Verlinde Bundles

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Contents

1 Verlinde Bundles on Pencils of Quartics

The thesis [Hem15] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in $\mathbb{P}^3$, after summarizing some of its properties. We work over a field k, but omit it in most notation[1], e.g. we write $\mathbb{P}^3$ for $\mathbb{P}^3_k$.

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. The quartics $\mathfrak{X}_t \subseteq \mathbb{P}^3$ parametrized by the $t \in |\mathcal{O}(4)|$ form a universal family $\pi\colon \mathfrak{X} \to |\mathcal{O}(4)|$ with fibers $\mathfrak{X}_t$. The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$. The morphism π is projective and flat.

Throughout, the homogeneous coordinates of $\mathbb{P}^3$ will be denoted by x_i, $i = 0, \dots, 4$.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$. In other words[2], the bundle $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \to \mathbb{P}^3$.

Proposition 1.1. *Let $k \geq 1$. The following statements hold:*

(i) *If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this number is independent of the point q.*

[1] Most instances of the letter k will be used to denote a natural number instead.

[2] For a fiber product $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ and sheaves $\mathcal{F}$ and $\mathcal{G}$ on X resp. Y, write $\mathcal{F} \boxtimes \mathcal{G} := p^*\mathcal{F} \otimes q^*\mathcal{G}$.

(ii) *The sheaf* $\pi_*\mathcal{L}^{\otimes k}$ *is locally free of rank* $\binom{k+3}{3} - \binom{k-1}{3}$*.*

(iii) *For all cartesian diagrams of the form*

$$\begin{tikzcd} \mathfrak{X}_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\rho"'] & |\mathcal{O}(4)| \end{tikzcd}$$

we have $\rho^*\pi_*\mathcal{L}^{\otimes k} \simeq (\pi_Z)_*\mathcal{L}_Z^{\otimes k}$*.*

Proof. For the first statement, see the proof of [Hem15, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. □

Let $T \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1_K \to |\mathcal{O}(4)|$, with K an extension field of k. We call T a *pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1_K}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}^1_K}$. The situation is summarized in the picture below:

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{P}^1_K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathbb{P}^1_K \arrow[r] & |\mathcal{O}(4)| \end{tikzcd}$$

Definition 1.2. For $k \geq 1$, we define the k-th Verlinde bundles $V_k := \pi_*\mathcal{L}^{\otimes k}$ and $V_{k,T} := (\pi_{\mathbb{P}^1})_*\mathcal{L}_{\mathbb{P}^1}^{\otimes k}$. These bundles are related by $V_k|_T = V_{k,T}$ using Proposition 1.1.

Proposition 1.3. *There exists a short exact sequence of coherent* $\mathcal{O}_{|\mathcal{O}(4)|}$*-modules*

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_k \to 0.$$

Let I_d *range over the tuples of the form* $(i_0, \ldots, i_3)$ *with* $\sum i_j = d$*. The first map is then given by* $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$*.*

Proof. See [Hem15, Proposition 4.2]. □

Remark 1.4. Let T be a pencil of quartics.

(i) The sequence from Proposition 1.3 restricts to a sequence

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_{k,T} \to 0 \tag{1.1}$$

over $\mathbb{P}^1$.

(ii) The vector bundle $V_{k,T}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

(iii) Let $V_{k,T} \simeq \bigoplus_i \mathcal{O}(d_i)$ be a splitting of $V_{k,T}$ over $\mathbb{P}^1$. By the sequence (1.1), we have $d_i \geq 0$.

Definition 1.5. Let $k \geq 1$.

(i) A *type candidate* for V_k is a non-increasing tuple $(d_1, \ldots, d_{r^{(k)}})$ of non-negative integers with $r^{(k)} = \binom{k+3}{3} - \binom{k-1}{3}$ and $\sum d_i = \binom{k-1}{3}$.

(ii) The *general type candidate* for V_k is the unique type candidate for V_k of the form $(b^{(k)} + 1, \ldots, b^{(k)} + 1, b^{(k)}, \ldots, b^{(k)})$. The integer $b^{(k)}$ is determined by the equation $\binom{k-1}{3} = b^{(k)} r^{(k)} + a$, with $a < r^{(k)}$ becoming the number of occurences of $b^{(k)} + 1$.

(iii) Let E be a locally free sheaf on $\mathbb{P}^1$. The *type* of E is the unique non-increasing tuple $(d_1, \ldots, d_{r^{(k)}})$ such that $E \simeq \bigoplus_i \mathcal{O}(d_i)$.

(iv) We say that $V_{k,T}$ has *general type* if its type is a general type candidate.

Definition 1.6. Let P denote the universal $\mathbb{P}^1$-bundle of the Grassmannian of lines $\mathrm{Gr}(2, H^0(\mathcal{O}(4))) = \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$, let $\varphi\colon P \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ be the universal map and $p\colon P \to |\mathcal{O}(4)|$ the canonical projection.

$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & |\mathcal{O}(4)| \\ \downarrow{\scriptstyle\varphi} & & \\ \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|) & & \end{array}$$

The mapping $t \mapsto P_t$ gives a canonical bijection between the points of $\mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ and the pencils of quartics in $|\mathcal{O}(4)|$. For such t, we write $V_{k,t} \coloneqq V_{k,p(P_t)}$.

Definition 1.7. Let $k \geq 1$ and (d_i) be a type candidate for V_k. We define the set $Z_{(d_i)}$ of all points $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ such that $V_{k,t}$ has type (d_i). For the set of points t where $V_{k,t}$ has general type, we also write Z_{gen}.

Proposition 1.8. *The set Z_{gen} is Zariski open. Its complement is the union*

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-b^{(k)} - 1)) \cup \mathrm{Supp}(R^1\varphi_*(p^* V_k(-b^{(k)})^\vee)).$$

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and $b \coloneqq b^{(k)}$. We have $t \in Z_{\mathrm{gen}}$ if and only if $b \leq d_i \leq b + 1$ for all i, which holds if and only if $H^1(P_t, V_{k,t}(-b-1)) = H^1(P_t, V_{k,t}(-b)^\vee) = 0$.

Next, we want to apply the Cohomology and Base Change Theorem [Vak17, 28.1.6] to the map $\varphi\colon P \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$, which is a $\mathbb{P}^1$-bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$.

For all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ we have $h^2(P_t, p^* V_{k,t}(-b-1)) = 0$ and $h^2(P_t, p^* V_{k,t}(-b)^\vee) = 0$. Since the sheaves $p^* V_{k,t}(-b-1)$ and $p^* V_{k,t}(-b)^\vee$ are locally free and coherent, we have

$$(R^1\varphi_* p^* V_k(-b-1))_t = H^1(P_t, V_{k,t}(-b-1))$$

and

$$(R^1\varphi_*(p^*V_k(-b)^\vee))_t = H^1(P_t, V_{k,t}(-b)^\vee).$$

By the previous characterization, we have

$$\mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|) \setminus Z_{\mathrm{gen}} = \mathrm{Supp}(R^1\varphi_* p^*V_k(-b-1)) \cup \mathrm{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)),$$

which is a Zariski closed set. □

Proposition 1.9. *The sets* $\mathrm{Supp}(R^1\varphi_*p^*V_k(-b^{(k)}-1))$ *and* $\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee))$ *are determinantal varieties in the sense of [Arb+13, Ch. II, §4]*

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k)), r_2 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4))$ and $b := b^{(k)}$, and rewrite the exact sequence from Proposition 1.3 as

$$0 \to \mathcal{O}(-1)^{r_2} \to \mathcal{O}^{r_1} \to V_k \to 0. \tag{1.2}$$

Twisting the sequence (1.2) with $\mathcal{O}(-b-1)$ and pulling back to P gives an exact sequence

$$0 \to p^*\mathcal{O}(-b-2)^{r_2} \to p^*\mathcal{O}(-b-1)^{r_1} \to p^*V_k(-b-1) \to 0.$$

For all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ we have $h^2(P_t, \mathcal{O}(-b-2)^{r_2}) = 0$, hence $R^2\varphi_*p^*\mathcal{O}(-b-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_*p^*\mathcal{O}(-b-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_*p^*\mathcal{O}(-b-1)^{r_1} \to R^1\varphi_*p^*V_k(-b-1) \to 0.$$

Note that since the numbers $h^1_2 := h^1(P_t, \mathcal{O}(-b-2)^{r_2})$ and $h^1_1 := h^1(P_t, \mathcal{O}(-b-1)^{r_1})$ do not depend on the point t, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h^2_1 and h^1_1, respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_*p^*V_k(-b-1))_t \neq 0$ if and only if $\mathrm{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b-1))) = \{t : \mathrm{rk}(\alpha_t) \leq h^1_1 - 1\}.$$

As a final remark, note that $h^1_1 = br_1 = b\binom{k+3}{3}$.

The proof for the second assertion is similar. We start with the sequence (1.2), twist with $\mathcal{O}(-b)$, take duals, pull back to P, and apply φ_*. Since for all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ we have $h^1(P_t, \mathcal{O}(b)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_*p^*\mathcal{O}(b)^{r_1} \xrightarrow{\beta} \varphi_*p^*\mathcal{O}(b+1)^{r_2} \to R^1\varphi_*(p^*V_k(-b)^\vee) \to 0.$$

Since the numbers $h^0_1 := h^0(P_t, \mathcal{O}(b)^{r_1})$ and $h^0_2 := h^0(P_t, \mathcal{O}(b+1)^{r_2})$ do not depend on the point t, again by Grauert's Theorem the first two terms of the sequence are locally free of rank h^0_1 and h^0_2, respectively. As before, we obtain the characterization

$$\mathrm{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)) = \{t : \mathrm{rk}(\beta_t) \leq h^0_2 - 1\}.$$

Here, we have $h^0_2 = (b+2)r_2 = (b+2)\binom{k-1}{3}$. □

Definition 1.10. For type candidates (d_i) and (d'_i) we define the expression $(d'_i) \geq (d_i)$ to mean
$$\sum_{i=1}^{s} d'_i \geq \sum_{i=1}^{s} d_i \text{ for all } s = 1, \dots, r^{(k)}.$$

Proposition 1.11. *Let (d_i) be a type candidate for V_k. The set $\widehat{Z}_{(d_i)} \coloneqq \bigcup_{(d'_i) \geq (d_i)} Z_{(d'_i)}$ is Zariski-closed. In particular, the set $Z_{(d_i)}$ is locally closed.*

Proof. Let $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ and $V_{k,t} = \bigoplus_{i=1}^{r^{(k)}} \mathcal{O}(d'_i)$. We have
$$\bigwedge^{s} V_{k,t} = \bigoplus_{I} \mathcal{O}(d'_I),$$
where I runs over the subsets of $\{1, \dots, r^{(k)}\}$ of size s and $d'_I \coloneqq \sum_{i \in I} d'_i$. For every type candidate (d'_i), the sum $\sum_{i=1}^{s} d'_i$ is the largest sum of s entries of (d'_i). Since $d'_i \geq 0$, the condition $\sum_{i=1}^{s} d'_i \geq \sum_{i=1}^{s} d_i$ is equivalent to the condition $h^0((\bigwedge^s V_{t,k})(-\sum^s d_i)) > 0$.
$$\widehat{Z}_{(d_i)} = \bigcap_{s=1}^{r^{(k)}} \{t : h^0((\textstyle\bigwedge^s V_{t,k})(-\sum^s d_i)) > 0\}.$$
With Serre duality and the Cohomology and Base Change theorem we write the sets of the intersection as
$$\mathrm{Supp}(R^1 \varphi_*(p^*(\textstyle\bigwedge^s V_k^\vee)(\sum^s d_i - 2))),$$
which is Zariski-closed. □

Proposition 1.12. *Of the five type candidates*
$$(1,1,1,1,0,\dots,0),\ (2,1,1,0,\dots,0),\ (2,2,0,\dots,0),\ (3,1,0,\dots,0),\ (4,0,\dots,0)$$
for V_5, only the first two occur as types of some $V_{5,t}$.

Proof. This is a special case of Proposition 2.8. □

1.1 Calculations in the Chow Ring

To perform calculations in the Chow ring A of $\mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$, we follow the conventions found in [EH16]. We assume $\mathrm{char}(k) = 0$ for simplicity. The ring A is generated by the Schubert classes $\sigma_{a,b}$ of the Schubert cycles
$$\Sigma_{a,b}(\mathcal{H}) \coloneqq \{T \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|) : T \cap H \neq \varnothing, T \subseteq H'\},$$
where $\mathcal{H} = (H \subset H')$ is a flag of linear subspaces of dimension $33 - a$ resp. $34 - b$ in the 34-dimensional projective space $|\mathcal{O}(4)|$. The class $\Sigma_{a,b}$ has codimension $a + b$, and we use the convention $\sigma_a \coloneqq \sigma_{a,0}$.

Proposition 1.13. *Let $Z := Z_{(2,1,1,0,\dots)}$ be locus of points $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$ such that $V_{5,t}$ is not generic. In the Chow ring A, we have*

$$[Z] = \sum_{a=0}^{11} \left(\binom{24-a}{3}\binom{a+1}{3} - \binom{25-a}{3}\binom{a}{3} \right) \sigma_{33-a,10+a}. \tag{1.3}$$

Proof. Define the subvariety $Q \subset |\mathcal{O}(4)|$ as the image of the map

$$f\colon |\mathcal{O}(1)| \times |\mathcal{O}(3)| \to |\mathcal{O}(4)|$$

defined by $f(g, h) = gh$. The map f is birational on its image, since a general point of Q has the form gh with h irreducible. By Proposition 2.6, the variety Z is the image of the finite map

$$\varphi\colon \mathbb{G}\mathrm{r}(1, |\mathcal{O}(1)|) \times |\mathcal{O}(3)| \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|)$$

defined by $\varphi((sg_1 + tg_2)_{(s:t)}, h) = (sg_1h + tg_2h)_{(s:t)}$. In particular, a line $T \subset |\mathcal{O}(4)|$ belonging to Z lies in Q.

Since f and φ are finite, we have $\dim(Q) = 22$ and $\dim(Z) = 23$, while $\operatorname{codim}(Z) = 43$. The Chow group A^{43} is generated by the classes $\sigma_{33,10}, \sigma_{32,11}, \dots, \sigma_{22,21}$, while the complementary group A^{23} is generated by $\sigma_{23,0}, \sigma_{22,1}, \dots, \sigma_{12,11}$. Write

$$[Z] = \sum_{a=0}^{11} \alpha_{33-a,10+a}\sigma_{33-a,10+a}.$$

We have $\sigma_{33-a,10+a}\sigma_{23-a',a'} = 1$ if $a = a'$ and 0 else. Hence, multiplying the above equation with $\sigma_{33-a',10+a'}$ and taking degrees gives $\alpha_{33-a,10+a} = \deg([Z] \cdot \sigma_{23-a,a})$.

Using Giambelli's formula $\sigma_{23-a,a} = \sigma_{23-a}\sigma_a - \sigma_{24-a}\sigma_{a-1}$ [EH16, Prop. 4.16], we reduce to computing $\deg([Z] \cdot \sigma_{23-a}\sigma_a)$ for $0 \leq a \leq 11$. By Kleiman transversality, we have

$$\deg([Z] \cdot \sigma_{23-a}\sigma_a) = |\{T \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(4)|) : T \cap H \neq \varnothing, T \cap H' \neq \varnothing\}|,$$

where H and H' are general linear subspaces of $|\mathcal{O}(4)|$ of dimension $10 + a$ and $33 - a$, respectively.

To a point $p = g_p h_p \in Q$ with $g_p \in |\mathcal{O}(1)|$ and $h_p \in |\mathcal{O}(3)|$, associate a closed reduced subscheme Λ_p containing p as follows. If h_p is irreducible, let Λ_p be the image of the linear embedding $|\mathcal{O}(1)| \times \{h_p\} \to |\mathcal{O}(4)|$ given by $g \mapsto gh_p$. If $h_p = g'_p h'_p$ with $h'_p \in |\mathcal{O}(2)|$ irreducible, let Λ_p be the union of the images of the linear embeddings $|\mathcal{O}(1)| \times \{h_p\} \to |\mathcal{O}(4)|$ and $|\mathcal{O}(1)| \times \{g_p h'_p\} \to |\mathcal{O}(4)|$. These two linear subspaces meet exactly at p. Similarly, if p is the product of four linear forms, define the space Λ_p as the union $\bigcup_h \operatorname{im}(|\mathcal{O}(1)| \times \{h\} \to |\mathcal{O}(4)|)$, where h runs over the four cubics arising as products of the linear factors of p.

By the definition of Z, all lines $T \in Z$ lie in Q and if T meets the point p, then $T \subseteq \Lambda_p$. For $H \subseteq |\mathcal{O}(4)|$ a linear subspace of dimension $10 + a$, define $Q' := H \cap Q$. For general

H, the subscheme Q' is a smooth subvariety of dimension $a-2$ such that a general point $p = gh$ of Q' with $h \in |\mathcal{O}(4)|$ has h irreducible. For $a = 0, 1$, a general linear subspace H does not intersect Q at all, from which follows $\deg([Z] \cdot \sigma_{23}\sigma_0) = \deg([Z] \cdot \sigma_{22}\sigma_1) = 0$.

Next, we show that for general H, for each point $p \in Q'$ we have $\Lambda_p \cap H = \{p\}$. Let $\mathcal{H}$ denote the parameter space for H, i. e. the Grassmannian $\mathbb{G}\mathrm{r}(10+a, 34)$. Define the closed subset $X \subseteq Q \times \mathcal{H}$ by $X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}$. The fibers of the induced map $X \to \mathcal{H}$ have dimension at least one. Hence, to prove that the desired condition on H is an open condition, it suffices to prove $\dim(X) \leq \dim(\mathcal{H})$. The fiber of the map $X \to Q$ over a point p consists of the union of finitely many closed subsets of the form $X'_p = \{H \in \mathcal{H} : \dim(H \cap \Lambda'_p) \geq 1\}$, where $\Lambda'_p \simeq \mathbb{P}^3 \subseteq |\mathcal{O}(4)|$ is one of the components of Λ_p. The space X'_p is a Schubert cycle

$$\Sigma_{22-a,22-a} = \{H \in \mathrm{Gr}(11, 35) : \dim(H \cap H_4) \geq 2\},$$

with H_4 a four-dimensional subspace of $H^0(\mathcal{O}(4))$. The codimension of the cycle is $2(22-a)$, hence also $\mathrm{codim}(X_p) = 2(22-a)$. Finally, we have $\dim(\mathcal{H}) - \dim(X) = \mathrm{codim}(X_p) - \dim(Q) = 22 - 2a \geq 0$ for $0 \leq a \leq 11$.

Next, let $\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|\mathcal{O}(1)| \otimes \mathrm{pr}_2 f^{-1}(Q'))$ and $\Lambda'' := |\mathcal{O}(1)| \otimes \mathrm{pr}_2 f^{-1}(Q')$. By the choice of H, the map $f^{-1}(Q') \to Q'$ is birational and the map $f^{-1}(Q) \to \mathrm{pr}_2 f^{-1}(Q)$ is even bijective. It follows that Λ'' and hence Λ have dimension $a+1$. The intersection of Λ with a general linear subspace H' of dimension $33-a$ is a finite set of points. For each point $p \in Q'$, the linear subspace H' intersects each component Λ'_p of Λ_p in at most one point. For each point $p' \in H' \cap \Lambda$ there exists a unique p such that $p' \in \Lambda_p$. Furthermore, the only line $T \in Z$ meeting both p and H' is the one through p and p'. If the intersection $H' \cap \Lambda_p$ is empty, then there will be no line meeting p and H'. Hence, $\deg([Z] \cdot \sigma_{22-a}\sigma_a)$ is the number of intersection points of Λ with a general H'.

Finally, the pre-image $f^{-1}(Q') = f^{-1}(H)$ is smooth for a general H by Bertini's Theorem. If ζ is the class of a hyperplane section of $|\mathcal{O}(4)|$ we have $f^*(\zeta) = \alpha + \beta$, , where α and β are classes of hyperplane sections of $|\mathcal{O}(1)|$ and $|\mathcal{O}(3)|$, respectively. Since pr_2 and f have degree one we compute:

$$\begin{aligned}
[\Lambda''] &= [\mathrm{pr}_2^{-1}\mathrm{pr}_2 f^{-1}(H)] \\
&= \mathrm{pr}_2^*[\mathrm{pr}_2 f^{-1}(H)] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*}[f^{-1}(H)] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*} f^*[H] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*}(\alpha+\beta)^{24-a} \\
&= \tbinom{24-a}{3}\mathrm{pr}_2^*\beta^{21-a} \\
&= \tbinom{24-a}{3}\beta^{21-a}
\end{aligned}$$

Hence, by the push-pull formula:

$$\deg([\Lambda] \cdot H') = \deg([\Lambda''] \cdot (\alpha+\beta)^{a+1}) = \binom{24-a}{3}\binom{a+1}{3}$$

We then use Giambelli's formula to obtain Equation (1.3).

□

2 Verlinde Bundles on Pencils of Hypersurfaces

Definition 2.1. Let $\pi\colon \mathfrak{X} \to |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the universal family of hypersurfaces of degree d in $\mathbb{P}^n$. Let $\mathcal{L}$ be the restriction of the bundle $\mathcal{O}(1)\boxtimes\mathcal{O}$ under the inclusion $\mathfrak{X} \subseteq \mathbb{P}^n \times |\mathcal{O}(d)|$. The vector bundle $V_k \coloneqq \pi_*(\mathcal{L}^{\otimes k})$ is the k-th *Verlinde Bundle* of the universal family π.

The vector bundle V_k is the cokernel of the map

$$M\colon \mathcal{O}_{|\mathcal{O}(d)|}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \to \mathcal{O}_{|\mathcal{O}(d)|} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k))$$

given by multiplication by $\sum_I \alpha_I \otimes x^I \in H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$, where the α_I are the homogeneous coordinates on $|\mathcal{O}(d)|$. The vector bundle V_k has rank $r^{(k)} \coloneqq h^0(\mathbb{P}^n, \mathcal{O}(k)) - h^0(\mathbb{P}^n, \mathcal{O}(k-d))$ and determinant $\mathcal{O}(d^{(k)})$ with $d^{(k)} \coloneqq h^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We study the restriction of V_k to lines $T \subseteq |\mathcal{O}(d)|$.

Lemma 2.2. *Let $\mathcal{E}$ be a finite free $\mathcal{O}_{\mathbb{P}^1}$-module, and let*

$$0 \to \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \to \mathcal{E}'' \to 0$$

be a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules. Given a splitting $\mathcal{E}'' = \mathcal{E}_1'' \oplus \mathcal{O}$, we may construct a splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$ such that the image of φ is contained in $\mathcal{E}_1$.

Proof. Define $\mathcal{E}_1 \coloneqq \ker(\mathrm{pr}_2 \circ \psi)$, which is a locally free sheaf on $\mathbb{P}^1$. By comparing determinants in the short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{O} \to 0$ we see that $\mathcal{E}_1$ is free, hence by an Ext^1 computation the sequence splits. The property $\mathrm{im}(\varphi) \subseteq \mathcal{E}_1$ follows from the definition. □

Proposition 2.3. *Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$ and $\mathrm{coker}(M|_T) \simeq \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^s \mathcal{O}(d_i)$. Define $U \coloneqq H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We have*

$$d_0 = \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1U + f_2U),$$

or, equivalently,

$$s = \dim(f_1U + f_2U) - d^{(k)}.$$

Proof. Note that the map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$. In particular, the image of $\mathcal{O}(-1) \otimes U$ is contained in $\mathcal{O} \otimes (f_1U + f_2U)$. It follows that $d_0 \geq \dim(f_1U + f_2U)$.

To prove the other inequality, consider the induced sequence

$$0 \to \mathcal{O}(-1) \otimes U \xrightarrow{M|_T} \mathcal{O} \otimes (f_1U + f_2U) \to \mathcal{E}'' \to 0$$

and assume that $\mathcal{E}'' \simeq \mathcal{E}_1'' \oplus \mathcal{O}$. By Lemma 2.2, we have a splitting $\mathcal{O} \otimes (f_1 U + f_2 U) \simeq \mathcal{E}_1 \oplus \mathcal{O}$ such that $\operatorname{im}(M|_T) \subseteq \mathcal{E}_1$.

Consider the map $\widetilde{M}|_T : (\mathcal{O} \otimes U) \oplus (\mathcal{O} \otimes U) \to \mathcal{O} \otimes (f_1 U + f_2 U)$ defined by

$$\widetilde{M}|_T(a \otimes \theta_1, b \otimes \theta_2) = a \otimes f_1\theta_1 + b \otimes f_2\theta_2.$$

We obtain the matrix description of $\widetilde{M}|_T$ from the matrix description of $M|_T$ as follows. If $M|_T$ is represented by the matrix A with coefficients $A_{i,j} = \lambda_{i,j} s + \mu_{i,j} t$, $i \leq \dim(f_1 U + f_2 U)$, $j \leq \dim U$, then $\widetilde{M}|_T$ is represented by a block matrix

$$B = \left(\begin{array}{c|c} A' & A'' \end{array} \right)$$

with $A'_{i,j} = \lambda_{i,j}$ and $A''_{i,j} = \mu_{i,j}$.

The property $\operatorname{im}(M|_T) \subseteq \mathcal{E}_1$ implies that after some row operations, the matrix A has a zero row. By the construction of $\widetilde{M}|_T$, the same row operations lead to the matrix B having a zero row, but this is a contradiction, since the map $\widetilde{M}|_T$ is surjective. $\square$

Remark 2.4. The general type candidate for V_k takes the form $(b^{(k)} + 1, \ldots, b^{(k)} + 1, b^{(k)}, \ldots, b^{(k)})$, where the number of entries is $r^{(k)} = \binom{n+k}{n} - \binom{n+k-d}{n}$ and their sum $d^{(k)} = \binom{n+k-d}{n}$, while $b^{(k)} = \lfloor d^{(k)}/r^{(k)} \rfloor$. Note that the degrees of $d^{(k)}$ and $r^{(k)}$ as polynomials in k are n and $n-1$, respectively. Hence, $b^{(k)} \to \infty$ for $k \to \infty$.

Corollary 2.5. *Let $t \in \operatorname{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2. Let (θ_j) be a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. Let k be such that $b^{(k)} = 0$, that is such that in the general type, only ones and zeroes appear. The bundle $V_{k,t}$ has general type if and only if $(f_1\theta_j, f_2\theta_j)_j$ is a linearly independent set in $H^0(\mathbb{P}^n, \mathcal{O}(k))$.*

Proof. Since $b^{(k)} = 0$, the type of $V_{k,t}$ is the general type if and only if it has $d^{(k)}$ many nonzero entries. By Proposition 2.3, this is the case if and only if $\dim \langle f_1\theta_j, f_2\theta_j \rangle_j = 2d^{(k)}$. $\square$

Proposition 2.6. *Let $t \in \operatorname{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2, and let k be such that $b^{(k)} = 0$. The bundle $V_{k,t}$ has nongeneral type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $b^{(k)} = 0$ but $k > 2d$ then the general type never occurs.*

Proof. By Corollary 2.5, the bundle $V_{k,t}$ has non-general type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ such that $g_1 f_1 + g_2 f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h, respectively.

On the other hand, given such g_1 and g_2, we have $f_1 \mid g_2 f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. $\square$

Example 2.7. For $n = 2, d = 2$, and $k = 3$, we have $d^{(k)} = 3$ and $r^{(k)} = 10$. We show that the only types of V_k that occur are $(1_3, 0_7)$ and $(2_1, 1_1, 0_8)$. The first type occurs e. g. for $f_1 = x_0^2, f_2 = x_1^2$, and the second for $f_1 = x_0^2, f_2 = x_0x_1$. Assume that the type $(3_1, 0_9)$ occurs for some $f_1, f_2 \in H^0(\mathbb{P}^2, \mathcal{O}(2))$. By Proposition 2.3 we then have $\dim \langle f_1x_j, f_2x_j\rangle_{j=0}^2 = 4$. Hence, we find $g_1, g_2, g_1', g_2' \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ and two linearly independent equations

$$g_1f_1 + g_2f_2 = 0$$
$$g_1'f_1 + g_2'f_2 = 0,$$

with both sets $(g_1, g_2), (g_1', g_2')$ linearly independent. From the first equation it follows that $f_1 = g_2h$ and $f_2 = -g_1h$, for some common linear factor h. Applying this to the second equation, we find $g_1'g_2 = g_2'g_1$, hence $g_1' = \alpha g_1$ and $g_2' = \alpha g_2$ for some scalar α, a contradiction.

Proposition 2.8. *Let $k = d + 1$. No types of V_k other than $(1, \ldots, 1, 0, \ldots, 0)$ and $(2, 1, \ldots, 1, 0, \ldots, 0)$ occur.*

Proof. The proof follows the lines of Example 2.7. Assume that the type of V_k at some line (f_1, f_2) is other than the two above. Then the type has two more zero entries than the general type, corresponding to two equations of the form

$$g_1f_1 + g_2f_2 = 0$$
$$g_1'f_1 + g_2'f_2 = 0,$$

with $g_i, g_i' \in H^0(\mathbb{P}^n, \mathcal{O}(1))$. We use the irreducibility of the g_i to produce a contradiction just like in the cited example, the only difference being that the common factor h of f_1 and f_2 need not be linear. □

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