Verlinde Bundles

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Contents

1. Universal Families of Extensions

Let X and S be Noetherian schemes over a field k. Let $f\colon X \to S$ be a flat, projective morphism, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, flat over $\mathcal{O}_X$.

Recall that an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences, or *extensions*, of the form

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on $\mathcal{F}$ and $\mathcal{G}$. The set of these equivalence classes can be given the structure of an $H^0(S, \mathcal{O}_S)$-module, see for example [Wei95, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$-module structure.

Explicitely, the sum of two elements of Ext^1 corresponds to the Baer sum of the associated extensions, while the multiplication of an extension as above by a scalar $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

The next proposition shows there exists a k-scheme V that parametrizes the points of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$.

Proposition 1.1. *Let* $V \coloneqq \mathbb{V}(\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee)$. *There exists an extension*

$$\xi_{\mathrm{univ}}\colon \quad 0 \to \mathrm{pr}_1^*\mathcal{G} \to \mathcal{E} \to \mathrm{pr}_1^*\mathcal{F} \to 0$$

over $X \times V$ *such that for all Noetherian affine* k*-schemes* Y*, the map* $\mathrm{Mor}_k(Y,V) \to \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y,\mathcal{G}_Y)$ *defined by* $\alpha \mapsto (\mathrm{id}_X \times \alpha)^*\xi_{\mathrm{univ}}$ *is a bijection, functorial in* Y*. In particular, pulling back* ξ_{univ} *gives a bijection* $\mathrm{Mor}_k(\mathrm{Spec}(k),V) \xrightarrow{\sim} \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$.

Proof. Write $Y = \mathrm{Spec}(A)$. We aim to construct a functorial isomorphim

$$\mathrm{Mor}_k(Y,V) \simeq \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y,\mathcal{G}_Y).$$

Given such an isomorphism for all Y, the required universal extension is the image of $\mathrm{id} \in \mathrm{Mor}_k(V,V)$.

Note that there exist functorial isomorphisms

$$\begin{aligned}\mathrm{Mor}_k(Y,V) &\simeq \mathrm{Hom}_{\text{k-alg}}(\mathrm{Sym}\,\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee, A) \simeq \mathrm{Hom}_{\text{k-mod}}(\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee, A)\\ &\simeq A \otimes_k \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}).\end{aligned}$$

For the final isomorphism $A \otimes_k \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \simeq \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y,\mathcal{G}_Y)$, it suffices to prove that the δ-functors[1] $A \otimes_k \mathrm{Hom}_X(\mathcal{F},-)$ and $\mathrm{Hom}_{X_Y}(\mathcal{F}_Y,-_Y)$ are canonically isomorphic.

In fact, there exists a canonical homomorphism $A \otimes_k \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{X\times Y}(\mathcal{F}_Y,\mathcal{G}_Y)$, functorial in $\mathcal{G}$, that sends an elementary tensor $a \otimes u$ to the homomorphism $a \otimes u$. This is an isomorphism on stalks by [Bou72, Ch. I, §2.10]. There, we need the assumptions that $\mathcal{F}$ is coherent and X Noetherian. □

The scheme V of Proposition 1.1 is a special case of the solution to the more general moduli problem of classifying relative families of sheaves. The rest of this section sketches the more general situation. The material is taken from the article [Lan83].

Definition 1.2. (i) The *i-th relative Ext module* $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ is the image of $\mathcal{G}$ under the right-derived functor $R^i(f_*\mathcal{H}om(\mathcal{F},\mathcal{G}))\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_S}$.

(ii) For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s,\mathcal{F},\mathcal{G}}\colon\ \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s)$$

by restricting extensions of $\mathcal{F}$ by $\mathcal{G}$ to the fiber X_s. This is well-defined, since $\mathcal{F}$ is flat over S.

(iii) A *family of extensions* of $\mathcal{F}$ by $\mathcal{G}$ over S is a family

$$\xi_s \in \mathrm{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s), \quad s \in S$$

such that there exists an open covering $\mathfrak{U}$ of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \mathrm{Ext}^1_{f^{-1}(U)}(\mathcal{F}_U,\mathcal{G}_U)$ with $\Phi_{s,\mathcal{F}_U,\mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

[1] Recall that $A \otimes_k -$ is exact.

Remark 1.3. If S is affine, then we have $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G}) = \operatorname{Ext}^i_X(\mathcal{F},\mathcal{G})^\sim$.

Proposition 1.4. *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. There exists a number $N \geq 0$ depending on $\mathcal{G}$ such that for all quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$, all $i \geq 1$ and $n \geq N$ we have*

$$\mathcal{E}xt^i_{f_Y}(\mathcal{O}_{X_Y}(-n), \mathcal{G} \boxtimes \mathcal{M}) = 0$$

Proposition 1.5. *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism*

$$\tau^i_g\colon g^*\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G}) \to \mathcal{E}xt^i_{f_Y}(g^*_X\mathcal{F}, g^*_X\mathcal{G}).$$

Furthermore, if g is flat, then τ^i_g is an isomorphism for all $i \geq 0$.

Definition 1.6. We say that $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ *commutes with base change* if for all morphisms of Noetherian schemes $g\colon Y \to S$, the base change homomorphism τ^i_g is an isomorphism.

Proposition 1.7. *Let $s \in S$ be a point such that τ^i_s is surjective. Then there exists an open neighborhood U of s such that $\tau^i_{s'}$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ^{i-1}_s is surjective if and only if $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ is locally free on an open neighborhood of s.*

Remark 1.8. (i) If τ^i_s is an isomorphism for all $s \in S$, then $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change.

(ii) From Proposition 1.7 we conclude that if $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ is locally free.

(iii) In case S is reduced, if $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ is locally free then $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 1.9. Let $u\colon Y' \to Y$ be a morphism of Noetherian S-schemes.

(i) We define a functoriality map $H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \to H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1\otimes \mathrm{id}} H^0(Y', u^*\mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau^1_u)} H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(u^*_{X_Y}\mathcal{F}_{Y'}, u^*_{X_Y}\mathcal{G}_{Y'})). \end{aligned}$$

(ii) Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of $\mathcal{F}_Y$ by $\mathcal{G}_Y$ over Y, we set $(u^*\xi)_{y'} := u^*\xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^*\xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y'.

(iii) We define the functors

$$\begin{aligned} &E, E'\colon (\mathrm{NoethSch}/S) \to (\mathrm{Sets}); \\ &E(Y) := H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)), \\ &E'(Y) := \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

Remark 1.10. The spectral sequence $E_2^{p,q} = H^p(S, \mathcal{E}xt_f^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$ gives an exact sequence

$$\begin{aligned} 0 \to & H^1(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\varepsilon} \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})) \\ \xrightarrow{d_2} & H^2(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})). \end{aligned}$$

Proposition 1.11. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S-schemes, the functors E and E' are isomorphic.*

Proposition 1.12. *Suppose that $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the $\mathcal{O}_S$-module $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.13. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S-schemes, the functor E' is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.14. *Suppose that S is affine and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. The functor*

$$(\mathrm{Aff}/S) \to (\mathrm{Sets})\colon Y \mapsto \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$$

is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.

Remark 1.15. As a special case of the above, we recover Proposition 1.1.

Remark 1.16. The article [Lan83] continues on to define a (projectivized) version of the problem, so that over $\mathrm{Spec}(k)$, the scheme $\mathbb{P}(\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of $\mathcal{F}$ by $\mathcal{G}$, modulo the action of $k^\times$. See also [HL10, Example 2.1.12].

2. Verlinde Bundles on Pencils of Quartics

The thesis [Hem15] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in $\mathbb{P}^3$, after summarizing some of its properties. We work over a field k, but omit it in most notation[2], e.g. we write $\mathbb{P}^3$ for $\mathbb{P}_k^3$.

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. The quartics $\mathfrak{X}_t \subseteq \mathbb{P}^3$ parametrized by the $t \in |\mathcal{O}(4)|$ form a universal family $\pi\colon \mathfrak{X} \to |\mathcal{O}(4)|$ with fibers $\mathfrak{X}_t$. The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$. The morphism π is projective and flat.

Throughout, the homogeneous coordinates of $\mathbb{P}^3$ will be denoted by x_i, $i = 0, \ldots, 4$.

[2] Most instances of the letter k will be used to denote a natural number instead.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$. In other words[3], the bundle $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \to \mathbb{P}^3$.

Proposition 2.1. *Let $k \geq 1$. The following statements hold:*

(i) If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this number is independent of the point q.

(ii) The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.

(iii) For all cartesian diagrams of the form

$$\begin{tikzcd} \mathfrak{X}_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\rho"'] & |\mathcal{O}(4)| \end{tikzcd}$$

we have $\rho^* \pi_* \mathcal{L}^{\otimes k} \simeq (\pi_Z)_* \mathcal{L}_Z^{\otimes k}$.

Proof. For the first statement, see the proof of [Hem15, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. □

Let $T \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1_K \to |\mathcal{O}(4)|$, with K an extension field of k. We call T a *pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1_K}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}^1_K}$. The situation is summarized in the picture below:

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{P}^1_K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathbb{P}^1_K \arrow[r] & |\mathcal{O}(4)| \end{tikzcd}$$

For $k \geq 1$, we define the k-th Verlinde bundles $V_k := \pi_* \mathcal{L}^{\otimes k}$ and $V_{k,T} := (\pi_{\mathbb{P}^1})_* \mathcal{L}_{\mathbb{P}^1}^{\otimes k}$. These bundles are related by $V_k|_T = V_{k,T}$ using Proposition 2.1.

Proposition 2.2. *There exists a short exact sequence of coherent $\mathcal{O}_{|\mathcal{O}(4)|}$-modules*

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_k \to 0.$$

Let I_d range over the tuples of the form $(i_0, \ldots, i_3)$ with $\sum i_j = d$. The first map is then given by $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.

Proof. See [Hem15, Proposition 4.2]. □

Remark 2.3. Let T be a pencil of quartics.

[3]For a fiber product $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ and sheaves $\mathcal{F}$ and $\mathcal{G}$ on X resp. Y, write $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$.

(i) The sequence from Proposition 2.2 restricts to a sequence

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_{k,T} \to 0$$

over $\mathbb{P}^1$.

(ii) The vector bundle $V_{k,T}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

Definition 2.4. Let $k \geq 1$.

(i) A *type candidate* for V_k is a non-increasing tuple $(d_1, \ldots, d_{r^{(k)}})$ of non-negative integers with $r^{(k)} = \binom{k+3}{3} - \binom{k-1}{3}$ and $\sum d_i = \binom{k-1}{3}$.

(ii) The *general type candidate* for V_k is the unique type candidate for V_k of the form $(b^{(k)}+1, \ldots, b^{(k)}+1, b^{(k)}, \ldots, b^{(k)})$. The integer $b^{(k)}$ is determined by the equation $\binom{k-1}{3} = b^{(k)} r^{(k)} + a$, with $a < r^{(k)}$ becoming the number of occurences of $b^{(k)}+1$.

(iii) Let T be a pencil of quartics. The *type* of $V_{k,T}$ is the unique type candidate (d_i) such that $V_{k,T} \simeq \bigoplus_i \mathcal{O}(d_i)$.

(iv) We say that $V_{k,T}$ has *general type* if its type is a general type candidate.

The points of $\mathrm{Gr}(2, 35)$ correspond to the pencils of quartics $T \subseteq |\mathcal{O}(4)|$ in the following way. Let P the universal $\mathbb{P}^1$-bundle over $\mathrm{Gr}(2, 35)$. It comes equipped with a projection map $P \to |\mathcal{O}(4)|$ such that for all pencils of quartics T there exists a unique point $t \in \mathrm{Gr}(2, 35)$ such that the image of the fiber P_t in $|\mathcal{O}(4)|$ is T.

$$\begin{tikzcd}
P_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & P \arrow[r, "p"] \arrow[d, "\varphi"] & {|\mathcal{O}(4)|} \\
\mathrm{Spec}(\kappa(t)) \arrow[r] & \mathrm{Gr}(2,35) &
\end{tikzcd}$$

For $t \in \mathrm{Gr}(2, 35)$ corresponding to the pencil T, we write $V_{k,t} := V_{k,T}$.

Definition 2.5. Let $k \geq 1$ and (d_i) be a type candidate for V_k. We define the set $Z_{(d_i)}$ of all points $t \in \mathrm{Gr}(2, 35)$ such that $V_{k,t}$ has type (d_i). For the set of points t where $V_{k,t}$ has general type, we also write Z_{gen}.

Proposition 2.6. *The set Z_{gen} is Zariski open. Its complement is the union*

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-b^{(k)} - 1)) \cup \mathrm{Supp}(R^1\varphi_*(p^* V_k(-b^{(k)})^\vee)).$$

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \mathrm{Gr}(2, 35)$, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and $b := b^{(k)}$. We have $t \in Z_{\mathrm{gen}}$ if and only if $b \leq d_i \leq b+1$ for all i, which holds if and only if $H^1(P_t, V_{k,t}(-b-1)) = H^1(P_t, V_{k,t}(-b)^\vee) = 0$.

Next, we want to apply the Cohomology and Base Change Theorem [Vak17, 28.1.6] to the map $\varphi\colon P \to \mathrm{Gr}(2, 25)$, which is a $\mathbb{P}^1$-bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\mathrm{Gr}(2, 35)$.

For all $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, p^*V_{k,t}(-b-1)) = 0$ and $h^2(P_t, p^*V_{k,t}(-b)^\vee) = 0$. Since the sheaves $p^*V_{k,t}(-b-1)$ and $p^*V_{k,t}(-b)^\vee$ are locally free and coherent, we have

$$(R^1\varphi_* p^* V_k(-b-1))_t = H^1(P_t, V_{k,t}(-b-1))$$

and

$$(R^1\varphi_*(p^*V_k(-b)^\vee))_t = H^1(P_t, V_{k,t}(-b)^\vee).$$

By the previous characterization, we have

$$\mathrm{Gr}(2,35) \setminus Z_{\mathrm{gen}} = \mathrm{Supp}(R^1\varphi_* p^* V_k(-b-1)) \cup \mathrm{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)),$$

which is a Zariski closed set. □

Proposition 2.7. *The sets* $\mathrm{Supp}(R^1\varphi_* p^* V_k(-b^{(k)}-1))$ *and* $\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee))$ *are determinantal varieties.*

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k)), r_2 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4))$ and $b := b^{(k)}$, and rewrite the exact sequence from Proposition 2.2 as

$$0 \to \mathcal{O}(-1)^{r_2} \to \mathcal{O}^{r_1} \to V_k \to 0. \tag{$\star$}$$

Twisting the sequence ($\star$) with $\mathcal{O}(-b-1)$ and pulling back to P gives an exact sequence

$$0 \to p^*\mathcal{O}(-b-2)^{r_2} \to p^*\mathcal{O}(-b-1)^{r_1} \to p^*V_k(-b-1) \to 0.$$

For all $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, \mathcal{O}(-b-2)^{r_2}) = 0$, hence $R^2\varphi_* p^*\mathcal{O}(-b-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_* p^*\mathcal{O}(-b-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_* p^*\mathcal{O}(-b-1)^{r_1} \to R^1\varphi_* p^* V_k(-b-1) \to 0.$$

Note that since the numbers $h_2^1 := h^1(P_t, \mathcal{O}(-b-2)^{r_2})$ and $h_1^1 := h^1(P_t, \mathcal{O}(-b-1)^{r_1})$ do not depend on the point t, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h_1^2 and h_1^1, respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_* p^* V_k(-b-1))_t \neq 0$ if and only if $\mathrm{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b-1))) = \{t : \mathrm{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = br_1 = b\binom{k+3}{3}$.

The proof for the second assertion is similar. We start with the sequence ($\star$), twist with $\mathcal{O}(-b)$, take duals, pull back to P, and apply φ_*. Since for all $t \in \mathrm{Gr}(2,35)$ we have $h^1(P_t, \mathcal{O}(b)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_* p^*\mathcal{O}(b)^{r_1} \xrightarrow{\beta} \varphi_* p^*\mathcal{O}(b+1)^{r_2} \to R^1\varphi_*(p^*V_k(-b)^\vee) \to 0.$$

Since the numbers $h_1^0 \coloneqq h^0(P_t, \mathcal{O}(b)^{r_1})$ and $h_2^0 \coloneqq h^0(P_t, \mathcal{O}(b+1)^{r_2})$ do not depend on the point t, again by Grauert's Theorem the first two terms of the sequence are locally free of rank h_1^0 and h_2^0, respectively. As before, we obtain the characterization

$$\operatorname{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)) = \{t : \operatorname{rk}(\beta_t) \le h_2^0 - 1\}.$$

Here, we have $h_2^0 = (b+2)r_2 = (b+2)\binom{k-1}{3}$. $\square$

Definition 2.8. For type candidates (d_i) and (d_i') we define the expression $(d_i') \ge (d_i)$ to mean

$$\sum_{i=1}^{s} d_i' \ge \sum_{i=1}^{s} d_i \text{ for all } s = 1, \dots, r^{(k)}.$$

Proposition 2.9. *Let (d_i) be a type candidate for V_k. The set $\widehat{Z}_{(d_i)} \coloneqq \bigcup_{(d_i')\ge(d_i)} Z_{(d_i')}$ is the intersection of (at most) $r^{(k)}$ determinantal varieties.*

Proof. For every type candidate (d_i'), the sum $\sum_{i=1}^{s} d_i'$ is the largest sum of s entries of (d_i'). Hence we have

$$\widehat{Z}_{(d_i)} = \bigcap_{s=1}^{r^{(k)}} \{t : h^0((\textstyle\bigwedge^s V_{t,k})(-\sum^s d_i)) > 0\}.$$

With Serre duality and the Cohomology and Base Change theorem we write the sets in the intersection as

$$\operatorname{Supp}(R^1\varphi_*(p^*(\textstyle\bigwedge^s V_k^\vee)(\sum^s d_i - 2))),$$

which is a determinantal variety by an argument similar to the second part of the proof of Proposition 2.7. One just has to note that $h^1(\mathbb{P}^1, \mathcal{O}(\sum^s d_i - 2)) = 0$ for all s. $\square$

Proposition 2.10. *Of the five type candidates*

$$(1,1,1,1,0,\dots,0),\ (2,1,1,0,\dots,0),\ (2,2,0,\dots,0),\ (3,1,0,\dots,0),\ (4,0,\dots,0)$$

for V_5, only the first two occur as types of some $V_{5,t}$.

Proof. This is a special case of Proposition 3.6. $\square$

3. Verlinde Bundles on Pencils of Hypersurfaces

Let V_k be the k-th Verlinde bundle of the universal family of hypersurfaces of degree d in $\mathbb{P}^n$. The vector bundle V_k is the cokernel of the map

$$M \colon \mathcal{O}_{|\mathcal{O}(d)|}(-1) \otimes H^0(\mathcal{O}(k-d), \mathbb{P}^n) \to \mathcal{O}_{|\mathcal{O}(d)|} \otimes H^0(\mathcal{O}(k), \mathbb{P}^n)$$

given by multiplication by $\sum_I \alpha_I \otimes x^I \in H^0(\mathcal{O}(1), |\mathcal{O}(d)|) \otimes H^0(\mathcal{O}(d), \mathbb{P}^n)$, where the α_I are the homogeneous coordinates on $|\mathcal{O}(d)|$. We study the restriction of V_k to lines $T \subseteq |\mathcal{O}(d)|$.

Proposition 3.1. *Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$, let* $\operatorname{coker}(M|_T) \simeq \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^{s} \mathcal{O}(\lambda_i)$. *Define $U := H^0(\mathcal{O}(k-d), \mathbb{P}^n)$. We have*

$$\lambda_0 = \dim H^0(\mathcal{O}(k), \mathbb{P}^n) - \dim(f_1 U + f_2 U),$$

or, equivalently,

$$s = \dim(f_1 U + f_2 U) - d^{(k)}.$$

Proof. Note that the map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$. In particular, the image of $\mathcal{O}(-1) \otimes U$ is contained in $\mathcal{O} \otimes (f_1 U + f_2 U)$. It follows that $\lambda_0 \geq \dim(f_1 U + f_2 U)$.

On the other hand, there are no direct summands $\mathcal{G} = \mathcal{O} \otimes \langle \theta_1 f_1 + \theta_2 f_2 \rangle$ of $\mathcal{O} \otimes (f_1 U + f_2 U)$ such that the image of M is contained in $\mathcal{G}^{\perp}$. This shows that $\lambda_0 = \dim H^0(\mathcal{O}(k), \mathbb{P}^n) - \dim(f_1 U + f_2 U)$. □

Remark 3.2. The general type candidate for V_k takes the form $(b^{(k)} + 1, \ldots, b^{(k)} + 1, b^{(k)}, \ldots, b^{(k)})$, where the number of entries is $r^{(k)} := \binom{n+k}{n} - \binom{n+k-d}{n}$ and their sum $d^{(k)} := \binom{n+k-d}{n}$, while $b^{(k)} = \lfloor d^{(k)} / r^{(k)} \rfloor$. Note that the degrees of $d^{(k)}$ and $r^{(k)}$ as polynomials in k are n and $n-1$, respectively. Hence, $b^{(k)} \to \infty$ for $k \to \infty$.

Corollary 3.3. *Let $t \in \mathrm{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2. Let (θ_j) be a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. Let k be such that $b^{(k)} = 0$, i. e. such that in the general type, only ones and zeroes appear. The bundle $V_{k,t}$ has general type if and only if $(f_1\theta_j, f_2\theta_j)_j$ is a linearly independent set in $H^0(\mathcal{O}(k), \mathbb{P}^n)$.*

Proof. Since $b^{(k)} = 0$, the type of $V_{k,t}$ is the general type if and only if it has $d^{(k)}$ many nonzero entries. By Proposition 3.1, this is the case if and only if $\dim \langle f_1\theta_j, f_2\theta_j \rangle_j = 2d^{(k)}$. □

Proposition 3.4. *Let $t \in \mathrm{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2, and let k be such that $b^{(k)} = 0$. The bundle $V_{k,t}$ has nongeneral type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $b^{(k)} = 0$ but $k > 2d$ then the general type never occurs.*

Proof. By Corollary 3.3, the bundle $V_{k,t}$ has non-general type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathcal{O}(k-d), \mathbb{P}^n)$ such that $g_1 f_1 + g_2 f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h, respectively.

On the other hand, given such g_1 and g_2, we have $f_1 \mid g_2 f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. □

Example 3.5. For $n = 2, d = 2$, and $k = 3$, we have $d^{(k)} = 3$ and $r^{(k)} = 10$. We show that the only types of V_k that occur are $(1_3, 0_7)$ and $(2_1, 1_1, 0_8)$. The first type occurs

e. g. for $f_1 = x_0^2, f_2 = x_1^2$, and the second for $f_1 = x_0^2, f_2 = x_0 x_1$. Assume that the type $(3_1, 0_9)$ occurs for some $f_1, f_2 \in H^0(\mathcal{O}(2), \mathbb{P}^2)$. By Proposition 3.1 we then have $\dim \langle f_1 x_j, f_2 x_j \rangle_{j=0}^2 = 4$. Hence, we find $g_1, g_2, g_1', g_2' \in H^0(\mathcal{O}(1), \mathbb{P}^2)$ and two linearly independent equations

$$g_1 f_1 + g_2 f_2 = 0$$
$$g_1' f_1 + g_2' f_2 = 0,$$

with both sets $(g_1, g_2), (g_1', g_2')$ linearly independent. From the first equation it follows that $f_1 = g_2 h$ and $f_2 = -g_1 h$, for some common linear factor h. Applying this to the second equation, we find $g_1' g_2 = g_2' g_1$, hence $g_1' = \alpha g_1$ and $g_2' = \alpha g_2$ for some scalar α, a contradiction.

Proposition 3.6. *Let $k = d + 1$. No types of V_k other than $(1, \ldots, 1, 0, \ldots, 0)$ and $(2, 1, \ldots, 1, 0, \ldots, 0)$ occur.*

Proof. The proof follows the lines of Example 3.5. Assume that the type of V_k at some line (f_1, f_2) is other than the two above. Then the type has two more zero entries than the general type, corresponding to two equations of the form

$$g_1 f_1 + g_2 f_2 = 0$$
$$g_1' f_1 + g_2' f_2 = 0,$$

with $g_i, g_i' \in H^0(\mathcal{O}(1), \mathbb{P}^n)$. We use the irreducibility of the g_i to produce a contradiction just like in the cited example, the only difference being that the common factor h of f_1 and f_2 need not be linear. $\square$

4. Specialization

Definition 4.1. Let $\mathcal{V}$ and $\mathcal{V}'$ be vector bundles on a projective k-scheme X. We say that $\mathcal{V}$ *specializes* to $\mathcal{V}'$ if there exists an affine k-scheme Y, spectrum of a discrete valuation ring, with generic point η and closed point η_0, and a vector bundle $\mathcal{W}$ on $Y \times X$ such that $\mathcal{W}|_{\eta \times X} \simeq \kappa(\eta) \boxtimes \mathcal{V}$ and $\mathcal{W}|_{\eta_0 \times X} \simeq \kappa(\eta_0) \boxtimes \mathcal{V}'$.

Remark 4.2. This definition reflects specialization of points in the moduli stack Vect_X of vector bundles over X, where e.g. the bundles $\mathcal{V}$ and $\kappa(\eta) \boxtimes \mathcal{V}$ define the same point. The stack is locally Noetherian, hence discrete valuation rings suffice. This notion generalizes the notion of specialization of points on a scheme, see e.g. [ÉGA II, Prop. 7.1.9].

Remark 4.3. Let

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \to 0$$

be a short exact sequence of coherent sheaves over a k-scheme X and let $\xi \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \to 0.$$

Example 4.4. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$ specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. The elements of $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})$ correspond to extensions of the form

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(2) \to 0$$

up to equivalence. The zero element corresponds to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have $\mathcal{E}$ locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Proposition 1.1, there exists an extension of the form

$$0 \to \mathcal{O} \boxtimes \mathcal{O}_{\mathbb{A}^1} \to \mathcal{E}_{\mathrm{univ}} \to \mathcal{O}(2) \boxtimes \mathcal{O}_{\mathbb{A}^1} \to 0$$

such that for all nonzero rational points $\xi \in \mathbb{A}^1 = \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee)$ we have the isomorphisms $\mathcal{E}_{\mathrm{univ}}|_{\xi \times \mathbb{A}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and $\mathcal{E}_{\mathrm{univ}}|_{0 \times \mathbb{A}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\mathrm{univ}}$ is locally free as the end terms of the sequence are.

Remark 4.5. If $\mathcal{V}$ specializes to $\mathcal{V}'$ and $\mathcal{W}$ specializes to $\mathcal{W}'$, then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

Remark 4.6. Let $b_1, \dots, b_m$ be non-negative integers, let $a \coloneqq \sum b_i$, and let s, t denote the homogeneous coordinates on $\mathbb{P}^1$. The sequence

$$0 \to \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \to 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & \\ t^{b_2} & s^{b_2} & & \\ & t^{b_3} & \ddots & \\ & & \ddots & s^{b_{m-1}} \\ & & & t^{b_m} \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \cdots & (-1)^m s^{b_1+\cdots+b_{m-1}} t^{a-b_1-\cdots-b_m} \end{pmatrix}$$

is exact.

Proposition 4.7. *Let $b_1, \dots, b_m$ be non-negative integers and π a partition of the set $\{1, \dots, m\}$. For a set of indices $I \in \pi$, let $b'_I \coloneqq \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m - |\pi|}$.*

Proof. By Remark 4.5 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 4.6, there exists a representative $\xi \in \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \to \mathcal{O}^{m-1} \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \to \mathcal{O}(a) \to 0. \tag{$\star$}$$

By Remark 4.3, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the form $\star$. Consider the associated closed embedding $\alpha\colon \mathbb{A}^1 \to \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let $\mathcal{E}$ be the universal extension from Proposition 1.1. Then, the vector bundle $(\mathrm{id}_{\mathbb{P}^1} \times \alpha)^*\mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. □

Remark 4.8. By twisting the exact sequence ($\star$) in the proof of Proposition 4.7 and using the same argument, we see that for every integer n and with b_i, π, and b_I as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m - |\pi|}$.

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