22nd ~~March~~ August 2024

Express $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ in the form

$z = a + ib$; $2\cos 60 + 2i\sin 60$

~~Conjugate~~

Conjugate of Complex Numbers in Polar Form

That is: $z = r(\cos\theta + i\sin\theta)$ -- the complex no

$\bar{z} = r(\cos\theta - i\sin\theta)$ --- the conjugate

Find (i) $z + \bar{z}$

(i) $z + \bar{z} = r(\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta})$

$= \underline{2r\cos\theta}$.

(ii) $z - \bar{z} = r(\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta)$

$= \underline{2ir\sin\theta}$

(iii) $z \cdot \bar{z} = (r\cos\theta + ri\sin\theta)(r\cos\theta - ri\sin\theta)$

$= r^2$ (iv) $\frac{z}{\bar{z}} = \frac{z^2}{r^2}$

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find the value of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

60
90

$\cos\frac{\pi}{2}$

Assignment:

Cancelled

1. ~~Using Integration by parts and reduction formular, if~~

~~$I_{m,p} = \int x^m (a + bx$~~

1. Express $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ in the polar form. Then evaluate: (i) $z\bar{z}$ (ii) $\frac{z}{\bar{z}}$ (iii) $z + \bar{z}$ (iv) $z - \bar{z}$

Solution

1. $z\bar{z} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{4} - \cancel{\frac{i\sqrt{3}}{4}} + \cancel{\frac{i\sqrt{3}}{4}} + \frac{3}{4}$

$= \frac{4}{4}$

$= \underline{\underline{1}}$

(ii) $\frac{z}{\bar{z}} = \frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{1}{2} - i\frac{\sqrt{3}}{2}} \times \frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$= \frac{\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^2 - \left(i\frac{\sqrt{3}}{2}\right)^2}$

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$$= \frac{\frac{1}{4} + 2 \cdot \frac{1}{2} \cdot i\frac{\sqrt{3}}{2} + \frac{3}{4}}{\frac{1}{4} + \frac{3}{4}}$$

$$= 1 + \frac{i\sqrt{3}}{2}$$

(iii) $Z + \bar{Z} = \frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} - \frac{i\sqrt{3}}{2}$

$$= 1$$

(iv) $Z - \bar{Z} = \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$= \frac{2i\sqrt{3}}{2}$$

$$= i\sqrt{3}$$

Operations on Complex Numbers 22nd August 2024

(a) Addition & subtraction

$Z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$Z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$Z_1 \pm Z_2 = (x_1 \pm x_2) \pm i(y_1 \pm y_2)$

$= r_1 \pm r_2 (\cos\theta_1 \pm \cos\theta_2 + i(\sin\theta_1 \sin\theta_2)$

(b) Multiplication and Division.

$Z_1 \times Z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 = x_1x_2 + i(x_1y_2 + x_2y_1) - y_1y_2$

$Z_1 \cdot Z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)]$

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$= r_1 r_2 [\cos\theta_1 + \cos\theta_2 + i \cos\theta_1 \sin\theta_2 + i \sin\theta_1 \cos\theta_2 +$
$i^2 \sin\theta_1 \sin\theta_2]$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)]$

$\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2} i$

Example 1

Let $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $z_2 = 1+i$. Then find –

(i) $z_1 + z_2$ (ii) $z_1 - z_2$ (iii) $z_1 z_2$

Solution

(i) $z_1 + z_2 = \frac{3}{2} + \frac{2+\sqrt{3}}{2} i$

(ii) $z_1 - z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i - (1+i) = \frac{1}{2} - 1 + \frac{\sqrt{3}}{2} i - i$

$= -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2} - 1\right) = -\frac{1}{2} + i\left(\frac{\sqrt{3}-2}{2}\right)$

(iii) $z_1 z_2 =$

$r_1 = |z_1| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{4}{4}} = 1$; $\tan\theta_1 = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

$r_2 = |z_2| = \sqrt{1^2+1^2} = \sqrt{2}$ $\qquad \theta_1 = \tan\sqrt{3} = 60°$

So, $z_1 z_2 = \sqrt{2}(\cos 105 + i \sin 105)$

Remark: If z_1, z_2 & z_3 with

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ & $z_3 = x_3 + iy_3$

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Then: $z_1 \cdot z_2 \cdot z_3 = r_1 r_2 r_3 \left(\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\right)$

Then for z

$$\prod_{j=1}^{n} z_j = \prod_{j=1}^{n} r_j \left(\cos \sum_{j=1}^{n}\theta_j + i\sin \sum_{j=1}^{n}\theta_j\right)$$

Division of Complex Numbers:

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2} = \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2+y_2^2}$$

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{(\cos\theta_2)^2 + \sin^2\theta_2}\right]$$

$$= r_1/r_2\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$

$$= \cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$$

$$[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] = r_3[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

where $r_3 = r_1/r_2$.

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De' Movier's Theorem

Let $z = x + iy$ then (i) z^2 (ii) z^3

$z^2 = (x+iy)(x+iy) = x^2 + 2ixy - y^2$

Let $z = r(\cos\theta + i\sin\theta)$

$z^2 = r(\cos\theta + i\sin\theta)\, r(\cos\theta + i\sin\theta)$

$= r^2[\cos^2\theta - \sin^2\theta \cos\theta + i\sin\theta\cos\theta + i^2\sin^2\theta]$

$= r^2[\cos 2\theta + i\sin 2\theta]$

Remark $z^3 = r^3(\cos 3\theta + i\sin 3\theta]$

$\cos^2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$

So, for $\underbrace{z \cdot z \cdot \cdot z}_{n\,times} = z^2 = r^n(\cos n\theta + i\sin n\theta)$

Thus De' Moivre's Theorem

(i) $\prod_{j=1}^{n} z_j = \prod_{j=1}^{n} r_j \left(\cos \sum_{j=1}^{n}\theta_j + i\sin \sum_{j=1}^{n}\theta_j\right)$

(ii) $z^n = r^n(\cos n\theta + i\sin n\theta)$

Find z^3 if $z = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

<u>Solution.</u>

$z^3 = 2^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$\boxed{z^{-n} = \frac{1}{z^n}}$ Important

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$$Z^{-3} = \frac{1}{8(\cos\frac{\pi}{2} + i\sin\frac{3\pi}{2})}$$

Example

Using De'Movire's theorem find:

(i) $\cos 2\theta$ (ii) $\sin 2\theta$ (iii) $\cos 2\theta$ (iv) $\sin 2\theta$.

Solution.

$$Z^n = (c+is)^n = c^2 + ics + ics + i^2 s^2$$

$$= c^2 + i(2cs) - s^2$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta)$$

$$= \cos^2\theta + \cos^2\theta - 1$$

$$= 2\cos^2\theta - 1$$

(ii) $\sin 2\theta = 2i\cos\theta\sin\theta = 2i(\sqrt{1-\sin^2\theta})\sin\theta$

$$= 2i(1-\sin^2\theta)^{\frac{1}{2}}\sin\theta$$

Do (iii) & (iv) on your own. {see page 116}

Exponential and Circular Functions.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{2!} + \cdots$$

find $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^{34}}{4!} + \cdots$$

Let $z = 1+i$, then find $\underline{\underline{e^z}}$

$$e^z = 1 + 1 + i + \frac{(1+i)^2}{2!} + \frac{(1+i)^3}{3!} + \cdots$$

$$= 2 + i + \frac{1 + 2i - 1}{2!} + \cdots$$

$$= 2 + 2i + \cdots$$

odd function Circular functions.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

If $z = 1+i$, then find (i) $\sin z$ (ii) $\cos z$ expand about 3.

$1 - (1+i)^2$

$$z = 1 + i - \frac{(1+i)^3}{3!} + \frac{(1+i)^5}{5!} + \cdots$$

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$$z = \frac{1+i - [1 + 3(1)^2 i + 3(1)(i)^2 +}{}$$

$z = re^{i\theta} = r(\cos\theta \; i\sin\theta)$

$r = e^{i(\theta + 2k\pi)}$ where K is a positive integer

$e^{-i\theta} = \cos\theta - i\sin\theta$

$x - iy$

Express $z = (4+3i)e^{i\pi/3}$ in the form z in $u+iv$

Hyperbolic Functions

$y = \sinh x$, $y = \cosh x$ $y = \tanh x$

$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$ $\cosh z = \frac{e^{z} + e^{-z}}{2}$

$\cosh\theta = e^{\theta} + e^{-\theta}$ $\tanh z = \frac{\sinh z}{\cosh z}$

~~Cosh θ~~ $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$

$\sinh z = \frac{e^{z} - e^{-z}}{2}$