

VERLINDE BUNDLES OF FAMILIES OF HYPERSURFACES

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1 Universal Families of Extensions

Let X and S be Noetherian schemes over a field k . Let $f: X \rightarrow S$ be a flat, projective morphism, and let \mathcal{F} and \mathcal{G} be coherent \mathcal{O}_X -modules, flat over \mathcal{O}_X .

Recall that an element $\xi \in \operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences, or *extensions*, of the form

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on \mathcal{F} and \mathcal{G} . The set of these equivalence classes can be given the structure of an $H^0(S, \mathcal{O}_S)$ -module, see for example [Wei95, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$ -module structure.

Explicitly, the sum of two elements of Ext^1 corresponds to the Baer sum of the associated extensions, while the multiplication of an extension as above by a scalar $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

In this section, we ask when it is possible to construct an S -scheme V and a universal extension on $X \times_S V$. This is quickly found to be true for \mathcal{F}, \mathcal{G} locally free and $S = \text{Spec}(k)$. For the more general situation, the article [Lan83] turns to the moduli problem of classifying relative extensions of sheaves and applies it to the global situation.

Remark 1.1. Let $\varphi: X_1 \rightarrow X_2$ be a morphism of schemes, let \mathcal{F}_1 and \mathcal{F}_2 be \mathcal{O}_{X_1} -modules. The Grothendieck spectral sequence [Vak17, Theorem 23.3.5] specializes to the Leray spectral sequence $E_2^{p,q} = H^q(X_2, R^p\varphi_*\mathcal{F}_1) \Rightarrow H^{p+q}(X_1, \mathcal{F}_1)$ and the local-to-global Ext spectral sequence $E_2^{p,q} = H^p(X_1, \mathcal{E}xt^q(\mathcal{F}_1, \mathcal{F}_2)) \Rightarrow \text{Ext}^{p+q}(\mathcal{F}_1, \mathcal{F}_2)$. The first few terms of the associated exact sequences in lower degrees are

$$0 \rightarrow H^1(X_2, \varphi_*\mathcal{F}_1) \rightarrow H^1(X_1, \mathcal{F}_1) \rightarrow H^0(X_2, R^1\varphi_*\mathcal{F}_1) \quad (1.1)$$

and

$$0 \rightarrow H^1(X_2, \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow \text{Ext}^1(\mathcal{F}_1, \mathcal{F}_2) \rightarrow H^0(X_2, \mathcal{E}xt^1(\mathcal{F}_1, \mathcal{F}_2)). \quad (1.2)$$

Proposition 1.2. *Let \mathcal{F} and \mathcal{G} be locally free, and let $V := \mathbb{V}(\text{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$. There exists an extension*

$$\xi_{\text{univ}}: \quad 0 \rightarrow \text{pr}_1^*\mathcal{G} \rightarrow \mathcal{E} \rightarrow \text{pr}_1^*\mathcal{F} \rightarrow 0$$

over $X \times_k V$ such that for all Noetherian k -schemes Y , the map

$$\text{Mor}_k(Y, V) \rightarrow \text{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$$

defined by $\alpha \mapsto (\text{id}_X \times \alpha)^\xi_{\text{univ}}$ is a bijection, functorial in Y . In particular, pulling back ξ_{univ} gives a bijection $\text{Mor}_k(\text{Spec}(k), V) \xrightarrow{\sim} \text{Ext}_X^1(\mathcal{F}, \mathcal{G})$.*

Proof. We find functorial isomorphisms

$$\text{Mor}_k(Y, V) \simeq H^0(\mathcal{O}_Y \otimes \text{Ext}_X^1(\mathcal{F}, \mathcal{G})) \quad (1.3)$$

$$\simeq H^0(s_2^*H^1(\mathcal{H}om(\mathcal{F}, \mathcal{G}))) \quad (1.4)$$

$$\simeq H^0(R^1\text{pr}_{2,*}(\text{pr}_1^*\mathcal{H}om(\mathcal{G}, \mathcal{F}))) \quad (1.5)$$

$$\simeq H^0(R^1\text{pr}_{2,*}(\mathcal{H}om(\mathcal{F}_Y, \mathcal{G}_Y))) \quad (1.6)$$

$$\simeq H^1(\mathcal{H}om(\mathcal{G}_Y, \mathcal{F}_Y)) \quad (1.7)$$

$$\simeq \text{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y). \quad (1.8)$$

The required universal extension is then the image of $\text{id} \in \text{Mor}_k(V, V)$ in $\text{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$.

The isomorphism (1.3) comes from the universal property of $\mathbb{V}(\text{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$. The isomorphisms (1.4) and (1.8) come from the sequence (1.2), whose third term is zero since \mathcal{F} and \mathcal{G} are locally free. We have (1.5) by the Cohomology and Base Change Theorem [Vak17, 28.1.6], and (1.6) since \mathcal{F} and \mathcal{G} are locally free. For the isomorphism (1.7), we use the sequence (1.1), whose third term is found to be zero after applying the Cohomology and Base Change Theorem. \square

Definition 1.3. (i) The i -th relative Ext module $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ is the image of \mathcal{G} under the i -th right-derived functor of $f_*\mathcal{H}om(\mathcal{F}, -): \text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_{\mathcal{O}_S}$.

(ii) For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s, \mathcal{F}, \mathcal{G}}: \text{Ext}_X^1(\mathcal{F}, \mathcal{G}) \rightarrow \text{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s)$$

by restricting extensions of \mathcal{F} by \mathcal{G} to the fiber X_s . This is well-defined, since \mathcal{F} is flat over S .

(iii) A family of extensions of \mathcal{F} by \mathcal{G} over S is a family

$$\xi_s \in \text{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s), \quad s \in S$$

such that there exists an open covering \mathfrak{U} of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \text{Ext}_{f^{-1}(U)}^1(\mathcal{F}_U, \mathcal{G}_U)$ with $\Phi_{s, \mathcal{F}_U, \mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

Remark 1.4. If S is affine, then we have $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G}) = \text{Ext}_X^i(\mathcal{F}, \mathcal{G})^\sim$.

Proposition 1.5. Let $g: Y \rightarrow S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism

$$\tau_g^i: g^*\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{E}xt_{f_Y}^i(g_X^*\mathcal{F}, g_X^*\mathcal{G}).$$

Furthermore, if g is flat, then τ_g^i is an isomorphism for all $i \geq 0$.

Proof. See [Lan83, Prop. 1.3] □

Definition 1.6. We say that $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change if for all morphisms of Noetherian schemes $g: Y \rightarrow S$, the base change homomorphism τ_g^i is an isomorphism.

Proposition 1.7. Let $s \in S$ be a point such that τ_s^i is surjective. Then there exists an open neighborhood U of s such that $\tau_{s'}^i$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ_s^{i-1} is surjective if and only if $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ is locally free on an open neighborhood of s .

Proof. See [Lan83, Thm. 1.4] □

Remark 1.8. (i) If τ_s^i is an isomorphism for all $s \in S$, then $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change.

(ii) From Proposition 1.7 we conclude that if $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ is locally free.

(iii) In case S is reduced, if $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ is locally free then $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 1.9. Let $u: Y' \rightarrow Y$ be a morphism of Noetherian S -schemes.

(i) We define a functoriality map $H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) \rightarrow H^0(Y', \mathcal{E}xt_{f_{Y'}}^1(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1 \otimes \text{id}} H^0(Y', u^* \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau_u^1)} H^0(Y', \mathcal{E}xt_{f_{Y'}}^1(u_{X_Y}^* \mathcal{F}_{Y'}, u_{X_Y}^* \mathcal{G}_{Y'})). \end{aligned}$$

(ii) Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of \mathcal{F}_Y by \mathcal{G}_Y over Y , we set $(u^* \xi)_{y'} := u^* \xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^* \xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y' .

(iii) We define the functors

$$\begin{aligned} E, E' &: (\text{NoethSch}/S) \rightarrow (\text{Sets}); \\ E(Y) &:= H^0(Y, \mathcal{E}xt_{f_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)), \\ E'(Y) &:= \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

Remark 1.10. The Grothendieck spectral sequence for the sequence of functors

$$\text{Mod}_{\mathcal{O}_X} \xrightarrow{\mathcal{H}om(\mathcal{F}, -)} \text{Mod}_{\mathcal{O}_X} \xrightarrow{f_*} \text{Mod}_{\mathcal{O}_S}$$

is the spectral sequence with $E_2^{p,q} = H^p(S, \mathcal{E}xt_f^q(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$. This gives the exact sequence

$$\begin{aligned} 0 \rightarrow H^1(S, f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})) &\xrightarrow{\varepsilon} \text{Ext}_X^1(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})) \\ &\xrightarrow{d_2} H^2(S, f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})). \end{aligned} \tag{1.9}$$

Proposition 1.11. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S -schemes, the functors E and E' are isomorphic.*

Proof. See [Lan83, Prop. 2.3]. □

Proposition 1.12. *Suppose that $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the \mathcal{O}_S -module $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Proof. See [Lan83, Prop. 3.1]. □

Corollary 1.13. *Suppose that S is reduced and $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S -schemes, the functor E' is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.14. *Suppose that for all Noetherian S -schemes Y we have*

$$H^i(Y, f_{Y,*} \mathcal{H}om_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) = 0$$

for $i = 1, 2$. The functor $Y \mapsto \text{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$ is representable by the S -scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.

Proof. Use the sequence (1.9) and Proposition 1.12. \square

Remark 1.15. As a special case of the above, we recover Proposition 1.2.

Remark 1.16. The article [Lan83] continues on to define a (projectivized) version of the problem, so that over $\text{Spec}(k)$, the scheme $\mathbb{P}(\text{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of \mathcal{F} by \mathcal{G} , modulo the action of k^\times . See also [HL10, Example 2.1.12].

2 Verlinde Bundles on Pencils of Quartics

The thesis [Hem15] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in \mathbb{P}^3 , after summarizing some of its properties. We work over a field k , but omit it in most notation¹, e.g. we write \mathbb{P}^3 for \mathbb{P}_k^3 .

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in \mathbb{P}^3 . The quartics $\mathfrak{X}_t \subseteq \mathbb{P}^3$ parametrized by the $t \in |\mathcal{O}(4)|$ form a universal family $\pi: \mathfrak{X} \rightarrow |\mathcal{O}(4)|$ with fibers \mathfrak{X}_t . The family \mathfrak{X} is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$. The morphism π is projective and flat.

Throughout, the homogeneous coordinates of \mathbb{P}^3 will be denoted by x_i , $i = 0, \dots, 4$.

We define the line bundle \mathcal{L} on \mathfrak{X} as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to \mathfrak{X} . In other words², the bundle \mathcal{L} is the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \rightarrow \mathbb{P}^3$.

Proposition 2.1. *Let $k \geq 1$. The following statements hold:*

- (i) *If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this number is independent of the point q .*
- (ii) *The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.*
- (iii) *For all cartesian diagrams of the form*

$$\begin{array}{ccc} \mathfrak{X}_Z & \longrightarrow & \mathfrak{X} \\ \pi_Z \downarrow & \times & \downarrow \pi \\ Z & \xrightarrow{\rho} & |\mathcal{O}(4)| \end{array}$$

we have $\rho^ \pi_* \mathcal{L}^{\otimes k} \simeq (\pi_Z)_* \mathcal{L}_Z^{\otimes k}$.*

Proof. For the first statement, see the proof of [Hem15, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. \square

¹Most instances of the letter k will be used to denote a natural number instead.

²For a fiber product $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ and sheaves \mathcal{F} and \mathcal{G} on X resp. Y , write $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$.

Let $T \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}_K^1 \rightarrow |\mathcal{O}(4)|$, with K an extension field of k . We call T a *pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}_K^1}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}_K^1}^1$. The situation is summarized in the picture below:

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{P}_K^1} & \longrightarrow & \mathfrak{X} \\ \downarrow & \times & \downarrow \pi \\ \mathbb{P}_K^1 & \longrightarrow & |\mathcal{O}(4)| \end{array}$$

Definition 2.2. For $k \geq 1$, we define the k -th Verlinde bundles $V_k := \pi_* \mathcal{L}^{\otimes k}$ and $V_{k,T} := (\pi_{\mathbb{P}^1})_* \mathcal{L}_{\mathbb{P}^1}^{\otimes k}$. These bundles are related by $V_k|_T = V_{k,T}$ using Proposition 2.1.

Proposition 2.3. *There exists a short exact sequence of coherent $\mathcal{O}_{|\mathcal{O}(4)|}$ -modules*

$$0 \rightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \rightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \rightarrow V_k \rightarrow 0.$$

Let I_d range over the tuples of the form (i_0, \dots, i_3) with $\sum i_j = d$. The first map is then given by $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.

Proof. See [Hem15, Proposition 4.2]. □

Remark 2.4. Let T be a pencil of quartics.

(i) The sequence from Proposition 2.3 restricts to a sequence

$$0 \rightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \rightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \rightarrow V_{k,T} \rightarrow 0 \quad (2.1)$$

over \mathbb{P}^1 .

(ii) The vector bundle $V_{k,T}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

(iii) Let $V_{k,T} \simeq \bigoplus_i \mathcal{O}(d_i)$ be a splitting of $V_{k,T}$ over \mathbb{P}^1 . By the sequence (2.1), we have $d_i \geq 0$.

Definition 2.5. Let $k \geq 1$.

(i) A *type candidate* for V_k is a non-increasing tuple $(d_1, \dots, d_{r(k)})$ of non-negative integers with $r^{(k)} = \binom{k+3}{3} - \binom{k-1}{3}$ and $\sum d_i = \binom{k-1}{3}$.

(ii) The *general type candidate* for V_k is the unique type candidate for V_k of the form $(b^{(k)} + 1, \dots, b^{(k)} + 1, b^{(k)}, \dots, b^{(k)})$. The integer $b^{(k)}$ is determined by the equation $\binom{k-1}{3} = b^{(k)} r^{(k)} + a$, with $a < r^{(k)}$ becoming the number of occurrences of $b^{(k)} + 1$.

(iii) Let E be a locally free sheaf on \mathbb{P}^1 . The *type* of E is the unique non-increasing tuple $(d_1, \dots, d_{r(k)})$ such that $E \simeq \bigoplus_i \mathcal{O}(d_i)$.

(iv) We say that $V_{k,T}$ has *general type* if its type is a general type candidate.

Definition 2.6. Let P denote the universal \mathbb{P}^1 -bundle of the Grassmannian of lines $\mathrm{Gr}(2, H^0(\mathcal{O}(4))) = \mathrm{Gr}(1, |\mathcal{O}(4)|)$, let $\varphi: P \rightarrow \mathrm{Gr}(1, |\mathcal{O}(4)|)$ be the universal map and $p: P \rightarrow |\mathcal{O}(4)|$ the canonical projection.

$$\begin{array}{ccc} P & \xrightarrow{p} & |\mathcal{O}(4)| \\ \downarrow \varphi & & \\ \mathrm{Gr}(1, |\mathcal{O}(4)|) & & \end{array}$$

The mapping $t \mapsto P_t$ gives a canonical bijection between the points of $\mathrm{Gr}(1, |\mathcal{O}(4)|)$ and the pencils of quartics in $|\mathcal{O}(4)|$. For such t , we write $V_{k,t} := V_{k,p(P_t)}$.

Definition 2.7. Let $k \geq 1$ and (d_i) be a type candidate for V_k . We define the set $Z_{(d_i)}$ of all points $t \in \mathrm{Gr}(1, |\mathcal{O}(4)|)$ such that $V_{k,t}$ has type (d_i) . For the set of points t where $V_{k,t}$ has general type, we also write Z_{gen} .

Proposition 2.8. *The set Z_{gen} is Zariski open. Its complement is the union*

$$\mathrm{Supp}(R^1\varphi_*p^*V_k(-b^{(k)} - 1)) \cup \mathrm{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee)).$$

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \mathrm{Gr}(1, |\mathcal{O}(4)|)$, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and $b := b^{(k)}$. We have $t \in Z_{\mathrm{gen}}$ if and only if $b \leq d_i \leq b+1$ for all i , which holds if and only if $H^1(P_t, V_{k,t}(-b-1)) = H^1(P_t, V_{k,t}(-b)^\vee) = 0$.

Next, we want to apply the Cohomology and Base Change Theorem [Vak17, 28.1.6] to the map $\varphi: P \rightarrow \mathrm{Gr}(1, |\mathcal{O}(4)|)$, which is a \mathbb{P}^1 -bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\mathrm{Gr}(1, |\mathcal{O}(4)|)$.

For all $t \in \mathrm{Gr}(1, |\mathcal{O}(4)|)$ we have $h^2(P_t, p^*V_{k,t}(-b-1)) = 0$ and $h^2(P_t, p^*V_{k,t}(-b)^\vee) = 0$. Since the sheaves $p^*V_{k,t}(-b-1)$ and $p^*V_{k,t}(-b)^\vee$ are locally free and coherent, we have

$$(R^1\varphi_*p^*V_k(-b-1))_t = H^1(P_t, V_{k,t}(-b-1))$$

and

$$(R^1\varphi_*(p^*V_k(-b)^\vee))_t = H^1(P_t, V_{k,t}(-b)^\vee).$$

By the previous characterization, we have

$$\mathrm{Gr}(1, |\mathcal{O}(4)|) \setminus Z_{\mathrm{gen}} = \mathrm{Supp}(R^1\varphi_*p^*V_k(-b-1)) \cup \mathrm{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)),$$

which is a Zariski closed set. □

Proposition 2.9. *The sets $\mathrm{Supp}(R^1\varphi_*p^*V_k(-b^{(k)} - 1))$ and $\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee))$ are determinantal varieties in the sense of [Arb+13, Ch. II, §4]*

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k))$, $r_2 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4))$ and $b := b^{(k)}$, and rewrite the exact sequence from Proposition 2.3 as

$$0 \rightarrow \mathcal{O}(-1)^{r_2} \rightarrow \mathcal{O}^{r_1} \rightarrow V_k \rightarrow 0. \quad (2.2)$$

Twisting the sequence (2.2) with $\mathcal{O}(-b-1)$ and pulling back to P gives an exact sequence

$$0 \rightarrow p^* \mathcal{O}(-b-2)^{r_2} \rightarrow p^* \mathcal{O}(-b-1)^{r_1} \rightarrow p^* V_k(-b-1) \rightarrow 0.$$

For all $t \in \mathbb{G}r(1, |\mathcal{O}(4)|)$ we have $h^2(P_t, \mathcal{O}(-b-2)^{r_2}) = 0$, hence $R^2 \varphi_* p^* \mathcal{O}(-b-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1 \varphi_* p^* \mathcal{O}(-b-2)^{r_2} \xrightarrow{\alpha} R^1 \varphi_* p^* \mathcal{O}(-b-1)^{r_1} \rightarrow R^1 \varphi_* p^* V_k(-b-1) \rightarrow 0.$$

Note that since the numbers $h_2^1 := h^1(P_t, \mathcal{O}(-b-2)^{r_2})$ and $h_1^1 := h^1(P_t, \mathcal{O}(-b-1)^{r_1})$ do not depend on the point t , Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h_2^1 and h_1^1 , respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1 \varphi_* p^* V_k(-b-1))_t \neq 0$ if and only if $\text{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\text{Supp}(R^1 \varphi_*(p^* V_k(-b-1))) = \{t : \text{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = br_1 = b \binom{k+3}{3}$.

The proof for the second assertion is similar. We start with the sequence (2.2), twist with $\mathcal{O}(-b)$, take duals, pull back to P , and apply φ_* . Since for all $t \in \mathbb{G}r(1, |\mathcal{O}(4)|)$ we have $h^1(P_t, \mathcal{O}(b)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_* p^* \mathcal{O}(b)^{r_1} \xrightarrow{\beta} \varphi_* p^* \mathcal{O}(b+1)^{r_2} \rightarrow R^1 \varphi_*(p^* V_k(-b)^\vee) \rightarrow 0.$$

Since the numbers $h_1^0 := h^0(P_t, \mathcal{O}(b)^{r_1})$ and $h_2^0 := h^0(P_t, \mathcal{O}(b+1)^{r_2})$ do not depend on the point t , again by Grauert's Theorem the first two terms of the sequence are locally free of rank h_1^0 and h_2^0 , respectively. As before, we obtain the characterization

$$\text{Supp}(R^1 \varphi_*(p^* V_k(-b)^\vee)) = \{t : \text{rk}(\beta_t) \leq h_2^0 - 1\}.$$

Here, we have $h_2^0 = (b+2)r_2 = (b+2) \binom{k-1}{3}$. □

Definition 2.10. For type candidates (d_i) and (d'_i) we define the expression $(d'_i) \geq (d_i)$ to mean

$$\sum_{i=1}^s d'_i \geq \sum_{i=1}^s d_i \text{ for all } s = 1, \dots, r^{(k)}.$$

Proposition 2.11. Let (d_i) be a type candidate for V_k . The set $\widehat{Z}_{(d_i)} := \bigcup_{(d'_i) \geq (d_i)} Z_{(d'_i)}$ is Zariski-closed. In particular, the set $Z_{(d_i)}$ is locally closed.

Proof. Let $t \in \mathbb{G}r(1, |\mathcal{O}(4)|)$ and $V_{k,t} = \bigoplus_{i=1}^{r(k)} \mathcal{O}(d'_i)$. We have

$$\bigwedge^s V_{k,t} = \bigoplus_I \mathcal{O}(d'_I),$$

where I runs over the subsets of $\{1, \dots, r(k)\}$ of size s and $d'_I := \sum_{i \in I} d'_i$. For every type candidate (d'_i) , the sum $\sum_{i=1}^s d'_i$ is the largest sum of s entries of (d'_i) . Since $d'_i \geq 0$, the condition $\sum_{i=1}^s d'_i \geq \sum_{i=1}^s d_i$ is equivalent to the condition $h^0((\bigwedge^s V_{t,k})(-\sum^s d_i)) > 0$.

$$\widehat{Z}_{(d_i)} = \bigcap_{s=1}^{r(k)} \{t : h^0((\bigwedge^s V_{t,k})(-\sum^s d_i)) > 0\}.$$

With Serre duality and the Cohomology and Base Change theorem we write the sets of the intersection as

$$\text{Supp}(R^1 \varphi_*(p^*(\bigwedge^s V_k^\vee)(\sum^s d_i - 2))),$$

which is Zariski-closed. □

Proposition 2.12. *Of the five type candidates*

$$(1, 1, 1, 1, 0, \dots, 0), (2, 1, 1, 0, \dots, 0), (2, 2, 0, \dots, 0), (3, 1, 0, \dots, 0), (4, 0, \dots, 0)$$

for V_5 , only the first two occur as types of some $V_{5,t}$.

Proof. This is a special case of Proposition 3.8. □

2.1 Calculations in the Chow Ring

To perform calculations in the Chow ring A of $\mathbb{G}r(1, |\mathcal{O}(4)|)$, we follow the conventions found in [EH16]. We assume $\text{char}(k) = 0$ for simplicity. The ring A is generated by the Schubert classes $\sigma_{a,b}$ of the Schubert cycles

$$\Sigma_{a,b}(\mathcal{H}) := \{T \in \mathbb{G}r(1, |\mathcal{O}(4)|) : T \cap H \neq \emptyset, T \subseteq H'\},$$

where $\mathcal{H} = (H \subset H')$ is a flag of linear subspaces of dimension $33 - a$ resp. $34 - b$ in the 34-dimensional projective space $|\mathcal{O}(4)|$. The class $\Sigma_{a,b}$ has codimension $a + b$, and we use the convention $\sigma_a := \sigma_{a,0}$.

Proposition 2.13. *Let $Z := Z_{(2,1,1,0,\dots)}$ be locus of points $t \in \mathbb{G}r(1, |\mathcal{O}(4)|)$ such that $V_{5,t}$ is not generic. In the Chow ring A , we have*

$$[Z] = \sum_{a=0}^{11} \left(\binom{24-a}{3} \binom{a+1}{3} - \binom{25-a}{3} \binom{a}{3} \right) \sigma_{33-a,10+a}. \quad (2.3)$$

Proof. Define the subvariety $Q \subset |\mathcal{O}(4)|$ as the image of the map

$$f: |\mathcal{O}(1)| \times |\mathcal{O}(3)| \rightarrow |\mathcal{O}(4)|$$

defined by $f(g, h) = gh$. The map f is birational on its image, since a general point of Q has the form gh with h irreducible. By Proposition 3.6, the variety Z is the image of the finite map

$$\varphi: \mathbb{G}r(1, |\mathcal{O}(1)|) \times |\mathcal{O}(3)| \rightarrow \mathbb{G}r(1, |\mathcal{O}(4)|)$$

defined by $\varphi((sg_1 + tg_2)_{(s:t)}, h) = (sg_1h + tg_2h)_{(s:t)}$. In particular, a line $T \subset |\mathcal{O}(4)|$ belonging to Z lies in Q .

Since f and φ are finite, we have $\dim(Q) = 22$ and $\dim(Z) = 23$, while $\text{codim}(Z) = 43$. The Chow group A^{43} is generated by the classes $\sigma_{33,10}, \sigma_{32,11}, \dots, \sigma_{22,21}$, while the complementary group A^{23} is generated by $\sigma_{23,0}, \sigma_{22,1}, \dots, \sigma_{12,11}$. Write

$$[Z] = \sum_{a=0}^{11} \alpha_{33-a,10+a} \sigma_{33-a,10+a}.$$

We have $\sigma_{33-a,10+a} \sigma_{23-a',a'} = 1$ if $a = a'$ and 0 else. Hence, multiplying the above equation with $\sigma_{33-a',10+a'}$ and taking degrees gives $\alpha_{33-a,10+a} = \deg([Z] \cdot \sigma_{23-a,a})$.

Using Giambelli's formula $\sigma_{23-a,a} = \sigma_{23-a} \sigma_a - \sigma_{24-a} \sigma_{a-1}$ [EH16, Prop. 4.16], we reduce to computing $\deg([Z] \cdot \sigma_{23-a} \sigma_a)$ for $0 \leq a \leq 11$. By Kleiman transversality, we have

$$\deg([Z] \cdot \sigma_{23-a} \sigma_a) = |\{T \in \mathbb{G}r(1, |\mathcal{O}(4)|) : T \cap H \neq \emptyset, T \cap H' \neq \emptyset\}|,$$

where H and H' are general linear subspaces of $|\mathcal{O}(4)|$ of dimension $10 + a$ and $33 - a$, respectively.

To a point $p = g_p h_p \in Q$ with $g_p \in |\mathcal{O}(1)|$ and $h_p \in |\mathcal{O}(3)|$, associate a closed reduced subscheme Λ_p containing p as follows. If h_p is irreducible, let Λ_p be the image of the linear embedding $|\mathcal{O}(1)| \times \{h_p\} \rightarrow |\mathcal{O}(4)|$ given by $g \mapsto gh_p$. If $h_p = g'_p h'_p$ with $h'_p \in |\mathcal{O}(2)|$ irreducible, let Λ_p be the union of the images of the linear embeddings $|\mathcal{O}(1)| \times \{h_p\} \rightarrow |\mathcal{O}(4)|$ and $|\mathcal{O}(1)| \times \{g_p h'_p\} \rightarrow |\mathcal{O}(4)|$. These two linear subspaces meet exactly at p . Similarly, if p is the product of four linear forms, define the space Λ_p as the union $\bigcup_h \text{im}(|\mathcal{O}(1)| \times \{h\} \rightarrow |\mathcal{O}(4)|)$, where h runs over the four cubics arising as products of the linear factors of p .

By the definition of Z , all lines $T \in Z$ lie in Q and if T meets the point p , then $T \subseteq \Lambda_p$. For $H \subseteq |\mathcal{O}(4)|$ a linear subspace of dimension $10 + a$, define $Q' := H \cap Q$. For general H , the subscheme Q' is a smooth subvariety of dimension $a - 2$ such that a general point $p = gh$ of Q' with $h \in |\mathcal{O}(4)|$ has h irreducible. For $a = 0, 1$, a general linear subspace H does not intersect Q at all, from which follows $\deg([Z] \cdot \sigma_{23} \sigma_0) = \deg([Z] \cdot \sigma_{22} \sigma_1) = 0$.

Next, we show that for general H , for each point $p \in Q'$ we have $\Lambda_p \cap H = \{p\}$. Let \mathcal{H} denote the parameter space for H , i.e. the Grassmannian $\mathbb{G}r(10 + a, 34)$. Define the closed subset $X \subseteq Q \times \mathcal{H}$ by $X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}$. The fibers of the induced

map $X \rightarrow \mathcal{H}$ have dimension at least one. Hence, to prove that the desired condition on H is an open condition, it suffices to prove $\dim(X) \leq \dim(\mathcal{H})$. The fiber of the map $X \rightarrow Q$ over a point p consists of the union of finitely many closed subsets of the form $X'_p = \{H \in \mathcal{H} : \dim(H \cap \Lambda'_p) \geq 1\}$, where $\Lambda'_p \simeq \mathbb{P}^3 \subseteq |\mathcal{O}(4)|$ is one of the components of Λ_p . The space X'_p is a Schubert cycle

$$\Sigma_{22-a, 22-a} = \{H \in \text{Gr}(11, 35) : \dim(H \cap H_4) \geq 2\},$$

with H_4 a four-dimensional subspace of $H^0(\mathcal{O}(4))$. The codimension of the cycle is $2(22 - a)$, hence also $\text{codim}(X_p) = 2(22 - a)$. Finally, we have $\dim(\mathcal{H}) - \dim(X) = \text{codim}(X_p) - \dim(Q) = 22 - 2a \geq 0$ for $0 \leq a \leq 11$.

Next, let $\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|\mathcal{O}(1)| \otimes \text{pr}_2 f^{-1}(Q'))$ and $\Lambda'' := |\mathcal{O}(1)| \otimes \text{pr}_2 f^{-1}(Q')$. By the choice of H , the map $f^{-1}(Q') \rightarrow Q'$ is birational and the map $f^{-1}(Q) \rightarrow \text{pr}_2 f^{-1}(Q)$ is even bijective. It follows that Λ'' and hence Λ have dimension $a + 1$. The intersection of Λ with a general linear subspace H' of dimension $33 - a$ is a finite set of points. For each point $p \in Q'$, the linear subspace H' intersects each component Λ'_p of Λ_p in at most one point. For each point $p' \in H' \cap \Lambda$ there exists a unique p such that $p' \in \Lambda_p$. Furthermore, the only line $T \in Z$ meeting both p and H' is the one through p and p' . If the intersection $H' \cap \Lambda_p$ is empty, then there will be no line meeting p and H' . Hence, $\deg([Z] \cdot \sigma_{22-a} \sigma_a)$ is the number of intersection points of Λ with a general H' .

Finally, the pre-image $f^{-1}(Q') = f^{-1}(H)$ is smooth for a general H by Bertini's Theorem. If ζ is the class of a hyperplane section of $|\mathcal{O}(4)|$ we have $f^*(\zeta) = \alpha + \beta$, where α and β are classes of hyperplane sections of $|\mathcal{O}(1)|$ and $|\mathcal{O}(3)|$, respectively. Since pr_2 and f have degree one we compute:

$$\begin{aligned} [\Lambda''] &= [\text{pr}_2^{-1} \text{pr}_2 f^{-1}(H)] \\ &= \text{pr}_2^* [\text{pr}_2 f^{-1}(H)] \\ &= \text{pr}_2^* \text{pr}_{2,*} [f^{-1}(H)] \\ &= \text{pr}_2^* \text{pr}_{2,*} f^*[H] \\ &= \text{pr}_2^* \text{pr}_{2,*} (\alpha + \beta)^{24-a} \\ &= \binom{24-a}{3} \text{pr}_2^* \beta^{21-a} \\ &= \binom{24-a}{3} \beta^{21-a} \end{aligned}$$

Hence, by the push-pull formula:

$$\deg([\Lambda] \cdot H') = \deg([\Lambda''] \cdot (\alpha + \beta)^{a+1}) = \binom{24-a}{3} \binom{a+1}{3}$$

We then use Giambelli's formula to obtain Equation (2.3).

□

3 Verlinde Bundles on Pencils of Hypersurfaces

Definition 3.1. Let $\pi: \mathfrak{X} \rightarrow |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the universal family of hypersurfaces of degree d in \mathbb{P}^n . Let \mathcal{L} be the restriction of the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}$ under the inclusion $\mathfrak{X} \subseteq \mathbb{P}^n \times |\mathcal{O}(d)|$. The vector bundle $V_k := \pi_*(\mathcal{L}^{\otimes k})$ is the k -th *Verlinde Bundle* of the universal family π .

The vector bundle V_k is the cokernel of the map

$$M: \mathcal{O}_{|\mathcal{O}(d)|}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \rightarrow \mathcal{O}_{|\mathcal{O}(d)|} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k))$$

given by multiplication by $\sum_I \alpha_I \otimes x^I \in H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$, where the α_I are the homogeneous coordinates on $|\mathcal{O}(d)|$. The vector bundle V_k has rank $r^{(k)} := h^0(\mathbb{P}^n, \mathcal{O}(k)) - h^0(\mathbb{P}^n, \mathcal{O}(k-d))$ and determinant $\mathcal{O}(d^{(k)})$ with $d^{(k)} := h^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We study the restriction of V_k to lines $T \subseteq |\mathcal{O}(d)|$.

Lemma 3.2. Let \mathcal{E} be a finite free $\mathcal{O}_{\mathbb{P}^1}$ -module, and let

$$0 \rightarrow \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$$

be a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$ -modules. Given a splitting $\mathcal{E}'' = \mathcal{E}_1'' \oplus \mathcal{O}$, we may construct a splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$ such that the image of φ is contained in \mathcal{E}_1 .

Proof. Define $\mathcal{E}_1 := \ker(\text{pr}_2 \circ \psi)$, which is a locally free sheaf on \mathbb{P}^1 . By comparing determinants in the short exact sequence $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$ we see that \mathcal{E}_1 is free, hence by an Ext^1 computation the sequence splits. The property $\text{im}(\varphi) \subseteq \mathcal{E}_1$ follows from the definition. \square

Proposition 3.3. Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$ and $\text{coker}(M|_T) \simeq \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^s \mathcal{O}(d_i)$. Define $U := H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We have

$$d_0 = \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1 U + f_2 U),$$

or, equivalently,

$$s = \dim(f_1 U + f_2 U) - d^{(k)}.$$

Proof. Note that the map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1 \theta + t\xi \otimes f_2 \theta$. In particular, the image of $\mathcal{O}(-1) \otimes U$ is contained in $\mathcal{O} \otimes (f_1 U + f_2 U)$. It follows that $d_0 \geq \dim(f_1 U + f_2 U)$.

To prove the other inequality, consider the induced sequence

$$0 \rightarrow \mathcal{O}(-1) \otimes U \xrightarrow{M|_T} \mathcal{O} \otimes (f_1 U + f_2 U) \rightarrow \mathcal{E}'' \rightarrow 0$$

and assume that $\mathcal{E}'' \simeq \mathcal{E}_1'' \oplus \mathcal{O}$. By Lemma 3.2, we have a splitting $\mathcal{O} \otimes (f_1 U + f_2 U) \simeq \mathcal{E}_1 \oplus \mathcal{O}$ such that $\text{im}(M|_T) \subseteq \mathcal{E}_1$.

Consider the map $\widetilde{M}|_T: (\mathcal{O} \otimes U) \oplus (\mathcal{O} \otimes U) \rightarrow \mathcal{O} \otimes (f_1U + f_2U)$ defined by

$$\widetilde{M}|_T(a \otimes \theta_1, b \otimes \theta_2) = a \otimes f_1\theta_1 + b \otimes f_2\theta_2.$$

We obtain the matrix description of $\widetilde{M}|_T$ from the matrix description of $M|_T$ as follows. If $M|_T$ is represented by the matrix A with coefficients $A_{i,j} = \lambda_{i,j}s + \mu_{i,j}t$, $i \leq \dim(f_1U + f_2U)$, $j \leq \dim U$, then $\widetilde{M}|_T$ is represented by a block matrix

$$B = \left(\begin{array}{c|c} A' & A'' \end{array} \right)$$

with $A'_{i,j} = \lambda_{i,j}$ and $A''_{i,j} = \mu_{i,j}$.

The property $\text{im}(M|_T) \subseteq \mathcal{E}_1$ implies that after some row operations, the matrix A has a zero row. By the construction of $\widetilde{M}|_T$, the same row operations lead to the matrix B having a zero row, but this is a contradiction, since the map $\widetilde{M}|_T$ is surjective. \square

Remark 3.4. The general type candidate for V_k takes the form $(b^{(k)} + 1, \dots, b^{(k)} + 1, b^{(k)}, \dots, b^{(k)})$, where the number of entries is $r^{(k)} = \binom{n+k}{n} - \binom{n+k-d}{n}$ and their sum $d^{(k)} = \binom{n+k-d}{n}$, while $b^{(k)} = \lfloor d^{(k)}/r^{(k)} \rfloor$. Note that the degrees of $d^{(k)}$ and $r^{(k)}$ as polynomials in k are n and $n-1$, respectively. Hence, $b^{(k)} \rightarrow \infty$ for $k \rightarrow \infty$.

Corollary 3.5. *Let $t \in \text{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2 . Let (θ_j) be a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. Let k be such that $b^{(k)} = 0$, that is such that in the general type, only ones and zeroes appear. The bundle $V_{k,t}$ has general type if and only if $(f_1\theta_j, f_2\theta_j)_j$ is a linearly independent set in $H^0(\mathbb{P}^n, \mathcal{O}(k))$.*

Proof. Since $b^{(k)} = 0$, the type of $V_{k,t}$ is the general type if and only if it has $d^{(k)}$ many nonzero entries. By Proposition 3.3, this is the case if and only if $\dim \langle f_1\theta_j, f_2\theta_j \rangle_j = 2d^{(k)}$. \square

Proposition 3.6. *Let $t \in \text{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2 , and let k be such that $b^{(k)} = 0$. The bundle $V_{k,t}$ has nongeneral type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $b^{(k)} = 0$ but $k > 2d$ then the general type never occurs.*

Proof. By Corollary 3.5, the bundle $V_{k,t}$ has non-general type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ such that $g_1f_1 + g_2f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h , respectively.

On the other hand, given such g_1 and g_2 , we have $f_1 \mid g_2f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. \square

Example 3.7. For $n = 2, d = 2$, and $k = 3$, we have $d^{(k)} = 3$ and $r^{(k)} = 10$. We show that the only types of V_k that occur are $(1_3, 0_7)$ and $(2_1, 1_1, 0_8)$. The first type occurs e.g. for $f_1 = x_0^2, f_2 = x_1^2$, and the second for $f_1 = x_0^2, f_2 = x_0x_1$. Assume that the type $(3_1, 0_9)$ occurs for some $f_1, f_2 \in H^0(\mathbb{P}^2, \mathcal{O}(2))$. By Proposition 3.3 we then have $\dim \langle f_1x_j, f_2x_j \rangle_{j=0}^2 = 4$. Hence, we find $g_1, g_2, g'_1, g'_2 \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ and two linearly independent equations

$$\begin{aligned} g_1f_1 + g_2f_2 &= 0 \\ g'_1f_1 + g'_2f_2 &= 0, \end{aligned}$$

with both sets $(g_1, g_2), (g'_1, g'_2)$ linearly independent. From the first equation it follows that $f_1 = g_2h$ and $f_2 = -g_1h$, for some common linear factor h . Applying this to the second equation, we find $g'_1g_2 = g'_2g_1$, hence $g'_1 = \alpha g_1$ and $g'_2 = \alpha g_2$ for some scalar α , a contradiction.

Proposition 3.8. *Let $k = d + 1$. No types of V_k other than $(1, \dots, 1, 0, \dots, 0)$ and $(2, 1, \dots, 1, 0, \dots, 0)$ occur.*

Proof. The proof follows the lines of Example 3.7. Assume that the type of V_k at some line (f_1, f_2) is other than the two above. Then the type has two more zero entries than the general type, corresponding to two equations of the form

$$\begin{aligned} g_1f_1 + g_2f_2 &= 0 \\ g'_1f_1 + g'_2f_2 &= 0, \end{aligned}$$

with $g_i, g'_i \in H^0(\mathbb{P}^n, \mathcal{O}(1))$. We use the irreducibility of the g_i to produce a contradiction just like in the cited example, the only difference being that the common factor h of f_1 and f_2 need not be linear. \square

4 Specialization

Definition 4.1. Let \mathcal{V} and \mathcal{V}' be vector bundles on a projective k -scheme X . We say that \mathcal{V} *specializes* to \mathcal{V}' if there exists an affine k -scheme Y , spectrum of a discrete valuation ring, with generic point η and closed point η_0 , and a vector bundle \mathcal{W} on $Y \times X$ such that $\mathcal{W}|_{\eta \times X} \simeq \kappa(\eta) \boxtimes \mathcal{V}$ and $\mathcal{W}|_{\eta_0 \times X} \simeq \kappa(\eta_0) \boxtimes \mathcal{V}'$.

Remark 4.2. Specialization is transitive for $X = \mathbb{P}^1$: if \mathcal{V} specializes to \mathcal{V}' and \mathcal{V}' specializes to \mathcal{V}'' , then \mathcal{V} specializes to \mathcal{V}'' , see e.g. [Ram83, Cor. 6.14].

Remark 4.3. This definition reflects specialization of points in the moduli stack Vect_X of vector bundles over X , where e.g. the bundles \mathcal{V} and $\kappa(\eta) \boxtimes \mathcal{V}$ define the same point. The stack is locally Noetherian, hence discrete valuation rings suffice. This notion generalizes the notion of specialization of points on a scheme, see e.g. [ÉGA II, Prop. 7.1.9].

Remark 4.4. Let

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \rightarrow 0$$

be a short exact sequence of coherent sheaves over a k -scheme X and let $\xi \in \text{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \rightarrow 0.$$

Example 4.5. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on \mathbb{P}^1 specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. The elements of $\text{Ext}^1(\mathcal{O}(2), \mathcal{O})$ correspond to extensions of the form

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{O}(2) \rightarrow 0$$

up to equivalence. The zero element corresponds to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have \mathcal{E} locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\text{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Proposition 1.2 and using $\mathbb{V}(\text{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee) \simeq \mathbb{A}^1$, there exists an extension of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O} \rightarrow \mathcal{E}_{\text{univ}} \rightarrow \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}(2) \rightarrow 0$$

on $\mathbb{A}^1 \times \mathbb{P}^1$ such that $\mathcal{E}_{\text{univ}}|_{\xi \times \mathbb{P}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ for generic $\xi \in \mathbb{A}^1$ and $\mathcal{E}_{\text{univ}}|_{0 \times \mathbb{P}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\text{univ}}$ is locally free as the end terms of the sequence are.

Remark 4.6. If \mathcal{V} specializes to \mathcal{V}' and \mathcal{W} specializes to \mathcal{W}' , then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

Remark 4.7. Let b_1, \dots, b_m be non-negative integers, let $a := \sum b_i$, and let s, t denote the homogeneous coordinates on \mathbb{P}^1 . The sequence

$$0 \rightarrow \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \rightarrow 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & & \\ t^{b_2} & s^{b_2} & & & \\ & t^{b_3} & \ddots & & \\ & & \ddots & s^{b_{m-1}} & \\ & & & t^{b_m} & \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \dots & (-1)^m s^{b_1+\dots+b_{m-1}} t^{a-b_1-\dots-b_m} \end{pmatrix}$$

is exact.

Proposition 4.8. Let b_1, \dots, b_m be non-negative integers and π a partition of the set $\{1, \dots, m\}$. For a set of indices $I \in \pi$, let $b'_I := \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on \mathbb{P}^1 specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m-|\pi|}$.

Proof. By Remark 4.6 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 4.7, there exists a representative $\xi \in \text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \rightarrow \mathcal{O}^{m-1} \rightarrow \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_m) \rightarrow \mathcal{O}(a) \rightarrow 0. \quad (4.1)$$

By Remark 4.4, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the form (4.1). Consider the associated closed embedding $\alpha: \mathbb{A}^1 \rightarrow \mathbb{V}(\text{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let \mathcal{E} be the universal extension from Proposition 1.2. Then, the vector bundle $(\text{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. \square

Remark 4.9. By twisting the exact sequence (4.1) in the proof of Proposition 4.8 and using the same argument, we see that for every integer n and with b_i, π , and b_I as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m - |\pi|}$.

Proposition 4.10. *Let $(b_i)_{i=1}^m$ and $(b'_i)_{i=1}^m$ be tuples of integers such that $\sum b_i = \sum b'_i$ and $(b_i) \leq (b'_i)$ in the sense of Definition 2.10. Then $\bigoplus \mathcal{O}(b_i)$ specializes to $\bigoplus \mathcal{O}(b'_i)$.*

Proof. The tuple (b_i) can be transformed into the tuple (b'_i) by a finite sequence of substitutions of the form

$$b_i \leftarrow b_i - 1, \quad b_j \leftarrow b_j + 1$$

for appropriate i and j . By Remark 4.9, $\mathcal{O}(b_i - 1) \oplus \mathcal{O}(b_j + 1)$ specializes to $\mathcal{O}(b_i) \oplus \mathcal{O}(b_j)$. Hence, each step k gives an intermediate tuple $(b_i^{(k)})$ such that $\bigoplus \mathcal{O}(b_i^{(k-1)})$ specializes to $\bigoplus \mathcal{O}(b_i^{(k)})$. This proves the proposition since specialization is transitive. \square

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