N-dimensional Laplace Equation

otissaint74561

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Consider the N-dimensional Laplace equation

$$\nabla^2 f = \sum_{n=1}^{N} \partial_n^2 f = 0 \tag{1}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is some scalar field. Suppose f can be written as a product of scalar valued functions, i.e

$$f(x_i : i = 1, ..., N) = \prod_{j=1}^{N} \Phi_j(x_j) \tag{2}$$

so that the derivatives become

$$\partial_n^2 f = \partial_n^2 \Phi_n(x_n) \prod_{\substack{j=1 \\ j\neq n}}^{N} \Phi_j(x_j) \ . \tag{3}$$

Hence, the Laplace equation can now be written as

$$\sum_{n=1}^{N} \partial_n^2 \Phi_n(x_n) \prod_{\substack{j=1 \\ j\neq n}}^{N} \Phi_j(x_j) = 0 \ . \tag{4}$$

Dividing both sides of this equation by (2) then yields,

$$\sum_{n=1}^{N} \frac{\partial_n^2 \Phi_n}{\Phi_n} = 0 \ . \tag{5}$$

This equation is satisfied iff

$$\frac{\partial_n^2 \Phi_n}{\Phi_n} = -\lambda_n \implies \partial_n^2 \Phi_n + \lambda_n \Phi_n = 0, \qquad n < N \tag{6}$$

$$\frac{\partial_N^2 \Phi_N}{\Phi_N} = \sum_{k=1}^{N-1} \lambda_k \implies \partial_N^2 \Phi_N - \Phi_N \sum_{k=1}^{N-1} \lambda_k = 0 \tag{7}$$

The corresponding characteristic equations are thus

$$K_n^2 + \lambda_n = 0 \implies K_n = \pm i\sqrt{\lambda_n} \tag{8}$$

$$K_N^2 - \sum_{k=1}^{N-1} \lambda_k = 0 \implies K_N = \pm\sqrt{\sum_{k=1}^{N-1} \lambda_k} \tag{9}$$

Hence, the solutions are

$$\Phi_n(x_n) = C_{1,n} \cos(x_n\sqrt{\lambda_n}\ ) + C_{2,n} \sin(x_n\sqrt{\lambda_n}) \tag{10}$$

$$\Phi_N(x_N) = D_{1,N} \sinh\left(x_N\sqrt{\sum_{k=1}^{N-1} \lambda_k}\right) + D_{2,N} \cosh\left(x_N\sqrt{\sum_{k=1}^{N-1} \lambda_k}\right) \tag{11}$$

We now introduce some boundary conditions

$$\begin{cases} f(x_n = 0) & = 0 \implies \Phi_n(0) = 0, \ \forall n \\ f(x_n = L_n) & = 0 \implies \Phi_n(L_n) = 0, \ \forall n < N \\ f(x_N = \pi) & = \Psi(x_n : n < N) \end{cases} \tag{12}$$

where $\Psi : \mathbb{R}^{N-1} \to \mathbb{R}$. The first boundary condition then yield

$$\Phi_n(0) = C_1 = 0 \tag{13}$$

$$\Phi_N(0) = D_2 = 0 \tag{14}$$

The second boundary condition together with Φ_n then gives the following Sturm-Liouville boundary value problem

$$\Phi_n(L_n) = C_{2,n} \sin(L_n \sqrt{\lambda_n}) = 0 \tag{15}$$

which is solved by the eigenvalue

$$\sqrt{\lambda_n} = \frac{\alpha_n \pi}{L_n} \tag{16}$$

where $\alpha_n \in \mathbb{Z}$. Thus,

$$\Phi_n(x_n) = C_{\alpha_n} \sin\left(\frac{\alpha_n \pi x_n}{L_n}\right), \quad n < N \tag{17}$$

$$\Phi_N(X_n) = D_{\alpha_1,\dots,\alpha_{N-1}} \sinh\left(x_N \sqrt{\sum_{k=1}^{N-1} \frac{\alpha_k^2 \pi^2}{L_k^2}}\right) \tag{18}$$

To this end, we can write one linearly independent solution to the N-dimensional Laplace equation as

$$f_{\alpha_1,\dots,\alpha_{N-1}} = C_{\alpha_1,\dots,\alpha_{N-1}} \sinh\left(x_N \sqrt{\sum_{k=1}^{N-1} \frac{\alpha_k^2 \pi^2}{L_k^2}}\right) \prod_{j=1}^{N-1} \sin\left(\frac{\alpha_j \pi x_j}{L_j}\right)$$

and hence, by the superposition principle we can write the full solution as the infinite series

$$f = \sum_{\alpha_1=1}^{\infty} \dots \sum_{\alpha_{N-1}=1}^{\infty} C_{\alpha_1,\dots,\alpha_{N-1}} \sinh\left(x_N \sqrt{\sum_{k=1}^{N-1} \frac{\alpha_k^2 \pi^2}{L_k^2}}\right) \prod_{j=1}^{N-1} \sin\left(\frac{\alpha_j \pi x_j}{L_j}\right)$$

The final boundary condition then leads to the following equation

$$\sum_{\alpha_1=1}^{\infty} \dots \sum_{\alpha_{N-1}=1}^{\infty} C_{\alpha_1,\dots,\alpha_{N-1}} \sinh\left(\pi \sqrt{\sum_{k=1}^{N-1} \frac{\alpha_k^2 \pi^2}{L_k^2}}\right) \prod_{j=1}^{N-1} \sin\left(\frac{\alpha_j \pi x_j}{L_j}\right) = \Psi(x_n)$$

Without justification, we now multiply both sides of this equation by

$$\prod_{j=1}^{N-1} \sin\left(\frac{\tilde{\alpha}_j \pi x_j}{L_j}\right)$$

to give

$$\sum_{\alpha_1=1}^{\infty}...\sum_{\alpha_{N-1}=1}^{\infty} C_{\alpha_1,...,\alpha_{N-1}} \sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right) = \Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)$$

We can now integrate both sides over the region $\{0 < x_n < L_n\}$

$$\int_0^{L_1}...\int_0^{L_{N-1}}\sum_{\alpha_1=1}^{\infty}...\sum_{\alpha_{N-1}=1}^{\infty} C_{\alpha_1,...,\alpha_{N-1}} \sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$
$$=\int_0^{L_1}...\int_0^{L_{N-1}}\Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$

moving the integral inside the sum then gives

$$\sum_{\alpha_1=1}^{\infty}...\sum_{\alpha_{N-1}=1}^{\infty} C_{\alpha_1,...,\alpha_{N-1}} \sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\int_0^{L_1}...\int_0^{L_{N-1}}\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$
$$=\int_0^{L_1}...\int_0^{L_{N-1}}\Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\tilde{\alpha}_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$

Notice that due to the orthogonality of the sine function, every term in the series on the LHS will vanish, except for when $\alpha_j = \tilde{\alpha}_j$. Thus,

$$C_{\alpha_1,...,\alpha_{N-1}} \sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\int_0^{L_1}...\int_0^{L_{N-1}}\prod_{j=1}^{N-1}\sin^2\left(\frac{\alpha_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$
$$=\int_0^{L_1}...\int_0^{L_{N-1}}\Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)dx_1...dx_{N-1}$$

Hence, we can express the coefficient $C_{\alpha_1,...,\alpha_{N-1}}$ as

$$C_{\alpha_1,...,\alpha_{N-1}} = \frac{\displaystyle\int_0^{L_1}...\int_0^{L_{N-1}}\Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)\,dx_1...dx_{N-1}}{\displaystyle\sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\int_0^{L_1}...\int_0^{L_{N-1}}\prod_{j=1}^{N-1}\sin^2\left(\frac{\alpha_j\pi x_j}{L_j}\right)\,dx_1...dx_{N-1}}$$

Thus, we can finally write the full solution to the Laplace equation as

$$f(x_1,..,x_N) = \sum_{\alpha_1=1}^{\infty}...\sum_{\alpha_{N-1}=1}^{\infty}\frac{\displaystyle\int_0^{L_1}...\int_0^{L_{N-1}}\Psi(x_n)\prod_{j=1}^{N-1}\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)\,dx_1...dx_{N-1}\,\sinh\left(x_N\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\prod_{j=1}^{N-1}\sin\left(\frac{\alpha_j\pi x_j}{L_j}\right)}{\displaystyle\sinh\left(\pi\sqrt{\sum_{k=1}^{N-1}\frac{\alpha_k^2\pi^2}{L_k^2}}\right)\int_0^{L_1}...\int_0^{L_{N-1}}\prod_{j=1}^{N-1}\sin^2\left(\frac{\alpha_j\pi x_j}{L_j}\right)\,dx_1...dx_{N-1}}$$