Probability Refresher **September 2023**
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PC 4 – Convergence & Conditioning

Convergence

Exercise 1

Let $\{X_i\}_{i\geq 0}$ be a sequence of i.i.d. Bernoulli variables with parameter θ.

1. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, \theta(1-\theta))$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$.
2. Show that $\bar{X}_n\left(1-\bar{X}_n\right) \xrightarrow{P} \theta(1-\theta)$.
3. Show that $\sqrt{n}\left(\bar{X}_n - \theta\right)^2 \xrightarrow{P} 0$.
4. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n\left(1-\bar{X}_n\right) - \theta(1-\theta)\right)$.

Exercise 2

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. square-integrable random variables with mean m and variance $\sigma^2 > 0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^2$.

1. Show that $\hat{\sigma}_n^2$ converges in probability to σ^2 as $n \to \infty$.
2. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n - m\right)/\hat{\sigma}_n$.

Exercise 3

(Poisson model). Let $(X_1, \ldots, X_n)$ be an i.i.d. sample from the Poisson distribution with unknown parameter $\lambda > 0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

1. Show that $\bar{X}_n$ is an unbiased estimator of λ, that is $\mathbb{E}\left[\bar{X}_n\right] = \lambda$.
2. Show that $\bar{X}_n$ converges in probability to λ when n tends to infinity.
3. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n - \lambda\right)/\sqrt{\bar{X}_n}$.
4. Find an appropriate function g such that $\sqrt{n}\left(g\left(\bar{X}_n\right) - g(\lambda)\right) \xrightarrow{d} \mathcal{N}(0,1)$.

Exercise 4

Define the random variable

$$Y = \mathbb{1}\{\theta > X\}$$

where $\theta \in \mathbb{R}$ and X is a random variable with standard normal distribution $\mathcal{N}(0,1)$. We observe a sample $Y_1, \ldots, Y_n$ of i.i.d. realizations of Y and suppose that parameter θ is unknown. Denote by Φ the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. An estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n = \Phi^{-1}\left(\bar{Y}_n\right)$$

where $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$

1. Determine the distribution of Y.
2. Study the convergence in probability of $\hat{\theta}_n$ towards θ when n tends to infinity.
3. Study the limit distribution of $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$.

Conditionning

Exercise 5

Let (X,Y) be a couple of random variables admitting a density on $\mathbb{R}^2$ such that
(i) X has Gamma distribution $\gamma(2,\lambda)$ (with density $f_X(x) = \lambda^2 x \mathrm{e}^{-\lambda x} \mathbb{1}_{\{x \geq 0\}}$),
(ii) the conditional distribution of Y given X is the uniform distribution on the segment $[0,X]$ (or, to put it differently, the conditional density of Y given $X=x$ is $f_{Y|X=x}(y) = \frac{1}{x}\mathbb{1}_{\{0<y<x\}}$).

1. Determine the density of (X,Y) and the distribution of Y.
2. Compute the conditional density of X given Y.
3. Evaluate the following quantities:

(a) $\mathbb{E}[XY]$ (one may use that $\mathbb{E}\left[X^2\right] = \frac{6}{\lambda^2}$),
(b) $\mathbb{E}[Y \mid X]$,
(c) $\mathbb{E}[X \mid Y]$,
(d) $\mathbb{E}[X + XY \mid Y]$,
(e) $\mathbb{E}[\mathbb{E}[Y \mid X]]$

Exercise 6

We consider an electronic device. Denote by T the life time of the component, which is the amount of time in years such that it works properly until it breaks down. We assume that T is exponentially distributed with parameter $1/2$.

1. What is the probability that the device breaks down during the first year?
2. We know that the electronic device has been used during two years without any problems. What is the probability that it breaks down during the third year?

Exercise 7

We set

$$f(x,y) = \frac{\alpha y^2}{x}\mathbf{1}_{0<y<x<1}.$$

1. Find α such that f is a probability density on $\mathbb{R}^2$ of some pair of random variables (X,Y) and compute the marginal laws of X and Y.
2. Prove that the conditional density of Y given $X=x$ is given by

$$f_{Y|X=x}(y) = \frac{3y^2}{x^3}, 0 < y < x < 1$$

and deduce that $\mathbb{E}[Y \mid X] = \frac{3}{4}X$.