Test 4 Unitary Transformations & Multi-qubits system

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1. Define Hadamard gate H:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Show H is unitary.

Ans: Consider

$$\begin{aligned} H^T H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}^2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I \end{aligned}$$

So $H^T H = H H^T = I$

Therefore H be unitary.

2. What is a joint 3-qubit state of $|0\rangle \otimes |01\rangle$? Is it entangled?
Definition: An entangled state of a composite system is a state that cannot be written as a product state of the component systems.

Consider

$$|0\rangle \otimes |01\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$
$$= \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}$$
$$= |001\rangle$$

$|001\rangle$ is not entangled. Because $|001\rangle$ can be written as a product state of

$$(1|0\rangle + 0|1\rangle)(1|0\rangle + 0|1\rangle)(0|0\rangle + 1|1\rangle)$$

3. What is the EPR pairs? Is it an entangled state?
Ans:
EPR pairs (or Bell states) are specific quantum states of two qubits that represent the simplest examples of quantum entanglement.

For the four basic two-qubit inputs, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the circuit outputs a final Bell state in accordance with the equation

$$|\beta(x,y)\rangle = \frac{|0,y\rangle + (-1)^x |1,Y\rangle}{\sqrt{2}}$$

where Y is the negation of y
Let

$$|\Psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$$
$$|\Psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$$

Consider

$$|\beta(x,y)\rangle = |\Psi_1\Psi_2\rangle$$
$$= \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

case $(x, y) = (0, 0)$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

$\alpha_1\alpha_2 = \beta_1\beta_2 = \frac{1}{\sqrt{2}} \Longrightarrow \alpha_1\beta_2 \neq 0 \quad \text{and} \quad \beta_1\alpha_2 \neq 0$ Contradiction!

case $(x, y) = (0, 1)$

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}} = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

$\alpha_1\beta_2 = \beta_1\alpha_2 = \frac{1}{\sqrt{2}} \Longrightarrow \alpha_1\alpha_2 \neq 0 \quad \text{and} \quad \beta_1\beta_2 \neq 0$ Contradiction!

case $(x, y) = (1, 0)$

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad \beta_1\beta_2 = -\frac{1}{\sqrt{2}} \Longrightarrow \alpha_1\beta_2 \neq 0 \quad \text{and} \quad \beta_1\alpha_2 \neq 0$
Contradiction!

case $(x, y) = (1, 1)$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

$\alpha_1\beta_2 = \frac{1}{\sqrt{2}} \quad \text{and} \quad \beta_1\alpha_2 = -\frac{1}{\sqrt{2}} \Longrightarrow \alpha_1\alpha_2 \neq 0 \quad \text{and} \quad \beta_1\beta_2 \neq 0$
Contradiction!

So, $|\beta(x, y)\rangle$ cannot be written as a product state of the component systems. Therefore $|\beta(x, y)\rangle$ is entangled.