Probability Refresher
Master X-HEC
École Polytechnique

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PC 2 - Probability distributions

Exercise 1

(Uniform distribution). Let X be a random variable with uniform distribution on $[0, 1]$. We define $Y = \min(X, 1 - X)$ and $Z = \max(X, 1 - X)$. Determine the distributions of Y and Z. Compute $\mathbb{E}[YZ]$.

Solution 1 *La variable aléatoire Y prend ses valeurs dans $[1/2, 1]$ et pour tout $t \in [1/2, 1]$,*

$$F_Y(t) = \mathbb{P}(U \leq t, 1 - U \leq t) = \mathbb{P}(U \leq t, U \geq 1 - t) = t - (1 - t) = 2t - 1$$

donc Y suit la loi uniforme sur $[1/2, 1]$. On remarque que $X = 1 - Y$ et on en déduit que X suit la loi uniforme sur $[0, 1/2]$. Pour calculer $\mathbb{E}[XY]$, on remarque que $XY = U(1 - U)$ et donc

$$\mathbb{E}[XY] = \mathbb{E}[U(1 - U)] = \int_0^1 (t - t^2)\, dt = [t^2/2 - t^3/3]_0^1 = 1/2 - 1/3 = 1/6.$$

Exercise 2

One says that $X \in (0, +\infty)$ follows the log-normal distribution if $\log(X) \sim \mathcal{N}(0, 1)$. What is the density of X ?

Solution 2 *The density of $\log(X)$ is a log normal distribution :*

$$f_{\log(X)}(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log(x))^2}{2}\right)$$

Exercise 3

Consider a random variable X having exponential distribution with parameter 1 . Let $a > 0$ be a positive real number.

1. Compute the cumulative distribution function of $Y = \min(X, a)$. Plot the function.
2. What can you say about the existence of a density for the distribution of Y ?
3. Compute $\mathbb{E}[Y]$. Hint: Use $Y = X\mathbb{1}_{X \leq a} + a\mathbb{1}_{X > a}$.

Exercise 4

Let V be a random variable with uniform distribution on $[0, \pi/2]$. Define the random variable $W = \sin(V)$.

1. Determine the distributions of W.
2. How does the distribution of W change when V has uniform distribution on $[0, \pi]$?

Exercise 5

(Cauchy distribution). Let X be a random variable with Cauchy distribution whose density is given by $f(x) = \left(\pi\left(1+x^2\right)\right)^{-1}$. Determine the distribution of $1/X$ using a change of variables.

Solution 3 *Soit $f : \mathbb{R} \longrightarrow \mathbb{R}$ continue bornée. On a*

$$\mathbb{E}[f(\frac{1}{X})] = \int_{\mathbb{R}} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx.$$

On a envie de faire le changement $u = 1/x$ mais pas bijectif sur $\mathbb{R}$! on scinde en deux

$$\mathbb{E}[f(\frac{1}{X})] = \int_0^{+\infty} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx + \int_{-\infty}^0 f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx.$$

On pose la variable $u = 1/x$ donc $du = -u^2 dx$ ainsi

$$\begin{aligned}\int_0^{+\infty} f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx &= \int_0^{+\infty} f(u) \frac{1}{u^2} \frac{1}{\pi(1+u^{-2})} du \\ &= \int_0^{+\infty} f(u) \frac{1}{\pi(1+u^2)} du.\end{aligned}$$

De plus en faisant $u = 1/x$ dans l'integrale sur $\mathbb{R}^-$ on a de même

$$\int_{-\infty}^0 f(\frac{1}{x}) \frac{1}{\pi(1+x^2)} dx = \int_{-\infty}^0 f(u) \frac{1}{\pi(1+u^2)} du.$$

Donc $\frac{1}{X}$ a même loi que X.

Exercise 6

* Let $p > 0$ and an integer n such that $n > p$. Consider random variables Y_n such that nY_n has a geometric distribution $\text{Geo}\left(\frac{p}{n}\right)$ with parameter $\frac{p}{n}$. Show that the characteristic function of Y_n tends to the characteristic function of an exponentially distributed random variable with parameter p.

Exercise 7

Let $\alpha > 1$ be fixed. Consider the random variable X with density given by

$$f(x) = c_\alpha x^{-\alpha} \mathbb{1}_{x \geq 1}$$

1. Determine the constant c_α.
2. For which values of p we have X belongs to L^p ?

Exercise 8

Let X and Y be two independent random variables such that X (resp. Y) has geometric distribution with parameter p (resp. q).

1. Compute $\mathbb{P}(X > n)$ for any $n \in \mathbb{N}$.
2. What is the distribution of the random variable $Z = \min(X, Y)$?

Exercise 9

Assume that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

1. Show that $Y = (X - \mu)/\sigma$ has standard normal distribution $\mathcal{N}(0, 1)$.
2. Compute $\mathbb{E}[|Y|]$ and $\mathbb{E}\left[Y^{2019}\right]$.

Solution 4 *1. If $Y = (X - \mu)/\sigma$, then :*

$$
\begin{aligned}
\mathbb{P}[a \leq Y \leq b] &= \mathbb{P}[a \leq \frac{X-\mu}{\sigma} \leq b] \\
&= \mathbb{P}[a\sigma + \mu \leq X \leq b\sigma + \mu] \\
&= \int_{a\sigma+\mu}^{b\sigma+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \text{ with change of variable } y = \frac{x-\mu}{\sigma}
\end{aligned}
$$

Therefore, Y follows a normal distribution.

2. Let compute $\mathbb{E}[|Y|]$.

$$
\begin{aligned}
\mathbb{E}[|Y|] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{\frac{-x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} .2. \int_{\mathbb{R}+} x e^{\frac{-x^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} \left[e^{\frac{-x^2}{2}} \right]_0^\infty \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}
$$