Verlinde Bundles

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Contents

1. Universal Families of Extensions

The material of this section is taken from the article [Lan83].

Let X and S be Noetherian schemes over a field k. Let $f\colon X \longrightarrow S$ be a flat, projective morphism, let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, flat over $\mathcal{O}_X$.

Recall that an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences of the form

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on $\mathcal{F}$ and $\mathcal{G}$. The set of these equivalence classes can be given the structure of a $\Gamma(S, \mathcal{O}_S)$-module, see for example [Wei95, sec. 3.4]. This correspondence is functorial in both arguments, and preserves $\Gamma(S, \mathcal{O}_S)$-module structure.

Definition 1.1. 1. The *i-th relative Ext module* $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ is the image of $\mathcal{G}$ under the right-derived functor $R^i(f_*\mathcal{H}om(\mathcal{F}, \mathcal{G}))\colon \operatorname{Mod}_{\mathcal{O}_X} \longrightarrow \operatorname{Mod}_{\mathcal{O}_S}$.

2. For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s,\mathcal{F},\mathcal{G}}\colon\ \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G}) \longrightarrow \operatorname{Ext}^1_{X_s}(\mathcal{F}_s, \mathcal{G}_s)$$

by restricting extensions of $\mathcal{F}$ by $\mathcal{G}$ to the fiber X_s. This is possible since $\mathcal{F}$ is flat over S.

3. A *family of extensions* of $\mathcal{F}$ by $\mathcal{G}$ over S is a family

$$\xi_s \in \operatorname{Ext}^1_{X_s}(\mathcal{F}_s, \mathcal{G}_s) \quad (s \in S)$$

such that there exists an open covering $\mathfrak{U}$ of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \operatorname{Ext}^1_{f^{-1}(U)}(\mathcal{F}_U, \mathcal{G}_U)$ with $\Phi_{s,\mathcal{F}_U,\mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

Remark 1.2. If S is affine, then we have $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) = \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G})^\sim$.

Proposition 1.3. *Let $g \colon Y \longrightarrow S$ be a morphism of Noetherian schemes. There exists a number $N \geq 0$ dependent on $\mathcal{G}$ such that for all quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$, all $i \geq 1$ and $n \geq N$ we have*

$$\mathcal{E}xt^i_{f_Y}(\mathcal{O}_{X_Y}(-n), \mathcal{G} \boxtimes \mathcal{M}) = 0$$

Proposition 1.4. *Let $g \colon Y \longrightarrow S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism*

$$\tau^i_g \colon g^* \mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{E}xt^i_{f_Y}(g^*_X \mathcal{F}, g^*_X \mathcal{G}).$$

Furthermore, if g is flat, then τ^i_g is an isomorphism for all $i \geq 0$.

Definition 1.5. We say that $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ *commutes with base change* if for all morphisms of Noetherian schemes $g \colon Y \longrightarrow S$, the base change homomorphism τ^i_g is an isomorphism.

Proposition 1.6. *Let $s \in S$ be a point such that τ^i_s is surjective. Then there exists an open neighborhood U of s such that $\tau^i_{s'}$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ^{i-1}_s is surjective if and only if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ is locally free on an open neighborhood of s.*

Remark 1.7. 1. If τ^i_s is an isomorphism for all $s \in S$, then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change.

2. We have directly from Proposition 1.6 that if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free.

3. In case S is reduced, if $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 1.8. Let $u \colon Y' \longrightarrow Y$ be a morphism of Noetherian S-schemes.

1. We define a functoriality map $H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \longrightarrow H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1 \otimes \mathrm{id}} H^0(Y', u^* \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau^1_u)} H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(u^*_{X_Y} \mathcal{F}_{Y'}, u^*_{X_Y} \mathcal{G}_{Y'})). \end{aligned}$$

2. Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of $\mathcal{F}_Y$ by $\mathcal{G}_Y$ over Y, we set $(u^* \xi)_{y'} \coloneqq u^* \xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^* \xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y'. Moreover, if the family ξ is globally defined, then so is its pullback $u^* \xi$.

3. We define thus functors

$$\begin{aligned} &E, E' \colon \{\text{Noeth. schemes over } S\} \longrightarrow \{\text{Sets}\}; \\ &E(Y) \coloneqq H^0(Y, \operatorname{Ext}^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)), \\ &E'(Y) \coloneqq \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

Remark 1.9. The spectral sequence $H^p(S, \mathcal{E}xt_f^q(\mathcal{F}, \mathcal{G})) \Rightarrow \operatorname{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G})$ gives an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\varepsilon} \operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})) \\ \xrightarrow{d_2} H^2(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})). \end{aligned}$$

Proposition 1.10. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S-schemes, the functors E and E' are isomorphic.*

Proposition 1.11. *Suppose that $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the $\mathcal{O}_S$-module $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.12. *Suppose that S is reduced and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S-schemes, the functor E' is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.13. *Suppose that S is affine and $\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. The functor*

$$Y \longmapsto \operatorname{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y) \colon \{\textit{Affine } S\textit{-schemes}\} \longrightarrow \{\textit{Sets}\}$$

is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt_f^1(\mathcal{F}, \mathcal{G})^\vee)$.

Corollary 1.14. *Finally, let $S = \operatorname{Spec}(k)$, let $V := \mathbb{V}(\operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$. On the scheme $X \times V$ there exists an extension*

$$\xi_{\mathrm{univ}}\colon \quad 0 \longrightarrow \mathrm{pr}_1^*\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathrm{pr}_1^*\mathcal{G} \longrightarrow 0$$

which is universal on the category of affine k-schemes. In particular, pulling back ξ_{univ} defines an isomorphism $\operatorname{Hom}(\operatorname{Spec}(k), V) \xrightarrow{\sim} \operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G})$.

Remark 1.15. The article [Lan83] continues on to define a "projectivized" version of the problem, so that over $\operatorname{Spec}(k)$, the scheme $\mathbb{P}(\operatorname{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of $\mathcal{F}$ by $\mathcal{G}$, modulo the action of $k^\times$. See also [HL10, Example 2.1.12].

2. Verlinde bundles on Lefschetz pencils

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. Consider the universal family $\mathfrak{X} \longrightarrow |\mathcal{O}(4)|$, given by

$$\mathfrak{X} = \{(x, q) \in \mathbb{P}^3 \times |\mathcal{O}(4)| : x \in q\}.$$

The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$, which can be seen as follows. Let the index I range over the tuples of the form (i_0, i_1, i_2, i_3) with $\sum i_j = 4$, and let Q_I denote the I-th projective coordinate of $|\mathcal{O}(4)|$. For $j = 0, \ldots, 3$, let X_j denote the j-th coordinate of $\mathbb{P}^3$. Then the family $\mathfrak{X}$ is cut out by the section $\sum_I Q_I X^I$ of the line bundle $\mathcal{O}(4) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^3 \times |\mathcal{O}(4)|$.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$, in other words as the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \longrightarrow \mathbb{P}^3$.

Let $l \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1 \longrightarrow |\mathcal{O}(4)|$. We call l a *Lefschetz pencil* of quartics. Its universal family is the scheme $l \times_{|\mathcal{O}(4)|} \mathfrak{X}$, which comes equipped with the pullback line bundle $\mathcal{L}_l$.

3. Specialization

In this section we collect some facts about specialization phenomena. We say that a vector bundle $\mathcal{V}$ on a k-scheme X *specializes* to another vector bundle $\mathcal{V}'$ over the same scheme if there exists a vector bundle $\mathcal{W}$ on $\mathbb{A}^1 \times X$ such that $\mathcal{W}|_{0\times X} \simeq \mathcal{V}'$ and $\mathcal{W}|_{t\times X} \simeq \mathcal{V}$ for all rational $t \in \mathbb{A}^1$.

Remark 3.1. Let

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \longrightarrow 0$$

be a short exact sequence of coherent sheaves over a k-scheme X and let $\xi \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{ag} \mathcal{G} \longrightarrow 0.$$

To see this, consider the two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "f"] \arrow[d, "a"'] & \mathcal{E} \arrow[r, "g"] \arrow[d, equal] & \mathcal{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r, "a^{-1}f"'] & \mathcal{E} \arrow[r, "g"] & \mathcal{G} \arrow[r] & 0.
\end{tikzcd}$$

Let $\delta, \delta' \colon \operatorname{Hom}(\mathcal{G}, \mathcal{G}) \longrightarrow \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})$ be the boundary homomorphisms associated to the upper and lower sequence respectively. We have on the one hand $\delta'(\mathrm{id}) = a\delta(\mathrm{id})$ by naturality, and on the other hand $\delta(\mathrm{id}) = \xi$ since this is the way extensions are identified with elements of Ext^1. But the lower sequence is equivalent to the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{ag} \mathcal{G} \longrightarrow 0,$$

hence our claim holds.

Example 3.2. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$ specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. Consider $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})$, whose elements correspond to extensions of the form

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(2) \longrightarrow 0$$

up to equivalence, the zero element corresponding to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have $\mathcal{E}$ locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the non-split extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Corollary 1.14, there exists an extension of the form

$$0 \longrightarrow \mathcal{O} \boxtimes \mathcal{O}_{\mathbb{A}^1} \longrightarrow \mathcal{E}_{\text{univ}} \longrightarrow \mathcal{O}(2) \boxtimes \mathcal{O}_{\mathbb{A}^1} \longrightarrow 0$$

such that for all nonzero rational points $\xi \in \mathbb{A}^1 = \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee)$ we have $\mathcal{E}_{\mathrm{univ}}|_{\xi \times \mathbb{A}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and $\mathcal{E}_{\mathrm{univ}}|_{0 \times \mathbb{A}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\mathrm{univ}}$ has to be locally free as the end terms of the sequence are.

Remark 3.3. If $\mathcal{V}$ specializes to $\mathcal{V}'$ and $\mathcal{W}$ specializes to $\mathcal{W}'$, then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

Remark 3.4. Let $b_1, \dots, b_m$ be non-negative numbers, let $a \coloneqq \sum b_i$. The sequence

$$0 \longrightarrow \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \longrightarrow 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & \\ t^{b_2} & s^{b_2} & & \\ & t^{b_3} & \ddots & \\ & & \ddots & \\ & & & s^{b_{m-1}} \\ & & & t^{b_m} \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \cdots & (-1)^m s^{b_1+\cdots+b_{m-1}} t^{a-b_1-\cdots-b_m} \end{pmatrix}$$

is exact.

Proposition 3.5. *Let $b_1, \dots, b_m$ be non-negative numbers and π a partition of the set $\{1, \dots, m\}$. For a set of indices $I \in \pi$, let $b'_I \coloneqq \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m - |\pi|}$.*

Proof. By Remark 3.3 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 3.4, there exists a representative $\xi \in \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1}$ of an exact sequence of the form

$$0 \longrightarrow \mathcal{O}^{m-1} \longrightarrow \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \longrightarrow \mathcal{O}(a) \longrightarrow 0.$$

By Remark 3.1, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the same form. Consider the associated closed embedding $\alpha \colon \mathbb{A}^1 \longrightarrow \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let $\mathcal{E}$ be the universal extension from Corollary 1.14. Then, the vector bundle $(\mathrm{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. □

References

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