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PC 2 - Probability distributions

Exercise 1

(Uniform distribution). Let X be a random variable with uniform distribution on $[0,1]$. We define $Y=\min(X,1-X)$ and $Z=\max(X,1-X)$. Determine the distributions of Y and Z. Compute $\mathbb{E}[YZ]$.

Exercise 2

One says that $X\in(0,+\infty)$ follows the log-normal distribution if $\log(X)\sim\mathcal{N}(0,1)$. What is the density of X ?

Exercise 3

Consider a random variable X having exponential distribution with parameter 1 . Let $a>0$ be a positive real number.

1. Compute the cumulative distribution function of $Y=\min(X,a)$. Plot the function.
2. What can you say about the existence of a density for the distribution of Y ?
3. Compute $\mathbb{E}[Y]$. Hint: Use $Y=X\mathbb{1}_{X\leq a}+a\mathbb{1}_{X>a}$.

Exercise 4

Let V be a random variable with uniform distribution on $[0,\pi/2]$. Define the random variable $W=\sin(V)$.

1. Determine the distributions of W.
2. How does the distribution of W change when V has uniform distribution on $[0,\pi]$?

Exercise 5

(Cauchy distribution). Let X be a random variable with Cauchy distribution whose density is given by $f(x)=\left(\pi\left(1+x^2\right)\right)^{-1}$. Determine the distribution of $1/X$ using a change of variables.

Exercise 6

* Let $p>0$ and an integer n such that $n>p$. Consider random variables Y_n such that nY_n has a geometric distribution Geo $\left(\frac{p}{n}\right)$ with parameter $\frac{p}{n}$. Show that the characteristic function of Y_n tends to the characteristic function of an exponentially distributed random variable with parameter p.

Exercise 7

Let $\alpha>1$ be fixed. Consider the random variable X with density given by

$$f(x)=c_\alpha x^{-\alpha}\mathbb{1}_{x\geq 1}$$

1. Determine the constant c_α.

2. For which values of p we have X belongs to L^p ?

Exercise 8

Let X and Y be two independent random variables such that X (resp. Y) has geometric distribution with parameter p (resp. q).

1. Compute $\mathbb{P}(X > n)$ for any $n \in \mathbb{N}$.
2. What is the distribution of the random variable $Z = \min(X, Y)$?

Exercise 9

Assume that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

1. Show that $Y = (X - \mu)/\sigma$ has standard normal distribution $\mathcal{N}(0, 1)$.
2. Compute $\mathbb{E}[|Y|]$ and $\mathbb{E}\left[Y^{2019}\right]$.