Probability Refresher **September 2023**
Master X-HEC
École Polytechnique

PC 3 – Random vectors & Convergence

Gamma distribution

Exercise 1

(Gamma distribution). One says that X has Gamma distribution with parameters $p > 0$ et $\theta > 0$, denoted by $\gamma(p, \theta)$, if its density is given by

$$f(x) = \frac{\theta^p}{\Gamma(p)} \exp(-\theta x) x^{p-1} \mathbb{1}_{[0,+\infty[}(x).$$

The associated characteristic function is given by

$$\Phi_X(t) = \frac{1}{(1 - it/\theta)^p}, \quad t \in \mathbb{R}.$$

Here $\Gamma(\cdot)$ denotes the Gamma function defined as

$$\forall \alpha > 0, \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \mathrm{d}x, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(1/2) = \sqrt{\pi}.$$

1. Compute $\mathbb{E}\left[X^k\right]$ for $k \geq 1$. Deduce that $\mathbb{E}[X] = p/\theta$ and $\mathrm{Var}(X) = p/\theta^2$.
2. Let $a > 0$. Show that $X/a \sim \gamma(p, a\theta)$.
3. Let X and Y be two independent random variables with Gamma distribution $\gamma(p_1, \theta)$ and $\gamma(p_2, \theta)$, respectively. Show that $X + Y \sim \gamma(p_1 + p_2, \theta)$.
4. Let Z have standard normal distribution $\mathcal{N}(0, 1)$. What is the distribution of Z^2 ?
5. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with exponential distribution $\mathrm{Exp}(\theta)$. Determine the distribution of the sum $S_n = X_1 + \ldots + X_n$. Compute $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$.
6. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with standard normal distribution $\mathcal{N}(0, 1)$. Determine the distribution of the sum $S'_n = X_1^2 + \ldots + X_n^2$. Compute $\mathbb{E}[S'_n]$ and $\mathrm{Var}(S'_n)$.

Random vectors

Exercise 2

Denote

$$f(x, y) = c\mathrm{e}^{-x} \mathbb{1}_{|y| \leq x}.$$

1. Find c such that f is a probability density function of a pair (X, Y) of random variables.
2. Compute the marginal distributions of X and Y.

3. Conclude on the independence of X and Y.

Exercise 3

Let X and Y be two random variables taking their values in $\mathbb{N}$. Consider the joint probability mass function of (X, Y) given by

$$\mathbb{P}(X = i, Y = j) = \frac{a}{2^{i+j}}, i, j \in \mathbb{N}, a \in \mathbb{R}.$$

1. Compute a.
2. Give the marginal distributions of X and Y.
3. Are X and Y independent?

Exercise 4

Denote

$$f(x, y) = a\left(x^2 + y^2\right) \mathbb{1}_{(x,y)\in[-1,1]^2}.$$

1. Find a such that f is a probability density. We denote (X, Y) the pair of random variables with joint distribution f.
2. Compute the marginal distributions of X and Y.
3. Compute the covariance of X and Y.
4. Are X and Y independent?

Exercise 5

Let $\mathbf{X} = (X_1, X_2, X_3)$ be a random vector with the following covariance matrix

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

1. Give the variance of X_2 and the covariance between X_1 and X_3.
2. Compute the variance of $Z = X_3 - \alpha_1 X_1 - \alpha_2 X_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}$.
3. Deduce that X_3 is almost surely a linear combination of X_1 and X_2.
4. More generally, let $\mathbf{Y}$ be a random vector. Give a necessary and sufficient condition on the covariance matrix of $\mathbf{Y}$ ensuring that one of the components of $\mathbf{Y}$ is almost surely a linear combination of the components of $\mathbf{Y}$.

Convergence

Exercise 6

Let $\{X_i\}_{i\geq 0}$ be a sequence of i.i.d. Bernoulli variables with parameter θ.

1. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, \theta(1-\theta))$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$.
2. Show that $\bar{X}_n\left(1 - \bar{X}_n\right) \xrightarrow{P} \theta(1-\theta)$.
3. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right)^2 \xrightarrow{P} 0$.
4. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n\left(1 - \bar{X}_n\right) - \theta(1-\theta)\right)$.

Exercise 7

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. square-integrable random variables with mean m and variance $\sigma^2 > 0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2$.

1. Show that $\hat{\sigma}_n^2$ converges in probability to σ^2 as $n \to \infty$.
2. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n - m\right)/\hat{\sigma}_n$.

Exercise 8

(Poisson model). Let $(X_1, \ldots, X_n)$ be an i.i.d. sample from the Poisson distribution with unknown parameter $\lambda > 0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

1. Show that $\bar{X}_n$ is an unbiased estimator of λ, that is $\mathbb{E}\left[\bar{X}_n\right] = \lambda$.
2. Show that $\bar{X}_n$ converges in probability to λ when n tends to infinity.
3. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n - \lambda\right)/\sqrt{\bar{X}_n}$.
4. Find an appropriate function g such that $\sqrt{n}\left(g\left(\bar{X}_n\right) - g(\lambda)\right) \xrightarrow{d} \mathcal{N}(0,1)$.

Exercise 9

Define the random variable

$$Y = \mathbb{1}\{\theta > X\}$$

where $\theta \in \mathbb{R}$ and X is a random variable with standard normal distribution $\mathcal{N}(0,1)$. We observe a sample $Y_1, \ldots, Y_n$ of i.i.d. realizations of Y and suppose that parameter θ is unknown. Denote by Φ the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. An estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n = \Phi^{-1}\left(\bar{Y}_n\right)$$

where $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$

1. Determine the distribution of Y.
2. Study the convergence in probability of $\hat{\theta}_n$ towards θ when n tends to infinity.
3. Study the limit distribution of $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$.