Probability Refresher **September 2023**
Master X-HEC
École Polytechnique

PC 3 – Random vectors & Convergence

Gamma distribution

Exercise 1

(Gamma distribution). One says that X has Gamma distribution with parameters $p > 0$ et $\theta > 0$, denoted by $\gamma(p, \theta)$, if its density is given by

$$f(x) = \frac{\theta^p}{\Gamma(p)} \exp(-\theta x) x^{p-1} \mathbb{1}_{[0,+\infty[}(x)$$

The associated characteristic function is given by

$$\Phi_X(t) = \frac{1}{(1 - it/\theta)^p}, \quad t \in \mathbb{R}.$$

Here $\Gamma(\cdot)$ denotes the Gamma function defined as

$$\forall \alpha > 0, \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \mathrm{d}x, \quad \Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \Gamma(1/2) = \sqrt{\pi}$$

1. Compute $\mathbb{E}\left[X^k\right]$ for $k \geq 1$. Deduce that $\mathbb{E}[X] = p/\theta$ and $\mathrm{Var}(X) = p/\theta^2$.
2. Let $a > 0$. Show that $X/a \sim \gamma(p, a\theta)$.
3. Let X and Y be two independent random variables with Gamma distribution $\gamma(p_1, \theta)$ and $\gamma(p_2, \theta)$, respectively. Show that $X + Y \sim \gamma(p_1 + p_2, \theta)$.
4. Let Z have standard normal distribution $\mathcal{N}(0, 1)$. What is the distribution of Z^2?
5. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with exponential distribution $\mathrm{Exp}(\theta)$. Determine the distribution of the sum $S_n = X_1 + \cdots + X_n$. Compute $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$.
6. Let $X_1, \ldots, X_n$ be n i.i.d. random variables aléatoires with standard normal distribution $\mathcal{N}(0, 1)$. Determine the distribution of the sum $S'_n = X_1^2 + \cdots + X_n^2$. Compute $\mathbb{E}[S'_n]$ and $\mathrm{Var}(S'_n)$.

Solution 1 *1. We have:*

$$\begin{aligned}
\mathbb{E}[X^k] &= \frac{\theta^p}{\Gamma(p)} \int_0^\infty x^k e^{-\theta x} x^{p-1} dx \\
&= \frac{\theta^p}{\Gamma(p)} \frac{\Gamma(k+p)}{\theta^{k+p}} \\
&= \frac{(k+p-1) \times \cdots \times p}{\theta^k}
\end{aligned}$$

Therefore,

$$\mathbb{E}[X] = p/\theta$$

and

$$\mathrm{Var}(X) = p/\theta^2$$

2. *For $a > 0$, we have:*

$$\begin{aligned}\Phi_{X/a}(t) &= \mathbb{E}[e^{itX/a}] \\ &= \mathbb{E}[e^{i\frac{t}{a}X}] \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^p}\end{aligned}$$

Therefore $X/a \sim \gamma(p, a\theta)$ identifying random variables from characteristic functions.

3. *We have*

$$\begin{aligned}\Phi_{X+Y}(t) &= \Phi_X(t)\Phi_Y(t) \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^{p_1}} \frac{1}{(1 - i\frac{t}{a\theta})^{p_2}} \\ &= \frac{1}{(1 - i\frac{t}{a\theta})^{p_1+p_2}}\end{aligned}$$

Which concludes.

4. *The distribution of Z is exactly a Chi-squared distribution χ_1^2 with one degree of liberty:*

$$f_{Z^2}(x) = \frac{1}{\sqrt{2\pi x}e^{-x/2}}$$

5. *For X following an exponential distribution,*

$$\Phi_X(t) = \frac{1}{1 - it/\theta}$$

Therefore,

$$\Phi_{S_n}(t) = \left(\frac{1}{1 - it/\theta}\right)^n$$

And finally $S_n \sim \gamma(n, \theta)$ with same expected value and variance than in 1.

6.

Random vectors

Exercise 2

Denote

$$f(x, y) = c\mathrm{e}^{-x}\mathbb{1}_{|y|\leq x}$$

1. Find c such that f is a probability density function of a pair (X, Y) of random variables.
2. Compute the marginal distributions of X and Y.
3. Conclude on the independence of X and Y.

Solution 2 1. *We have:*

$$\begin{aligned} f \textit{ is a density} &\iff \int_{\mathbb{R}^2} ce^{-x}\mathbb{1}_{|y|\leq x} d(x,y) = 1 \\ &\iff \int_{\mathbb{R}}\int_{\mathbb{R}+} ce^{-x}\mathbb{1}_{|y|\leq x} dxdy = 1 \\ &\iff \int_{\mathbb{R}+} 2xce^{-x} dx = 1 \\ &\iff 2c\int_{\mathbb{R}+} xe^{-x} dx = 1 \\ &\iff 2c = 1 \iff c = \frac{1}{2} \end{aligned}$$

2. *Moreover,*

$$f_X(x) = xe^{-x}$$

And

$$f_Y(y) = \frac{1}{2}e^{-y}$$

3. *We finally have:*

$$f(x,y) \neq f_X(x)f_Y(y)$$

and the random variables therefore are not independents.

Exercise 3

Let X and Y be two random variables taking their values in $\mathbb{N}$. Consider the joint probability mass function of (X,Y) given by

$$\mathbb{P}[(X=i)\cap(Y=j)] = \frac{a}{2^{i+j}}, i,j \in \mathbb{N}, a \in \mathbb{R}.$$

1. Compute a.
2. Give the marginal distributions of X and Y.
3. Are X and Y independent?

Solution 3 1. *We have:*

$$\sum_{i,j=0}^{\infty} \frac{a}{2^{i+j}} = a\left(\sum_{i=0}^{\infty}\frac{1}{2^i}\right)^2 = a.2.2 = 4a$$

Therefore, $4a = 1$ *and finally* $a = \frac{1}{4}$.

2. *We have:*

$$\begin{aligned} \mathbb{P}[X=i] &= \sum_{j=0}^{\infty} \mathbb{P}[(X=i)\cap(Y=j)] \\ &= \sum_{j=0}^{\infty} \frac{1}{4.2^i.2^j} \\ &= \frac{1}{2^{i+1}} \end{aligned}$$

In the same way:

$$\mathbb{P}[Y=i] = \frac{1}{2^{i+1}}$$

3. *We have:*

$$\mathbb{P}[(X=i)\cap(Y=j)] = \frac{1}{2^{i+j+2}} = \left(\frac{1}{2^{i+1}}\right)\left(\frac{1}{2^{j+1}}\right) = \mathbb{P}[X=i]\mathbb{P}[Y=j]$$

And the random variables are therefore independents.

Exercise 4

Denote

$$f(x,y) = a\left(x^2+y^2\right)\mathbb{1}_{(x,y)\in[-1,1]^2}.$$

1. Find a such that f is a probability density. We denote (X,Y) the pair of random variables with joint distribution f.
2. Compute the marginal distributions of X and Y.
3. Compute the covariance of X and Y.
4. Are X and Y independent?

Solution 4 1. *We have*

$$\int_{[-1,1]^2} x^2+y^2 = \pi$$

Therefore $a = \frac{1}{\pi}$.

2.

Exercise 5

Let $\mathbf{X} = (X_1, X_2, X_3)$ be a random vector with the following covariance matrix

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

1. Give the variance of X_2 and the covariance between X_1 and X_3.
2. Compute the variance of $Z = X_3 - \alpha_1 X_1 - \alpha_2 X_2$ for $\alpha_1, \alpha_2 \in \mathbb{R}$.
3. Deduce that X_3 is almost surely a linear combination of X_1 and X_2.
4. More generally, let $\mathbf{Y}$ be a random vector. Give a necessary and sufficient condition on the covariance matrix of $\mathbf{Y}$ ensuring that one of the components of $\mathbf{Y}$ is almost surely a linear combination of the components of $\mathbf{Y}$.

Solution 5

Convergence

Exercise 6

Let $\{X_i\}_{i\geq 0}$ be a sequence of i.i.d. Bernoulli variables with parameter θ.

1. Show that $\sqrt{n}\left(\bar{X}_n-\theta\right) \xrightarrow{d} \mathcal{N}(0,\theta(1-\theta))$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$.
2. Show that $\bar{X}_n\left(1-\bar{X}_n\right) \xrightarrow{P} \theta(1-\theta)$.
3. Show that $\sqrt{n}\left(\bar{X}_n-\theta\right)^2 \xrightarrow{P} 0$.
4. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n\left(1-\bar{X}_n\right)-\theta(1-\theta)\right)$.

Solution 6 *1. Par le TCL, on a* $\sqrt{n}(\bar{X}_n-\theta) = \sqrt{n}(\bar{X}_n - \mathbb{E}[][X_1]) \xrightarrow{d} \mathcal{N}(0, Var(X_1)) = \mathcal{N}(0,\theta(1-\theta))$.

2. *Par la LGN, on a* $\bar{X}_n \xrightarrow{P} \mathbb{E}[][X_1] = \theta$. *La fonction* $h(x) = x(1-x)$ *étant continue, on obtient par le théorème de continuité,* $\bar{X}_n(1-\bar{X}_n) = h(\bar{X}_n) \xrightarrow{P} h(\theta) = \theta(1-\theta)$.
3. *On a*
$$\sqrt{n}(\bar{X}_n-\theta)^2 = \underbrace{\sqrt{n}(\bar{X}_n-\theta)}_{\xrightarrow{d}\mathcal{N}(0,\theta(1-\theta))}\underbrace{(\bar{X}_n-\theta)}_{\xrightarrow{P}0} \xrightarrow{d} 0\times\mathcal{N}(0,\theta(1-\theta)) = 0.$$
La convergence en loi vers une constante est équivalente à la convergence en probabilité, d'où le résultat.
4. *On écrit*
$$\begin{aligned}
\sqrt{n}\left(\bar{X}_n(1-\bar{X}_n)-\theta(1-\theta)\right) &= \sqrt{n}\left((\bar{X}_n-\theta)(1-\bar{X}_n)+\theta(1-\bar{X}_n)-\theta(1-\theta)\right)\\
&= \sqrt{n}\left((\bar{X}_n-\theta)(1-\bar{X}_n)-\theta(\bar{X}_n-\theta)\right)\\
&= \underbrace{\sqrt{n}(\bar{X}_n-\theta)}_{\xrightarrow{d}\mathcal{N}(0,\theta(1-\theta))}\underbrace{(1-\bar{X}_n-\theta)}_{\xrightarrow{P}1-2\theta}\\
&\xrightarrow{d} (1-2\theta)\mathcal{N}(0,\theta(1-\theta)) = \mathcal{N}(0,(1-2\theta)^2\theta(1-\theta)),
\end{aligned}$$
par le lemme de Slutsky.

Exercise 7

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. square-integrable random variables with mean m and variance $\sigma^2>0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2$.

1. Show that $\hat{\sigma}_n^2$ converges in probability to σ^2 as $n\to\infty$.
2. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n-m\right)/\hat{\sigma}_n$.

Solution 7 *Commençons par étudier le comportement limite de* $\hat{\sigma}_n^2$ *quand* $n\to+\infty$.

$$\begin{aligned}
(n-1)\hat{\sigma}_n^2 &= \sum_{k=1}^n (X_k-\bar{X}_n)^2\\
&= \sum_{k=1}^n (X_k-m)^2 + 2\sum_{k=1}^n (X_k-m)(m-\bar{X}_n) + n(m-\bar{X}_n)^2\\
&= \sum_{k=1}^n (X_k-m)^2 - n(m-\bar{X}_n)^2.
\end{aligned}$$

Donc

$$\frac{n-1}{n}\hat{\sigma}_n^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k-m)^2 - (m-\bar{X}_n)^2$$
$$\xrightarrow{p.s.} \mathbb{E}[][(X_1-m)^2] - 0 = \mathrm{Var}(X_1) =: \sigma^2,$$

où la limite est donnée par la loi des grands nombres. Par suite, $\hat{\sigma}_n \to \sigma$ presque sûrement. Notons $Z_n = \sqrt{n}(\bar{X}_n - m)$ qui converge en loi, d'après le théorème limite central vers une variable aléatoire gaussienne $Z \sim \mathcal{N}(0,\sigma^2)$. D'après le lemme de Slutsky, le couple $(Z_n, \hat{\sigma}_n^{-1})$ converge en loi vers (Z, σ^{-1}). En particulier, la fonction produit étant continue, $\frac{Z_n}{\hat{\sigma}_n} \xrightarrow{d} Z/\sigma \sim \mathcal{N}(0,1)$.

Exercise 8

(Poisson model). Let $(X_1,\ldots,X_n)$ be an i.i.d. sample from the Poisson distribution with unknown parameter $\lambda > 0$. Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

1. Show that $\bar{X}_n$ is an unbiased estimator of λ, that is $\mathbb{E}\left[\bar{X}_n\right] = \lambda$.
2. Show that $\bar{X}_n$ converges in probability to λ when n tends to infinity.
3. Determine the limit distribution of $\sqrt{n}\left(\bar{X}_n - \lambda\right)/\sqrt{\bar{X}_n}$.
4. Find an appropriate function g such that $\sqrt{n}\left(g\left(\bar{X}_n\right) - g(\lambda)\right) \xrightarrow{d} \mathcal{N}(0,1)$.

Solution 8 *1. On rappelle, pour $X \sim \mathcal{P}(\lambda)$, $\lambda > 0$, on a $\mathbb{E}[][X] = \mathrm{Var}[(]X) = \lambda$. Alors l'estimateur $\bar{X}_n$ est alors sans biais ($\mathbb{E}[][\bar{X}_n] = \lambda$), consistant en vertu de la LFGN ($\bar{X}_n \longrightarrow \mathbb{E}[][X_1] = \lambda$ p.s.), et enfin, $\bar{X}_n$ est asymptotiquement normal par le TCL:*

$$\sqrt{n}(\bar{X}_n - \lambda) = \sqrt{n}(\bar{X}_n - \mathbb{E}[][X_1]) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathrm{Var}[(]X_1)) = \mathcal{N}(0,\lambda), \quad \textit{lorsque } n \to \infty.$$

2. En utilisant la question a et le lemme de Slutsky, on obtient

$$\sqrt{n}\left(\frac{\bar{X}_n - \lambda}{\sqrt{\bar{X}_n}}\right) = \underbrace{\sqrt{n}\left(\frac{\bar{X}_n-\lambda}{\sqrt{\lambda}}\right)}_{\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)} \underbrace{\frac{\sqrt{\lambda}}{\sqrt{\bar{X}_n}}}_{\xrightarrow{P}\frac{\sqrt{\lambda}}{\sqrt{\mathbb{E}[][X_1]}}=1} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

3. D'après la delta méthode, pour toute fonction g continument dérivable sur $\mathbb{R}_+$, on a

$$\sqrt{n}\left(g(\bar{X}_n) - g(\lambda)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (g'(\lambda))^2\,\mathrm{Var}[(]X)).$$

Nous cherchons donc une fonction g telle que la variance limite vaut 1. Ce qui veut dire

$$(g'(\lambda))^2\,\mathrm{Var}[(]X) = 1 \Leftrightarrow (g'(\lambda))^2 = \frac{1}{\lambda}.$$

On peut alors choisir $g(u) = 2\sqrt{u}$ avec dérivée $g'(u) = 1/\sqrt{u}$ et on obtient

$$\sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\lambda}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \left(\frac{1}{\sqrt{\lambda}}\right)^2\lambda\right) = \mathcal{N}(0,1).$$

Exercise 9

Define the random variable

$$Y = \mathbb{1}\{\theta > X\}$$

where $\theta \in \mathbb{R}$ and X is a random variable with standard normal distribution $\mathcal{N}(0,1)$. We observe a sample $Y_1, \ldots, Y_n$ of i.i.d. realizations of Y and suppose that parameter θ is unknown. Denote by Φ the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. An estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n = \Phi^{-1}\left(\bar{Y}_n\right)$$

where $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$

1. Determine the distribution of Y.
2. Study the convergence in probability of $\hat{\theta}_n$ towards θ when n tends to infinity.
3. Study the limit distribution of $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$.

Solution 9 *1. Comme Y prend ses valeurs dans $\{0,1\}$, Y suit une loi de Bernoulli avec paramètre $¶(Y=1) = ¶(\theta > \xi) = \Phi(\theta)$.*

2. *Puisque $\frac{1}{n}\sum_{i=1}^n Y_i \to \mathbb{E}[][Y_1] = \Phi(\theta)$ p.s. et Φ^{-1} est une fonction continue, on a $\hat{\theta}_n = \Phi^{-1}(\frac{1}{n}\sum_{i=1}^n Y_i) \to \Phi^{-1}(\Phi(\theta)) = \theta$ p.s.. Donc $\hat{\theta}_n$ est consistant pour θ.*

3. *En vertu du TCL (car $\mathbb{E}[][Y_1^2] < \infty$), on a $\sqrt{n}(\frac{1}{n}\sum_{i=1}^n Y_i - \Phi(\theta)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathrm{Var}[(]Y_1)) = \mathcal{N}(0, \Phi(\theta)(1-\Phi(\theta)))$. La fonction $\Phi^{-1}(\theta)$ est continument dérivable avec dérivée $(\Phi^{-1})'(\theta) = 1/\varphi(\Phi^{-1}(\theta))$. On obtient par la delta-méthode*

$$\begin{aligned}\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}\left(\Phi^{-1}\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) - \Phi^{-1}(\Phi(\theta))\right) &\xrightarrow{\mathcal{L}} \mathcal{N}(0, ((\Phi^{-1})'(\Phi(\theta)))^2\Phi(\theta)(1-\Phi(\theta))) \\ &= \mathcal{N}\left(0, \frac{\Phi(\theta)(1-\Phi(\theta))}{\varphi^2(\theta)}\right).\end{aligned}$$