Probability Refresher **September 2023**
Master X-HEC **C. Deslandes, A. Loison,**
École Polytechnique **D. Métivier & T. Rebafka**

PC 3 – Random vectors & Convergence

Probability distributions

Exercise 1

(Uniform distribution). Let X be a random variable with uniform distribution on $[0,1]$. We define $Y=\min(X,1-X)$ and $Z=\max(X,1-X)$. Determine the distributions of Y and Z. Compute $\mathbb{E}[YZ]$.

Exercise 2

One says that $X \in (0,+\infty)$ follows the log-normal distribution if $\log(X) \sim \mathcal{N}(0,1)$. What is the density of X ?

Exercise 3

Consider a random variable X having exponential distribution with parameter 1 . Let $a>0$ be a positive real number.

1. Compute the cumulative distribution function of $Y=\min(X,a)$. Plot the function.
2. What can you say about the existence of a density for the distribution of Y ?
3. Compute $\mathbb{E}[Y]$. Hint: Use $Y=X\mathbb{1}_{X\leq a}+a\mathbb{1}_{X>a}$.

Exercise 4

Let V be a random variable with uniform distribution on $[0,\pi/2]$. Define the random variable $W=\sin(V)$.

1. Determine the distributions of W.
2. How does the distribution of W change when V has uniform distribution on $[0,\pi]$?

Exercise 5

(Cauchy distribution). Let X be a random variable with Cauchy distribution whose density is given by $f(x)=\left(\pi\left(1+x^2\right)\right)^{-1}$. Determine the distribution of $1/X$ using a change of variables.

Exercise 6

* Let $p>0$ and an integer n such that $n>p$. Consider random variables Y_n such that nY_n has a geometric distribution Geo $\left(\frac{p}{n}\right)$ with parameter $\frac{p}{n}$. Show that the characteristic function of Y_n tends to the characteristic function of an exponentially distributed random variable with parameter p.

Exercise 7

Let $\alpha>1$ be fixed. Consider the random variable X with density given by

$$f(x)=c_\alpha x^{-\alpha}\mathbb{1}_{x\geq 1}$$

1. Determine the constant c_α.
2. For which values of p we have X belongs to L^p ?

Exercise 8

Let X and Y be two independent random variables such that X (resp. Y) has geometric distribution with parameter p (resp. q).

1. Compute $\mathbb{P}(X > n)$ for any $n \in \mathbb{N}$.
2. What is the distribution of the random variable $Z = \min(X, Y)$?

Exercise 9

Assume that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

1. Show that $Y = (X - \mu)/\sigma$ has standard normal distribution $\mathcal{N}(0,1)$.
2. Compute $\mathbb{E}[|Y|]$ and $\mathbb{E}\left[Y^{2019}\right]$.

Gamma distribution

Exercise 10

(Gamma distribution). One says that X has Gamma distribution with parameters $p > 0$ et $\theta > 0$, denoted by $\gamma(p, \theta)$, if its density is given by

$$f(x) = \frac{\theta^p}{\Gamma(p)} \exp(-\theta x) x^{p-1} \mathbb{1}_{[0,+\infty[}(x).$$

The associated characteristic function is given by

$$\Phi_X(t) = \frac{1}{(1 - it/\theta)^p}, \quad t \in \mathbb{R}.$$

Here $\Gamma(\cdot)$ denotes the Gamma function defined as

$$\forall \alpha > 0, \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \mathrm{d}x, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(1/2) = \sqrt{\pi}.$$

1. Compute $\mathbb{E}\left[X^k\right]$ for $k \geq 1$. Deduce that $\mathbb{E}[X] = p/\theta$ and $\operatorname{Var}(X) = p/\theta^2$.
2. Let $a > 0$. Show that $X/a \sim \gamma(p, a\theta)$.
3. Let X and Y be two independent random variables with Gamma distribution $\gamma\left(p_1, \theta\right)$ and $\gamma\left(p_2, \theta\right)$, respectively. Show that $X + Y \sim \gamma\left(p_1 + p_2, \theta\right)$.
4. Let Z have standard normal distribution $\mathcal{N}(0,1)$. What is the distribution of Z^2 ?
5. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with exponential distribution $\operatorname{Exp}(\theta)$. Determine the distribution of the sum $S_n = X_1 + \ldots + X_n$. Compute $\mathbb{E}\left[S_n\right]$ and $\operatorname{Var}\left(S_n\right)$.
6. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with standard normal distribution $\mathcal{N}(0,1)$. Determine the distribution of the sum $S_n' = X_1^2 + \ldots + X_n^2$. Compute $\mathbb{E}\left[S_n'\right]$ and $\operatorname{Var}\left(S_n'\right)$.

Random vectors

Exercise 11

Denote

$$f(x,y) = c\mathrm{e}^{-x}\mathbb{1}_{|y|\leq x}.$$

1. Find c such that f is a probability density function of a pair (X,Y) of random variables.
2. Compute the marginal distributions of X and Y.
3. Conclude on the independence of X and Y.

Exercise 12

Let X and Y be two random variables taking their values in $\mathbb{N}$. Consider the joint probability mass function of (X,Y) given by

$$\mathbb{P}(X=i, Y=j) = \frac{a}{2^{i+j}}, i,j \in \mathbb{N}, a \in \mathbb{R}.$$

1. Compute a.
2. Give the marginal distributions of X and Y.
3. Are X and Y independent?

Exercise 13

Denote

$$f(x,y) = a\left(x^2+y^2\right)\mathbb{1}_{(x,y)\in[-1,1]^2}.$$

1. Find a such that f is a probability density. We denote (X,Y) the pair of random variables with joint distribution f.
2. Compute the marginal distributions of X and Y.
3. Compute the covariance of X and Y.
4. Are X and Y independent?

Exercise 14

Let $\mathbf{X} = (X_1, X_2, X_3)$ be a random vector with the following covariance matrix

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

1. Give the variance of X_2 and the covariance between X_1 and X_3.
2. Compute the variance of $Z = X_3 - \alpha_1 X_1 - \alpha_2 X_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}$.
3. Deduce that X_3 is almost surely a linear combination of X_1 and X_2.
4. More generally, let $\mathbf{Y}$ be a random vector. Give a necessary and sufficient condition on the covariance matrix of $\mathbf{Y}$ ensuring that one of the components of $\mathbf{Y}$ is almost surely a linear combination of the components of $\mathbf{Y}$.