VERLINDE BUNDLES

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Contents

1. Universal Families of Extensions

Let X and S be Noetherian schemes over a field k. Let $f\colon X \to S$ be a flat, projective morphism, let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, flat over $\mathcal{O}_X$.

Recall that an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences, or *extensions*, of the form

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on $\mathcal{F}$ and $\mathcal{G}$. The set of these equivalence classes can be given the structure of an $H^0(S, \mathcal{O}_S)$-module, see for example [Wei95, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$-module structure.

Explicitly, the sum of two elements of Ext^1 corresponds to the Baer sum of the associated extensions, while the multiplication of an extension as above by a scalar $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

The next proposition shows there exists a k-scheme V that parametrizes the points of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$.

Proposition 1.1. *Let $V := \mathbb{V}(\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee)$. There exists an extension*

$$\xi_{\mathrm{univ}} : \quad 0 \to \mathrm{pr}_1^* \mathcal{G} \to \mathcal{E} \to \mathrm{pr}_1^* \mathcal{F} \to 0$$

over $X \times V$ such that for all Noetherian affine k-schemes Y, the map $\mathrm{Hom}(Y, V) \to \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$ defined by $\alpha \mapsto (\mathrm{id}_X \times \alpha)^* \xi_{\mathrm{univ}}$ is a bijection, functorial in Y. In particular, pulling back ξ_{univ} gives a bijection $\mathrm{Hom}(\mathrm{Spec}(k), V) \xrightarrow{\sim} \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})$.

Proof. Write $Y = \mathrm{Spec}(A)$. We aim to construct a functorial isomorphim

$$\mathrm{Hom}(Y, V) \simeq \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y).$$

Given such an isomorphism for all Y, the required universal extension is the image of $\mathrm{id} \in \mathrm{Hom}(V, V)$.

Note that there exist functorial isomorphisms

$$\mathrm{Hom}(Y, V) \simeq \mathrm{Hom}_{\text{k-alg}}(\mathrm{Sym}\,\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee, A) \simeq \mathrm{Hom}_{\text{k-mod}}(\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G})^\vee, A)$$
$$\simeq A \otimes_k \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}).$$

For the final isomorphism $A \otimes_k \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}_{X_Y}^1(\mathcal{F}_Y, \mathcal{G}_Y)$, it suffices to prove that the δ-functors[1] $A \otimes_k \mathrm{Hom}_X(\mathcal{F}, -)$ and $\mathrm{Hom}_{X_Y}(\mathcal{F}_Y, -_Y)$ are canonically isomorphic.

In fact, there exists a canonical homomorphism $A \otimes_k \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{X \times Y}(\mathcal{F}_Y, \mathcal{G}_Y)$, functorial in $\mathcal{G}$, that sends an elementar tensor $a \otimes u$ to the homomorphism $a \otimes u$. This is an isomorphism on stalks by [Bou72, Ch. I, §2.10]. There, we need the assumptions that $\mathcal{F}$ is coherent and X Noetherian. $\quad\square$

The scheme V of Proposition 1.1 is a special case of the solution to the more general moduli problem of classifying relative families of sheaves. The rest of this section sketches the more general situation. The material is taken from the article [Lan83].

Definition 1.2. 1. The *i-th relative Ext module* $\mathcal{E}xt_f^i(\mathcal{F}, \mathcal{G})$ is the image of $\mathcal{G}$ under the right-derived functor $R^i(f_* \mathcal{H}om(\mathcal{F}, \mathcal{G})) : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_S}$.

2. For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s, \mathcal{F}, \mathcal{G}} : \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s)$$

by restricting extensions of $\mathcal{F}$ by $\mathcal{G}$ to the fiber X_s. This is possible since $\mathcal{F}$ is flat over S.

3. A *family of extensions* of $\mathcal{F}$ by $\mathcal{G}$ over S is a family

$$\xi_s \in \mathrm{Ext}_{X_s}^1(\mathcal{F}_s, \mathcal{G}_s) \quad (s \in S)$$

such that there exists an open covering $\mathfrak{U}$ of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \mathrm{Ext}_{f^{-1}(U)}^1(\mathcal{F}_U, \mathcal{G}_U)$ with $\Phi_{s, \mathcal{F}_U, \mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

---

[1] Recall that $A \otimes_k -$ is exact.

Remark 1.3. If S is affine, then we have $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G})^{\sim}$.

Proposition 1.4. *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. There exists a number $N \geq 0$ dependent on $\mathcal{G}$ such that for all quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$, all $i \geq 1$ and $n \geq N$ we have*

$$\mathcal{E}xt^i_{f_Y}(\mathcal{O}_{X_Y}(-n), \mathcal{G} \boxtimes \mathcal{M}) = 0$$

Proposition 1.5. *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism*

$$\tau^i_g \colon g^* \mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^i_{f_Y}(g^*_X \mathcal{F}, g^*_X \mathcal{G}).$$

Furthermore, if g is flat, then τ^i_g is an isomorphism for all $i \geq 0$.

Definition 1.6. We say that $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ *commutes with base change* if for all morphisms of Noetherian schemes $g\colon Y \to S$, the base change homomorphism τ^i_g is an isomorphism.

Proposition 1.7. *Let $s \in S$ be a point such that τ^i_s is surjective. Then there exists an open neighborhood U of s such that $\tau^i_{s'}$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ^{i-1}_s is surjective if and only if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ is locally free on an open neighborhood of s.*

Remark 1.8. 1. If τ^i_s is an isomorphism for all $s \in S$, then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change.

2. We have directly from Proposition 1.7 that if $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free.

3. In case S is reduced, if $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ is locally free then $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 1.9. Let $u\colon Y' \to Y$ be a morphism of Noetherian S-schemes.

1. We define a functoriality map $H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \to H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \xrightarrow{1 \otimes \mathrm{id}} H^0(Y', u^* \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y))$$
$$\xrightarrow{H^0(\tau^1_u)} H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(u^*_{X_Y}\mathcal{F}_{Y'}, u^*_{X_Y}\mathcal{G}_{Y'})).$$

2. Given a family of extensions $\xi = (\xi_y)_{y \in Y}$ of $\mathcal{F}_Y$ by $\mathcal{G}_Y$ over Y, we set $(u^*\xi)_{y'} := u^*\xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^*\xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y'.

3. We define thus the functors

$$E, E' \colon \{\text{Noeth. schemes over } S\} \to \{\text{Sets}\};$$
$$E(Y) := H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)),$$
$$E'(Y) := \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}.$$

Remark 1.10. The spectral sequence $H^p(S, \mathcal{E}xt^q_f(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G})$ gives an exact sequence

$$0 \to H^1(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\varepsilon} \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G}))$$
$$\xrightarrow{d_2} H^2(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})).$$

Proposition 1.11. *Suppose that S is reduced and $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S-schemes, the functors E and E' are isomorphic.*

Proposition 1.12. *Suppose that $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the $\mathcal{O}_S$-module $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.13. *Suppose that S is reduced and $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S-schemes, the functor E' is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 1.14. *Suppose that S is affine and $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. The functor*

$$\{\text{Affine } S\text{-schemes}\} \to \{\text{Sets}\} \colon Y \mapsto \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)$$

is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.

Remark 1.15. As a special case of the above, we recover Proposition 1.1.

Remark 1.16. The article [Lan83] continues on to define a "projectivized" version of the problem, so that over $\mathrm{Spec}(k)$, the scheme $\mathbb{P}(\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of $\mathcal{F}$ by $\mathcal{G}$, modulo the action of $k^\times$. See also [HL10, Example 2.1.12].

2. Verlinde Bundles on Lefschetz Pencils

The thesis [Hem] studies Verlinde bundles for families of polarized schemes. This section further discusses the example of the universal family of quartics in $\mathbb{P}^3$, after summarizing some of its properties.

Denote by $|\mathcal{O}(4)|$ the complete linear system $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ of quartics in $\mathbb{P}^3$. Consider the universal family $\pi \colon \mathfrak{X} \to |\mathcal{O}(4)|$, given by

$$\mathfrak{X} = \{(x, q) \in \mathbb{P}^3 \times |\mathcal{O}(4)| : x \in q\}.$$

The family $\mathfrak{X}$ is a closed subscheme of $\mathbb{P}^3 \times |\mathcal{O}(4)|$. The morphism π is projective and flat.

Throughout, the coordinates of $\mathbb{P}^3$ will be denoted by x_i, $i = 0, \dots, 4$.

We define the line bundle $\mathcal{L}$ on $\mathfrak{X}$ as the restriction of $\mathcal{O}(1) \boxtimes \mathcal{O}$ to $\mathfrak{X}$, in other words as the pullback of $\mathcal{O}(1)$ under the canonical projection $\mathfrak{X} \to \mathbb{P}^3$.

Proposition 2.1. *Let $k \geq 1$. The following statements hold:*

1. If $q \in |\mathcal{O}(4)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+3}{3} - \binom{k-1}{3}$. In particular this number is independent of the point q.

2. The sheaf $\pi_* \mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+3}{3} - \binom{k-1}{3}$.

3. For all cartesian diagrams of the form

$$
\begin{array}{ccc}
\mathfrak{X}_Z & \longrightarrow & \mathfrak{X} \\
\downarrow & \times & \downarrow{\scriptstyle \pi} \\
Z & \xrightarrow{\ \rho\ } & |\mathcal{O}(4)|
\end{array}
$$

we have $\rho^* \pi_* \mathcal{L}^{\otimes k} \simeq (\pi_Z)_* \mathcal{L}_Z^{\otimes k}$.

Proof. For the first statement, see the proof of [Hem, Proposition 4.1]. The others follow from Grauert's Theorem [Vak17, 28.1.5]. □

Let $t \subseteq |\mathcal{O}(4)|$ be the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1 \to |\mathcal{O}(4)|$. We call t a *Lefschetz pencil* of quartics. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}^1}$. The situation is summarized in the picture below:

$$
\begin{array}{ccc}
\mathfrak{X}_{\mathbb{P}^1} & \longrightarrow & \mathfrak{X} \\
\downarrow & \times & \downarrow{\scriptstyle \pi} \\
\mathbb{P}^1 & \longrightarrow & |\mathcal{O}(4)|
\end{array}
$$

For $k \geq 1$, we define the k-th Verlinde bundles $V_k := \pi_* \mathcal{L}^k$ and $V_{k,t} := (\pi_{\mathbb{P}^1})_* \mathcal{L}_{\mathbb{P}^1}^k$. These bundles are related by $V_k|_t = V_{k,t}$ using Proposition 2.1.

Proposition 2.2. *There exists a short exact sequence of coherent $\mathcal{O}_{|\mathcal{O}(4)|}$-modules*

$$
0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_k \to 0.
$$

Let I_d range over the tuples of the form $(i_0, \ldots, i_3)$ with $\sum i_j = d$. The first map is then given by $\xi \otimes x^{I_{k-4}} \mapsto \sum_{I_4} \xi x^{I_4} \otimes x^{I_{k-4}+I_4}$.

Proof. See [Hem, Proposition 4.2]. □

Remark 2.3. Let t be a Lefschetz pencil of quartics.

1. The sequence from Proposition 2.2 restricts to a sequence

$$
0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^3, \mathcal{O}(k-4)) \to \mathcal{O} \otimes H^0(\mathbb{P}^3, \mathcal{O}(k)) \to V_{k,t} \to 0
$$

over $\mathbb{P}^1$.

2. The vector bundle $V_{k,t}$ has determinant $\mathcal{O}(\binom{k-1}{3})$ and rank $\binom{k+3}{3} - \binom{k-1}{3}$.

Definition 2.4. Let $k \geq 1$.

1. A *type candidate* for V_k is a collection[2] $(d_1, \ldots, d_r)$ of non-negative integers with $r = \operatorname{rk} V_k$ and $\sum d_i = \binom{k-1}{3}$.

2. The *general type candidate* for V_k is the unique[3] type candidate for V_k of the form $(d, \ldots, d, d+1, \ldots d+1)$.

3. Let t be a Lefschetz pencil of quartics. The *type* of $V_{k,t}$ is the unique type candidate (d_i) such that $V_{k,t} \simeq \bigoplus_i \mathcal{O}(d_i)$.

4. We say that $V_{k,t}$ has *general type* if its type is a general type candidate.

The rational points of $\operatorname{Gr}(2, 35)$ correspond to the Lefschetz pencils of quartics $t \subseteq |\mathcal{O}(4)|$ in the following way. Let P the universal $\mathbb{P}^1$-bundle over $\operatorname{Gr}(2, 35)$. It comes equipped with a projection map $P \to |\mathcal{O}(4)|$ such that for all Lefschetz pencils of quartics t' there exists a unique rational point $t \in \operatorname{Gr}(2, 35)$ such that the image of the fiber P_t in $|\mathcal{O}(4)|$ is t'.

$$
\begin{array}{ccccc}
P_t & \longrightarrow & P & \overset{p}{\longrightarrow} & |\mathcal{O}(4)| \\
\downarrow & \times & \downarrow{\scriptstyle \varphi} & & \\
\operatorname{Spec}(\kappa(t)) & \longrightarrow & \operatorname{Gr}(2, 35) & &
\end{array}
$$

Definition 2.5. Let $k \geq 1$ and (d_i) be a type candidate for V_k. We define the set $Z_{(d_i)}$ of all rational points $t \in \operatorname{Gr}(2, 35)$ such that $V_{k,t}$ has type (d_i). For the set of points t where $V_{k,t}$ has generic type, we also write Z_{gen}.

Proposition 2.6. *The set Z_{gen} is Zariski open. Its complement is the union*

$$
\operatorname{Supp}(R^1 \varphi_* p^* V_k(-d-1)) \cup \operatorname{Supp}(R^1 \varphi_* p^* V_k(-d)^\vee),
$$

where d is the smaller of the two numbers appearing in the general type candidate $(d, \ldots, d, d+1, \ldots, d+1)$ *for V_k.*

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \operatorname{Gr}(2, 35)$ be a rational point, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(d_i)$. We have $t \in Z_{\mathrm{gen}}$ if and only if $d \leq d_i$ and $d_i \leq d+1$ for all i, which holds if and only if the equalities $H^1(P_t, V_{k,t}(-d-1)) = 0$ and $H^1(P_t, V_{k,t}(-d)^\vee) = 0$ both hold.

Next, we want to use the Cohomology and Base Change Theorem [Vak17, 28.1.6] on the map $\varphi\colon P \to \operatorname{Gr}(2, 25)$, which is a $\mathbb{P}^1$-bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\operatorname{Gr}(2, 35)$.

---

[2] i.e. an equivalence class of tuples up to reordering of the entries
[3] For fixed $r, s > 0$, the equation $dr + b = s$ has only one non-negative solution (d, b) with $b < s$.

For all rational $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, p^*V_{k,t}(-d-1)) = 0$ and $h^2(P_t, p^*V_{k,t}(-d)^\vee) = 0$. Since the sheaves $p^*V_{k,t}(-d-1)$ and $p^*V_{k,t}(-d)^\vee$ are locally free and coherent, the Cohomology and Base Change Theorem applies and we have

$$(R^1\varphi_*p^*V_k(-d-1))_t = H^1(P_t, V_{k,t}(-d-1))$$

and

$$(R^1\varphi_*p^*V_k(-d)^\vee)_t = H^1(P_t, V_{k,t}(-d)^\vee).$$

By the previous characterization, we have

$$\mathrm{Gr}(2,35) \setminus Z_{\mathrm{gen}} = \mathrm{Supp}(R^1\varphi_*p^*V_k(-d-1)) \cup \mathrm{Supp}(R^1\varphi_*p^*V_k(-d)^\vee),$$

which is a Zariski-closed set. $\qquad\square$

Proposition 2.7. *The closed subsets*

1. $\mathrm{Supp}(R^1\varphi_*p^*V_k(-d-1))$ *and*

2. $\mathrm{Supp}(R^1\varphi_*p^*V_k(-d)^\vee)$

are determinantal varieties.

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k))$ and $r_2 := \dim H^0(\mathbb{P}^3, \mathcal{O}(k-4))$, and rewrite the exact sequence from Proposition 2.2 as

$$0 \to \mathcal{O}(-1)^{r_2} \to \mathcal{O}^{r_1} \to V_k \to 0. \qquad (\star)$$

1. Twisting the sequence $\star$ with $\mathcal{O}(-d-1)$ and pulling back to P gives an exact sequence

$$0 \to p^*\mathcal{O}(-d-2)^{r_2} \to p^*\mathcal{O}(-d-1)^{r_1} \to p^*V_k(-d-1) \to 0.$$

For every rational $t \in \mathrm{Gr}(2,35)$ we have $h^2(P_t, \mathcal{O}(-d-2)^{r_2}) = 0$, hence $R^2\varphi_*p^*\mathcal{O}(-d-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_*p^*\mathcal{O}(-d-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_*p^*\mathcal{O}(-d-1)^{r_1} \to R^1\varphi_*p^*V_k(-d-1) \to 0.$$

Note that since the numbers $h_2^1 := h^1(P_t, \mathcal{O}(-d-2)^{r_2})$ and $h_1^1 := h^1(P_t, \mathcal{O}(-d-1)^{r_1})$ do not depend on the point t, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h_1^2 and h_1^1, respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_*p^*V_k(-d-1))_t \neq 0$ if and only if $\mathrm{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\mathrm{Supp}(R^1\varphi_*p^*V_k(-d-1)) = \{t : \mathrm{rk}(\alpha_t) \leq h_1^1 - 1\}.$$

As a final remark, note that $h_1^1 = dr_1 = d\binom{k+3}{3}$.

2. The proof for this point is analogous to the one for the first point. We start with the sequence $\star$, twist with $\mathcal{O}(-d)$, take duals, pull back to P, and apply φ_*. Since for each rational $t \in \mathrm{Gr}(2,35)$ we have $h^1(P_t, \mathcal{O}(d)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_*p^*\mathcal{O}(d)^{r_1} \xrightarrow{\beta} \varphi_*p^*\mathcal{O}(d+1)^{r_2} \to R^1\varphi_*p^*V_k(-d)^\vee \to 0.$$

Since the numbers $h_1^0 := h^0(P_t, \mathcal{O}(d)^{r_1})$ and $h_2^0 := h^0(P_t, \mathcal{O}(d+1)^{r_2})$ do not depend on the point t, again by Grauert's Theorem the first two terms of the sequence are locally free of rank h_1^0 and h_2^0, respectively. As before, we obtain the characterization

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-d)^\vee) = \{t : \mathrm{rk}(\beta_t) \leq h_2^0 - 1\}.$$

Here, we have $h_2^0 = (d+2)r_2 = (d+2)\binom{k-1}{3}$. $\qquad\square$

Proposition 2.8. *Of the four type candidates*

$$(0,\ldots,0,1,1,1,1),\ (0,\ldots,0,1,1,2),\ (0,\ldots,0,2,2),\ (0,\ldots,0,1,3),\ (0,\ldots,0,4)$$

for V_5, only the first two occur as types of some $V_{k,t}$.

Proof. [Lange Rechnung, wird noch hinzugefügt.] $\qquad\square$

3. Specialization

In this section we collect some facts about specialization phenomena. We say that a vector bundle $\mathcal{V}$ on a k-scheme X *specializes* to another vector bundle $\mathcal{V}'$ over the same scheme if there exists a vector bundle $\mathcal{W}$ on $\mathbb{A}^1 \times X$ such that $\mathcal{W}|_{0 \times X} \simeq \mathcal{V}'$ and $\mathcal{W}|_{t \times X} \simeq \mathcal{V}$ for all rational $t \in \mathbb{A}^1 \setminus 0$.

Remark 3.1. Let

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \to 0$$

be a short exact sequence of coherent sheaves over a k-scheme X and let $\xi \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \to 0.$$

Example 3.2. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$ specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. The elements of $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O})$ correspond to extensions of the form

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(2) \to 0$$

up to equivalence. The zero element corresponds to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have $\mathcal{E}$ locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k$. By Proposition 1.1, there exists an extension of the form

$$0 \to \mathcal{O} \boxtimes \mathcal{O}_{\mathbb{A}^1} \to \mathcal{E}_{\mathrm{univ}} \to \mathcal{O}(2) \boxtimes \mathcal{O}_{\mathbb{A}^1} \to 0$$

such that for all nonzero rational points $\xi \in \mathbb{A}^1 = \mathbb{V}(\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee)$ we have the isomorphisms $\mathcal{E}_{\mathrm{univ}}|_{\xi \times \mathbb{A}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and $\mathcal{E}_{\mathrm{univ}}|_{0 \times \mathbb{A}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Note that $\mathcal{E}_{\mathrm{univ}}$ is locally free as the end terms of the sequence are.

Remark 3.3. If $\mathcal{V}$ specializes to $\mathcal{V}'$ and $\mathcal{W}$ specializes to $\mathcal{W}'$, then $\mathcal{V} \oplus \mathcal{W}$ specializes to $\mathcal{V}' \oplus \mathcal{W}'$.

Remark 3.4. Let $b_1, \ldots, b_m$ be non-negative numbers, let $a := \sum b_i$. The sequence

$$0 \to \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \to 0$$

with

$$f = \begin{pmatrix} s^{b_1} & & & & \\ t^{b_2} & s^{b_2} & & & \\ & t^{b_3} & \ddots & & \\ & & \ddots & s^{b_{m-1}} & \\ & & & t^{b_m} & \end{pmatrix}$$

and

$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1}t^{a-b_1-b_2} & \cdots & (-1)^m s^{b_1+\cdots+b_{m-1}}t^{a-b_1-\cdots-b_m} \end{pmatrix}$$

is exact.

Proposition 3.5. *Let $b_1, \ldots, b_m$ be non-negative numbers and π a partition of the set $\{1, \ldots, m\}$. For a set of indices $I \in \pi$, let $b'_I := \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m - |\pi|}$.*

Proof. By Remark 3.3 it suffices to prove the special case $\pi = \{\{1, \ldots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(n) \oplus \mathcal{O}^{m-1}$. By Remark 3.4, there exists a representative $\xi \in \mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \to \mathcal{O}^{m-1} \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \to \mathcal{O}(a) \to 0. \tag{$\star$}$$

By Remark 3.1, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the form $\star$. Consider the associated closed embedding $\alpha \colon \mathbb{A}^1 \to \mathbb{V}(\mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let $\mathcal{E}$ be the universal extension from Proposition 1.1. Then, the vector bundle $(\mathrm{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. $\square$

Remark 3.6. By twisting the exact sequence $\star$ in the proof of Proposition 3.5 and using the same argument, we see that for every integer n and with b_i, π, and b_I as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m - |\pi|}$.

References

[Bou72] Nicolas Bourbaki. *Commutative algebra.* Vol. 8. Hermann Paris, 1972.
[Hem]   Christian Hemminghaus. *Families of polarized K3 surfaces and associated bundles.* Master thesis.

[HL10]   Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[Lan83]   Herbert Lange. "Universal families of extensions". In: *Journal of Algebra* 83.1 (1983), pp. 101–112.

[Vak17]   Ravi Vakil. *The rising sea. Foundations of algebraic geometry*. Feb. 2017.

[Wei95]   Charles A Weibel. *An introduction to homological algebra*. 38. Cambridge university press, 1995.