Probability Refresher
Master X-HEC
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C. Deslandes, A. Loison,
D. Métivier & T. Rebafka

PC 2 - Probability distributions

Exercise 1

(Uniform distribution). Let X be a random variable with uniform distribution on $[0,1]$. We define $Y=\min(X,1-X)$ and $Z=\max(X,1-X)$. Determine the distributions of Y and Z. Compute $\mathbb{E}[YZ]$.

Solution 1 *La variable aléatoire Y prend ses valeurs dans $[1/2,1]$ et pour tout $t\in[1/2,1]$,*

$$F_Y(t)=\mathbb{P}(U\leq t,1-U\leq t)=\mathbb{P}(U\leq t,U\geq 1-t)=t-(1-t)=2t-1$$

donc Y suit la loi uniforme sur $[1/2,1]$. On remarque que $X=1-Y$ et on en déduit que X suit la loi uniforme sur $[0,1/2]$. Pour calculer $\mathbb{E}[XY]$, on remarque que $XY=U(1-U)$ et donc

$$\mathbb{E}[XY]=\mathbb{E}[U(1-U)]=\int_0^1(t-t^2)\,dt=[t^2/2-t^3/3]_0^1=1/2-1/3=1/6.$$

Exercise 2

One says that $X\in(0,+\infty)$ follows the log-normal distribution if $\log(X)\sim\mathcal{N}(0,1)$. What is the density of X ?

Exercise 3

Consider a random variable X having exponential distribution with parameter 1 . Let $a>0$ be a positive real number.

1. Compute the cumulative distribution function of $Y=\min(X,a)$. Plot the function.
2. What can you say about the existence of a density for the distribution of Y ?
3. Compute $\mathbb{E}[Y]$. Hint: Use $Y=X\mathbb{1}_{X\leq a}+a\mathbb{1}_{X>a}$.

Exercise 4

Let V be a random variable with uniform distribution on $[0,\pi/2]$. Define the random variable $W=\sin(V)$.

1. Determine the distributions of W.
2. How does the distribution of W change when V has uniform distribution on $[0,\pi]$?

Exercise 5

(Cauchy distribution). Let X be a random variable with Cauchy distribution whose density is given by $f(x)=\left(\pi\left(1+x^2\right)\right)^{-1}$. Determine the distribution of $1/X$ using a change of variables.

Solution 2 *Soit $f : \mathbb{R} \longrightarrow \mathbb{R}$ continue bornée. On a*

$$\mathbb{E}[f(\frac{1}{X})] = \int_{\mathbb{R}} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx.$$

On a envie de faire le changement $u = 1/x$ mais pas bijectif sur $\mathbb{R}$! on scinde en deux

$$\mathbb{E}[f(\frac{1}{X})] = \int_0^{+\infty} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx + \int_{-\infty}^0 f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx.$$

On pose la variable $u = 1/x$ donc $du = -u^2 dx$ ainsi

$$\begin{aligned}\int_0^{+\infty} f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx &= \int_0^{+\infty} f(u)\frac{1}{u^2}\frac{1}{\pi(1+u^{-2})}du \\ &= \int_0^{+\infty} f(u)\frac{1}{\pi(1+u^2)}du.\end{aligned}$$

De plus en faisant $u = 1/x$ dans l'integrale sur $\mathbb{R}^-$ on a de même

$$\int_{-\infty}^0 f(\frac{1}{x})\frac{1}{\pi(1+x^2)}dx = \int_{-\infty}^0 f(u)\frac{1}{\pi(1+u^2)}du.$$

Donc $\frac{1}{X}$ a même loi que X.

Exercise 6

* Let $p > 0$ and an integer n such that $n > p$. Consider random variables Y_n such that nY_n has a geometric distribution Geo $\left(\frac{p}{n}\right)$ with parameter $\frac{p}{n}$. Show that the characteristic function of Y_n tends to the characteristic function of an exponentially distributed random variable with parameter p.

Exercise 7

Let $\alpha > 1$ be fixed. Consider the random variable X with density given by

$$f(x) = c_\alpha x^{-\alpha}\mathbb{1}_{x\geq 1}$$

1. Determine the constant c_α.
2. For which values of p we have X belongs to L^p ?

Exercise 8

Let X and Y be two independent random variables such that X (resp. Y) has geometric distribution with parameter p (resp. q).

1. Compute $\mathbb{P}(X > n)$ for any $n \in \mathbb{N}$.
2. What is the distribution of the random variable $Z = \min(X, Y)$?

Exercise 9

Assume that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

1. Show that $Y = (X - \mu)/\sigma$ has standard normal distribution $\mathcal{N}(0, 1)$.
2. Compute $\mathbb{E}[|Y|]$ and $\mathbb{E}\left[Y^{2019}\right]$.

Solution 3 *1. If* $Y = (X-\mu)/\sigma$*, then :*

$$\begin{aligned}
\mathbb{P}[a \leq Y \leq b] &= \mathbb{P}[a \leq \frac{X-\mu}{\sigma} \leq b] \\
&= \mathbb{P}[a\sigma + \mu \leq X \leq b\sigma + \mu] \\
&= \int_{a\sigma+\mu}^{b\sigma+\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \text{ with change of variable } y = \frac{x-\mu}{\sigma}
\end{aligned}$$

Therefore, Y *follows a normal distribution.*

2. Let compute $\mathbb{E}[|Y|]$*.*

$$\begin{aligned}
\mathbb{E}[|Y|] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{\frac{-x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}.2. \int_{\mathbb{R}+} x e^{\frac{-x^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} \left[e^{\frac{-x^2}{2}} \right]_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$