Verlinde Bundles

Orlando Marigliano

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Contents

1. Verlinde bundles on Lefschetz pencils

The thesis [Hem] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in $\mathbb{P}^3$, after summarizing some of its properties.

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. Consider the universal family $\pi\colon \mathfrak{X} \longrightarrow |\mathcal{O}(4)|$, given by

$$\mathfrak{X} = \{(x, q) \in \mathbb{P}^3 \times |\mathcal{O}(4)| : x \in q\}.$$

The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$.

Throughout, the coordinates of $\mathbb{P}^3$ will be denoted by x_i, $i = 0, \ldots, 4$.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$, in other words as the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \longrightarrow \mathbb{P}^3$.

Proposition 1.1. *Let $k \geq 1$. The following statements hold:*

1. *If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this dimension is independent of the rank q.*

2. *The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.*

3. *For all cartesian diagrams of the form*

$$\begin{tikzcd} \mathfrak{X}_Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\rho"'] & |\mathcal{O}(4)| \end{tikzcd}$$

we have $\rho^*\pi_*\mathcal{L}^{\otimes k} \simeq (\pi_Z)_*\mathcal{L}_Z^{\otimes k}$.

Proof. For the first statement, see the proof of [Hem, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. □

Let $t \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1 \longrightarrow |\mathcal{O}(4)|$. We call t a *Lefschetz pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1}$, which comes equipped with the pullback line bundle $\mathcal{L}_{\mathbb{P}^1}$. The situation is summarized in the picture below:

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{P}^1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r] & |\mathcal{O}(4)| \end{tikzcd}$$

For $k \geq 1$, we define the k-th Verlinde bundles $V_k := \pi_*\mathcal{L}^k$ and $V_{k,t} := (\pi_{\mathbb{P}^1})_*\mathcal{L}^k_{\mathbb{P}^1}$. These bundles are related by $V_k|_t = V_{k,t}$ using Proposition 1.1.

Proposition 1.2. *There exists a short exact sequence of coherent* $\mathcal{O}_{|\mathcal{O}(4)|}$*-modules*

$$0 \longrightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \longrightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \longrightarrow V_k \longrightarrow 0.$$

Let I_d *range over the tuples of the form* $(i_0, \ldots, i_3)$ *with* $\sum i_j = d$. *The first map is then given by* $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.

Proof. See [Hem, Proposition 4.2]. □

Remark 1.3. Let t be a Lefschetz pencil of quartics.

1. The sequence from Proposition 1.2 restricts to a sequence

$$0 \longrightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \longrightarrow \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \longrightarrow V_{k,t} \longrightarrow 0$$

over $\mathbb{P}^1$.

2. The vector bundle $V_{k,t}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

Definition 1.4. Let $k \geq 1$.

1. A *type candidate* for V_k is a non-decreasing tuple $(d_1, \dots, d_r)$ of non-negative integers with $r = \operatorname{rk} V_k$ and $\sum d_i = \binom{k-1}{3}$.

2. The *general type candidate* for V_k is the unique[1] type candidate for V_k of the form $(d, \dots, d, d+1, \dots d+1)$.

3. Let t be a Lefschetz pencil of quartics. The *type* of $V_{k,t}$ is the unique type candidate (d_i) such that $V_{k_t} \simeq \bigoplus \mathcal{O}(d_i)$.

4. We say that $V_{k,t}$ has *general type* if its type (d_i) is a general type candidate.

The rational points of $\mathrm{Gr}(2, 35)$ correspond to the Lefschetz pencils of quartics $t \subseteq |\mathcal{O}(4)|$ in the following way. Let P the universal $\mathbb{P}^1$-bundle over $\mathrm{Gr}(2, 35)$. It comes equipped with a projection map $P \longrightarrow \mathbb{P}^3$ such that for all Lefschetz pencils of quartics t' there exists a unique rational point $t \in \mathrm{Gr}(2, 35)$ and a commutative diagram

$$\begin{tikzcd}
P_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & P \arrow[r, "p"] \arrow[d, "\varphi"] & {|\mathcal{O}(4)|} \\
\operatorname{Spec}(\kappa(t)) \arrow[r] & \mathrm{Gr}(2,35) &
\end{tikzcd}$$

such that the image of the fiber P_t in $|\mathcal{O}(4)|$ is t'.

Definition 1.5. Let $k \geq 1$ and (d_i) be a type candidate for V_k. We define the set $Z_{(d_i)}$ of all rational points $t \in \mathrm{Gr}(2, 35)$ such that $V_{k,t}$ has type (d_i). For the set of points t where $V_{k,t}$ has generic type, we also write Z_{gen}.

Proposition 1.6. *The set Z_{gen} is Zariski open. Its complement is the union*

$$\mathrm{Gr}(2, 35) \setminus Z_{gen} = \operatorname{Supp}(R^1\varphi_* p^* V_k(-d-1)) \cup \operatorname{Supp}(R^1\varphi_* p^* V_k(-d)^\vee),$$

where d is the smaller of the two numbers appearing in the general type candidate $(d, \dots, d, d+1, \dots, d+1)$ for V_k.

Proof. We begin by finding a characterization of the set Z_{gen} via cohomology.

Let $t \in \mathrm{Gr}(2, 35)$ be a rational point, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$. The conditions that for all i we have $d \leq d_i$ and $d_i \leq d+1$ are equivalent to the conditions

$$H^1(P_t, V_{k,t}(-d-1)) = 0 \text{ and } H^1(P_t, V_{k,t}(-d)^\vee) = 0,$$

[1] The equations $ad + bd + b = \binom{k-1}{3}$ and $a + b = \operatorname{rk} V_k$ have an unique solution (a, b).

respectively. Both conditions together are in turn equivaleng to $t \in Z_{\text{gen}}$.

Next, we want to use the Cohomology and Base Change Theorem [Vak17, 28.1.6] on the map $\varphi \colon P \longrightarrow \operatorname{Gr}(2, 25)$, which is a $\mathbb{P}^1$-bundle, in particular proper and flat. The last property ensures that locally free sheaves on P are flat over $\operatorname{Gr}(2, 35)$.

For all rational $t \in \operatorname{Gr}(2, 35)$ we have

$$h^2(P_t, p^* V_{k,t}(-d-1)) = 0 \text{ and } h^2(P_t, p^* V_{k,t}(-d)^\vee) = 0.$$

Since the sheaves $p^* V_{k,t}(-d-1)$ and $p^* V_{k,t}(-d)^\vee$ are locally free and coherent, the Cohomology and Base Change Theorem applies and we have

$$(R^1 \varphi_* p^* V_k(-d-1))_t = H^1(P_t, V_{k,t}(-d-1))$$

and

$$(R^1 \varphi_* p^* V_k(-d)^\vee)_t = H^1(P_t, V_{k,t}(-d)^\vee).$$

By the previous characterization, we have

$$\operatorname{Gr}(2, 35) \setminus Z_{\text{gen}} = \operatorname{Supp}(R^1 \varphi_* p^* V_k(-d-1)) \cup \operatorname{Supp}(R^1 \varphi_* p^* V_k(-d)^\vee),$$

which is a Zariski-closed set. □

Proposition 1.7. *The closed subsets*

1. $\operatorname{Supp}(R^1 \varphi_* p^* V_k(-d-1))$ *and*

2. $\operatorname{Supp}(R^1 \varphi_* p^* V_k(-d)^\vee)$

are determinantal varieties.

Proof. To simplify notation, we set

$$r_1 \coloneqq \dim H^0(\mathbb{P}^3, \mathcal{O}(k)) \text{ and } r_2 \coloneqq \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4)).$$

Rewrite the exact sequence from Proposition 1.2 as

$$0 \longrightarrow \mathcal{O}(-1)^{r_2} \longrightarrow \mathcal{O}^{r_1} \longrightarrow V_k \longrightarrow 0. \tag{1}$$

1. Twisting the sequence (1) with $\mathcal{O}(-d-1)$ and pulling back to P gives an exact sequence

$$0 \longrightarrow p^* \mathcal{O}(-d-2)^{r_2} \longrightarrow p^* \mathcal{O}(-d-1)^{r_1} \longrightarrow p^* V_k(-d-1) \longrightarrow 0.$$

For every rational $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, \mathcal{O}(-d-2)^{r_2}) = 0$, hence $R^2\varphi_* p^* \mathcal{O}(-d-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_* p^* \mathcal{O}(-d-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_* p^* \mathcal{O}(-d-1)^{r_1} \longrightarrow R^1\varphi_* p^* V_k(-d-1) \longrightarrow 0.$$

Note that since the numbers

$$h_2^1 \coloneqq h^1(P_t, \mathcal{O}(-d-2)^{r_2}) \text{ and } h_1^1 \coloneqq h^1(P_t, \mathcal{O}(-d-1)^{r_1})$$

do not depend on the point t, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h_1^2 and h_1^1, respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_* p^* V_k(-d-1))_t \neq 0$ if and only if $\mathrm{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-d-1)) = \{t : \mathrm{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = dr_1 = d\binom{k+3}{3}$.

2. The proof for this point is analogous to the first point. We start with the sequence (1), twist with $\mathcal{O}(-d)$, take duals, pull back to P, and apply φ_*. Since for each rational $t \in \mathrm{Gr}(2,35)$ we have $h^1(P_t, \mathcal{O}(d)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_* p^* \mathcal{O}(d)^{r_1} \xrightarrow{\beta} \varphi_* p^* \mathcal{O}(d+1)^{r_2} \longrightarrow R^1\varphi_* p^* V_k(-d)^\vee \longrightarrow 0.$$

Since the numbers

$$h_1^0 \coloneqq h^0(P_t, \mathcal{O}(d)^{r_1}) \text{ and } h_2^0 \coloneqq h^0(P_t, \mathcal{O}(d+1)^{r_2})$$

do not depend on the point t, again by Grauert's Theorem the first two terms of the sequence are locally free of rank h_1^0 and h_2^0, respectively. As before, we obtain the characterization

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-d)^\vee) = \{t : \mathrm{rk}(\beta_t) \leq h_2^0 - 1\}.$$

Here, we have $h_2^0 = (d+2)r_2 = (d+2)\binom{k-1}{3}$. $\square$

References

[Hem] Christian Hemminghaus. *Families of polarized K3 surfaces and associated bundles.*

[Vak17] Ravi Vakil. *The rising sea. Foundations of algebraic geometry.* Feb. 2017.