Probability Refresher
Master X-HEC
École Polytechnique

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C. Deslandes, A. Loison,
D. Métivier & T. Rebafka

PC 2 — Probability distributions

Exercise 1

(Uniform distribution). Let X be a random variable with uniform distribution on $[0,1]$. We define $Y = \min(X, 1-X)$ and $Z = \max(X, 1-X)$. Determine the distributions of Y and Z. Compute $\mathbb{E}[YZ]$.

Solution 1 *La variable aléatoire* Y *prend ses valeurs dans* $[1/2, 1]$ *et pour tout* $t \in [1/2, 1]$,

$$F_Y(t) = \mathbb{P}(U \leq t, 1-U \leq t) = \mathbb{P}(U \leq t, U \geq 1-t) = t - (1-t) = 2t - 1$$

donc Y *suit la loi uniforme sur* $[1/2, 1]$. *On remarque que* $X = 1 - Y$ *et on en déduit que* X *suit la loi uniforme sur* $[0, 1/2]$. *Pour calculer* $\mathbb{E}[XY]$, *on remarque que* $XY = U(1-U)$ *et donc*

$$\mathbb{E}[XY] = \mathbb{E}[U(1-U)] = \int_0^1 (t - t^2)\, dt = [t^2/2 - t^3/3]_0^{\;1} = 1/2 - 1/3 = 1/6.$$

Exercise 2

One says that $X \in (0, +\infty)$ follows the log-normal distribution if $\log(X) \sim \mathcal{N}(0,1)$. What is the density of X?

Solution 2 *The density of* X *is a log normal distribution:*

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log(x))^2}{2}\right)$$

Indeed,

$$\begin{aligned} F_{\log(X)}(x) &= \mathbb{P}[\log(X) \leq x] \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dt \end{aligned}$$

Therefore,

$$\begin{aligned} F_X(x) &= \mathbb{P}[X \leq x] \\ &= \mathbb{P}[\log(X) \leq \log(x)] \\ &= \int_{-\infty}^{\log(x)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dt \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\log(u)^2}{2}\right) \frac{du}{u} \;\; \textit{with new variable } u = e^x \end{aligned}$$

And Therefore

$$f_X(x) = \frac{d}{dx}\left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\log(u)^2}{2}\right) \frac{du}{u}\right) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{\log(x)^2}{2}\right)$$

Exercise 3

Consider a random variable X having exponential distribution with parameter 1. Let $a > 0$ be a positive real number.

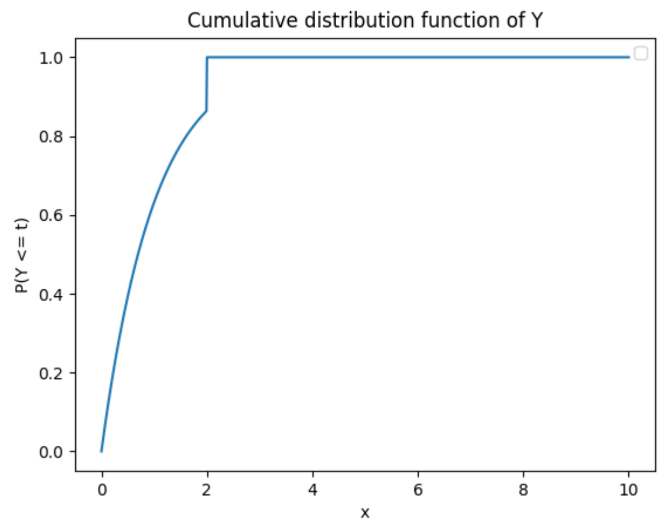
1. Compute the cumulative distribution function of $Y = \min(X, a)$. Plot the function.
2. What can you say about the existence of a density for the distribution of Y?
3. Compute $\mathbb{E}[Y]$. Hint: Use $Y = X\mathbb{1}_{X\leq a} + a\mathbb{1}_{X>a}$.

Solution 3 *1. We have,* $\forall t \in \mathbb{R}^+$*:*

$$\begin{aligned}\mathbb{P}[Y \geq t] &= \mathbb{P}[\min(X, a) \geq t] \\ &= \mathbb{P}[(X \geq t) \cap (a \geq t)] \\ &= \mathbb{1}_{a\geq t}\mathbb{P}[X \geq t]\end{aligned}$$

Therefore,

$$\mathbb{P}[Y \leq t] = 1 - \mathbb{1}_{a\geq t}\mathbb{P}[X \geq t]$$



2. *As the cumulative distribution function is not continuous, the density could not exist : if X has a density f_X, F_X should be a continuous function.*
3. *We have:*

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[X\mathbb{1}_{X\leq a}] + a\mathbb{E}[\mathbb{1}_{X>a}] \\ &= \int_0^a te^{-t}dt + a\mathbb{P}[X > a] \\ &= 1 - (a + 1)e^{-a} + ae^{-a} \\ &= 1 - e^{-a}\end{aligned}$$

Exercise 4

Let V be a random variable with uniform distribution on $[0, \pi/2]$. Define the random variable $W = \sin(V)$.

1. Determine the distributions of W.
2. How does the distribution of W change when V has uniform distribution on $[0, \pi]$?

Solution 4 *1. We have:*

$$\begin{aligned}\mathbb{P}[W \leq t] &= \mathbb{P}[\sin(V) \leq t] \\ &= \mathbb{P}[V \leq \arcsin(t)] \\ &= \frac{2\arcsin(t)}{\pi}\end{aligned}$$

2. Let $0 \leq t \leq 1$ and $\theta \in [0, \frac{\pi}{2}]$ such that $\sin(\theta) = t$. If V is uniform on $[0, \pi]$:

$$\begin{aligned}\mathbb{P}[W \leq t] &= \mathbb{P}[\sin(V) \leq t] \\ &= \mathbb{P}[V \leq \theta] + \mathbb{P}[V \geq \pi - \theta] \\ &= \frac{\theta + \pi - (\pi - \theta)}{\pi} \\ &= \frac{2\theta}{\pi}\end{aligned}$$

Therefore, the distribution of $\sin(V)$ has not changed.

Exercise 5

(Cauchy distribution). Let X be a random variable with Cauchy distribution whose density is given by $f(x) = \left(\pi\left(1 + x^2\right)\right)^{-1}$. Determine the distribution of $1/X$ using a change of variables.

Solution 5 *Soit $f : \mathbb{R} \longrightarrow \mathbb{R}$ continue bornée. On a*

$$\mathbb{E}[f(\frac{1}{X})] = \int_{\mathbb{R}} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx.$$

On a envie de faire le changement $u = 1/x$ (mais pas bijectif sur $\mathbb{R}$) on scinde en deux

$$\mathbb{E}[f(\frac{1}{X})] = \int_0^{+\infty} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx + \int_{-\infty}^0 f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx.$$

On pose la variable $u = 1/x$ donc $du = -u^2 dx$ ainsi

$$\begin{aligned}\int_0^{+\infty} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx &= \int_0^{+\infty} f(u)\frac{1}{u^2}\frac{1}{\pi(1+u^{-2})}du \\ &= \int_0^{+\infty} f(u)\frac{1}{\pi(1+u^2)}du.\end{aligned}$$

De plus en faisant $u = 1/x$ dans l'integrale sur $\mathbb{R}^-$ on a de même

$$\int_{-\infty}^0 f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx = \int_{-\infty}^0 f(u)\frac{1}{\pi(1+u^2)}du.$$

Donc $\frac{1}{X}$ a même loi que X.

Exercise 6

Let $p > 0$ and an integer n such that $n > p$. Consider random variables Y_n such that nY_n has a geometric distribution Geo$\left(\frac{p}{n}\right)$ with parameter $\frac{p}{n}$. Show that the characteristic function of Y_n tends to the characteristic function of an exponentially distributed random variable with parameter p.

Solution 6 *Let $p_n = \frac{p}{n}$. We have:*

$$\begin{aligned}\phi_{Y_n}(t) &= \mathbb{E}[e^{itY_n}]\\ &= \mathbb{E}[e^{i\frac{t}{n}nY_n}]\\ &= \sum_{k=1}^{\infty} p_n e^{i\frac{t}{n}k}(1-p_n)^{k-1}\\ &= \frac{p_n}{1-(1-p_n)e^{i\frac{t}{n}}}\end{aligned}$$

But:

$$\begin{aligned}(1-p_n)e^{i\frac{t}{n}} &\sim (1-p_n)(1+i\frac{t}{n})\\ &= 1-p_n+i\frac{t}{n}-p_n i\frac{t}{n}\end{aligned}$$

Therefore:

$$1-(1-p_n)e^{i\frac{t}{n}} = p_n - i\frac{t}{n} + p_n i\frac{t}{n}$$

And finally

$$\frac{p_n}{1-(1-p_n)e^{i\frac{t}{n}}} \to \frac{p}{p-it}$$

And if $X \sim Exp(p)$:

$$\begin{aligned}\phi_X(t) &= \mathbb{E}[e^{itX}]\\ &= \sum_{k=0}^{\infty} pe^{itk}e^{-p}\frac{p^k}{k!}\\ &= \frac{p}{p-it}\end{aligned}$$

which conclude the execise.

Exercise 7

Let $\alpha > 1$ be fixed. Consider the random variable X with density given by

$$f(x) = c_\alpha x^{-\alpha}\mathbb{1}_{x\geq 1}$$

1. Determine the constant c_α.
2. For which values of p we have X belongs to L^p?

Solution 7 *1. Necessarily, $\int_{\mathbb{R}} f(x) = 1$. So:*

$$\int_{\mathbb{R}} c_\alpha x^{-\alpha}\mathbb{1}_{x\geq 1}dx = 1 \iff \qquad c_\alpha\int_1^{\infty} x^{-\alpha}dx = 1 \iff c_\alpha = \frac{-1}{\frac{1}{1-\alpha}}$$

$$\iff c_\alpha = \alpha - 1$$

2. *We have:*

$$\begin{aligned}
X \in L^p &\iff \mathbb{E}[X^p] < \infty \\
&\iff \int_{\mathbb{R}} x^p x^{-\alpha} \mathbb{1}_{x\geq 1} dx < \infty \qquad \iff \int_1^{\infty} x^{p-\alpha} dx < \infty \\
&\iff p - \alpha < -1
\end{aligned}$$

To understant the last step, see **https://boilley.ovh/cours/integrale-generalisee.html**. *And finally the nessary and sufficient condition is* $p - \alpha < -1$.

Exercise 8

Let X and Y be two independent random variables such that X (resp. Y) has geometric distribution with parameter p (resp. q).

1. Compute $\mathbb{P}(X > n)$ for any $n \in \mathbb{N}$.
2. What is the distribution of the random variable $Z = \min(X, Y)$?

Solution 8 1. *We have:*

$$\begin{aligned}
\mathbb{P}[X > n] &= \sum_{k=n+1}^{\infty} p(1-p)^{k-1} \\
&= (1-p)^n
\end{aligned}$$

using a formula for the sum of geometric terms.

2. *Let determine the value of* $\mathbb{P}[\min(X, Y) = k]$.

$$\begin{aligned}
\mathbb{P}[\min(X, Y) = k] &= \mathbb{P}[(X = k \cap Y > k) \cup (X > k \cap Y = k) \cup (X = k \cap Y = k)] \\
&= \mathbb{P}[X = k \cap Y > k] + \mathbb{P}[X > k \cap Y = k] + \mathbb{P}[X = k \cap Y = k] \\
&= p(1-p)^k(1-q)^k + (1-p)^k q(1-q)^k + p(1-p)^k q(1-q)^k \\
&= (1-p)^k(1-q)^k(p + q + pq)
\end{aligned}$$

Exercise 9

Assume that $X \sim \mathcal{N}(\mu, \sigma^2)$.

1. Show that $Y = (X - \mu)/\sigma$ has standard normal distribution $\mathcal{N}(0, 1)$.
2. Compute $\mathbb{E}[|Y|]$ and $\mathbb{E}\left[Y^{2019}\right]$.

Solution 9 1. *If* $Y = (X - \mu)/\sigma$, *then:*

$$\begin{aligned}
\mathbb{P}[a \leq Y \leq b] &= \mathbb{P}[a \leq \frac{X-\mu}{\sigma} \leq b] \\
&= \mathbb{P}[a\sigma + \mu \leq X \leq b\sigma + \mu] \\
&= \int_{a\sigma+\mu}^{b\sigma+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \text{ with change of variable } y = \frac{x-\mu}{\sigma}
\end{aligned}$$

Therefore, Y *follows a normal distribution.*

2. Let compute $\mathbb{E}[|Y|]$.

$$\begin{aligned}
\mathbb{E}[|Y|] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{\frac{-x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}.2. \int_{\mathbb{R}+} x e^{\frac{-x^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} \left[e^{\frac{-x^2}{2}} \right]_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$