PC 1 – Sets, Measures and Random Variables

Set theory

Exercise 1

For $n \geq 1$, let
$$A_n = \left[-\frac{1}{n}; 2 + \frac{1}{n}\right], \quad B_n = \left[-\frac{5}{n}; n^2\right].$$

1. Compute $\bigcup_{n\geq 1} A_n, \bigcap_{n\geq 1} A_n$ and $\limsup_n A_n$, where $\limsup_n A_n$ is defined as
$$\limsup A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k = \{x \text{ such that " } x \in A_n \text{ for infinitely many } n \text{ " }\}.$$
2. Compute $\bigcup_{n\geq 1} B_n, \bigcap_{n\geq 1} B_n$ and $\limsup_n B_n$.
3. Evaluate the following set
$$\left\{x \text{ such that } \sum_{n\geq 1} \mathbf{1}_{A_n}(x) = +\infty\right\}.$$

Solution 1 *1. Rappel de la définition :* $\limsup_{n\to\infty} A_n = \cap_{k\geq 1} \cup_{n\geq k} A_n$*. Il s'agit de l'événement où :*
$$\omega \in \limsup_{n\to\infty} A_n \iff \text{il existe une infinité de } n \text{ tels que } \omega \in A_n.$$

La suite $(A_n)_{n\geq 1}$ *étant monotone décroissante (*$A_n \supset A_{n+1}$ *pour tout* $n \geq 1$*), on a pour tout* $k \geq 1$, $\cup_{n\geq k} A_n = [-1/k, 3 + 1/k]$*. D'une part, on voit que* $[0, 3] \subset A_k \subset \cup_{n\geq k} A_n$ *pour tout* k*. D'autre part, pour tout* $s < 0$ *et pour tout* $t > 3$ *il existe* k *tel que* $s < -1/k$ *et* $t > 3 + 1/k$*. Donc,* $\limsup_{n\to\infty} A_n = [0, 3]$.

Independence

Exercise 2 (*Independent events*)

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ equipped with the uniform probability distribution $\mathbb{P}$. Define the events $A = \{\omega_1, \omega_2\}$, $B = \{\omega_1, \omega_3\}$ and $C = \{\omega_2, \omega_3\}$. Show that A, B and C are pairwise independent. Compare $\mathbb{P}(A \cap B \cap C)$ and $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Solution 2 *On a* $\mathbb{P}(\{w_i\}) = 1/|\Omega| = 1/4$ *pour* $i = 1, \dots, 4$*. D'une part,* $\mathbb{P}(A) = \mathbb{P}(\{\omega_1\}) + \mathbb{P}(\{\omega_2\}) = 1/2$*. De même,* $\mathbb{P}(B) = \mathbb{P}(C) = 1/2$*. D'autre part,* $A \cap B = \{\omega_1\}$ *et donc* $\mathbb{P}(A \cap B) = 1/4$*. Donc, on a montré que* $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$*, d'où l'indépendance de* A *et* B*. De même, on montre que* A *et* C *sont indépendants et* B *et* C *sont indépendants.*

Comme $A \cap B \cap C = \emptyset$*, on a* $\mathbb{P}(A \cap B \cap C) = 0$*. En revanche,* $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 1/8$*. Cela implique que* A*,* B *et* C *ne sont pas mutuellement indépendants.*

Exercise 3

Let $A_1, \ldots, A_n$ be n events from a probability space $(\Omega, \mathbb{P})$. Suppose that they are mutually independent. Find an explicit expression for $\mathbb{P}(A_1 \cup \cdots \cup A_n)$ depending on the $\mathbb{P}(A_i)$.

Exercise 4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $(A_n)_{n\geq 0}$ a series of independent events. We note $A = \limsup_n A_n$. Let assume that $\sum_n \mathbb{P}(A_n) = +\infty$ and we want to prove that $\mathbb{P}(A) = 1$.

1. Preliminary. Justify that for all $x > -1, \ln(1+x) \leq x$.
2. Let $n \leq N$. We note $E_{n,N} = \bigcap_{k=n}^{N} A_k^c$ and $E_n = \bigcap_{k\geq n} A_k^c$.
   (a) Prove that (n fixed), $\lim_{N\to+\infty} \ln(\mathbb{P}(E_{n,N})) = -\infty$.
   (b) Deduce that $\mathbb{P}(E_n) = 0$.
   (c) Deduce that $\mathbb{P}(A) = 1$.

Solution 3 *1. La fonction ln est concave. Sa courbe représentative est en-dessous de sa tangente au point d'abscisse 1. L'inégalité demandée est juste la traduction analytique de cette propriété géométrique.*

2. (a) Les événements A_k étant indépendants, il en est de même des événements $\overline{A_k}$, et donc

$$P(E_{n,N}) = \prod_{k=n}^{N} P\left(\overline{A_k}\right) = \prod_{k=n}^{N} (1 - P(A_k)).$$

En utilisant l'inégalité précédente, on a

$$\ln(P(E_{n,N})) \leq -\sum_{k=n}^{N} P(A_k).$$

Puisque $\sum_{k\geq n} P(A_k) = +\infty$, on en déduit le résultat.

(b) Par composition par la fonction exponentielle, $(P(E_{n,N}))$ tend vers 0 lorsque N tend vers l'infini (et n reste fixé). Mais, la suite $(E_{n,N})_N$ est décroissante et

$$E_n = \bigcap_{N\geq n} E_{n,N}$$

Ainsi,

$$P(E_n) = \lim_N P(E_{n,N}) = 0.$$

(c) A s'écrit $A = \bigcap_n \overline{E_n}$. La suite $\left(\overline{E_n}\right)$ est décroissante et $P\left(\overline{E_n}\right) = 1$. Ainsi, on trouve que

$$P(A) = \lim_n P\left(\overline{E_n}\right) = 1.$$

Random variables

Exercise 5

Find two random variables X and Y on a probability space $(\Omega, \mathbb{P})$ (to be specified) having the same distribution, but that are not equal.

Exercise 6

In an oil region, the probability that one drilling leads to an oil slick is 0.1 .

1. Justify that one drilling can be modeled using a Bernoulli distribution.

2. We made 10 oil drillings. Let X be the number of drillings that led to an oil slick.

   (a) Under which assumptions X can be modeled using a binomial distribution? Precise the parameters.

   (b) Assume that X follows a binomial distribution. Compute

      i. the probability that exactly two drillings lead to oil slicks.

      ii. the probability that at least one drilling leads to an oil slick.

Exercise 7

Let $\lambda > 0$ be fixed. Let $X_n, n \geq 1$ be random variables with binomial distribution with parameters n and λ/n, and Y be a random variable with Poisson distribution with parameter λ. Show that, for any $k \in \mathbb{N}$,

$$\lim_{n\to+\infty} \mathbb{P}\left(X_n = k\right) = \mathbb{P}(Y = k).$$

Hint: Use Striling's approximation: $n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$.

We will later see that this result means that X_n converges in distribution to Y, or, to put it differently, that the binomial distribution with parameters n and λ/n converges to the Poisson distribution with parameter λ.

Expectation

Exercise 8

Compute the mean, variance and cumulated distribution function of

1. the binomial distribution $\mathrm{Bin}(n, p)$ with $n \geq 1$ and $p > 0$.
2. the Poisson distribution $\mathrm{Poi}(\lambda)$ with $\lambda > 0$.
3. the uniform distribution $U[a, b]$ with $a < b$.
4. the exponential distribution $\mathrm{Exp}(\lambda)$ with $\lambda > 0$.
5. the normal distribution $\mathcal{N}\left(\mu, \sigma^2\right)$ of probability density function

$$f(x) : x \to \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   with $\mu \in \mathbb{R}$ and $\sigma > 0$.

Exercise 9

1. Show that if X exponential distribution $\mathrm{Exp}(\lambda)$ with $\lambda > 0$, then $\mathbb{E}\left[X^n\right] = \frac{n!}{\lambda^n}$;
2. Show that if X follows $\mathcal{N}(0,1)$ then $\mathbb{E}\left[X^{2n}\right] = \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n n!}$.

Solution 4 *1. Par intégration par parties :*

$$\mathbb{E}\left(X^n\right) = \int_0^{+\infty} x^n \lambda e^{-\lambda x} dx = \int_0^{+\infty} x^{n-1} e^{-\lambda x} dx = \frac{n}{\lambda}\mathbb{E}\left(X^{n-1}\right).$$

On en déduit le résultat par récurence immédiate.

2. *Par intégration par parties:*

$$\mathbb{E}\left(X^{2n}\right) = \int_{\mathbb{R}} x^{2n} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{\mathbb{R}} \frac{x^{2n+2}}{2n+1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{1}{2n+1} \mathbb{E}\left(X^{2(n+1)}\right).$$

On en déduit le résultat par récurence immédiate.

Exercise 10

* Let $X : \Omega \to [0; +\infty]$ (note that $+\infty$ is allowed) be a random variable such that $\mathbb{E}[X] < \infty$.

1. Prove that X is finite almost surely (proceed by contradiction).
2. Assume that $\mathbb{E}[X] = 0$. Prove that $X = 0$ almost surely. Hint: use that $X \geq \frac{1}{n}\mathbf{1}_{X \geq 1/n}$.

Variance Inequalities

Exercise 11

Let X be a random variable such that $\mathbb{E}\left[X^2\right] < +\infty$. Prove that :

1. $0 \leq \operatorname{Var}(X) < \infty$
2. $\operatorname{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$.
3. $\operatorname{Var}(X) = 0 \iff \mathbb{P}(X = c) = 1$ for some constant c.
4. For any constants $a, b, \operatorname{Var}(aX + b) = \operatorname{Var}(aX) = a^2 \operatorname{Var}(X)$.

Exercise 12

Let X be non-negative ($X \geq 0$ a.s.) and $a > 0$ be a constant.

1. Justify that
$$\forall \omega \in \Omega, \quad a\mathbf{1}_{\{X(\omega) \geqslant a\}} \leqslant X(\omega)\mathbf{1}_{\{X(\omega) \geqslant a\}} \leqslant X(\omega)$$
2. Prove the Markov's inequality
$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Exercise 13

Assume that $\mathbb{E}\left[X^2\right] < +\infty$. Applying Markov's inequality to $(X - \mathbb{E}[X])^2$ prove that, for any constant $a > 0$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \frac{\operatorname{Var}(X)}{a^2}$$