Probability Refresher **September 2023**
Master X-HEC
École Polytechnique

PC 1 – Sets, Measures and Random Variables

Set theory

Exercise 1

For $n \geq 1$, let

$$A_n = \left[-\frac{1}{n}; 2 + \frac{1}{n}\right], \quad B_n = \left[-\frac{5}{n}; n^2\right].$$

1. Compute $\bigcup_{n\geq 1} A_n, \bigcap_{n\geq 1} A_n$ and $\limsup_n A_n$, where $\limsup_n A_n$ is defined as

$$\limsup A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k = \{x \text{ such that " } x \in A_n \text{ for infinitely many } n \text{ " }\}.$$

2. Compute $\bigcup_{n\geq 1} B_n, \bigcap_{n\geq 1} B_n$ and $\limsup_n B_n$.

3. Evaluate the following set

$$\left\{x \text{ such that } \sum_{n\geq 1} \mathbf{1}_{A_n}(x) = +\infty\right\}.$$

Solution 1 *1. Rappel de la définition:* $\limsup_{n\to\infty} A_n = \cap_{k\geq 1} \cup_{n\geq k} A_n$. *Il s'agit de l'événement où:*

$$\omega \in \limsup_{n\to\infty} A_n \iff \textit{il existe une infinité de } n \textit{ tels que } \omega \in A_n$$

La suite $(A_n)_{n\geq 1}$ *étant monotone décroissante* ($A_n \supset A_{n+1}$ *pour tout* $n \geq 1$), *on a pour tout* $k \geq 1$, $\cup_{n\geq k} A_n = [-1/k, 3 + 1/k]$. *D'une part, on voit que* $[0, 3] \subset A_k \subset \cup_{n\geq k} A_n$ *pour tout* k. *D'autre part, pour tout* $s < 0$ *et pour tout* $t > 3$ *il existe* k *tel que* $s < -1/k$ *et* $t > 3 + 1/k$. *Donc,* $\limsup_{n\to\infty} A_n = [0, 3]$.

Independence

Exercise 2 (*Independent events*)

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ equipped with the uniform probability distribution $\mathbb{P}$. Define the events $A = \{\omega_1, \omega_2\}$, $B = \{\omega_1, \omega_3\}$ and $C = \{\omega_2, \omega_3\}$. Show that A, B and C are pairwise independent. Compare $\mathbb{P}(A \cap B \cap C)$ and $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Solution 2 *We have:*

$$A \cap B = \{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\} = \{\omega_1\}$$

Therefore

$$\mathbb{P}[A \cap B] = \mathbb{P}[\omega_1] = \frac{1}{4}$$

And

$$\mathbb{P}[A].\mathbb{P}[B] = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

Therefore:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A].\mathbb{P}[B]$$

And finally A and B are independent. For similar reasons, B and C are independent and so A and C.

As $A \cap B \cap C = \emptyset$, we have $\mathbb{P}[(]A \cap B \cap C) = 0$. However, $\mathbb{P}[(]A)\mathbb{P}[(]B)\mathbb{P}[(]C) = 1/8$. This implies that A, B et C are not mutually independent.

Exercise 3

Let $A_1, \ldots, A_n$ be n events from a probability space $(\Omega, \mathbb{P})$. Suppose that they are mutually independent. Find an explicit expression for $\mathbb{P}(A_1 \cup \cdots \cup A_n)$ depending on the $\mathbb{P}(A_i)$.

Solution 3

$$\begin{aligned}
\mathbb{P}[A_1 \cup \cdots \cup A_n] &= 1 - \mathbb{P}[A_1^c \cap \cdots \cap A_n^c] \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}[A_i^c] \\
&= 1 - \prod_{i=1}^{n} (1 - \mathbb{P}[A_i])
\end{aligned}$$

Exercise 4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $(A_n)_{n\geq 0}$ a series of independent events. We note $A = \limsup_n A_n$. Let assume that $\sum_n \mathbb{P}(A_n) = +\infty$ and we want to prove that $\mathbb{P}(A) = 1$.

1. Preliminary. Justify that for all $x > -1, \ln(1+x) \leq x$.
2. Let $n \leq N$. We note $E_{n,N} = \bigcap_{k=n}^{N} A_k^c$ and $E_n = \bigcap_{k\geq n} A_k^c$.
   (a) Prove that (n fixed), $\lim_{N\to+\infty} \ln(\mathbb{P}(E_{n,N})) = -\infty$.
   (b) Deduce that $\mathbb{P}(E_n) = 0$.
   (c) Deduce that $\mathbb{P}(A) = 1$.

Solution 4

1. *La fonction ln est concave. Sa courbe représentative est en-dessous de sa tangente au point d'abscisse 1. L'inégalité demandée est juste la traduction analytique de cette propriété géométrique.*
2. *(a) Les événements A_k étant indépendants, il en est de même des événements $\overline{A_k}$, et donc*

$$P(E_{n,N}) = \prod_{k=n}^{N} P\left(\overline{A_k}\right) = \prod_{k=n}^{N} (1 - P(A_k)).$$

En utilisant l'inégalité précédente, on a

$$\ln(P(E_{n,N})) \leq -\sum_{k=n}^{N} P(A_k).$$

Puisque $\sum_{k\geq n} P(A_k) = +\infty$, on en déduit le résultat.

(b) *Par composition par la fonction exponentielle,* $(P(E_{n,N}))$ *tend vers 0 lorsque* N *tend vers l'infini (et* n *reste fixé). Mais, la suite* $(E_{n,N})_N$ *est décroissante et*

$$E_n = \bigcap_{N \geq n} E_{n,N}$$

Ainsi,

$$P(E_n) = \lim_N P(E_{n,N}) = 0$$

(c) A *s'écrit* $A = \bigcap_n \overline{E_n}$. *La suite* $\left(\overline{E_n}\right)$ *est décroissante et* $P\left(\overline{E_n}\right) = 1$. *Ainsi, on trouve que*

$$P(A) = \lim_n P\left(\overline{E_n}\right) = 1$$

Random variables

Exercise 5

Find two random variables X and Y on a probability space $(\Omega, \mathbb{P})$ (to be specified) having the same distribution, but that are not equal.

Solution 5 *Let* X *a random variable on* $[1/2, 2]$ *with p.d.f.* $f : x \to \frac{2}{3x}$ *and* $Y = 1/X$.

1. X *and* $1/X$ *have same p.d.f.*
2. X *and* $1/X$ *are not the same random variable. Indeed:*

$$\frac{X}{1/X} = X^2$$

and X^2 *is not equal to* 1 *almost surely.*

Exercise 6

In an oil region, the probability that one drilling leads to an oil slick is 0.1.

1. Justify that one drilling can be modeled using a Bernoulli distribution.
2. We made 10 oil drillings. Let X be the number of drillings that led to an oil slick.
   (a) Under which assumptions X can be modeled using a binomial distribution? Precise the parameters.
   (b) Assume that X follows a binomial distribution. Compute
      i. the probability that exactly two drillings lead to oil slicks.
      ii. the probability that at least one drilling leads to an oil slick.

Solution 6 1. *Bernoulli is a Success/Failure model with a given probability of succes.*

2. *Using definition of a binomial law:*

   (a)

$$\mathbb{P}[X = 2] = \binom{10}{2} .1^2 \times .9^8$$
$$\approx 0.194$$

(b)

$$\begin{aligned}\mathbb{P}[X \geq 1] &= 1 - \mathbb{P}[X = 0]\\ &= 1 - \binom{10}{0} 0.1^0 0.9^{10}\\ &\approx 0.651\end{aligned}$$

Exercise 7

Let $\lambda > 0$ be fixed. Let $X_n, n \geq 1$ be random variables with binomial distribution with parameters n and λ/n, and Y be a random variable with Poisson distribution with parameter λ. Show that, for any $k \in \mathbb{N}$,

$$\lim_{n \to +\infty} \mathbb{P}\left(X_n = k\right) = \mathbb{P}(Y = k).$$

Hint: Use Striling's approximation: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

We will later see that this result means that X_n converges in distribution to Y, or, to put it differently, that the binomial distribution with parameters n and λ/n converges to the Poisson distribution with parameter λ.

Solution 7 *We have:*

$$\begin{aligned}\mathbb{P}[X_n = k] &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}\\ &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}\\ &= \frac{\lambda^k}{k!}\frac{n!}{(n-k)!n^k}\left(1 - \frac{\lambda}{n}\right)^{n-k}\end{aligned}$$

And:

$$\left(1 - \frac{\lambda}{n}\right)^{n-k} = \left(1 - \frac{\lambda}{n}\right)^{k}\left(1 - \frac{\lambda}{n}\right)^{n}$$

with

$$\left(1 - \frac{\lambda}{n}\right)^{k} \xrightarrow[n \to \infty]{} 0$$

and

$$\begin{aligned}\left(1 - \frac{\lambda}{n}\right)^{n} &= e^{\log\left(1 - \frac{\lambda}{n}\right)^n}\\ &= e^{n.\log\left(1 - \frac{\lambda}{n}\right)}\\ &\sim e^{n\left(-\frac{\lambda}{n}\right)}\\ &= e^{-\lambda}\end{aligned}$$

Also,

$$\frac{n!}{(n-k)!} = n.(n-1).(n-2).\cdot.(n-k+1) \sim n^k$$

Finally,

$$\binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \sim \frac{\lambda^k}{k!}1.e^{-\lambda}$$

This conclude the exercise.

Expectation

Exercise 8

Compute the mean, variance and cumulated distribution function of

1. the binomial distribution $\mathrm{Bin}(n, p)$ with $n \geq 1$ and $p > 0$.
2. the Poisson distribution $\mathrm{Poi}(\lambda)$ with $\lambda > 0$.
3. the uniform distribution $U[a, b]$ with $a < b$.
4. the exponential distribution $\mathrm{Exp}(\lambda)$ with $\lambda > 0$.
5. the normal distribution $\mathcal{N}\left(\mu, \sigma^2\right)$ of probability density function

$$f(x) : x \to \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$.

Solution 8 *1. As any binomial random variable can be expressed as the sum of independant Bernoulli random variables, we have*

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$$

and

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{Var}[\sum_{i=1}^{n} X_i] \\
&= \sum_{i=1}^{n} \mathrm{Var}[X_i] \\
&= \sum_{i=1}^{n} p(1-p) \\
&= np(1-p)
\end{aligned}$$

2. We have

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\lambda} \int_{\mathbb{R}+} x.e^{-\lambda.x} dx \\
&= \frac{1}{\lambda} \left[x \frac{e^{-\lambda.x}}{-\lambda} \right]_0^{\infty} - \frac{1}{\lambda} \int_{\mathbb{R}+} \frac{e^{-\lambda.x}}{-\lambda} dx \\
&= \frac{1}{\lambda}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[X] &= \frac{1}{\lambda} \int_{\mathbb{R}+} x^2.e^{-\lambda.x} dx - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} \int_{\mathbb{R}+} x.e^{-\lambda.x} dx - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

3. *We have*

$$\begin{aligned}\mathbb{E}[X] &= \int_a^b \frac{x}{b-a}dx \\ &= \frac{1}{b-a}\int_a^b xdx \\ &= \frac{1}{b-a}(\frac{b^2}{2}-\frac{a^2}{2}) \\ &= \frac{a+b}{2}\end{aligned}$$

Exercise 9

1. Show that if X exponential distribution $\text{Exp}(\lambda)$ with $\lambda > 0$, then $\mathbb{E}[X^n] = \frac{n!}{\lambda^n}$;
2. Show that if X follows $\mathcal{N}(0,1)$ then $\mathbb{E}\left[X^{2n}\right] = \prod_{k=1}^n (2k-1) = \frac{(2n)!}{2^n n!}$.

Solution 9 1. *Par intégration par parties:*

$$\mathbb{E}\left(X^n\right) = \int_0^{+\infty} x^n \lambda e^{-\lambda x} dx = \int_0^{+\infty} x^{n-1} e^{-\lambda x} dx = \frac{n}{\lambda}\mathbb{E}\left(X^{n-1}\right).$$

On en déduit le résultat par récurence immédiate.

2. *Par intégration par parties:*

$$\mathbb{E}\left(X^{2n}\right) = \int_{\mathbb{R}} x^{2n}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \int_{\mathbb{R}} \frac{x^{2n+2}}{2n+1}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \frac{1}{2n+1}\mathbb{E}\left(X^{2(n+1)}\right).$$

On en déduit le résultat par récurence immédiate.

Exercise 10

Let $X : \Omega \to [0; +\infty]$ (note that $+\infty$ is allowed) be a random variable such that $\mathbb{E}[X] < \infty$.

1. Prove that X is finite almost surely (proceed by contradiction).
2. Assume that $\mathbb{E}[X] = 0$. Prove that $X = 0$ almost surely. Hint: use that $X \geq \frac{1}{n}\mathbf{1}_{X \geq 1/n}$.

Solution 10 1. *Let assume that $\mathbb{P}[X = \infty] > 0$. Then:*

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}[\omega]$$

In the sum, there is $\omega \in \omega$ such that $X(\omega) = \infty$ and the sum cannot be finite.

2. *X is positive and therefore, as a sum of positive terms:*

$$\mathbb{E}[X] = 0 \implies \sum_{\omega \in \Omega} X(\omega)\mathbb{P}[\omega] = 0 \implies \forall \omega, X(\omega) = 0 \text{ or } P(\omega) = 0$$

Then "$\forall \omega, X(\omega) = 0$ or $P(\omega) = 0$" can be red $\mathbb{P}[X \geq 0] = 0$.

Variance Inequalities

Exercise 11

Let X be a random variable such that $\mathbb{E}\left[X^2\right] < +\infty$. Prove that:

1. $0 \leq \operatorname{Var}(X) < \infty$
2. $\operatorname{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$.
3. $\operatorname{Var}(X) = 0 \Longleftrightarrow \mathbb{P}(X = c) = 1$ for some constant c.
4. For any constants a, b, $\operatorname{Var}(aX + b) = \operatorname{Var}(aX) = a^2 \operatorname{Var}(X)$.

Solution 11 *1. We will show here:*

(a) $\mathbb{E}[X]$ *exists and is finite.*
(b) $\operatorname{Var}(X) \geq 0$
(c) $\operatorname{Var}(X) < \infty$

Let begin.

(a) $\mathbb{E}[X^2]$ *is finite so* $\mathbb{E}[(X+1)^2]$ *is. So* $\mathbb{E}[X^2 + 2X + 1]$ *is. As* $\mathbb{E}[X^2] + 1$ *is obviously finite,* $\mathbb{E}[X]$ *exists and is finite.*

(b) We have $\operatorname{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$ *which exists as* $\mathbb{E}[X]$ *is a constant. As* $(X - \mathbb{E}[X])^2$ *is almost surely positive, we have the wanted result.*

(c) We have:

$$\begin{aligned}
\operatorname{Var}[X] &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] + \mathbb{E}[-2.X.\mathbb{E}[X]] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - 2.\mathbb{E}[X].\mathbb{E}[X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2
\end{aligned}$$

The two term are finite. Therefore, $\operatorname{Var}[X]$ *is finite.*

2. Following the previous calculus, the wanted equality is true.

3. We have:

$$\begin{aligned}
\operatorname{Var}[X] = 0 &\iff \operatorname{Var}[(X - \mathbb{E}[X])^2] = 0 \\
&\iff X - \mathbb{E}[X] = 0 almost surely \\
&\iff X = \mathbb{E}[X] almost surely \\
&\iff \exists c \in \mathbb{R}, \mathbb{P}[X = c] = 1
\end{aligned}$$

4. We have:

$$\begin{aligned}
\operatorname{Var}[aX + b] &= \mathbb{E}[(aX + b - \mathbb{E}[aX + b])^2] \\
&= \mathbb{E}[(aX - \mathbb{E}[ax])^2] = \operatorname{Var}[aX] \\
&= a^2 \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= a^2 \operatorname{Var}[X]
\end{aligned}$$

Exercise 12

Let X be non-negative ($X \geq 0$ a.s.) and $a > 0$ be a constant.

1. Justify that
$$\forall \omega \in \Omega, \quad a\mathbf{1}_{\{X(\omega)\geqslant a\}} \leqslant X(\omega)\mathbf{1}_{\{X(\omega)\geqslant a\}} \leqslant X(\omega)$$
2. Prove the Markov's inequality
$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Solution 12 *1. Let prove that:*

(a) $\forall \omega \in \Omega, \quad a\mathbf{1}_{\{X(\omega)\geqslant a\}} \leqslant X(\omega)\mathbf{1}_{\{X(\omega)\geqslant a\}}$

(b) $\forall \omega \in \Omega, \quad X(\omega)\mathbf{1}_{\{X(\omega)\geqslant a\}} \leqslant X(\omega)$

Let us begin.

(a) Let $\omega \in \Omega$. Consider the two cases $X(\omega) \geq a$ and $X(\omega) < a$ and prove that the wanted inequality is true in the two cases.

(b) As the indicator function is below or equal to 1, the inequality is true.

2. We have from 1.:
$$\forall \omega \in \Omega, \quad a\mathbf{1}_{\{X(\omega)\geqslant a\}} \leqslant X(\omega)$$
Taking the expected value in this inequality:
$$\begin{aligned} & \mathbb{E}[a\mathbf{1}_{\{X(\omega)\geqslant a\}}] \leq \mathbb{E}[X] \\ \iff & a\mathbb{E}[\mathbf{1}_{\{X(\omega)\geqslant a\}}] \leq \mathbb{E}[X] \\ \iff & a\mathbb{P}[X \geq a] \leq \mathbb{E}[X] \\ \iff & \mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \end{aligned}$$

Exercise 13

Assume that $\mathbb{E}\left[X^2\right] < +\infty$. Applying Markov's inequality to $(X - \mathbb{E}[X])^2$ prove that, for any constant $a > 0$,
$$\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2}$$

Solution 13 *Applying markov inequality to $(X - \mathbb{E}[X])^2$ and $a^2 > 0$, we have:*
$$\begin{aligned} & \mathbb{P}[(X - \mathbb{E}[X])^2 \geq a^2] \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{a^2} \\ \iff & \mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2} \end{aligned}$$