

## PC 2 — Probability distributions

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### Exercise 1

(Uniform distribution). Let  $X$  be a random variable with uniform distribution on  $[0, 1]$ . We define  $Y = \min(X, 1 - X)$  and  $Z = \max(X, 1 - X)$ . Determine the distributions of  $Y$  and  $Z$ . Compute  $\mathbb{E}[YZ]$ .

**Solution 1** *La variable aléatoire  $Y$  prend ses valeurs dans  $[1/2, 1]$  et pour tout  $t \in [1/2, 1]$ ,*

$$F_Y(t) = \mathbb{P}(U \leq t, 1 - U \leq t) = \mathbb{P}(U \leq t, U \geq 1 - t) = t - (1 - t) = 2t - 1$$

*donc  $Y$  suit la loi uniforme sur  $[1/2, 1]$ . On remarque que  $X = 1 - Y$  et on en déduit que  $X$  suit la loi uniforme sur  $[0, 1/2]$ . Pour calculer  $\mathbb{E}[XY]$ , on remarque que  $XY = U(1 - U)$  et donc*

$$\mathbb{E}[XY] = \mathbb{E}[U(1 - U)] = \int_0^1 (t - t^2) dt = [t^2/2 - t^3/3]_0^1 = 1/2 - 1/3 = 1/6.$$

### Exercise 2

One says that  $X \in (0, +\infty)$  follows the log-normal distribution if  $\log(X) \sim \mathcal{N}(0, 1)$ . What is the density of  $X$ ?

**Solution 2** *The density of  $\log(X)$  is a log normal distribution:*

$$f_{\log(X)}(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log(x))^2}{2}\right)$$

### Exercise 3

Consider a random variable  $X$  having exponential distribution with parameter 1. Let  $a > 0$  be a positive real number.

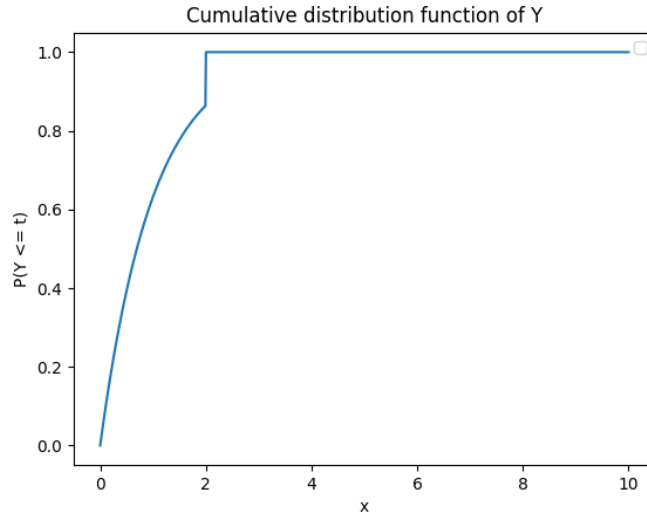
1. Compute the cumulative distribution function of  $Y = \min(X, a)$ . Plot the function.
2. What can you say about the existence of a density for the distribution of  $Y$ ?
3. Compute  $\mathbb{E}[Y]$ . Hint: Use  $Y = X\mathbb{1}_{X \leq a} + a\mathbb{1}_{X > a}$ .

**Solution 3** 1. *We have,  $\forall t \in \mathbb{R}^+$ :*

$$\begin{aligned} \mathbb{P}[Y \geq t] &= \mathbb{P}[\min(X, a) \geq t] \\ &= \mathbb{P}[(X \geq t) \cap (a \geq t)] \\ &= \mathbb{1}_{a \geq t} \mathbb{P}[X \geq t] \end{aligned}$$

*Therefore,*

$$\mathbb{P}[Y \leq t] = 1 - \mathbb{1}_{a \geq t} \mathbb{P}[X \geq t]$$



2. As the cumulative distribution function is not continuous, the density could not exist.
3. We have :

$$\begin{aligned}
 \mathbb{E}[Y] &= \mathbb{E}[X\mathbb{1}_{X \leq a}] + a\mathbb{E}[\mathbb{1}_{X > a}] \\
 &= \int_0^a te^{-t}dt + a\mathbb{P}[X > a] \\
 &= 1 - (a+1)e^{-a} + ae^{-a} \\
 &= 1 - e^{-a}
 \end{aligned}$$

#### Exercise 4

Let  $V$  be a random variable with uniform distribution on  $[0, \pi/2]$ . Define the random variable  $W = \sin(V)$ .

1. Determine the distributions of  $W$ .
2. How does the distribution of  $W$  change when  $V$  has uniform distribution on  $[0, \pi]$ ?

**Solution 4** 1. We have:

$$\begin{aligned}
 \mathbb{P}[W \leq t] &= \mathbb{P}[\sin(V) \leq t] \\
 &= \mathbb{P}[V \leq \arcsin(t)] \\
 &= \frac{2 \arcsin(t)}{\pi}
 \end{aligned}$$

2. Let  $0 \leq t \leq 1$  and  $\theta \in [0, \frac{\pi}{2}]$  such that  $\sin(\theta) = t$ . If  $V$  is uniform on  $[0, \pi]$ :

$$\begin{aligned}
 \mathbb{P}[W \leq t] &= \mathbb{P}[\sin(V) \leq t] \\
 &= \mathbb{P}[V \leq \theta] + \mathbb{P}[V \geq \pi - \theta] \\
 &= \frac{\theta + \pi - (\pi - \theta)}{\pi} \\
 &= \frac{2\theta}{\pi}
 \end{aligned}$$

Therefore, the distribution of  $\sin(V)$  has not changed.

## Exercise 5

(Cauchy distribution). Let  $X$  be a random variable with Cauchy distribution whose density is given by  $f(x) = (\pi(1+x^2))^{-1}$ . Determine the distribution of  $1/X$  using a change of variables.

**Solution 5** Soit  $f : \mathbb{R} \rightarrow \mathbb{R}$  continue bornée. On a

$$\mathbb{E}[f(\frac{1}{X})] = \int_{\mathbb{R}} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx.$$

On a envie de faire le changement  $u = 1/x$  (mais pas bijectif sur  $\mathbb{R}$ ) on scinde en deux

$$\mathbb{E}[f(\frac{1}{X})] = \int_0^{+\infty} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx + \int_{-\infty}^0 f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx.$$

On pose la variable  $u = 1/x$  donc  $du = -u^2 dx$  ainsi

$$\begin{aligned} \int_0^{+\infty} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx &= \int_0^{+\infty} f(u) \frac{1}{u^2} \frac{1}{\pi(1+u^{-2})} du \\ &= \int_0^{+\infty} f(u) \frac{1}{\pi(1+u^2)} du. \end{aligned}$$

De plus en faisant  $u = 1/x$  dans l'intégrale sur  $\mathbb{R}^-$  on a de même

$$\int_{-\infty}^0 f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx = \int_{-\infty}^0 f(u) \frac{1}{\pi(1+u^2)} du.$$

Donc  $\frac{1}{X}$  a même loi que  $X$ .

## Exercise 6

Let  $p > 0$  and an integer  $n$  such that  $n > p$ . Consider random variables  $Y_n$  such that  $nY_n$  has a geometric distribution  $\text{Geo}(\frac{p}{n})$  with parameter  $\frac{p}{n}$ . Show that the characteristic function of  $Y_n$  tends to the characteristic function of an exponentially distributed random variable with parameter  $p$ .

**Solution 6** Let  $p_n = \frac{p}{n}$ . We have:

$$\begin{aligned} \phi_{Y_n}(t) &= \mathbb{E}[e^{itY_n}] \\ &= \mathbb{E}[e^{i\frac{t}{n}nY_n}] \\ &= \sum_{k=1}^{\infty} p_n e^{i\frac{t}{n}k} (1-p_n)^{k-1} \\ &= \frac{p_n}{1 - (1-p_n)e^{i\frac{t}{n}}} \end{aligned}$$

But:

$$\begin{aligned} (1-p_n)e^{i\frac{t}{n}} &\sim (1-p_n)(1+i\frac{t}{n}) \\ &= 1-p_n + i\frac{t}{n} - p_n i\frac{t}{n} \end{aligned}$$

Therefore:

$$1 - (1-p_n)e^{i\frac{t}{n}} = p_n - i\frac{t}{n} + p_n i\frac{t}{n}$$

And finally

$$\frac{p_n}{1 - (1 - p_n)e^{i\frac{t}{n}}} \rightarrow \frac{p}{p - it}$$

And if  $X \sim \text{Exp}(p)$ :

$$\begin{aligned}\phi_X(t) &= \mathbb{E}[e^{itX}] \\ &= \sum_{k=0}^{\infty} p e^{itk} e^{-p} \frac{p^k}{k!} \\ &= \frac{p}{p - it}\end{aligned}$$

which conclude the exercise.

## Exercise 7

Let  $\alpha > 1$  be fixed. Consider the random variable  $X$  with density given by

$$f(x) = c_\alpha x^{-\alpha} \mathbb{1}_{x \geq 1}$$

1. Determine the constant  $c_\alpha$ .
2. For which values of  $p$  we have  $X$  belongs to  $L^p$ ?

**Solution 7** 1. Necessarily,  $\int_{\mathbb{R}} f(x) = 1$ . So:

$$\begin{aligned}\int_{\mathbb{R}} c_\alpha x^{-\alpha} \mathbb{1}_{x \geq 1} dx = 1 &\iff c_\alpha \int_1^\infty x^{-\alpha} dx = 1 \iff c_\alpha = \frac{-1}{\frac{1}{1-\alpha}} \\ &\iff c_\alpha = \alpha - 1\end{aligned}$$

2. We have:

$$\begin{aligned}X \in L^p &\iff \mathbb{E}[X^p] < \infty \\ &\iff \int_{\mathbb{R}} x^p x^{-\alpha} \mathbb{1}_{x \geq 1} dx < \infty &\iff \int_1^\infty x^{p-\alpha} dx < \infty \\ &\iff p - \alpha < -1\end{aligned}$$

To understand the last step, see <https://boilleys.ovh/cours/integrale-generalisee.html>. And finally the necessary and sufficient condition is  $p - \alpha < -1$ .

## Exercise 8

Let  $X$  and  $Y$  be two independent random variables such that  $X$  (resp.  $Y$ ) has geometric distribution with parameter  $p$  (resp.  $q$ ).

1. Compute  $\mathbb{P}(X > n)$  for any  $n \in \mathbb{N}$ .
2. What is the distribution of the random variable  $Z = \min(X, Y)$ ?

**Solution 8** 1. We have:

$$\begin{aligned}\mathbb{P}[X > n] &= \sum_{k=n+1}^{\infty} p(1-p)^{k-1} \\ &= (1-p)^n\end{aligned}$$

using a formula for the sum of geometric terms.

2. Let determine the value of  $\mathbb{P}[\min(X, Y) = k]$ .

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= \mathbb{P}[(X = k \cap Y > k) \cup (X > k \cap Y = k) \cup (X = k \cap Y = k)] \\
&= \mathbb{P}[X = k \cap Y > k] + \mathbb{P}[X > k \cap Y = k] + \mathbb{P}[X = k \cap Y = k] \\
&= p(1-p)^k(1-q)^k + (1-p)^k q(1-q)^k + p(1-p)^k q(1-q)^k \\
&= (1-p)^k(1-q)^k(p+q+pq)
\end{aligned}$$

## Exercise 9

Assume that  $X \sim \mathcal{N}(\mu, \sigma^2)$ .

1. Show that  $Y = (X - \mu)/\sigma$  has standard normal distribution  $\mathcal{N}(0, 1)$ .
2. Compute  $\mathbb{E}[|Y|]$  and  $\mathbb{E}[Y^{2019}]$ .

**Solution 9** 1. If  $Y = (X - \mu)/\sigma$ , then:

$$\begin{aligned}
\mathbb{P}[a \leq Y \leq b] &= \mathbb{P}[a \leq \frac{X - \mu}{\sigma} \leq b] \\
&= \mathbb{P}[a\sigma + \mu \leq X \leq b\sigma + \mu] \\
&= \int_{a\sigma + \mu}^{b\sigma + \mu} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \text{ with change of variable } y = \frac{x - \mu}{\sigma}
\end{aligned}$$

Therefore,  $Y$  follows a normal distribution.

2. Let compute  $\mathbb{E}[|Y|]$ .

$$\begin{aligned}
\mathbb{E}[|Y|] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot \int_{\mathbb{R}^+} x e^{-\frac{x^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} \left[ e^{-\frac{x^2}{2}} \right]_0^\infty \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$