2\varphi_*p^*\mathcal{O}(-b-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_*p^*\mathcal{O}(-b-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_*p^*\mathcal{O}(-b-1)^{r_1} \rightarrow R^1\varphi_*p^*V_k(-b-1) \rightarrow 0.$$

Note that since the numbers $h_2^1 := h^1(P_t, \mathcal{O}(-b-2)^{r_2})$ and $h_1^1 := h^1(P_t, \mathcal{O}(-b-1)^{r_1})$ do not depend on the point t , Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h_2^1 and h_1^1 , respectively. Since

taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_*p^*V_k(-b-1))_t \neq 0$ if and only if $\text{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\text{Supp}(R^1\varphi_*(p^*V_k(-b-1))) = \{t : \text{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = br_1 = b\binom{k+3}{3}$.

The proof for the second assertion is similar. We start with the sequence (\star) , twist with $\mathcal{O}(-b)$, take duals, pull back to P , and apply φ_* . Since for all $t \in \text{Gr}(2, 35)$ we have $h^1(P_t, \mathcal{O}(b)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_*p^*\mathcal{O}(b)^{r_1} \xrightarrow{\beta} \varphi_*p^*\mathcal{O}(b+1)^{r_2} \rightarrow R^1\varphi_*(p^*V_k(-b)^\vee) \rightarrow 0.$$

Since the numbers $h_1^0 := h^0(P_t, \mathcal{O}(b)^{r_1})$ and $h_2^0 := h^0(P_t, \mathcal{O}(b+1)^{r_2})$ do not depend on the point t , again by Grauert's Theorem the first two terms of the sequence are locally free of rank h_1^0 and h_2^0 , respectively. As before, we obtain the characterization

$$\text{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)) = \{t : \text{rk}(\beta_t) \leq h_2^0 - 1\}.$$

Here, we have $h_2^0 = (b+2)r_2 = (b+2)\binom{k-1}{3}$. □

Definition 2.8. For type candidates (d_i) and (d'_i) we define the expression $(d'_i) \leq (d_i)$ to mean

$$\sum_{i=1}^s d'_i \leq \sum_{i=1}^s d_i \text{ for all } s = 1, \dots, r^{(k)}.$$

Proposition 2.9. *Let (d_i) be a type candidate for V_k . The set $\widehat{Z}_{(d_i)} := \bigcup_{(d'_i) \geq (d_i)} Z_{(d'_i)}$ is the intersection of (at most) $r^{(k)}$ determinantal varieties.*

Proof. For every type candidate (d'_i) , the sum $\sum_{i=1}^s d'_i$ is the largest sum of s entries of (d'_i) . Hence we have

$$\widehat{Z}_{(d_i)} = \bigcap_{s=1}^{r^{(k)}} \{t : h^0((\wedge^s V_{t,k})(-\sum^s d_i)) > 0\}.$$

With Serre duality and the Cohomology and Base Change theorem we write the sets in the intersection as

$$\text{Supp}(R^1\varphi_*(p^*(\wedge^s V_k^\vee)(\sum^s d_i - 2))),$$

which is a determinantal variety by an argument similar to the second part of the proof of Proposition 2.7. One just has to note that $h^1(\mathbb{P}^1, \mathcal{O}(\sum^s d_i - 2)) = 0$ for all s . □

Proposition 2.10. *Of the five type candidates*

$$(1, 1, 1, 1, 0, \dots, 0), (2, 1, 1, 0, \dots, 0), (2, 2, 0, \dots, 0), (3, 1, 0, \dots, 0), (4, 0, \dots, 0)$$

for V_5 , only the first two occur as types of some $V_{5,t}$.

Proof. A K -point q of $\text{Gr}(2, 35)$ is given by a matrix of the form

$$\begin{pmatrix} \lambda_0 & \cdots & \lambda_{34} \\ \mu_0 & \cdots & \mu_{34} \end{pmatrix},$$

with $\lambda_i, \mu_i \in K$, up to elementary row operations. Restricting the sequence from Proposition 2.2 to the quartic defined by q , we see that the bundle $V_{5,q}$ is the cokernel of the matrix $A \in \text{Mat}(56 \times 4, \Gamma(\mathbb{P}^1, \mathcal{O}(1)))$, given as follows. Let s, t denote the homogeneous coordinates of \mathbb{P}^1 and let I_d range over the tuples of the form (i_0, \dots, i_3) with $\sum i_j = d$. The (I_5, j) -th entry of A is $s\lambda_{I_4} + t\mu_{I_4}$ if $x_j x^{I_4} = x^{I_5}$ and 0 else.

Because of the invariance under elementary row operation, the point q can be defined by a matrix of the form

$$\left(0 \mid \begin{array}{ccccccccc} 1 & \lambda_1 & \cdots & \lambda_k & 0 & \lambda_{k+1} & \cdots & \lambda_{k+l} \\ 0 & 0 & \cdots & 0 & 1 & \mu_1 & \cdots & \mu_l \end{array} \right).$$

We hence start with the matrix A having the form

$$\left(\begin{array}{c|cccc} s & 0 & 0 & 0 & 0 \\ \hline \lambda_1 s & & & & \\ \vdots & & & & \\ \lambda_k s & & * & & \\ t & & & & \\ \hline & * & & & \end{array} \right),$$

with $*$ for now not specified, but not arbitrary.

We may perform elementary row and column operations without changing $\text{coker}(A)$. Using Remark 4.6, the goal is to show that the matrix A can be modified to take one of the following forms:

$$\left(\begin{array}{cccc} s & & & \\ t & s & & \\ & t & & \\ & & s & \\ & & t & \\ & & & s \\ & & & t \\ \hline & & & 0 \end{array} \right), \quad \left(\begin{array}{cccc} s & & & \\ t & & & \\ & s & & \\ & t & & \\ & & s & \\ & & t & \\ & & & s \\ & & & t \\ \hline & & & 0 \end{array} \right).$$

We see that this is possible in Computation 2.11. □

Computation 2.11. [Vielleicht abstrakteres Argument hier]

3. Verlinde Bundles on Pencils of Hypersurfaces

Let V_k be the k -th Verlinde bundle of hypersurfaces of degree d in \mathbb{P}^n . The vector bundle is the cokernel of the map

$$M: \mathcal{O}_{|\mathcal{O}(d)|}(-1) \otimes H^0(\mathcal{O}(k-d), \mathbb{P}^n) \rightarrow \mathcal{O}_{|\mathcal{O}(d)|} \otimes H^0(\mathcal{O}(k), \mathbb{P}^n)$$

given by multiplication by $\sum_I \alpha_I \otimes x^I \in H^0(\mathcal{O}(1), |\mathcal{O}(d)|) \otimes H^0(\mathcal{O}(d), \mathbb{P}^n)$. We study the restriction of V_k to lines $T \subseteq |\mathcal{O}(d)|$.

Proposition 3.1. *Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$. The map $M|_T$ splits as a direct sum*

$$\bigoplus_{i=1}^s M_i: \bigoplus_{i=1}^s \mathcal{F}_i \rightarrow \bigoplus_{i=0}^s \mathcal{G}_i$$

with $\mathcal{F}_i \simeq \mathcal{O}(-1)^{\lambda_i}$, $\mathcal{G}_i \simeq \mathcal{O}^{\lambda_i+1}$ for $i \geq 1$, and $\mathcal{G}_0 \simeq \mathcal{O}^{\lambda_0}$. This splitting gives rise to a splitting of the cokernel $\text{coker}(M|_T) = \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^s \mathcal{O}(\lambda_i)$.

Proof (Construction). Let $(\theta_j)_{j \geq 1}$ be a monomial basis of $H^0(\mathcal{O}(k-d), \mathbb{P}^n)$. We construct $M_1: \mathcal{F}_1 \rightarrow \mathcal{G}_1$, the others can be defined similarly. The bundle \mathcal{G}_0 will then just be a direct complement of $\bigoplus_{i \geq 1} \mathcal{G}_i$. Note that for all $\theta \in H^0(\mathcal{O}(k-d), \mathbb{P}^n)$, the morphism $M|_T$ maps $\mathcal{O}(-1) \otimes \langle \theta \rangle$ into $\mathcal{O} \otimes \langle f_1 \theta, f_2 \theta \rangle$.

The bundle \mathcal{F}_1 shall have the form $\mathcal{O}(-1) \otimes U_1$, with U_1 the span of some of the basis elements θ_j , while $\mathcal{G}_1 = \mathcal{O} \otimes W_1$, with $W_1 := f_1 U_1 + f_2 U_1$. We begin the construction by including θ_1 in U_1 . If no other subbundle $\mathcal{O}(-1) \otimes \langle \theta_j \rangle$ maps into $\mathcal{O} \otimes \langle f_1 \theta_1, f_2 \theta_1 \rangle$ we stop, so that $U_1 = \langle \theta_1 \rangle$. Else we pick such a basis element θ_j , and add it to U_1 . We iterate this process until $f_1 U_1^\perp + f_2 U_1^\perp$ does not intersect W_1 . At that point, we may split $M_1: \mathcal{F}_1 \rightarrow \mathcal{G}_1$ as a direct summand of $M|_T$.

Note that after each iterative step in the construction, apart from the first, the dimension of W_1 could increase either by zero or by one. To show that only the second case occurs, we exhibit a linearly independent set of size $j+1$ in $\langle f_1 \theta_1, f_2 \theta_1, \dots, f_1 \theta_j, f_2 \theta_j \rangle$. Here, the indices of the $\theta_{j'}$ are simplified for convenience. Having shown this, it follows that $\text{rk } \mathcal{G}_1 = \lambda_1 + 1$.

Consider an ordering of the monomial basis of $H^0(\mathcal{O}(1), \mathbb{P}^n)$ and accordingly order the monomial bases of all $H^0(\mathcal{O}(m), \mathbb{P}^n)$ lexicographically. Let θ_{\max} denote the greatest basis element among the $(\theta_{j'})_{j'=1}^j$. Assume without loss of generality that the greatest monomial appearing in f_2 is strictly greater than all monomials appearing in f_1 . Then $(f_1 \theta_1, \dots, f_1 \theta_j, f_2 \theta_{\max})$ is a linearly independent set.

Finally, the direct summands $\text{coker}(M_i)$ of the locally free sheaf $\text{coker}(M)$ are themselves locally free, and their form can be found by comparing ranks and determinants. \square

Remark 3.2. The general type candidate for V_k takes the form $(b^{(k)} + 1, \dots, b^{(k)} + 1, b^{(k)}, \dots, b^{(k)})$, where the number of entries is $r^{(k)} := \binom{n+k}{n} - \binom{n+k-d}{n}$ and their sum

$d^{(k)} := \binom{n+k-d}{n}$, while $b^{(k)} = \lfloor d^{(k)}/r^{(k)} \rfloor$. Note that the degrees of $d^{(k)}$ and $r^{(k)}$ as polynomials in k are n and $n-1$, respectively. Hence, $b^{(k)} \rightarrow \infty$ for $k \rightarrow \infty$.

Corollary 3.3. *Let $t \in \text{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2 . Let (θ_j) be a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. Let k be such that $b^{(k)} = 0$, i. e. such that in the general type, only ones and zeroes appear. The bundle $V_{k,t}$ has general type if and only if $(f_1\theta_j, f_2\theta_j)_j$ is a linearly independent set in $H^0(\mathcal{O}(k), \mathbb{P}^n)$.*

Proof. By following the proof of Proposition 3.1, we see that the only way that the assumption is met is if the construction of each direct summand \mathcal{F}_i stops after the first step. \square

Proposition 3.4. *Let $t \in \text{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2 , and let k be such that $b^{(k)} = 0$. The bundle $V_{k,t}$ has nongeneral type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $b^{(k)} = 0$ but $k > 2d$ then the general type never occurs.*

Proof. By Corollary 3.3, the bundle $V_{k,t}$ has non-general type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathcal{O}(k-d), \mathbb{P}^n)$ such that $g_1f_1 + g_2f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h , respectively.

On the other hand, given such g_1 and g_2 , we have $f_1 \mid g_2f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. \square

Example 3.5. For $n = 2, d = 2$, and $k = 3$, we have $d^{(k)} = 3$ and $r^{(k)} = 10$. We show that the only types of V_k that occur are $(1_3, 0_7)$ and $(2_1, 1_2, 0_7)$. The first type occurs e. g. for $f_1 = x_0^2, f_2 = x_1^2$, and the second for $f_1 = x_0^2, f_2 = x_0x_1$. Assume that the type $(3_1, 0_7)$ occurs for some $f_1, f_2 \in H^0(\mathcal{O}(2), \mathbb{P}^2)$. By the proof of Proposition 3.1 we then have $\dim \langle f_1x_j, f_2x_j \rangle_{j=0}^2 = 4$. Hence, we find $g_1, g_2, g'_1, g'_2 \in H^0(\mathcal{O}(1), \mathbb{P}^2)$ and two linearly independent equations

$$\begin{aligned} g_1f_1 + g_2f_2 &= 0 \\ g'_1f_1 + g'_2f_2 &= 0, \end{aligned}$$

with both sets $(g_1, g_2), (g'_1, g'_2)$ linearly independent. From the first equation it follows that $f_1 = g_2h$ and $f_2 = -g_1h$, for some common linear factor h . Applying this to the second equation, we find $g'_1g_2 = g'_2g_1$, hence $g'_1 = \alpha g_1$ and $g'_2 = \alpha g_2$ for some scalar α , a contradiction.

Proposition 3.6. *For $n = 2, d = 2$, and $k = 3$, we have $d^{(k)} = 3$ and $r^{(k)} = 10$. We show that the only types of V_k that occur are $(1_3, 0_7)$ and $(2_1, 1_2, 0_7)$. The first type occurs e. g. for $f_1 = x_0^2, f_2 = x_1^2$, and the second for $f_1 = x_0^2, f_2 = x_0x_1$. Assume that the type $(3_1, 0_7)$ occurs for some $f_1, f_2 \in H^0(\mathcal{O}(2), \mathbb{P}^2)$. By the proof of Proposition 3.1 we*

then have $\dim \langle f_1 x_j, f_2 x_j \rangle_{j=0}^2 = 4$. Hence, we find $g_1, g_2, g'_1, g'_2 \in H^0(\mathcal{O}(1), \mathbb{P}^2)$ and two linearly independent equations

$$\begin{aligned} g_1 f_1 + g_2 f_2 &= 0 \\ g'_1 f_1 + g'_2 f_2 &= 0, \end{aligned}$$

with both sets $(g_1, g_2), (g'_1, g'_2)$ linearly independent. From the first equation it follows that $f_1 = g_2 h$ and $f_2 = -g_1 h$, for some common linear factor h . Applying this to the second equation, we find $g'_1 g_2 = g'_2 g_1$, hence $g'_1 = \alpha g_1$ and $g'_2 = \alpha g_2$ for some scalar α , a contradiction.

4. Specialization

Definition 4.1. Let \mathcal{V} and \mathcal{V}' be vector bundles on a projective k -scheme X . We say that \mathcal{V} *specializes* to \mathcal{V}' if there exists an affine k -scheme Y , spectrum of a discrete valuation ring, with generic point η and closed point η_0 , and a vector bundle \mathcal{W} on $Y \times X$ such that $\mathcal{W}|_{\eta \times X} \simeq \kappa(\eta) \boxtimes \mathcal{V}$ and $\mathcal{W}|_{\eta_0 \times X} \simeq \kappa(\eta_0) \boxtimes \mathcal{V}'$.

Remark 4.2. This definition reflects specialization of points in the moduli stack Vect_X of vector bundles over X , where e.g. the bundles \mathcal{V} and $\kappa(\eta) \boxtimes \mathcal{V}$ define the same point. The stack is locally Noetherian, hence discrete valuation rings suffice. This notion generalizes the notion of specialization of points on a scheme, see e.g. [ÉGA II, Prop. 7.1.9].

Remark 4.3. Let

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \rightarrow 0$$

be a short exact sequence of coherent sheaves over a k -scheme X and let $\xi \in \text{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \rightarrow 0.$$

Example 4.4. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on \mathbb{P}^1 specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. The elements of $\text{Ext}^1(\mathcal{O}(2), \mathcal{O})$ correspond to extensions of the form

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O}(2) \rightarrow 0$$

up to equivalence. The zero element corresponds to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have \mathcal{E} locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\text{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Proposition 1.1, there exists an extension of the form

$$0 \rightarrow \mathcal{O} \boxtimes \mathcal{O}_{\mathbb{A}^1} \rightarrow \mathcal{E}_{\text{univ}} \rightarrow \mathcal{O}(2) \boxtimes \mathcal{O}_{\mathbb{A}^1} \rightarrow 0$$

such that for all nonzero rational points $\xi \in \mathbb{A}^1 = \mathbb{V}(\text{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee)$ we have the isomorphisms $\mathcal{E}_{\text{univ}}|_{\xi \times \mathbb{A}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and $\mathcal{E}_{\text{univ}}|_{0 \times \mathbb{A}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\text{univ}}$ is locally free as the end terms of the sequence are.

Remark 4.5. If \mathcal{V} specializes to \mathcal{V}' and \mathcal{W} specializes to \mathcal{W}' , then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

Remark 4.6. Let b_1, \dots, b_m be non-negative integers, let $a := \sum b_i$, and let s, t denote the homogeneous coordinates on \mathbb{P}^1 . The sequence

$$0 \rightarrow \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \rightarrow 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & & \\ t^{b_2} & s^{b_2} & & & \\ & t^{b_3} & \ddots & & \\ & & \ddots & s^{b_{m-1}} & \\ & & & t^{b_m} & \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \dots & (-1)^m s^{b_1+\dots+b_{m-1}} t^{a-b_1-\dots-b_m} \end{pmatrix}$$

is exact.

Proposition 4.7. Let b_1, \dots, b_m be non-negative integers and π a partition of the set $\{1, \dots, m\}$. For a set of indices $I \in \pi$, let $b'_I := \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on \mathbb{P}^1 specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m-|\pi|}$.

Proof. By Remark 4.5 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 4.6, there exists a representative $\xi \in \text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \rightarrow \mathcal{O}^{m-1} \rightarrow \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_m) \rightarrow \mathcal{O}(a) \rightarrow 0. \quad (\star)$$

By Remark 4.3, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the form \star . Consider the associated closed embedding $\alpha: \mathbb{A}^1 \rightarrow \mathbb{V}(\text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let \mathcal{E} be the universal extension from Proposition 1.1. Then, the vector bundle $(\text{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. \square

Remark 4.8. By twisting the exact sequence (\star) in the proof of Proposition 4.7 and using the same argument, we see that for every integer n and with b_i, π , and b_I as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m-|\pi|}$.

5. Misc./ Notes

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