1 The Real Number System

1.1 Definitions and Theorems

We firstly introduce all the definitions and theorems from the body of the chapter that will be required to prove the exercises.

Definition 1.1. *An order on a set S is a relation ("$<$") that satisfies the following properties:*

1. *Completeness: For all $x, y \in S$, $x < y$, $y < x$, or $x = y$*
2. *Transitivity: For all $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$*

A set S with a defined order is an "ordered set".

Definition 1.2. *Suppose we have a set $E \subset S$ where S is an ordered set. An element $\beta \in S$ is an* ***upper bound*** *if $\beta > x : \forall x \in E$. A set that has an upper bound is "bounded above". Let the set of all upper bounds of E be $U(E)$.*

Furthermore, β is a ***least upper bound*** *for E if for all $\gamma \in U(E)$:*

1. *β is an upper bound ($\beta \in U(E)$)*
2. *β is the smallest upper bound ($\beta \leq \gamma$)*

In this case β is also called the supremum of E ($\sup(E)$).

Definition 1.3. *A set S has the least upper bound property if for each $E \subset S$, if:*

1. *E is nonempty*
2. *E is bounded above*

Then $\sup(E) \in S$. The real numbers $\mathbb{R}$ have the least upper bound property by construction.

Definition 1.4. *A field $\mathbb{F}$ is a set with two defined operations, addition and multiplication, which satisfy the 11 field axioms (5 addition axioms, 5 multiplication axioms, 1 distributive axiom).*

An ordered field is a field $\mathbb{F}$ with a defined order, so that the following hold:

1. *$x + y < x + z$ if $x, y, z \in \mathbb{F}$ and $x < y$*
2. *$xy > 0$ if $x, y \in \mathbb{F}$ and $x, y > 0$*

Theorem 1.5. *(a) Archimedean principle: if $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$*

(b) Density of $\mathbb{Q} \subset \mathbb{R}$: if $x, y \in \mathbb{R}$ and $x < y$, there exists some rational q such that $x < q < y$

Proof. Archimedean principle:

- Let $A = \{nx : n \in \mathbb{N}\}$.
- Suppose the principle does not hold. Then $y > nx\ \forall n$.
- By Definition 1.2, y is an upper bound of A, and A is therefore bounded.
- A is a non-empty, bounded subset of $\mathbb{R}$. Therefore, by Definition 1.3, A possesses a least upper bound. Call this $\beta = \sup(A)$.
- Consider $\gamma = \beta - x < \beta$. By definition γ is not an upper bound of A, which means that there exists some $mx \in A$ such that $mx > \gamma$.
- Now by Definition 1.4 $mx + x > \gamma + x \Rightarrow (m+1)x > \beta$.
- However this is a contradiction, as β is no longer an upper bound

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Proof. Density of $\mathbb{Q} \subset \mathbb{R}$:

- **Lemma 1**: For any $x, y \in \mathbb{R}$ where $x < y$, there exists $n \in \mathbb{N}$ s.t. $x + 1/n < y$.
- Proof of Lemma 1:
  - $y - x > 0$
  - $n(y - x) > 1$ for some $n \in \mathbb{N}$ (by Archimedean principle)
  - $y - x > 1/n$
  - $y > x + 1/n$
- Therefore if either $x, y \in \mathbb{Q}$ then by Lemma 1 there is a rational $(x + 1/n)$ or $(y - 1/n)$ in between them, and the result follows. So assume neither are rational.
- **Lemma 2**: All real numbers $x \in \mathbb{R}$ are the supremum for the set of rationals smaller than them.
- Proof of Lemma 2:
  - Denote A as the set of all rationals smaller than x. This set is bounded above and non-empty[1], so there is a least upper bound β.
  - If $\beta > x$ then there is a rational between x and β, which is a contradiction.
  - If $\beta < x$, then by Lemma 1, $\beta + 1/m < x$.
  - Then consider also $(\beta - 1/m)$. Since this is smaller than the least upper bound β, that means there is some $a \in A$ such that $a > \beta - 1/m$.
  - Therefore $a + 1/m > \beta$

[1] Non-emptiness also follows trivially from Archimedean principle

- But note also that $a < \beta$ so $a + 1/m < \beta + 1/m < x$
  - Therefore $x > a + 1/m > \beta$
  - However $a + 1/m$ is a rational, therefore violating the fact that β is a least upper bound. Therefore $\beta = x$.

- Therefore, x and y are the supremum for the set of rationals smaller than them respectively. It follows that there must be a rational smaller than y and greater than x. If there was not, then y would not be the supremum of all smaller rationals, x would be.

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Theorem 1.6. *For every $x > 0 \in \mathbb{R}$ and every integer $n > 0$, there is a unique positive real y such that $y^n = x$. This number is written $y = x^{\frac{1}{n}}$.*

Proof.

- Let E contain all positive numbers t such that $t^n < x$
- Clearly, E is bounded above (for example, by $(x+1)$, since $(x+1)^n > x$)
- E is also non-empty (If $x \geq 1$, then choose any $t < 1$. For $x < 1$, note that by the Archimedean principle $\frac{1}{m} < x$, and as such $(\frac{1}{m})^n < x$).
- Therefore by Definition 1.3, E has a least upper bound y. We wish to show $y^n = x$.
- **Lemma 3**: For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, $(a+b)^n = a^n + br$ for some $r \in \mathbb{R}$.
- Proof of Lemma 3:
  - Suppose Lemma 3 holds for $(n-1)$. Then $(a+b)^{n-1} = a^{n-1} + br$.
  - Then $(a+b)(a+b)^{n-1} = a^n + bar + ba^{n-1} + b^2r = a^n + bs$ where $s \in \mathbb{R}$.
  - Lemma 3 holds for $n = 1$. As such, the result follows by induction.
- Now suppose $y^n < x$. Then by Lemma 1, there exists some m such that $y^n + 1/m < x$.
- Now we can choose some $(y + 1/p)^n = y^n + \frac{1}{p}r$.
- By the Archimedean principle, there exists some $p \in \mathbb{N}$ so that $p\frac{1}{m} > r$. As such $\frac{r}{p} < \frac{1}{m}$ and $(y + 1/p)^n < y^n + 1/m < x$. As such $y^n < x$ cannot be the supremum of E.
- By an analogous process replace $1/p$ with a negative p to show that $y^n > x$ cannot be a supremum. As such, $y^n = x$.

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Corollary 1.6.1. *If $a, b > 0 \in \mathbb{R}$ and $n \in \mathbb{N}$, then $(ab)^{1/n} = a^{1/n}b^{1/n}$.*

Proof.

- Let $\alpha = a^{1/n}$ and $\beta = b^{1/n}$.
- Then by Theorem 1.6, $\alpha^n = a$ and $\beta^n = b$.
- Then $(ab) = \alpha^n\beta^n = (\alpha\beta)^n$ (the latter by the commutative property of the real field).

- Therefore by 1.6 again, $(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}$.

$\blacksquare$

Remarks:

- It's interesting how useful the Archimedean property is; especially once one recognizes that any positive real expression x and any real y can be written as $nx > y$. The introduction of indexing by natural numbers adds a lot of ease.
- Note Lemma 1 - it keeps coming up. Worth bearing in mind.
- The set of upper bounds of a set $U(E)$ and lower bounds $L(E)$ are useful constructs worth revisiting.
- When proving whether two things are equal, always try to define them as x and y and show x and y must be equal (usually by showing they cannot be greater or less than each other).

1.2 Exercises

Exercise 1.1. *If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.*

Proof.

- First note $(r + x) = \frac{m}{n} + x = \frac{m+n}{x}$.
- Suppose $(r + x)$ is rational. Then $\frac{m+n}{x} = \frac{p}{q}$.
- Then $x = \frac{q(m+n)}{p}$, which makes x rational. [Contradiction].
- Now note $(rx) = \frac{mx}{n}$.
- Suppose (rx) is rational. Then $\frac{mx}{n} = \frac{p}{q}$.
- Then $x = \frac{np}{mq}$, which makes x rational. [Contradiction].

$\blacksquare$

Exercise 1.2. *Prove there is no rational number whose square is 12.*

Proof.

- First we will show that $\sqrt{3}$ is irrational. The results follow quickly from there.
- Suppose by contradiction $\sqrt{3} = \frac{p}{q}$. Then $p^2 = 3q^2$.
- Then $p^2 - q^2 = 2q^2$, and $\Rightarrow (p + q)(p - q) = 2q^2$.
- Then product $(p + q)(p - q)$ must be even, which only arises when p, q are both odd or both even
- Since they cannot be both even, it follows they are both odd. However this means that $(p + q)(p - q) = 2k.2l = 4m = 2q^2$.

- Therefore $q^2 = 2m$, which means q is even and results in a contradiction.
- To finalise note that if $(\frac{p}{q})^2 = 12$, then by rearranging terms one obtains $(\frac{p}{2q})^2 = 3$.

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Exercise 1.3. *Prove Proposition 1.15.*

Proof. TBC ■

Exercise 1.4. *Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.*

Proof.

- Choose some $e \in E$. By Definition 1.2, $\beta \geq e$ and $e \geq \alpha$.
- By the transitive property of ordered relations (Definition 1.1), $\beta \geq \alpha$.

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Exercise 1.5. *Let A be a nonempty subset of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that* $\inf(A) = -\sup(-A)$.

Proof.

- Let $\beta = \inf(A)$.
- For any $-x \in -A$, it follows there exists $x \in A$.
- By definition, $\beta \leq x$.
- By re-arranging, one obtains $-x \leq -\beta$.
- Therefore, for an element in $a \in -A$, $-\beta \geq a$. $-\beta = \sup(-A)$, and $-\sup(-A) = \beta = \inf(A)$.

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Exercise 1.6. *Fix $b > 1$.*

Exercise 1.6.1. *If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$.*

Proof.

- Let $(b^m)^{1/n} = x$ and $(b^p)^{1/q} = y$.
- Rearranging both using Theorem 1.6 yields $b^m = x^n$ and $b^p = y^q$.
- Taking powers results in $b^{mq} = x^{nq}$ and $b^{np} = y^{nq}$.
- Note that $m/n = p/q \Rightarrow mq = np$. Therefore $b^{mq} = b^{np}$.
- Therefore, $x^{nq} = y^{nq}$ and $x = y$. As a result $(b^m)^{1/n} = (b^p)^{1/q}$.

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Exercise 1.6.2. *(Cont). Hence it makes sense to define $b^{m/n} = (b^m)^{1/n}$. Prove that $b^{r+s} = b^r b^s$ if r and s are rational.*

Proof.

- Let $b^{r+s} = x$. It follows $b^{r+s} = b^{m/n+p/q} = b^{(mq+np)/nq} = x$.

- Therefore from Theorem 1.6, $b^{mq+np} = x^{nq}$.
- Let $b^r b^s = y$. It follows $b^r b^s = b^{m/n} b^{p/q} = y$.
- Taking both sides by nq and using Ex 1.6.1 to simplify results in $b^{mq}.b^{np} = y^{nq} = b^{mq+np}$.
- Therefore $x^{nq} = y^{nq}$ and $x = y$. The result $b^{r+s} = b^r b^s$ follows.

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Exercise 1.6.3. *(Cont). If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that $b^r = \sup(B(r))$ when r is rational.*

Proof.

- Let $\beta = \sup(B(r))$. Suppose $b^r > \beta$. Then β cannot be an upper bound as $r \in B(r)$.
- Suppose $b^r < \beta$. Then for any $t \in B(r)$, $b^t \leq b^r$. Therefore b^r is an upper bound for $B(r)$. But if $b^r < \beta$ then β cannot be least upper bound.

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Exercise 1.6.4. *(Cont). Hence it makes sense to define $b^x = \sup(B(x))$ for every real x. Prove that $b^{x+y} = b^x b^y$ for all real x and y.*

Proof.

- **Lemma 4:** The set $C = \{b^s : s \in \mathbb{Q}\}$ is dense in $\mathbb{R}$ for $b > 1$.
- Proof of Lemma 4:
  - Suppose the Lemma is not true. Let there be some $x, y \in \mathbb{R}$ such that $x < y$ and there is no b^s such that $x < b^s < y$.
  - Let $B(x) = \{b^t : b^t \leq x \ \wedge t \in \mathbb{Q}\}$ and $A(y) = \{b^t : b^t \geq y \ \wedge t \in \mathbb{Q}\}$. Let $\beta = \sup(B(x))$ and $\alpha = \inf(A(y))$.
  - Now choose some $n \in \mathbb{N}$ such that $(\frac{\alpha}{\beta})^n > b$. (This n is guaranteed to exist as shown in the footnote[2]). Re-arranging, one obtains $b^{1/n}\beta < \alpha$.
  - Now consider $\beta/b^{1/n}$. Since $\beta/b^{1/n} < \beta$ it is not an upper bound and as such there is some $b^s \in B(x)$ such that $(\beta/b^{1/n}) < b^s < \beta$. Rearranging, one obtains $b^{s+1/n} > \beta$.
  - Since $b^s < \beta$, then $b^{1/n} b^s < b^{1/n}\beta < \alpha$.
  - Therefore, $\beta < b^{s+1/n} < \alpha$. Since s is rational so is $(s + 1/n)$ so the result follows from contradiction.
- **Lemma 5:** For any $s < (x + y) \wedge s \in \mathbb{Q}$, $b^s < b^x b^y$.
- Proof of Lemma 5:

[2] Let A be the set of x^n for all $n \in \mathbb{N}$. Suppose $x^n \ngeq b$ for some n. Then A is bounded above and non-empty, so possesses a supremum β. Now consider $\beta/x < \beta$. Therefore β/x is not an upper bound and there is some $x^n > \beta/x$. Rearranging, one obtains $x^{n+1} > \beta$. Therefore β is not an upper bound of A, and cannot be its supremum. ■

- If $s < (x+y), s = x + y - \epsilon_0$ for some $\epsilon_0 \in \mathbb{R}$
- Choose rational s_1 such that $x > s_1 > x - 1/m$ for any arbitrarily large m (the existence of s_1 follows from the density of the rationals).
- Define $\epsilon_1 = x - s_1$. Note from the previous point ϵ_1 can be made arbitrarily small.
- Now define $s_2 = s - s_1$ (so $s = s_1 + s_2$). Note that s_2 is rational.
- Substitutions result in $s_2 + (\epsilon_0 - \epsilon_1) = y$. Therefore if $\epsilon_1 < \epsilon_0$ it follows that $s_2 < y$. From previously we noted that ϵ_1 can be made arbitrarily small. So we can choose $s_1 < x$ such that $s_2 < y$.
- If $x > s_1$ and $y > s_2$, then it folows[3] that $b^x > b^{s_1}$ and $b^y > b^{s_2} \Rightarrow b^x b^y > b^{s_1} b^{s_2} = b^{s_1+s_2} = b^s$.

- **Corollary 5:** For any $s > (x+y) \wedge s \in \mathbb{Q}$, $b^s > b^x b^y$. This follows by reversing the previous argument.

- Now suppose $b^x b^y < b^{x+y}$. By Lemma 4, there exists some $b^x b^y < b^s < b^{x+y}$. If $s < (x+y)$, then by Lemma 5, $b^s < b^x b^y$, which is a contradiction. If $s > (x+y)$ then $b^s > b^{x+y}$, which is a contradiction.

- Now suppose $b^x b^y > b^{x+y}$. By Lemma 4, there exists some $b^x b^y > b^s > b^{x+y}$. If $s > (x+y)$, then by Lemma 5, $b^s > b^x b^y$, which is a contradiction. If $s < (x+y)$ then $b^s < b^{x+y}$, which is a contradiction.

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Exercise 1.7. *Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the logarithm of y to the base b).*

Exercise 1.7.1. *For any positive integer n, $b^n - 1 \geq n(b-1)$.*

Proof. ■

[3]To be rigorous, we show that for $x, y \in \mathbb{R}$, if $x > y$ then $b^x > b^y$. By the density of the rationals, there is some $y < p < x$ and therefore $b^p \in B(x) \notin B(y)$. If $b^p \leq b^y$ then $b^p \in B(y)$. So $b^p > b^y$. Furthermore $b^x \geq b^p$ by definition. Therefore $b^x > b^y$. ■