# Verlinde Bundles

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## Contents



## 1. Universal Families of Extensions

Let $X$ and $S$ be Noetherian schemes over a field $k$. Let $f\colon X \to S$ be a flat, projective morphism, let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, flat over $\mathcal{O}_X$.

Recall that an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences of the form

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on $\mathcal{F}$ and $\mathcal{G}$. The set of these equivalence classes can be given the structure of a $H^0(S, \mathcal{O}_S)$-module, see for example [Wei95, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$-module structure.

Explicitely, the sum of two elements of $\operatorname{Ext}^1$ corresponds to the Baer sum of the associated extensions, while the scalar multiplication of an extension as above by $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

The next proposition shows there exists a $k$-scheme $V$ that parametrizes the points of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$.

**Proposition 1.1.** *Let $V \coloneqq \mathbb{V}(\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})^\vee)$. On the scheme $X \times V$ there exists an extension*

$$\xi_{\text{univ}}\colon \quad 0 \to \operatorname{pr}_1^*\mathcal{F} \to \mathcal{E} \to \operatorname{pr}_1^*\mathcal{G} \to 0$$

*Such that for all affine $k$-schemes $Y$, the map* $\mathrm{Hom}(Y,V) \to \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y,\mathcal{G}_Y)$ *defined by* $\alpha \mapsto (\mathrm{id}_X \times \alpha)^*\xi_{\mathrm{univ}}$ *is an isomorphism, functorial in $Y$. In particular, we have an isomorphism* $\mathrm{Hom}(\mathrm{Spec}(k),V) \xrightarrow{\sim} \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})$.

*Proof.* Write $Y = \mathrm{Spec}(A)$. We aim to construct a functorial isomorphim

$$\mathrm{Hom}(Y,V) \simeq \mathrm{Ext}^1_{X\times Y}(\mathcal{F}_Y,\mathcal{G}_Y).$$

Given such an isomorphism for all $Y$, the required universal extension is the image of $\mathrm{id} \in \mathrm{Hom}(V,V)$.

Note that there exist functorial isomorphisms

$$\begin{aligned}\mathrm{Hom}(Y,V) &\simeq \mathrm{Hom}_{\text{k-alg}}(\mathrm{Sym}\,\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee, A) \simeq \mathrm{Hom}_{\text{k-mod}}(\mathrm{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee, A)\\ &\simeq A \otimes_k \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}).\end{aligned}$$

For the final isomorphism $A \otimes_k \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \simeq \mathrm{Ext}^1_{X\times Y}(\mathcal{F}_Y,\mathcal{G}_Y)$, it suffices to prove that the $\delta$-functors[1] $A \otimes_k \mathrm{Hom}_X(\mathcal{F},-)$ and $\mathrm{Hom}_{X\times Y}(\mathcal{F}_Y,-_Y)$ are canonically isomorphic.

In fact, there exists a canonical homomorphism $A \otimes_k \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{X\times Y}(\mathcal{F}_Y,\mathcal{G}_Y)$, functorial in $\mathcal{G}$, that sends an elementar tensor $a \otimes u$ to the homomorphism $a \otimes u$. This is an isomorphism on stalks by [Bou72, Ch. I, §2.10]. There, we need the assumptions that $\mathcal{F}$ is coherent and $X$ Noetherian. $\square$

The scheme $V$ of Proposition 1.1 is a special case of the solution to a more general moduli problem of classifying relative extensions of sheaves. The rest of this section sketches the more general situation. The material is taken from the article [Lan83].

**Definition 1.2. 1.** The *$i$-th relative Ext module* $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ is the image of $\mathcal{G}$ under the right-derived functor $R^i(f_*\mathcal{H}om(\mathcal{F},\mathcal{G}))\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_S}$.

**2.** For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s,\mathcal{F},\mathcal{G}}\colon\ \mathrm{Ext}^1_X(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s)$$

by restricting extensions of $\mathcal{F}$ by $\mathcal{G}$ to the fiber $X_s$. This is possible since $\mathcal{F}$ is flat over $S$.

**3.** A *family of extensions* of $\mathcal{F}$ by $\mathcal{G}$ over $S$ is a family

$$\xi_s \in \mathrm{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s) \quad (s \in S)$$

such that there exists an open covering $\mathfrak{U}$ of $S$ and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \mathrm{Ext}^1_{f^{-1}(U)}(\mathcal{F}_U,\mathcal{G}_U)$ with $\Phi_{s,\mathcal{F}_U,\mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

**Remark 1.3.** If $S$ is affine, then we have $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G}) = \mathrm{Ext}^i_X(\mathcal{F},\mathcal{G})^\sim$.

[1] Recall that $A \otimes_k -$ is exact.

**Proposition 1.4.** *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. There exists a number $N \geq 0$ dependent on $\mathcal{G}$ such that for all quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$, all $i \geq 1$ and $n \geq N$ we have*

$$\mathcal{E}xt^i_{f_Y}(\mathcal{O}_{X_Y}(-n), \mathcal{G} \boxtimes \mathcal{M}) = 0$$

**Proposition 1.5.** *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism*

$$\tau^i_g\colon g^*\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^i_{f_Y}(g^*_X\mathcal{F}, g^*_X\mathcal{G}).$$

*Furthermore, if $g$ is flat, then $\tau^i_g$ is an isomorphism for all $i \geq 0$.*

**Definition 1.6.** We say that $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ *commutes with base change* if for all morphisms of Noetherian schemes $g\colon Y \to S$, the base change homomorphism $\tau^i_g$ is an isomorphism.

**Proposition 1.7.** *Let $s \in S$ be a point such that $\tau^i_s$ is surjective. Then there exists an open neighborhood $U$ of $s$ such that $\tau^i_{s'}$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism $\tau^{i-1}_s$ is surjective if and only if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ is locally free on an open neighborhood of $s$.*

**Remark 1.8.** **1.** If $\tau^i_s$ is an isomorphism for all $s \in S$, then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change.

**2.** We have directly from Proposition 1.7 that if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free.

**3.** In case $S$ is reduced, if $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$.

**Definition 1.9.** Let $u\colon Y' \to Y$ be a morphism of Noetherian $S$-schemes.

**1.** We define a functoriality map $H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \to H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1 \otimes \mathrm{id}} H^0(Y', u^*\mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau^1_u)} H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(u^*_{X_Y}\mathcal{F}_{Y'}, u^*_{X_Y}\mathcal{G}_{Y'})). \end{aligned}$$

**2.** Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of $\mathcal{F}_Y$ by $\mathcal{G}_Y$ over $Y$, we set $(u^*\xi)_{y'} := u^*\xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^*\xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over $Y'$. Moreover, if the family $\xi$ is globally defined, then so is its pullback $u^*\xi$.

**3.** We define thus functors

$$\begin{aligned} &E, E'\colon \{\text{Noeth. schemes over } S\} \to \{\text{Sets}\}; \\ &E(Y) := H^0(Y, \mathrm{Ext}^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)), \\ &E'(Y) := \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

**Remark 1.10.** The spectral sequence $H^p(S, \mathcal{E}xt^q_f(\mathcal{F},\mathcal{G})) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{F},\mathcal{G})$ gives an exact sequence

$$\begin{aligned} 0 \to & H^1(S, f_*\mathcal{H}om(\mathcal{F},\mathcal{G})) \xrightarrow{\varepsilon} \operatorname{Ext}^1_X(\mathcal{F},\mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})) \\ \xrightarrow{d_2} & H^2(S, f_*\mathcal{H}om(\mathcal{F},\mathcal{G})). \end{aligned}$$

**Proposition 1.11.** *Suppose that $S$ is reduced and $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian $S$-schemes, the functors $E$ and $E'$ are isomorphic.*

**Proposition 1.12.** *Suppose that $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$. Then the $\mathcal{O}_S$-module $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})^\vee$ is locally free and the functor $E$ is representable by the $S$-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})^\vee)$.*

**Corollary 1.13.** *Suppose that $S$ is reduced and $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian $S$-schemes, the functor $E'$ is representable by the $S$-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})^\vee)$.*

**Corollary 1.14.** *Suppose that $S$ is affine and $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$. The functor*

$$\{\textit{Affine } S\textit{-schemes}\} \to \{\textit{Sets}\} \colon Y \mapsto \operatorname{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)$$

*is representable by the $S$-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})^\vee)$.*

**Remark 1.15.** As a special case of the above, we recover Proposition 1.1.

**Remark 1.16.** The article [Lan83] continues on to define a "projectivized" version of the problem, so that over $\operatorname{Spec}(k)$, the scheme $\mathbb{P}(\operatorname{Ext}^1_X(\mathcal{F},\mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of $\mathcal{F}$ by $\mathcal{G}$, modulo the action of $k^\times$. See also [HL10, Example 2.1.12].

## 2. Verlinde Bundles on Lefschetz Pencils

The thesis [Hem] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in $\mathbb{P}^3$, after summarizing some of its properties.

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. Consider the universal family $\pi\colon \mathfrak{X} \to |\mathcal{O}(4)|$, given by

$$\mathfrak{X} = \{(x, q) \in \mathbb{P}^3 \times |\mathcal{O}(4)| : x \in q\}.$$

The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$.

Throughout, the coordinates of $\mathbb{P}^3$ will be denoted by $x_i$, $i = 0, \ldots, 4$.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$, in other words as the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \to \mathbb{P}^3$.

**Proposition 2.1.** *Let $k \geq 1$. The following statements hold:*

**1.** *If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this dimension is independent of the rank $q$.*

**2.** *The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.*

**3.** *For all cartesian diagrams of the form*

$$\begin{tikzcd} \mathfrak{X}_Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\rho"'] & |\mathcal{O}(4)| \end{tikzcd}$$

*we have $\rho^* \pi_* \mathcal{L}^{\otimes k} \simeq (\pi_Z)_* \mathcal{L}_Z^{\otimes k}$.*

*Proof.* For the first statement, see the proof of [Hem, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. ☐

Let $t \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1 \to |\mathcal{O}(4)|$. We call $t$ a *Lefschetz pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1}$, which comes equipped with the pullback line bundle $\mathcal{L}_{\mathbb{P}^1}$. The situation is summarized in the picture below:

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{P}^1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r] & |\mathcal{O}(4)| \end{tikzcd}$$

For $k \geq 1$, we define the $k$-th Verlinde bundles $V_k := \pi_* \mathcal{L}^k$ and $V_{k,t} := (\pi_{\mathbb{P}^1})_* \mathcal{L}^k_{\mathbb{P}^1}$. These bundles are related by $V_k|_t = V_{k,t}$ using Proposition 2.1.

**Proposition 2.2.** *There exists a short exact sequence of coherent $\mathcal{O}_{|\mathcal{O}(4)|}$-modules*

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_k \to 0.$$

*Let $I_d$ range over the tuples of the form $(i_0, \ldots, i_3)$ with $\sum i_j = d$. The first map is then given by $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.*

*Proof.* See [Hem, Proposition 4.2]. ☐

**Remark 2.3.** Let $t$ be a Lefschetz pencil of quartics.

**1.** The sequence from Proposition 2.2 restricts to a sequence

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_{k,t} \to 0$$

over $\mathbb{P}^1$.

**2.** The vector bundle $V_{k,t}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

**Definition 2.4.** Let $k \geq 1$.

**1.** A *type candidate* for $V_k$ is a non-decreasing tuple $(d_1, \dots, d_r)$ of non-negative integers with $r = \operatorname{rk} V_k$ and $\sum d_i = \binom{k-1}{3}$.

**2.** The *general type candidate* for $V_k$ is the unique[2] type candidate for $V_k$ of the form $(d, \dots, d, d+1, \dots d+1)$.

**3.** Let $t$ be a Lefschetz pencil of quartics. The *type* of $V_{k,t}$ is the unique type candidate $(d_i)$ such that $V_{k_t} \simeq \bigoplus \mathcal{O}(d_i)$.

**4.** We say that $V_{k,t}$ has *general type* if its type $(d_i)$ is a general type candidate.

The rational points of $\mathrm{Gr}(2, 35)$ correspond to the Lefschetz pencils of quartics $t \subseteq |\mathcal{O}(4)|$ in the following way. Let $P$ the universal $\mathbb{P}^1$-bundle over $\mathrm{Gr}(2, 35)$. It comes equipped with a projection map $P \to \mathbb{P}^3$ such that for all Lefschetz pencils of quartics $t'$ there exists a unique rational point $t \in \mathrm{Gr}(2, 35)$ and a commutative diagram

$$\begin{tikzcd}
P_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & P \arrow[r, "p"] \arrow[d, "\varphi"] & {|\mathcal{O}(4)|} \\
\operatorname{Spec}(\kappa(t)) \arrow[r] & \mathrm{Gr}(2, 35) &
\end{tikzcd}$$

such that the image of the fiber $P_t$ in $|\mathcal{O}(4)|$ is $t'$.

**Definition 2.5.** Let $k \geq 1$ and $(d_i)$ be a type candidate for $V_k$. We define the set $Z_{(d_i)}$ of all rational points $t \in \mathrm{Gr}(2, 35)$ such that $V_{k,t}$ has type $(d_i)$. For the set of points $t$ where $V_{k,t}$ has generic type, we also write $Z_{\mathrm{gen}}$.

**Proposition 2.6.** *The set $Z_{gen}$ is Zariski open. Its complement is the union*

$$\mathrm{Gr}(2, 35) \setminus Z_{gen} = \operatorname{Supp}(R^1 \varphi_* p^* V_k(-d-1)) \cup \operatorname{Supp}(R^1 \varphi_* p^* V_k(-d)^\vee),$$

*where $d$ is the smaller of the two numbers appearing in the general type candidate $(d, \dots, d, d+1, \dots, d+1)$ for $V_k$.*

*Proof.* We begin by finding a characterization of the set $Z_{\mathrm{gen}}$ via cohomology.

Let $t \in \mathrm{Gr}(2, 35)$ be a rational point, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$. The conditions that for all $i$ we have $d \leq d_i$ and $d_i \leq d+1$ are equivalent to the conditions

$$H^1(P_t, V_{k,t}(-d-1)) = 0 \text{ and } H^1(P_t, V_{k,t}(-d)^\vee) = 0,$$

respectively. Both conditions together are in turn equivaleng to $t \in Z_{\mathrm{gen}}$.

Next, we want to use the Cohomology and Base Change Theorem [Vak17, 28.1.6] on the map $\varphi \colon P \to \mathrm{Gr}(2, 25)$, which is a $\mathbb{P}^1$-bundle, in particular proper and flat. The last property ensures that locally free sheaves on $P$ are flat over $\mathrm{Gr}(2, 35)$.

[2]The equations $ad + bd + b = \binom{k-1}{3}$ and $a + b = \operatorname{rk} V_k$ have an unique solution $(a, b)$.

For all rational $t \in \mathrm{Gr}(2,35)$ we have

$$h^2(P_t, p^*V_{k,t}(-d-1)) = 0 \text{ and } h^2(P_t, p^*V_{k,t}(-d)^\vee) = 0.$$

Since the sheaves $p^*V_{k,t}(-d-1)$ and $p^*V_{k,t}(-d)^\vee$ are locally free and coherent, the Cohomology and Base Change Theorem applies and we have

$$(R^1\varphi_*p^*V_k(-d-1))_t = H^1(P_t, V_{k,t}(-d-1))$$

and

$$(R^1\varphi_*p^*V_k(-d)^\vee)_t = H^1(P_t, V_{k,t}(-d)^\vee).$$

By the previous characterization, we have

$$\mathrm{Gr}(2,35) \setminus Z_{\mathrm{gen}} = \mathrm{Supp}(R^1\varphi_*p^*V_k(-d-1)) \cup \mathrm{Supp}(R^1\varphi_*p^*V_k(-d)^\vee),$$

which is a Zariski-closed set. □

**Proposition 2.7.** *The closed subsets*

***1.*** $\mathrm{Supp}(R^1\varphi_*p^*V_k(-d-1))$ *and*

***2.*** $\mathrm{Supp}(R^1\varphi_*p^*V_k(-d)^\vee)$

*are determinantal varieties.*

*Proof.* To simplify notation, we set

$$r_1 \coloneqq \dim H^0(\mathbb{P}^3, \mathcal{O}(k)) \text{ and } r_2 \coloneqq \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4)).$$

Rewrite the exact sequence from Proposition 2.2 as

$$0 \to \mathcal{O}(-1)^{r_2} \to \mathcal{O}^{r_1} \to V_k \to 0. \tag{1}$$

**1.** Twisting the sequence (1) with $\mathcal{O}(-d-1)$ and pulling back to $P$ gives an exact sequence

$$0 \to p^*\mathcal{O}(-d-2)^{r_2} \to p^*\mathcal{O}(-d-1)^{r_1} \to p^*V_k(-d-1) \to 0.$$

For every rational $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, \mathcal{O}(-d-2)^{r_2}) = 0$, hence $R^2\varphi_*p^*\mathcal{O}(-d-2)^{r_2} = 0$ and applying $\varphi_*$ to the above sequence gives an exact sequence

$$R^1\varphi_*p^*\mathcal{O}(-d-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_*p^*\mathcal{O}(-d-1)^{r_1} \to R^1\varphi_*p^*V_k(-d-1) \to 0.$$

Note that since the numbers

$$h_2^1 \coloneqq h^1(P_t, \mathcal{O}(-d-2)^{r_2}) \text{ and } h_1^1 \coloneqq h^1(P_t, \mathcal{O}(-d-1)^{r_1})$$

do not depend on the point $t$, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank $h_1^2$ and $h_1^1$, respectively. Since taking the fiber is right-exact, we see that for all $t$ we have $(R^1\varphi_* p^* V_k(-d-1))_t \neq 0$ if and only if $\operatorname{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\operatorname{Supp}(R^1\varphi_* p^* V_k(-d-1)) = \{t : \operatorname{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = dr_1 = d\binom{k+3}{3}$.

**2.** The proof for this point is analogous to the first point. We start with the sequence (1), twist with $\mathcal{O}(-d)$, take duals, pull back to $P$, and apply $\varphi_*$. Since for each rational $t \in \operatorname{Gr}(2,35)$ we have $h^1(P_t, \mathcal{O}(d)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_* p^* \mathcal{O}(d)^{r_1} \xrightarrow{\beta} \varphi_* p^* \mathcal{O}(d+1)^{r_2} \to R^1\varphi_* p^* V_k(-d)^\vee \to 0.$$

Since the numbers

$$h_1^0 := h^0(P_t, \mathcal{O}(d)^{r_1}) \text{ and } h_2^0 := h^0(P_t, \mathcal{O}(d+1)^{r_2})$$

do not depend on the point $t$, again by Grauert's Theorem the first two terms of the sequence are locally free of rank $h_1^0$ and $h_2^0$, respectively. As before, we obtain the characterization

$$\operatorname{Supp}(R^1\varphi_* p^* V_k(-d)^\vee) = \{t : \operatorname{rk}(\beta_t) \leq h_2^0 - 1\}.$$

Here, we have $h_2^0 = (d+2)r_2 = (d+2)\binom{k-1}{3}$. □

# 3. Specialization

In this section we collect some facts about specialization phenomena. We say that a vector bundle $\mathcal{V}$ on a $k$-scheme $X$ *specializes* to another vector bundle $\mathcal{V}'$ over the same scheme if there exists a vector bundle $\mathcal{W}$ on $\mathbb{A}^1 \times X$ such that $\mathcal{W}|_{0\times X} \simeq \mathcal{V}'$ and $\mathcal{W}|_{t\times X} \simeq \mathcal{V}$ for all rational $t \in \mathbb{A}^1$.

**Remark 3.1.** Let

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \to 0$$

be a short exact sequence of coherent sheaves over a $k$-scheme $X$ and let $\xi \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \to 0.$$

**Example 3.2.** The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$ specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. Consider $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})$, whose elements correspond to extensions of the form

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(2) \to 0$$

up to equivalence, the zero element corresponding to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have $\mathcal{E}$ locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Proposition 1.1, there exists an extension of the form

$$0 \to \mathcal{O} \boxtimes \mathcal{O}_{\mathbb{A}^1} \to \mathcal{E}_{\text{univ}} \to \mathcal{O}(2) \boxtimes \mathcal{O}_{\mathbb{A}^1} \to 0$$

such that for all nonzero rational points $\xi \in \mathbb{A}^1 = \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee)$ we have the isomorphisms $\mathcal{E}_{\text{univ}}|_{\xi \times \mathbb{A}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and $\mathcal{E}_{\text{univ}}|_{0 \times \mathbb{A}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\text{univ}}$ has to be locally free as the end terms of the sequence are.

**Remark 3.3.** If $\mathcal{V}$ specializes to $\mathcal{V}'$ and $\mathcal{W}$ specializes to $\mathcal{W}'$, then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

**Remark 3.4.** Let $b_1, \dots, b_m$ be non-negative numbers, let $a := \sum b_i$. The sequence

$$0 \to \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \to 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & \\ t^{b_2} & s^{b_2} & & \\ & t^{b_3} & \ddots & \\ & & \ddots & s^{b_{m-1}} \\ & & & t^{b_m} \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \cdots & (-1)^m s^{b_1+\cdots+b_{m-1}} t^{a-b_1-\cdots-b_m} \end{pmatrix}$$

is exact.

**Proposition 3.5.** *Let $b_1, \dots, b_m$ be non-negative numbers and $\pi$ a partition of the set $\{1, \dots, m\}$. For a set of indices $I \in \pi$, let $b'_I := \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m - |\pi|}$.*

*Proof.* By Remark 3.3 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 3.4, there exists a representative $\xi \in \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \to \mathcal{O}^{m-1} \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \to \mathcal{O}(a) \to 0.$$

By Remark 3.1, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the same form. Consider the associated closed embedding $\alpha\colon \mathbb{A}^1 \to \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let $\mathcal{E}$ be the universal extension from Proposition 1.1. Then, the vector bundle $(\mathrm{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. $\square$

**Remark 3.6.** By twisting the exact sequence in the proof of Proposition 3.5 and using the same argument, we see that for every integer $n$ and with $b_i, \pi$, and $b_I$ as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m - |\pi|}$.

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