Probability Refresher **September 2023**
Master X-HEC
École Polytechnique

PC 3 – Random vectors & Convergence

Gamma distribution

Exercise 1

(Gamma distribution). One says that X has Gamma distribution with parameters $p > 0$ et $\theta > 0$, denoted by $\gamma(p, \theta)$, if its density is given by

$$f(x) = \frac{\theta^p}{\Gamma(p)} \exp(-\theta x) x^{p-1} \mathbb{1}_{[0,+\infty[}(x)$$

The associated characteristic function is given by

$$\Phi_X(t) = \frac{1}{(1 - it/\theta)^p}, \quad t \in \mathbb{R}.$$

Here $\Gamma(\cdot)$ denotes the Gamma function defined as

$$\forall \alpha > 0, \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \mathrm{d}x, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(1/2) = \sqrt{\pi}$$

1. Compute $\mathbb{E}\left[X^k\right]$ for $k \geq 1$. Deduce that $\mathbb{E}[X] = p/\theta$ and $\mathrm{Var}(X) = p/\theta^2$.
2. Let $a > 0$. Show that $X/a \sim \gamma(p, a\theta)$.
3. Let X and Y be two independent random variables with Gamma distribution $\gamma(p_1, \theta)$ and $\gamma(p_2, \theta)$, respectively. Show that $X + Y \sim \gamma(p_1 + p_2, \theta)$.
4. Let Z have standard normal distribution $\mathcal{N}(0, 1)$. What is the distribution of Z^2?
5. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with exponential distribution $\mathrm{Exp}(\theta)$. Determine the distribution of the sum $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$.
6. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with standard normal distribution $\mathcal{N}(0, 1)$. Determine the distribution of the sum $S'_n = X_1^2 + \cdots + X_n^2$. Compute $\mathbb{E}[S'_n]$ and $\mathrm{Var}(S'_n)$.

Solution 1 *1. We have:*

$$\begin{aligned}
\mathbb{E}[X^k] &= \frac{\theta^p}{\Gamma(p)} \int_0^\infty x^k e^{-\theta x} x^{p-1} dx \\
&= \frac{\theta^p}{\Gamma(p)} \frac{\Gamma(k+p)}{\theta^{k+p}} \\
&= \frac{(k+p-1) \times \cdots \times p}{\theta^k}
\end{aligned}$$

Therefore,

$$\mathbb{E}[X] = p/\theta$$

and

$$\mathrm{Var}(X) = p/\theta^2$$

2. *For* $a > 0$, *we have:*

$$\begin{aligned}\Phi_{X/a}(t) &= \mathbb{E}[e^{itX/a}] \\ &= \mathbb{E}[e^{i\frac{t}{a}X}] \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^p}\end{aligned}$$

Therefore $X/a \sim \gamma(p, a\theta)$ *identifying random variables from characteristic functions.*

3. *We have*

$$\begin{aligned}\Phi_{X+Y}(t) &= \Phi_X(t)\Phi_Y(t) \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^{p_1}} \frac{1}{(1 - i\frac{t}{a\theta})^{p_2}} \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^{p_1+p_2}}\end{aligned}$$

Which concludes.

4. *The distribution of* Z *is exactly a Gamma distribution* $\gamma(1/2, 1/2)$ *(or a Chi-squared distribution* χ^2_1 *with one degree of liberty):*

$$f_{Z^2}(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}$$

Indeed, $\forall t \geq 0$,

$$\begin{aligned}\mathbb{P}[Z^2 \leq x] &= \mathbb{P}[Z \leq \sqrt{x}] \\ &= \int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\ &= \int_0^x \frac{1}{\sqrt{2\pi u} e^{-u/2}} \text{ with } u = \sqrt{t}\end{aligned}$$

which concludes.

5. *For* X *following an exponential distribution,*

$$\Phi_X(t) = \frac{1}{1 - it/\theta}$$

Therefore,

$$\Phi_{S_n}(t) = \left(\frac{1}{1 - it/\theta}\right)^n$$

And finally $S_n \sim \gamma(n, \theta)$ *with same expected value and variance than in 1.*

6. *Using question 3 and 4 for* n *variables:*

$$X_1^2 + \cdots + X_n^2 \sim \gamma(n/2, n/2)$$

Random vectors

Exercise 2

Denote

$$f(x, y) = c\mathrm{e}^{-x}\mathbb{1}_{|y| \leq x}$$

1. Find c such that f is a probability density function of a pair (X, Y) of random variables.

2. Compute the marginal distributions of X and Y.
3. Conclude on the independence of X and Y.

Solution 2 *1. We have:*

$$\begin{aligned} f \textit{ is a density} &\iff \int_{\mathbb{R}^2} \mathrm{c}\mathrm{e}^{-x}\mathbb{1}_{|y|\leq x} d(x,y) = 1 \\ &\iff \int_{\mathbb{R}}\int_{\mathbb{R}^+} \mathrm{c}\mathrm{e}^{-x}\mathbb{1}_{|y|\leq x} dxdy = 1 \\ &\iff \int_{\mathbb{R}^+} 2x\mathrm{c}\mathrm{e}^{-x} dx = 1 \\ &\iff 2c\int_{\mathbb{R}^+} x\mathrm{e}^{-x} dx = 1 \\ &\iff 2c = 1 \iff c = \frac{1}{2} \end{aligned}$$

2. Moreover,

$$f_X(x) = xe^{-x}$$

And

$$f_Y(y) = \frac{1}{2}e^{-y}$$

3. We finally have:

$$f(x,y) \neq f_X(x)f_Y(y)$$

and the random variables therefore are not independents.

Exercise 3

Let X and Y be two random variables taking their values in $\mathbb{N}$. Consider the joint probability mass function of (X, Y) given by

$$\mathbb{P}[(X = i) \cap (Y = j)] = \frac{a}{2^{i+j}}, i, j \in \mathbb{N}, a \in \mathbb{R}.$$

1. Compute a.
2. Give the marginal distributions of X and Y.
3. Are X and Y independent?

Solution 3 *1. We have:*

$$\sum_{i,j=0}^{\infty} \frac{a}{2^{i+j}} = a\left(\sum_{i=0}^{\infty}\frac{1}{2^i}\right)^2 = a.2.2 = 4a$$

Therefore, $4a = 1$ *and finally* $a = \frac{1}{4}$.

2. We have:

$$\begin{aligned} \mathbb{P}[X = i] &= \sum_{j=0}^{\infty} \mathbb{P}[(X = i) \cap (Y = j)] \\ &= \sum_{j=0}^{\infty} \frac{1}{4.2^i.2^j} \\ &= \frac{1}{2^{i+1}} \end{aligned}$$

In the same way:

$$\mathbb{P}[Y=i] = \frac{1}{2^{i+1}}$$

3. *We have:*

$$\mathbb{P}[(X=i)\cap(Y=j)] = \frac{1}{2^{i+j+2}} = \left(\frac{1}{2^{i+1}}\right)\left(\frac{1}{2^{j+1}}\right) = \mathbb{P}[X=i]\mathbb{P}[Y=j]$$

And the random variables are therefore independents.

Exercise 4

Denote

$$f(x,y) = a\left(x^2+y^2\right)\mathbb{1}_{(x,y)\in[-1,1]^2}.$$

1. Find a such that f is a probability density. We denote (X,Y) the pair of random variables with joint distribution f.
2. Compute the marginal distributions of X and Y.
3. Compute the covariance of X and Y.
4. Are X and Y independent?

Solution 4 1. *We have*

$$\begin{aligned}\int_{[-1,1]^2} x^2+y^2 dxdy &= 2\int_{[-1,1]^2} x^2 \ dx \ dy \\ &= 4\int_{[-1,1]^2} x^2 \ dx \\ &= 8\int_0^1 x^2 \ dx \\ &= \frac{8}{3}\end{aligned}$$

Therefore $a=\frac{3}{8}$.

2. *We have*

$$\begin{aligned}f_X(x) &= a\int_{-1}^1 x^2+y^2 dy \\ &= a\left(\int_{-1}^1 x^2 dy + \int_{-1}^1 y^2 dy\right) \\ &= 2a(x^2+\frac{1}{3}) \\ &= \frac{3}{4}(x^2+\frac{1}{3}) \\ &= \frac{3x^2+1}{4}\end{aligned}$$

And by symmetry $f_Y = f_X$.

3. *We have*

$$\mathbb{E}[X] = \int_{-1}^{1} x \frac{3x^2+1}{4} dx = 0$$

and

$$\begin{aligned}\mathrm{Cov}[X] &= \mathbb{E}[X^2] \\ &= \int_{-1}^{1} x^2 \frac{3x^2+1}{4} dx \\ &= \frac{3}{4} \int_{-1}^{1} x^4 + \frac{1}{2} \\ &= \frac{8}{10}\end{aligned}$$

and same for Y.

4. *As clearly $f \neq f_X \cdot f_Y$, X and Y are not independent.*

Exercise 5

Let $\mathbf{X} = (X_1, X_2, X_3)$ be a random vector with the following covariance matrix

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

1. Give the variance of X_2 and the covariance between X_1 and X_3.
2. Compute the variance of $Z = X_3 - \alpha_1 X_1 - \alpha_2 X_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}$.
3. Deduce that X_3 is almost surely a linear combination of X_1 and X_2.
4. More generally, let $\mathbf{Y}$ be a random vector. Give a necessary and sufficient condition on the covariance matrix of $\mathbf{Y}$ ensuring that one of the components of $\mathbf{Y}$ is almost surely a linear combination of the components of $\mathbf{Y}$.

Solution 5 1. *We have* $\mathrm{Var}[X_2] = 5$ *and* $\mathrm{Cov}[X_1, X_3] = 3$.

2. *We want to compute* $\mathrm{Var}[Z]$. *Let us note that* $Z = X.y$ *with* $y = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. *Therefore,*

$$\begin{aligned}\mathrm{Var}[Z] &= \mathrm{Var}[X.y] \\ &= y^T \,\mathrm{Var}[X] y \\ &= \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ &= 0\end{aligned}$$

Thus, $\mathrm{Var}[Z] = 0$.

3. *Thus, $Z = 0$ almost surely and finally X_3 is almost surely a linear combination of X_1 and X_2.*

4. *The necessary and sufficient condition on* $\mathrm{Cov}[Y]$ *is:*

$$\exists y \in \mathbb{R}^n, Y \cdot y = 0 \text{ almost surely} \iff \mathrm{Cov}[Y] \text{ singular matrix}$$

The proof is quite the same as in Question 2.:

$$\begin{aligned}\exists y \in \mathbb{R}^n, Y \cdot y = 0 \text{ almost surely} &\iff \mathrm{Var}[Y \cdot y] = 0 \\ &\iff y^T \mathrm{Var}[Y] y = 0\end{aligned}$$

As $\mathrm{Var}[Y]$ *is a symmetric matrix semi-definite positive, it can be written* $\mathrm{Var}[Y] = \Lambda^T \Lambda$ *(Cholesky decomposition). Therefore :*

$$\begin{aligned}\exists y \in \mathbb{R}^n, Y \cdot y = 0 \text{ almost surely} &\iff y^T \Lambda^T \Lambda y = 0 \\ &\iff \|\Lambda y\|_2^2 = 0 \\ &\iff \Lambda y = 0_n \\ &\iff \Lambda^T \Lambda y = 0_n \\ &\iff \exists y, \mathrm{Var}[Y].y = 0_n \\ &\iff \mathrm{Cov}[Y] \text{ singular matrix}\end{aligned}$$

Convergence

Exercise 6

Let $\{X_i\}_{i \geq 0}$ be a sequence of i.i.d. Bernoulli variables with parameter θ.

1. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, \theta(1-\theta))$, where $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$.
2. Show that $\bar{X}_n\left(1 - \bar{X}_n\right) \xrightarrow{P} \theta(1-\theta)$.
3. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right)^2 \xrightarrow{P} 0$.
4. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n\left(1 - \bar{X}_n\right) - \theta(1-\theta)\right)$.

Solution 6

1. *Using Central Limit Theorem, we have:*

$$\begin{aligned}\sqrt{n}(\bar{X}_n - \theta) &= \sqrt{n}(\bar{X}_n - \mathbb{E}[X_1]) \\ &\xrightarrow{d} \mathcal{N}(0, Var(X_1)) \\ &= \mathcal{N}(0, \theta(1-\theta))\end{aligned}$$

2. *Applying Law of Large Numbers, we have* $\bar{X}_n \xrightarrow{P} \mathbb{E}[X_1] = \theta$*. The function* $h(x) = x(1-x)$ *being continuous, we obtain*

$$\bar{X}_n(1 - \bar{X}_n) = h(\bar{X}_n) \xrightarrow{P} h(\theta) = \theta(1-\theta)$$

applying continuity theorem.

3. *We have*

$$\sqrt{n}(\bar{X}_n - \theta)^2 = \sqrt{n}(\bar{X}_n - \theta)(\bar{X}_n - \theta)$$

but

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta(1-\theta))$$

and $(\bar{X}_n - \theta) \xrightarrow{d} 0$*. Finally*

$$\sqrt{n}(\bar{X}_n - \theta)^2 \xrightarrow{d} 0$$

As convergence in law towards a constant is equivalent to the probability convergence, we can extrapolate the wanted result.

4. We write

$$\begin{aligned}\sqrt{n}\left(\bar{X}_n(1-\bar{X}_n)-\theta(1-\theta)\right) &= \sqrt{n}\left((\bar{X}_n-\theta)(1-\bar{X}_n)+\theta(1-\bar{X}_n)-\theta(1-\theta)\right)\\ &= \sqrt{n}\left((\bar{X}_n-\theta)(1-\bar{X}_n)-\theta(\bar{X}_n-\theta)\right)\end{aligned}$$

but

$$\sqrt{n}(\bar{X}_n-\theta) \xrightarrow{d} \mathcal{N}(0,\theta(1-\theta))$$

and

$$(1-\bar{X}_n-\theta) \xrightarrow{P} (1-2\theta)$$

Therefore,

$$\left(\bar{X}_n(1-\bar{X}_n)-\theta(1-\theta)\right) \xrightarrow{d} (1-2\theta)N(0,\theta(1-\theta)) = N(0,\theta(1-\theta)(1-2\theta)^2)$$

applying Slutsky theorem.

Exercise 7

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. square-integrable random variables with mean m and variance $\sigma^2>0$. Denote $\bar{X}_n=\frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\sigma}_n^2=\frac{1}{n}\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2$.

1. Show that $\hat{\sigma}_n^2$ converges in probability to σ^2 as $n\to\infty$.
2. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n-m\right)/\hat{\sigma}_n$.

Solution 7 *Let us study the limit case of $\hat{\sigma}_n^2$ when $n\to+\infty$. We have:*

$$\begin{aligned}(n-1)\hat{\sigma}_n^2 &= \sum_{k=1}^n (X_k-\bar{X}_n)^2\\ &= \sum_{k=1}^n (X_k-m)^2+2\sum_{k=1}^n(X_k-m)(m-\bar{X}_n)+n(m-\bar{X}_n)^2\\ &= \sum_{k=1}^n (X_k-m)^2-n(m-\bar{X}_n)^2.\end{aligned}$$

So

$$\begin{aligned}\frac{n-1}{n}\hat{\sigma}_n^2 &= \frac{1}{n}\sum_{k=1}^n(X_k-m)^2-(m-\bar{X}_n)^2\\ &\xrightarrow{a.s.} \mathbb{E}[(X_1-m)^2]-0=\mathrm{Var}(X_1)=:\sigma^2,\end{aligned}$$

Here, the limit is given by the Law of Large Numbers. As a result, $\hat{\sigma}_n\to\sigma$ almost surely. Let us note $Z_n:=\sqrt{n}(\bar{X}_n-m)$. Applying Central Limit Theorem, Z_n converges in law to a gaussian random variable $Z\sim\mathcal{N}(0,\sigma^2)$. According to Slutsky theorem, the couple $(Z_n,\hat{\sigma}_n^{-1})$ converges in law to (Z,σ^{-1}). In particular, the product function being continuous, $\frac{Z_n}{\hat{\sigma}_n}\xrightarrow{d} Z/\sigma\sim\mathcal{N}(0,1)$

Exercise 8

(Poisson model). Let $(X_1,\ldots,X_n)$ be an i.i.d. sample from the Poisson distribution with unknown parameter $\lambda>0$. Denote $\bar{X}_n=\frac{1}{n}\sum_{i=1}^n X_i$.

1. Show that $\bar{X}_n$ is an unbiased estimator of λ, that is $\mathbb{E}\left[\bar{X}_n\right]=\lambda$.

2. Show that $\bar{X}_n$ converges in probability to λ when n tends to infinity.

3. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n-\lambda\right)/\sqrt{\bar{X}_n}$.

4. Find an appropriate function g such that $\sqrt{n}\left(g\left(\bar{X}_n\right)-g(\lambda)\right) \xrightarrow{d} \mathcal{N}(0,1)$.

Solution 8 *1. We recall that, for $X \sim \mathcal{P}(\lambda)$ and $\lambda>0$, we have $\mathbb{E}[X]=\operatorname{Var}[X]=\lambda$.*

Then, the estimator $\bar{X}_n$ is therefore unbiased ($\mathbb{E}[\bar{X}_n]=\lambda$) and consistant using the Strong Law of Large Numbers ($\bar{X}_n \longrightarrow \mathbb{E}[X_1]=\lambda$ p.s). Finally, $\bar{X}_n$ is asymptotically normal using the Central Limit Theorem:

$$\sqrt{n}(\bar{X}_n-\lambda)=\sqrt{n}(\bar{X}_n-\mathbb{E}[X_1]) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\operatorname{Var}[X_1])=\mathcal{N}(0,\lambda) \text{ when } n \to \infty$$

2. Using the previous question and Slutsky theorem, we get

$$\sqrt{n}\left(\frac{\bar{X}_n-\lambda}{\sqrt{\bar{X}_n}}\right)=\underbrace{\sqrt{n}\left(\frac{\bar{X}_n-\lambda}{\sqrt{\lambda}}\right)}_{\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)}\ \underbrace{\frac{\sqrt{\lambda}}{\sqrt{\bar{X}_n}}}_{\xrightarrow{P}\frac{\sqrt{\lambda}}{\sqrt{\mathbb{E}[X_1]}}=1} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

3. Applying Delta method, for any function g continuously differentiable on $\mathbb{R}_+$, we got

$$\sqrt{n}\left(g(\bar{X}_n)-g(\lambda)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, g'(\lambda)^2 \operatorname{Var}[X]\right)$$

We are looking for a function g with 1 as a limit variance. This means that

$$(g'(\lambda))^2 \operatorname{Var}[(]X)=1 \Leftrightarrow (g'(\lambda))^2=\frac{1}{\lambda}$$

We therefore can choose $g(u)=2\sqrt{u}$ with, as a derivative, $g'(u)=1/\sqrt{u}$ and we get

$$\sqrt{n}\left(2\sqrt{\bar{X}_n}-2\sqrt{\lambda}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\left(\frac{1}{\sqrt{\lambda}}\right)^2\lambda\right)=\mathcal{N}(0,1).$$

Exercise 9

Let $X \sim \mathcal{N}(0,1)$. Let $Y := \mathbb{1}_{X<\theta}$ for $\theta \in \mathbb{R}$. We observe a sample $Y_1,\ldots,Y_n$ of i.i.d. realizations of Y and suppose that parameter θ is unknown. Denote by Φ the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. An estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n=\Phi^{-1}\left(\bar{Y}_n\right)$$

where $\bar{Y}_n=\frac{1}{n}\sum_{i=1}^n Y_i$

1. Determine the distribution of Y.

2. Study the convergence in probability of $\hat{\theta}_n$ towards θ when n tends to infinity.

3. Study the limit distribution of $\sqrt{n}\left(\hat{\theta}_n-\theta\right)$.

Solution 9 *1. As Y takes its values $\{0,1\}$, Y follows a Bernoulli law with parameter $\mathbb{P}[Y=1]=\mathbb{P}[\theta>\xi]=\Phi(\theta)$*

2. As $\frac{1}{n}\sum_{i=1}^n Y_i \to \mathbb{E}[][Y_1]=\Phi(\theta)$ a.s and Φ^{-1} is a continuous function, we have $\hat{\theta}_n=\Phi^{-1}(\frac{1}{n}\sum_{i=1}^n Y_i) \to \Phi^{-1}(\Phi(\theta))=\theta$ p.s. Therefore, $\hat{\theta}_n$ is consistant for θ.

3. *According to CLT, (as* $\mathbb{E}[Y_1^2] < \infty$*), we have* $\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} Y_i - \Phi(\theta)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathrm{Var}[(]Y_1)) = \mathcal{N}(0, \Phi(\theta)(1- \Phi(\theta)))$*. The function* $\Phi^{-1}(\theta)$ *is continuously differentiable with derivative* $(\Phi^{-1})'(\theta) = 1/\varphi(\Phi^{-1}(\theta))$*. Applying Delta method, we obtain :*

$$\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}\left(\Phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - \Phi^{-1}(\Phi(\theta))\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \left((\Phi^{-1})'(\Phi(\theta))\right)^2 \Phi(\theta)(1-\Phi(\theta)))$$
$$= \mathcal{N}\left(0, \frac{\Phi(\theta)(1-\Phi(\theta))}{\varphi^2(\theta)}\right)$$