OLD SHOP 2

Tonia's shop
OLD SHOP 2

LINEAR ALGEBRA

MTH 230

DEMYSTIFIED

SUMMARY, PAST QUESTIONS AND SOLUTIONS (UPDATED)

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OLD PASTS

COURSE OUTLINE

- SET
- FUNCTIONS
- MATRICES
  - Matrix algebra
  - Types of matrices
  - Inverse
  - Rank
  - Systems of equations
  - L U Decomposition
  - Eigenvalues and Eigenvectors
  - Quadratic forms
- VECTOR SPACES
- Basis
- Dimension
- LINEAR MAPPINGS/TRANSFORMATION

YEARS SOLVED

2015

2014

2013

2017 + TEST

SUMMARY (II) DETERMINANTS

The notation of determinants arises from the process of elimination of the unknowns of a simultaneous linear equation: i.e.

$$a_1x + b_1 = 0 \quad \Rightarrow x = \frac{-b_1}{a_1} \text{ (Sub. into ii)}$$
$$a_2x + b_2 = 0$$

(A) 2X2

gives $a_1b_2 - a_2b_1 = 0$

which is written as;

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \quad \text{i.e.} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1b_2 - a_2b_1 = 0$$

e.g.

Expand: $\begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$

$8 - 15 = -7$

$$\Rightarrow \begin{vmatrix} \overset{+}{8} & \overset{-}{5} \\ 3 & 1 \end{vmatrix} = (8)(1) - (3)(5) = 8 - 15 = 17$$

(This is for 2x2 det.)

(B) 3X3 Consider

Laplace Expansion method

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix} \text{ (Rows)}$$

$\downarrow C_1 \quad \downarrow C_2 \quad \downarrow C_3$ (Columns)

We evaluate the minors of a_1, b_1, c_1 with their sign conventions

$$\begin{vmatrix} \overset{+}{a_1} & \overset{-}{b_1} & \overset{+}{c_1} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Longrightarrow + a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

(1) i.e. to get the minors of "a_1" close the row & column of a_1

i.e. the minors of a_1, b_1 and c_1 are.

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & \cancel{b_2} & c_2 \\ a_3 & \cancel{b_3} & c_3 \end{vmatrix} \qquad \begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & b_2 & \cancel{c_2} \\ a_3 & b_3 & \cancel{c_3} \end{vmatrix}$$

for $a_1 \Rightarrow \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ for $b_1 \Rightarrow \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ for $c_1 \Rightarrow \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Example:-

Evaluate;

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} \Rightarrow \begin{vmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{5} \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$$

$$\Rightarrow +2\begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} - 3\begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} + 5\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix}$$

$$= +2(7-0) - 3(28-0) + 5(8-6) = -60$$

(C) 4X4

Consider

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \Rightarrow \begin{vmatrix} \overset{+}{a_1} & \overset{-}{b_1} & \overset{+}{c_1} & \overset{-}{d_1} \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Get minors a_1, b_1, c_1 & d_1

$$\Longrightarrow a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

Example: Evaluate $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 1 & -1 & 2 & 1 \\ 3 & 0 & 4 & 1 \\ -1 & -2 & 0 & 5 \end{vmatrix} \Rightarrow \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{0} & \overset{-}{3} \\ 1 & -1 & 2 & 1 \\ 3 & 0 & 4 & 1 \\ -1 & -2 & 0 & 5 \end{vmatrix}$

$$\Rightarrow 1\begin{vmatrix} -1 & 2 & 1 \\ 0 & 4 & 1 \\ -2 & 0 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -1 & 0 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 & 1 \\ 3 & 0 & 1 \\ -1 & -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -1 & -2 & 0 \end{vmatrix}$$

(multiplication by zero)

$$\Rightarrow 1\begin{vmatrix} -1 & 2 & 1 \\ 0 & 4 & 1 \\ -2 & 0 & 5 \end{vmatrix} \Rightarrow \begin{vmatrix} \overset{+}{-1} & \overset{-}{2} & \overset{+}{1} \\ 0 & 4 & 1 \\ -2 & 0 & 5 \end{vmatrix} \Rightarrow -1\begin{vmatrix} 4 & 1 \\ 0 & 5 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ -2 & 5 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix}$$

$$= -1(20) - 2(2) + 1(8)$$

$$= \underline{\underline{-16}}$$

And

$$\Rightarrow -3\begin{vmatrix} +1 & 2 & 2 \\ 3 & 0 & 4 \\ -1 & -2 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} \overset{+}{+1} & \overset{-}{-2} & \overset{+}{2} \\ 3 & 0 & 4 \\ -1 & -2 & 0 \end{vmatrix} \Rightarrow$$

Note that $-(-2) = +2$

$$= -3\left[1\begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix}\right]$$

$$= -3\left[1(8) + 2(4) + 2(-6)\right] = -3(4) = \underline{\underline{-12}}$$

Ans $= -16 + 0 + 0 - 12 = \underline{\underline{-28}}$

Other methods for evaluating determinants are

(1) Product [illegible]

(2) Using Sarru's Expansion method to solve the above 3X3 i.e $\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$ (Repeat the shaded columns to the left of the det.)

$$\Rightarrow \begin{vmatrix} \overset{+}{2} & \overset{+}{3} & \overset{+}{5} & \overset{-}{2} & \overset{-}{3} \\ 4 & 1 & 0 & 4 & 1 \\ 6 & 2 & 7 & 6 & 2 \end{vmatrix}$$

(and cross multiply mentaining the sign convention as shown)

$$\Rightarrow (2)(1)(7) + \cancel{(3)(0)(6)} + (5)(4)(2) - (3)(4)(7) - \cancel{(2)(0)(2)} - (5)(1)(6)$$

$$= 14 + 40 - 84 - 30 = +24 - 84 = \underline{\underline{-60}}$$

as before see page (1).

PROPERTIES OF DETERMINANTS

(1) The value of a determinant remains 'unaltered', if the rows are interchanged into columns or columns into rows i.e

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \simeq \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

All columns to rows & all rows to columns

Example as above

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} \qquad \begin{vmatrix} 2 & 4 & 6 \\ 3 & 1 & 2 \\ 5 & 0 & 7 \end{vmatrix}$$

Clearly $\begin{vmatrix} 2 & 5 \\ -3 & 6 \end{vmatrix} = (2)(6) - (-3)(5) = 12 + 15 = 27$

$= \begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix} = (2)(6) - (5)(-3) = 12 + 15 = 27$

(2) If two rows or two columns interchanged, the sign of the determinant's value changes.

i.e. $\begin{array}{c} R_1 \\ R_2 \\ \\ \end{array}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{array}{c} R_2 \\ R_1 \\ \\ \end{array} = (-ve).$

Note that $R_1 \Leftrightarrow R_2$ (was interchanged with R_2)

also $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}$ $C_2 \Leftrightarrow C_3$ Column (2) enterchanged with column (3).

Example

$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} \quad C_1 \Leftrightarrow C_3 \quad = - \begin{vmatrix} 5 & 3 & 2 \\ 0 & 1 & 4 \\ 7 & 2 & 6 \end{vmatrix} = -(60) = \underline{-60}$

(3) If two rows (or columns) are identical, or one is a multiple of the other, the value of the det is zero.

i.e. $\begin{array}{c} R_1 \\ R_2 \\ \\ \end{array}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$ and e.g. $\begin{array}{ccc} & C_2 & C_3 \end{array}$ $\begin{vmatrix} 6 & 1 & 1 \\ 7 & 2 & 2 \\ 9 & 5 & 5 \end{vmatrix} = 0$ because $C_2 = C_3$

because $R_1 = R_2$

(3) (4) If the elements of any row (or column) of a det. is each multiplied by the same number then the det. is multiplied by that number.

i.e $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

also $\begin{vmatrix} a_1 & kb_1 \\ a_2 & kb_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. i.e $\begin{vmatrix} 1 & 4 \\ 3 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$

(5) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + mc_3 & b_3 & c_3 \end{vmatrix}$ Not necessary (Skip).

Application:

Consider the 4X4 det. $\begin{vmatrix} 5 & 4 & 1 & 0 \\ -7 & 2 & 3 & 5 \\ 8 & 0 & 0 & 0 \\ 9 & 6 & 2 & 7 \end{vmatrix}\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$

Observe that Row (3) contains more zero's than the rest rows then interchange $R_1 \Leftrightarrow R_3$ so that we can easily evaluate on the element (entry) without zero.

$\Rightarrow R_1 \Leftrightarrow R_3 = - \begin{vmatrix} 8 & 0 & 0 & 0 \\ -7 & 2 & 3 & 5 \\ 5 & 4 & 1 & 0 \\ 9 & 6 & 2 & 7 \end{vmatrix}$ So that we expand only the first entry (8) and the rest are zeros

$-8 \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = -8(-60) = \underline{+480}$

APPLICATION OF DETERMINANT

CRAMMER'S RULE: (Solving systems of equ.)

1) $a_1x+b_1y=d_1$
$a_2x+b_2y=d_2$ $\therefore x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$ where

2) $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x=\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$, $D_y=\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

Example; Solve.

1) $3x-y=7$
$-5x+4y=-2$, $D=\begin{vmatrix} 3 & -1 \\ -5 & 4 \end{vmatrix}=\cancel{2}$, $D_x=\begin{vmatrix} 7 & -1 \\ -2 & 4 \end{vmatrix}=\underline{26}$

$D_y=\begin{vmatrix} 3 & 7 \\ -5 & -2 \end{vmatrix}=29$.

$\left(x=\frac{26}{7}, y=\frac{29}{7}\right)$ (Hint) Replace x column with d for Dx

3) Example ② Solve;

$a_1x+b_1y+c_1z=d_1$
$a_2x+b_2y+c_2z=d_2$
$a_3x+b_3y+c_3z=d_3$

$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D_x=\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \end{array}$, $D_y=\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \end{array}$, $D_z=\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \end{array}$

x-column ⟺ d column

y column ⟶ d column

z column ⟺ d-column

④ Example: Solve; $x+y+z=6$
$x+2z=7$
$3x+y+z=12$

$\Rightarrow$ $1x+1y+1z=6$
$1x+0y+2z=7$
$3x+1y+1z=12$

$\therefore D=\begin{vmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{1} \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix}=1(0-2)-1(1-6)+1(1-0)=-2+5+1=\underline{4}$

$D_x=\begin{vmatrix} \overset{+}{6} & \overset{-}{7} & \overset{+}{1} \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{vmatrix}=6(0-2)-1(7-24)+1(7-0)$
$=-12+17+7=\underline{12}$

$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 1 & 7 & 2 \\ 3 & 12 & 1 \end{vmatrix}=1(7-34)-6(1-6)+1(12-2)$
$=-17+30-9=\underline{4}$

$D_z=\begin{vmatrix} 1 & 1 & 6 \\ 1 & 0 & 7 \\ 3 & 1 & 12 \end{vmatrix}=1(0-7)-1(12-21)+6(1-0)$
$=-7+9+6$
$=\underline{8}$

$\frac{D_x}{D}=\frac{12}{4}=3$ $\frac{D_y}{D}=\frac{4}{4}=1$ $\frac{D_z}{D}=\frac{8}{4}=2$

$x=\frac{D_x}{D}=\frac{12}{4}=\underline{3}$, $y=\frac{D_y}{D}=\frac{4}{4}=1$, $z=\frac{D_z}{D}=\frac{8}{4}=\underline{2}$

$\therefore x=3, y=1, z=2$

Ans $=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

SUMMARY (I) MATRICES (ALGEBRA) (5)

A Matrix A over a field K is a rectangular array of scalars usually rep. as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \to \text{Rows } (m)$$

Columns (n)

∴ The Matrix size is $(m \times n)$ or (m by n).

Note:

The element a_{ij} called the ij-entry (element) appears in row (i) and column (j).

∴ If a Matrix A = Matrix B then A and B are of the same size and equal entries.

Definition:

(i) A Matrix with one row $(a_{11}\ a_{12} \cdots a_{1n})$ is called a row matrix (vector) while the matrix with one Column $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ is called a Column matrix (Vector).

(ii) A Matrix with zero entries is called a zero matrix denoted by 0.

(iii) Matrix whose entries are all real nos are called real Matrices i.e matrix over a field $\mathbb{R}$.

(iv) Matrix whose entries are all Complex nos are called Complex Matrices i.e matrix over $\mathbb{C}$ (Complex nos).

MATRIX ADDITION: (±)

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then

$$A \pm B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

SCALAR MULTIPLICATION:

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $kA = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

MATRIX MULTIPLICATION: (X)

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then

$$A \times B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Two matrices are multiplicable iff the number of Column in the first = the number of rows in the second matrix

Observe above that matrix A = 2-Columns and B = 2-rows

$$A \times B = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

1×4 : 4×3

Consider:

$A = [a\ b\ c\ d]$, $B = \begin{bmatrix} e & f & g \\ h & i & j \\ k & l & m \\ n & p & q \end{bmatrix}$

Column A = Row B i.e

∴ A×B is possible (4 = 4)

But B×A is not possible

Reason $B = \begin{bmatrix} e & f & g \\ h & j & i \\ k & l & m \\ n & p & q \end{bmatrix}$. $A = [a, b\ c\ d]$ (6)

Observe that Column $B \neq$ Row A

i.e $3 \neq 1$

Hence ($B \times A$ is undefined).

Example: (1)

If $A = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ find (i) $A \times B$ & (ii) $B \times A$

(1) $A \times B = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} = \begin{bmatrix} (2)(3)+(4)(1) & (2)(6)+(4)(2) \\ (1)(3)+(0)(1) & (1)(6)+(0)(2) \end{bmatrix}$

$= \begin{bmatrix} 10 & 20 \\ 3 & 6 \end{bmatrix}$

$B \times A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} (3)(2)+(6)(1) & (3)(4)+(6)(0) \\ (1)(2)+(2)(1) & (1)(4)+(2)(0) \end{bmatrix}$

$= \begin{bmatrix} 12 & 12 \\ 4 & 4 \end{bmatrix}$

Observe that

$A \times B \neq B \times A$ (Hence Matrix multiplication is not commutative.)

Example (II)

If $X = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$.

find (i) $(X \times Y)$ (ii) $(Y \times X)$ (iii) Y^2

Solution

(1) $X \times Y = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$

i.e 2 multiplies 1, 0, 3

1 multiplies 2, 1, −1

0 multiplies 4, 1, 0

$\Rightarrow \begin{pmatrix} 2+2+0 & 0+1+0 & 6+(-1)+0 \\ 3+4+4 & 0+2+1 & 9-2+0 \end{pmatrix} = \begin{bmatrix} 4 & 1 & 5 \\ 11 & 3 & 7 \end{bmatrix}$

(ii) $Y \times X = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ Not multiplicable $\because$ column $Y \neq$ Row X.

Y . X

(iii) $Y^2 = Y \times Y = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1+0+12 & 0+0+3 & 3+0+0 \\ 2+2-4 & 0+1-1 & 6-1+0 \\ 4+2+0 & 0+1+0 & 12-1+0 \end{pmatrix} = \begin{pmatrix} 13 & 3 & 3 \\ 0 & 0 & 5 \\ 6 & 1 & 11 \end{pmatrix}$

LAWS: for any given Matrices A, B, C and scalar k

(i) $A \times B \neq B \times A$

(ii) $(AB)C = A(BC)$ — associative

(iii) $A(B+C) = AB + AC$ — left distributive

(iv) $(B+C)A = BA + CA$ — right distributive

(v) $k(AB) = (kA)B = A(kB)$ for scalar $k \in F$

Note that [illegible]

TYPES OF MATRICE

① **Zero/Null Matrix**: i.e. Matrix with zero entries $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ etc.

② **Square matrix**: A Matrix in which number of columns and rows are equal i.e. an $(m \times m)$ or an $(n \times n)$ matrix (Matrix that forms a Square)

e.g $[a]$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ i.e. $A = M_{m \times m}$

(1X1) (2X2) (3X3)

③ **Transpose of a Matrix**: $(A^T)/(A')$, It is a Matrix obtained by writing the columns of A as rows. If $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = M_{nm}$ if $A = M_{mn}$.

Eg $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

BASIC PROPERTIES OF TRANSPOSE OPERATION

Let A & B be a matrix and let K be a scalar then;

(i) $(A+B)^T = A^T + B^T$ (De morgan's law)

(ii) $(A^T)^T = A$ Complementation (double)

(iii) $(KA)^T = KA^T$

(iv) $(AB)^T = B^T A^T$ [whenever the sum and product are defined]

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④ **Diagonal & Trace**: Let $A = M_{m \times m}$ i.e. $[a_{ij}]$ square matrix then the diagonal elements consist of elements $(a_{11}, a_{22}, a_{33} \ldots a_{mm})$ & Trace $= (a_{11} + a_{22} + \ldots + a_{mm})$

e.g $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (diagonal elements) (sum of the entries.)

leading/Main/principal diagonal

⑤ **Diagonal matrix**: A Matrix with only diagonal elements i.e. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ or $A = \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$

i.e. a Matrix that $[a_{ij}] = 0$ if $i \neq j$ (i.e. only $a_{11}, a_{22} \ldots a_{mm}$ are left in the matrix) that is if $i<j$ $[a_{ij}] = 0$ & if $i>j$ $[a_{ij}] = 0$

⑥ **Identity/Unit Matrix**: A diagonal Matrix with unit diagonal elements i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc. Denoted by (I or I_n). So that $A \cdot I = A$ $\forall$ matrix A.

Note: A diagonal Matrix is a square matrix. An identity matrix is a diagonal matrix.

Same of the transpose

(7) Singular Matrix: A Matrix with zero determinant
if $A=[a_{ij}]$ — square matrix and $\det(A)=|A|=0$
then A is singular.

(8) Triangular Matrix:

Lower triangular Matrix

$$A=\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$ → entries above the leading diagonal are zeros

i.e $[a_{ij}]=0$ if $i<j$

Upper triangular Matrix

$$B=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33}\end{bmatrix}$$ (entries below the leading diagonal are zeros)

i.e $[a_{ij}]=0$ if $i>j$

The above is usually called the (LU form).

RHEMA: The det. of a triangular Matrix is the multiplication of the entries in the leading diagonal [or a product of its diagonal (Main) elements]. i.e $(a_{11}\times a_{22}\times a_{33}\times\cdots\times a_{nn})$

e.g if $A=\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 6 \\ 0 & 0 & -7\end{bmatrix}$ ∴ $\det(A)=\begin{vmatrix} 1 & 9 & 7 \\ 0 & 5 & 6 \\ 0 & 0 & -7\end{vmatrix}$

$=(1)(5)(-7)=-35$

$1\times5\times-7=-35$

(9) Symmetric Matrix: A Matrix that is equal to its own transpose i.e if $A=A^T$ then A is a Symmetric Matrix.

e.g $A=\begin{bmatrix} 9 & f \\ f & b\end{bmatrix}$ ∴ $A^T=\begin{bmatrix} 9 & f \\ f & b\end{bmatrix}$ ∴ $A=A^T$

(10) Skew/Anti-Symmetric: A Matrix A is Skew-Symmetric if $A=-A^T$ or $-A=A^T$.

e.g if $A=\begin{bmatrix} 0 & -3 \\ 3 & 0\end{bmatrix}$; $A^T=\begin{bmatrix} 0 & 3 \\ -3 & 0\end{bmatrix}$ Observe that ($A^T=-A$)

∴ $-A=\begin{bmatrix} 0 & -(-3) \\ -3 & 0\end{bmatrix}=\begin{bmatrix} 0 & +3 \\ -3 & 0\end{bmatrix}=A^T$

(11) Invertible (Non-Singular Matrix): opp. of singular matrix

A Matrix whose determinant is not zero.
If A is Invertible if $\det(A)=|A|\neq 0$.

(12) Orthogonal Matrix: A real matrix is orthogonal if $A^T=A^{-1}$ i.e $AA^T=A^T.A=I$ where (A^{-1} & I denote the Inverse & Identity Matrix of A.

Note: A must necessarily be square & invertible

Lower triangular matrix

(13) Inverse of a Matrix (A^{-1}):

If $A, B = M_{m m}$ square & invertible matrix such that B = Inverse of A then $A \cdot B = B \cdot A = I$ (where I = Identity Matrix)

$\therefore$ Inverse of A is denoted by A^{-1} $\therefore$ $B = A^{-1}$.

Inverse of 2x2 Matrixes

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

where $|A| = \det(A)$.

Example

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ $\therefore$ $\det(A) = (2)(5) - (1)(3) = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}$

$= 7$

$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$ $\left[\begin{array}{l} \text{i.e Interchange } a \leftrightarrow d \\ \text{and Multiply } b \,\&\, c \text{ by } (-1) \end{array}\right]$

Observe that $A \cdot A^{-1} = A^{-1} \cdot A = I$

i.e

$A \cdot A^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$

$= \frac{1}{7}\begin{pmatrix} 10-3 & -6+6 \\ 5-5 & -3+10 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 7/7 & 0/7 \\ 0/7 & 7/7 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (Identity)

(14) Co-Factor of a Matrix:

Suppose A is a matrix, $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ the Cofactor of A written as A^{cf} or A^c =

$$\begin{bmatrix} +\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} & +\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & +\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ +\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & +\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{bmatrix}$$

where $+\begin{vmatrix} b_2 & c_2 \\ c_3 & c_3 \end{vmatrix}$ = the Cofactor of the entry a_1

and $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ = Cofactor of b_1 and so on.

(15) Adjoint of a matrix:

This is the transpose of the Cofactor of a matrix A. (A^{adj}) i.e

If $A = M_{m \times m} = [a_{ij}]$ — square then the adjoint of A: $A^{adj} = (A^c)^T = A^{cT}$

(16) A Scalar Matrix:

Is a diagonal matrix in which the diagonal elements are equal.

i.e $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ or $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ etc.

i.e the diagonals are some Scalar k etc.

$-35+0+0$

$a \quad f$

$f \quad b$

$a \quad f$

$f \quad b$

Symmetric matrix

INVERSE of 3X3 MATRICES

Suppose $A = M_{3\times 3}$ then the Inverse of A is

$$A^{-1} = \frac{1}{|A|}\cdot A^{adj} \quad \text{where } A^{adj} = \text{adjoint of } A.$$

and $|A| = \det.$ of A.

Example:

Find the Inverse of $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

1st we find Cofactor

$$A^c = \begin{bmatrix} +\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} +(1) & -(2) & +(-2) \\ -(1) & +(3) & -(-3) \\ +(0) & -(4) & +(-3) \end{bmatrix}$$

$$\therefore A^c = \begin{pmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{pmatrix} \quad \therefore A^{adj} = A^{cT} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) + 3(2-0) + 4(-2+0)$$
$$= 3 + 6 - 8 = \underline{1}$$

$$A^{-1} = \frac{1}{|A|} A^{adj} = \frac{1}{1}\cdot\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

OTHER METHODS: Consider the Matrix above)

$$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} +3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \\ +3 & -3 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$

then add to bottom

$$\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{pmatrix} \sim$$

add to the right

$$\sim \begin{pmatrix} 3 & -3 & 4 & 3 & -3 \\ 2 & -3 & 4 & 2 & -3 \\ 0 & -1 & 1 & 0 & -1 \\ 3 & -3 & 4 & 3 & -3 \\ 2 & -3 & 4 & 2 & -3 \end{pmatrix}$$

then evaluate Neglecting R_1 & C_1

$$\begin{pmatrix} 3 & -3 & 4 & 3 & -3 \\ 2 & -3 & 4 & 2 & -3 \\ 0 & -1 & 1 & 0 & -1 \\ 3 & -3 & 4 & 3 & -3 \\ 2 & -3 & 4 & 2 & -3 \end{pmatrix}$$

i.e

$$\begin{matrix} -3 & 4 & 2 & -3 \\ -1 & 1 & 0 & -1 \\ -3 & 4 & 3 & -3 \\ -3 & 4 & 2 & -3 \end{matrix}$$

Evaluate

write

$$\begin{pmatrix} -3+4 & 0-2 & -2+0 \\ -4+3 & 3-0 & 0+3 \\ -12+12 & 8-12 & -9+6 \end{pmatrix}$$

(Evaluate Row wise and write out)

$$A^c \Rightarrow \begin{pmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{pmatrix} \text{ then}$$

Transpose to get $A^{adj} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

$\therefore$ $A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 & 0 \\ \ldots \end{pmatrix}$

SOLVING SYSTEMS OF EQUATIONS USING INVERSE METHOD

Consider $3x_1 - 3x_2 + 4x_3 = -1$ Solve;

$2x_1 - 3x_2 + 4x_3 = -3$

$-x_2 + x_3 = -2$

To solve this write in the form $AX = B$ —— (*)

(Argumented Matrix).

$$\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$$

$A \cdot X = B$ where B = Constant(s) column.

Now since $AX = B$ we need to find $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Hence Multiply both sides of (*) by A^{-1} to get

$A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$.

Since $A^{-1}A = AA^{-1} = I$ (where A^{-1}/I are the inverse and Identity of A respectively.

$\therefore (A^{-1}A)X = A^{-1}B$

$\Rightarrow I \cdot X = A^{-1}B \quad \therefore \underline{X = A^{-1}B}$ —— (**)

Since $IX = X$, i.e. $AI = A$ $\forall$ (for all Matrix A)

$X = A^{-1}B$

So that the solution to the above $AX = B$ is $\boxed{X = A^{-1}B}$

we need to get A^{-1} from the matrix A.

recall from page (10) that $A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$

$\therefore X = A^{-1}B \Longrightarrow$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1+3+0 \\ 2-9+8 \\ 2-9+6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Hence $x_1 = 2,\ x_2 = 1,\ x_3 = -1$

ELEMENTARY TRANSFORMATION

(I) Interchanging any two rows (Column). This is denoted by $R_p \leftrightarrow R_q$ or $(C_p \leftrightarrow C_q)$

(II) Multiplication of the elements of a rows (column) by a non-zero scalar k. denoted by $k \cdot R_p$ or (kC_p).

(III) Addition of a constant multiplication of the elements of any row (Column) R_p (C_p) to the corresponding elements of another row R_q denoted by $(R_q + kR_p)$.

Hence the matrix B is obtained from a matrix A after any of those transformation is

Application: (triangular Form)

Example ① Transform the Matrix $P = \begin{pmatrix} 1 & 3 \\ 6 & -5 \end{pmatrix}$ into triangular form (upper) i.e. $P = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$

i.e we need to turn $a_{21} = 6$ to zero.

So we use $a_{11} = 1$ as pivot.

$\begin{pmatrix} 1 & 3 \\ 6 & -5 \end{pmatrix} \begin{matrix} \to \text{Row } 1 \\ \to \text{Row } 2 \end{matrix}$ (1 - pivot) (How can 6 be made zero. $6-6=0$. Note which ever value we use for the entry in a particular row, will also be used for the entire row.)

R_2 becomes.

$R_2 = R - \frac{6}{1} \cdot R_1$ so that,

$$\begin{pmatrix} 1 & 3 \\ 6 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 6-\frac{6}{1}\cdot 1 & -5-\frac{6}{1}\cdot 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 6-6 & -5-18 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & -23 \end{pmatrix} \text{(U-form)}$$

Example (ii)

Decompose $A = \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 \end{matrix} \\ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & -1 \\ 0 & 5 & 1 \end{pmatrix} \end{array} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$ into U-form i.e

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}$ - Upper T. form (Note that $a_{11} \neq 1$. To make the decomposition easy we make $a_{11} = 1$ by interchanging ...)

To

$C_1 \leftrightarrow C_2 \Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix}$ ∴ to change a_{21}, a_{31} & a_{32} to 0.

Use $a_{11} = 1$ as pivot

Where the measure for $R_2 \Rightarrow \frac{a_{21}}{a_{11}} = \frac{3}{1}$

Row ②, R_2 becomes $R_2 = R_2 - \frac{3}{1} \cdot R_1$

Row ③, R_3 becomes $R_3 = R_3 - \frac{5}{1} \cdot R_1$ (where $\frac{5}{1}$ is the measure for R_3)

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 3-\frac{3}{1}\cdot 1 & 1-\frac{3}{1}\cdot 3 & -1-\frac{3}{1}\cdot 2 \\ 5-\frac{5}{1}\cdot 1 & 0-\frac{5}{1}\cdot 3 & 1-\frac{5}{1}\cdot 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & -8 & -7 \\ 0 & -15 & -9 \end{pmatrix}$$

Now a_{21} & $a_{31} = 0$

Remaining a_{32}.

$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -8 & -7 \\ 0 & -15 & -9 \end{pmatrix}$ → we now use the entry directly above a_{32} as pivot for R_3 in order to make a_{32} become zero.

Dividing R_2 by 8 i.e. $R_2 = \frac{1}{-8} R_2$.

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -8 & -7 \\ 0 & -15 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ \frac{0}{-8} & \frac{-8}{-8} & \frac{-7}{-8} \\ 0 & -15 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & +1 & 7/8 \\ 0 & -15 & -9 \end{pmatrix}$$

Measure for $R_3 \Rightarrow \frac{a_{32}}{a_{22}} = \left(\frac{-15}{+1}\right)$

$R_3 = R_3 - (-15) R_2$

Subject

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 7/8 \\ 0 & -15 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 7/8 \\ 0-\left(\frac{-15}{1}\right)0 & -15-\left(\frac{-15}{1}\right)1 & -9-\left(\frac{-15}{1}\right)\frac{7}{8} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 7/8 \\ 0 & 0 & 33/8 \end{pmatrix} \text{— Upper (U)—Triangular form}$$

Application (II) GAUSSIAN ELIMINATION METHOD

Consider the system $3X_1 - 3X_2 + 4X_3 = -1$

Write the matrix $2X_1 - 3X_2 + 4X_3 = -3$

in the form $A \vdots B$ i.e. $-X_2 + X_3 = -2$.

$$\begin{array}{c} R_1 \to \\ R_2 \to \\ R_3 \to \end{array} \left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 2 & -3 & 4 & -3 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

And then transform A into U form, subjecting Column B to same operation.

Here $a_{31} = 0$ (already)

So we only need to make a_{21} & $a_{31} = 0$.

Since $a_{11} \neq 1$, then the measure for row $R_2 = \frac{a_{21}}{a_{11}} = \frac{2}{3}$

$\therefore R_2$ (becomes $R_2 = R_2 - \frac{2}{3}R_1$)

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 2 & -3 & 4 & -3 \\ 0 & -1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 2-\frac{2}{3}\cdot 3 & -3-\frac{2}{3}(-3) & 4-\frac{2}{3}(4) & -3-\frac{2}{3}(-1) \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 0 & -1 & 4/3 & -7/3 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

To change a_{32} to 0 in R_3 (row 3) the measure is

$$\frac{a_{32}}{a_{22}} = \frac{-1}{-1} = \frac{1}{1}$$

Hence we get ($\therefore R_3 = R_3 - \frac{1}{1}R_2$)

$$\sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 0 & -1 & 4/3 & -7/3 \\ 0-\frac{1}{1}\cdot 0 & -1-\frac{1}{1}(-1) & 1-\frac{1}{1}\cdot\frac{4}{3} & -2-\frac{1}{1}\left(-\frac{7}{3}\right) \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & -1 \\ 0 & -1 & 4/3 & -7/3 \\ 0 & 0 & -1/3 & 1/3 \end{array}\right) \Rightarrow \begin{aligned} 3X_1 - 3X_2 + 4X_3 &= -1 \quad (i) \\ -X_2 + \tfrac{4}{3}X_3 &= -\tfrac{7}{3} \quad (ii) \\ -\tfrac{X_3}{3} &= \tfrac{1}{3} \quad (iii) \end{aligned}$$

From (iii) $-\frac{X_3}{3} = \frac{1}{3}$ $\therefore X_3 = -1$ Sub into (ii)

$\therefore -X_2 + \frac{4}{3}(-1) = -\frac{7}{3}$ $\therefore -X_2 = -\frac{7}{3} + \frac{4}{3} = -1$

$\therefore X_2 = 1$. Sub (X_3 & X_2) into (i)

$\therefore 3X_1 - 3X_2 + 4X_3 = -1$

$\therefore 3X_1 - 3(1) + 4(-1) = -1$ $\therefore 3X_1 = 6$ $\therefore X_1 = 2$

$X_1 = 2,\ X_2 = +1,\ X_3 = -1$

See H.K. DASS (Adv.) Eng Math.

or see Schaum's outline (Linear Algebra).

Application (III) RANK of A MATRIX

The rank of a matrix is said to be r if;

(i) It has at least one non-zero minor of order r.

(ii) Every minor of A of higher order than r is zero.

(Non zero row is that row in which all the elements are not zero).

TRIANGULAR METHOD

To find the rank of a matrix, we transform the matrix to triangular form. then the number of non-zero rows gives the rank of the Matrix.

Consider example (ii) in page (12) after writing the Matrix in U-form we get

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 7/8 \\ 0 & 0 & 33/8 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

Note that each row has an entry $\neq 0$ (at least)

$R_1 = (1\ 3\ 2)$, $R_2 (0, 1\ 7/8)$, $R_3 = (0\ 0\ 33/8)$

$\therefore$ The rank of the Matrix is $\underline{3}$.

Example (i)

find the rank of
$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$
Transform to U-form ..

then : Rank = the number of non-zero rows in the the form.

Using a_{11} as pivot for Row (2) & Row (3).

To turn a_{21} (2) to zero we use $\frac{a_{21}}{a_{11}} = \frac{2}{1}$

To turn a_{31} (1) to zero we use $\frac{a_{31}}{a_{11}} = \frac{1}{1}$

Note: that the triangular (U) form of the above Matrix is of the form;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \to \text{Leading (Main) diagonal.}$$

(where a_{21}, a_{31} & a_{32} would be made zero)

$\therefore$ Row (2) $R_2 = R_2 - \frac{2}{1} R_1$ (Since a_{21} is in R_2)

and Row (3) $R_3 = R_3 - \frac{1}{1} R_1$ (Since a_{31} is in R_3)

(1) - pivot

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2-\frac{2}{1}\cdot 1 & 3-\frac{2}{1}\cdot 2 & 5-\frac{2}{1}\cdot 3 & 1-\frac{2}{1}\cdot 2 \\ 1-\frac{1}{1}\cdot 1 & 3-\frac{1}{1}\cdot 2 & 4-\frac{1}{1}\cdot 3 & 5-\frac{1}{1}\cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$ Multiplying R_2 by (-1) i.e $R_2 = (-1) R_2$ $$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

To turn a_{32} to zero we use a_{22} as pivot

$\therefore$ Measure becomes; $\frac{a_{32}}{a_{22}} = \frac{1}{1}$

$\therefore$ Row (3) ... $R_3 - \frac{1}{1} R_1$ (pivot for a_{32} ...)

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0-\frac{1}{1}\cdot 0 & 1-\frac{1}{1}\cdot 1 & 1-\frac{1}{1}\cdot 1 & 3-\frac{1}{1}\cdot 3 \end{pmatrix}$$

pivot for a_{32}

$$\approx \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Since R_3 = all zeros ∴ R_1 & R_2 are the non zero row

Hence rank = 2.

Observe that trying to turn an entry (a_{ij}) to zero using elementary operation might coincidently turn another entry to zero. eg turning a_{32} to zero caused a_{33} & $a_{34} = 0$. This is due to the nature of this matrix.

Example (11)

find the rank of $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$ Multiply R_1 by (−1).

Here $a_{11} = -1$, it's easy to work with $a_{11} = 1$ as a pivot

then R_1 becomes $R_1 = (-1)R_1$

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

Now $a_{11} = 1$ = pivot for a_{21}; a_{31} & a_{41}.

Measures

$R_2 = \frac{a_{21}}{a_{11}} = \frac{2}{1}$ $\quad R_3 = \frac{a_{31}}{a_{11}} = \frac{3}{1}$

$R_4 = \frac{a_{41}}{a_{11}} = \frac{5}{1}$ (these are the measures for reduction)

∴ $R_2 = R_2 - \frac{2}{1}R_1$, $R_3 = R_3 - \frac{3}{1}R_1$, $R_4 = R_4 - \frac{5}{1}R_1$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 2-\frac{2}{1}(1) & -5-\frac{2}{1}(-2) & 1-\frac{2}{1}(-3) & 2-\frac{2}{1}(2) \\ 3-\frac{3}{1}(1) & -8-\frac{3}{1}(-2) & 5-\frac{3}{1}(-3) & 2-\frac{3}{1}(2) \\ 5-\frac{5}{1}(1) & -12-\frac{5}{1}(-2) & -1-\frac{5}{1}(-3) & 6-\frac{5}{1}(2) \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{pmatrix}$$

Negative signs are always issues ∴ Multiply Row ①, R_2, R_3 & R_4 by (−1)

$R_2 = -R_2$, $R_3 = -R_3$, $R_4 = -R_4$

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 2 & -14 & 4 \\ 0 & 2 & -14 & 4 \end{pmatrix}$$

pivot for a_{32} & a_{42}.

turn a_{32}, a_{42} to zero. Using a_{22} (1) as pivot

Measure for R_3 becomes $\frac{a_{32}}{a_{22}} = \frac{2}{1}$

R_4 becomes $\frac{a_{42}}{a_{22}} = \frac{2}{1}$.

$\left(R_3 = R_3 - \frac{2}{1}R_2\right)$ & $\left(R_4 = R_4 - \frac{2}{1}R_2\right)$.

$$\begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 2 & -14 & 4 \\ 0 & 2 & -14 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 2-\frac{2}{1}(1) & -14-\frac{2}{1}(-7) & 4-\frac{2}{1}(2) \\ 0 & 2-\frac{2}{1}(1) & -14-\frac{2}{1}(-7) & 4-\frac{2}{1}(2) \end{pmatrix}$$

So that we get
$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{array}{l} R_1 \text{ — non zero} \\ R_2 \text{ — non zero.} \\ R_3 \text{ — zero row} \\ R_4 \longrightarrow \text{zero row} \end{array}$$

∴ Rank = 2

∴ No of non zero rows.

ROW ECHELON & CANONICAL FORM OF A MATRIX

/ A matrix
$$\begin{pmatrix} 4 & 3 & -1 & 2 \\ & 1 & 3 & -1 \\ & & & 6 \end{pmatrix}$$
is in row echelon form ∵ all the leading entry of any row are to the right of the leading entry in the preceeding row.

i,e
1 is to the right of 4 and 6 is to the right of 1.

Note that a (U) form (triangular (U) form) of a matrix is in Row-echelon form.

/ A matrix
$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 4 & 0 & -1 & 2 \\ & 1 & 3 & 0 \\ & & & 6 \end{pmatrix}$$
is said to be in Row-Canonical Form i.e an echelon matrix in which the element directly above the leading entry of any Row = 0.

i.e no element is above 4, 1 is the leading entry of R_2 and the element above it is zero and the element ... R_3 is zero.

EIGEN-VALUES & EIGENVECTORS

(By S.R WILLIAMS)

SOLVABILITY OF SYSTEMS OF EQUATIONS

There are three different kinds of solution (see i.e it's either a system have (i) Unique Solution (ii) Infinite Solution (iii) No-solution.

Example:

(i) Consider
$$3x - y = 7$$
$$-5x + 4y = -2$$
See page (4) which has $x = \frac{26}{7}$ & $y = \frac{29}{7}$.

∴ the system have a Unique solution

Use calculator (MODE ⟶ EQN) to get x & y.

(ii) Consider
$$3x - y = 0$$
$$\& \quad 6x - 2y = 0$$
(Using calculator to solve this gives (Math Error.))

because the system have an Infinite solution.

i.e $x = 1$, $y = 3$ is a solution so that $3(1) - 3 = 0$

also $x = 2$, $y = 6$ is a solution

also $x = 3$, $y = 9$ is a solution. (Hence the solutions are infinite.)

In this case to solve pick one equation say eqn. (1) $3x - y = 0 \Rightarrow y = 3x$

Hence Solution $= \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix} = \begin{pmatrix} k \\ 3k \end{pmatrix}$ $k \in \mathbb{Z}$

(iii) Consider
$$2x - 4y = \ldots$$
$$4x - \ldots$$
(This system has no solution) ...

EIGENVALUES:

Let $AX = Y$ where A is a Matrix, X - column Matrix and Y is a column vector.

Put $Y = \lambda X$ so that $AX = \lambda X \Rightarrow AX - \lambda X = 0$

So that $(A - \lambda I_n)X = 0$ where I_n = Identity Matrix

The Unknown Scalar λ is called the eigenvalue of the Matrix A.

Exg (1)

Given $A = \begin{pmatrix} 2 & 6 \\ +2 & 1 \end{pmatrix}$ Find the eigenvalues of A.

write $A - \lambda I = \begin{pmatrix} 2 & 6 \\ +2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$

$= \begin{pmatrix} 2 & 6 \\ +2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 6 \\ +2 & 1-\lambda \end{pmatrix}$ or simply subtract (λ) from the diagonal elements.

* characteristic Matrix

$\therefore |A - \lambda I_n| = \begin{vmatrix} 2-\lambda & 6 \\ +2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - (+2)(6)$
$= 2 - 3\lambda + \lambda^2 - 12$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0 \longrightarrow$ characteristic equation

i.e $|A - \lambda I_n| = 0$ —— called the characteristic equ.

$(\lambda+2)(\lambda-5) = 0 \quad \therefore \lambda_1 = -2, \lambda_2 = 5$

Hence λ_1 & λ_2 = characteristic roots which = eigenvalues of Matrix A

Example (2)

Find the eigen values of $B = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Therefore $|B - \lambda I_n| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$ which is an upper triangular matrix.

Evaluating gives $(3-\lambda)(2-\lambda)(5-\lambda) = 0$

$\therefore \lambda = 2, 3, 5$ (order does not matter here)

Note: that det. of a Triangular matrix = product of its diagonal elements.

EIGENVECTORS

Consider example (1) above. the Matrix $A = \begin{pmatrix} 2 & 6 \\ 2 & 1 \end{pmatrix}$ whose eigen values are $\lambda_1 = -2$ & $\lambda_2 = 5$

Note that $|A - \lambda I_n| = \begin{vmatrix} 2-\lambda & 6 \\ 2 & 1-\lambda \end{vmatrix} = 0$

we need to get the eigenvector corresponding to each eigenvalue

for ($\lambda = -2$) substitute into $(A - \lambda I_n)X = 0$.

i.e $\begin{pmatrix} 2-\lambda & 6 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2-(-2) & 6 \\ 2 & 1-(-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 4x_1+6x_2=0 \\ 2x_1+3x_2=0 \end{matrix}$$

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Note: that the systems of equation that develops under eigenvectors are mostly of Infinite Solution. See page (16) — Solvability of equ.

$\therefore$ pick only $4x_1+6x_2=0$ $\therefore$ $\cancel{4x_1 = -6x_2}$

$\therefore$ $2x_1 = -3x_2$ $\therefore$ $x_1 = -\frac{3x_2}{2}$

$\therefore$ Solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{-3\cancel{x_2}}{2} \\ \cancel{x_2} \end{pmatrix} = \begin{pmatrix} -\frac{3k}{2} \\ k \end{pmatrix} = k\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$$

$\therefore$ the eigenvector corresponding to $\lambda = -2$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \cdot k$ where $k \in \mathbb{Z}$

for $\lambda = 5$ substitute into $(A - \lambda I_n)X = 0$

$$\begin{pmatrix} 2-\lambda & 6 \\ 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2-5 & 6 \\ 2 & 1-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -3 & 6 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -3x_1+6x_2=0 \\ 2x_1-4x_2=0 \end{matrix}$$

pick one $-3x_1+6x_2=0 \Rightarrow 3x_1 = 6x_2$

$\therefore x_1 = 2x_2$

Solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2k \\ k \end{pmatrix} = k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ the eigenvectors corresponding to $\lambda = 5$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Now Consider example (11) above the Matrix

$$B = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$ having the eigenvalues $\lambda = 2, 3, 5$.

we need to get the eigenvectors corresponding to these eigenvalues.

for $\lambda = 2$ substitute λ into $(B - \lambda I_n)X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \text{which gives}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case we pick two equations

(i) $x_1 + x_2 + 4x_3 = 0$

(ii) $0x_1 + 0x_2 + 6x_3 = 0$

$\equiv$ $\begin{matrix} a_1x_1 + b_1x_2 + c_1x_3 = 0 \\ a_2x_1 + b_2x_2 + c_2x_3 = 0 \end{matrix}$

use

$$\frac{x_1}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = k$$

$\therefore$ we get $\dfrac{x_1}{\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}1&1\\0&0\end{vmatrix}} = k$

$\Rightarrow \dfrac{x_1}{6} = \dfrac{x_2}{-6} = \dfrac{x_3}{0} = k \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}6\\-6\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

dividing by $\underline{6}$ since it has infinite solution

for $\underline{\lambda = 3}$

$\begin{pmatrix}3-3 & 1 & 4\\0 & 2-3 & 6\\0 & 0 & 5-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1 & 4\\0 & -1 & 6\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0$

pick two equations;

$0x_1 + 1x_2 + 4x_3 = 0$
$0x_1 - 1x_2 + 6x_3 = 0$ $\Rightarrow \boxed{\dfrac{x_1}{\begin{vmatrix}1&4\\-1&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = k}$

$\therefore \dfrac{x_1}{10} = \dfrac{x_2}{0} = \dfrac{x_3}{0} = k \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}10\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

Dividing by $\underline{10}$.

for $\lambda = \underline{5}$

$\begin{pmatrix}3-5 & 1 & 4\\0 & 2-5 & 6\\0 & 0 & 5-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$

$\begin{pmatrix}-2 & 1 & 4\\0 & -3 & 6\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ pick two equations

$-2x_1 + x_2 + 4x_3 = 0$
$0x_1 - 3x_2 + 6x_3 = 0$

BY SIR WILLIAMS

$\Rightarrow \dfrac{x_1}{\begin{vmatrix}1&4\\-3&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}-2&4\\0&-6\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}-2&1\\0&-3\end{vmatrix}} = k$.

$\Rightarrow \dfrac{x_1}{18} = \dfrac{x_2}{12} = \dfrac{x_3}{6} = k; \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}18\\12\\6\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

i.e Dividing by $\underline{6}$.

$\therefore$ The corresponding eigenvectors are

$\lambda = 2 \Rightarrow \begin{pmatrix}1\\-1\\0\end{pmatrix}, \lambda = 3 \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \lambda = 5 \Rightarrow \begin{pmatrix}3\\2\\1\end{pmatrix}$

QUADRATIC FORMS

Consider $A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$ square matrix. Then the quadratic form of matrix A is given as X^TAX

where X = column matrix $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ and $X^T = (x_1 \; x_2)$

So that $X^TAX = (x_1 \; x_2)\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$

multiplying we get

$\Rightarrow X^TAX = (a_{11}x_1 + a_{21}x_2 \quad a_{12}x_1 + a_{22}x_2)\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$

again gives

$X^TAX = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_{21}x_1x_2$
$= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$

 BY SIR WILLIAMS

$\therefore$ we get $\dfrac{X_1}{\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{X_2}{-\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{X_3}{\begin{vmatrix}1&1\\0&0\end{vmatrix}} = k$

$\Rightarrow \dfrac{X_1}{6} = \dfrac{X_2}{-6} = \dfrac{X_3}{0} = k \quad \begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}6\\-6\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

dividing by $\underline{\underline{6}}$ since it has infinite solution

for $\underline{\underline{\lambda = 3}}$

$\begin{pmatrix}3-3 & 1 & 4\\ 0 & 2-3 & 6\\ 0 & 0 & 5-3\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1 & 4\\ 0 & -1 & 6\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = 0$

pick two equations;

$0X_1 + 1X_2 + 4X_3 = 0$
$0X_1 - 1X_2 + 6X_3 = 0$
$\Rightarrow \boxed{\dfrac{X_1}{\begin{vmatrix}1&4\\-1&6\end{vmatrix}} = \dfrac{X_2}{-\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = \dfrac{X_3}{\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = k}$

$\therefore \dfrac{X_1}{10} = \dfrac{X_2}{0} = \dfrac{X_3}{0} = k \quad \begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}10\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

Dividing by $\underline{\underline{10}}$.

for $\underline{\underline{\lambda = 5}}$

$\begin{pmatrix}3-5 & 1 & 4\\ 0 & 2-5 & 6\\ 0 & 0 & 5-5\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$

$\begin{pmatrix}-2 & 1 & 4\\ 0 & -3 & 6\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ pick two equations

$-2X_1 + X_2 + 4X_3 = 0$
$0X_1 - 3X_2 + 6X_3 = 0$

$\Rightarrow \dfrac{X_1}{\begin{vmatrix}1&4\\-3&6\end{vmatrix}} = \dfrac{X_2}{-\begin{vmatrix}-2&4\\0&-8\end{vmatrix}} = \dfrac{X_3}{\begin{vmatrix}-2&1\\0&-3\end{vmatrix}} = k$

$\Rightarrow \dfrac{X_1}{18} = \dfrac{X_2}{12} = \dfrac{X_3}{6} = k; \quad \begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} = \begin{pmatrix}18\\12\\6\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

i.e Dividing by $\underline{6}$.

$\therefore$ The corresponding eigenvectors are

$\lambda = 2 \Rightarrow \begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad \lambda = 3 \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \lambda = 5 \Rightarrow \begin{pmatrix}3\\2\\1\end{pmatrix}$

QUADRATIC FORMS

Consider $A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ Square matrix. Then the quadratic form of matrix A is given as X^TAX

where X = column matrix $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ and $X^T = (x_1 \ x_2)$

So that $X^TAX = (\cancel{x_1 \ x_2})\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\cdot\begin{bmatrix}X_1\\X_2\end{bmatrix}$

multiplying we get

$\Rightarrow X^TAX = (a_{11}X_1 + a_{21}X_2 \quad a_{12}X_1 + a_{22}X_2)\cdot\begin{bmatrix}X_1\\X_2\end{bmatrix}$

again gives

$X^TAX = a_{11}X_1^2 + a_{22}X_2^2 + a_{12}X_1X_2 + a_{21}X_1X_2$
$= a_{11}X_1^2 + (a_{12} + a_{21})X_1X_2 + a_{22}X_2^2$

By Sir Williams

$\therefore$ we get $\dfrac{x_1}{\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}1&4\\0&6\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}1&1\\0&0\end{vmatrix}} = k$

$\Rightarrow \dfrac{x_1}{6} = \dfrac{x_2}{-6} = \dfrac{x_3}{0} = k \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}6\\-6\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

dividing by $\underline{6}$ since it has infinite solution

for $\lambda = 3$

$$\begin{pmatrix}3-3 & 1 & 4\\ 0 & 2-3 & 6\\ 0 & 0 & 5-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1 & 4\\ 0 & -1 & 6\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0$$

pick two equations;

$0x_1 + 1x_2 + 4x_3 = 0$

$0x_1 - 1x_2 + 6x_3 = 0$

$\Rightarrow \dfrac{x_1}{\begin{vmatrix}1&4\\-1&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}0&1\\0&-1\end{vmatrix}} = k$

$\therefore \dfrac{x_1}{10} = \dfrac{x_2}{0} = \dfrac{x_3}{0} = k \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}10\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

Dividing by $\underline{10}$.

for $\lambda = \underline{5}$

$$\begin{pmatrix}3-5 & 1 & 4\\ 0 & 2-5 & 6\\ 0 & 0 & 5-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$$

$$\begin{pmatrix}-2 & 1 & 4\\ 0 & -3 & 6\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

pick two equations

$-2x_1 + x_2 + 4x_3 = 0$

$0x_1 - 3x_2 + 6x_3 = 0$

$\Rightarrow \dfrac{x_1}{\begin{vmatrix}1&4\\-3&6\end{vmatrix}} = \dfrac{x_2}{-\begin{vmatrix}-2&4\\0&-6\end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix}-2&1\\0&-3\end{vmatrix}} = k$

$\Rightarrow \dfrac{x_1}{18} = \dfrac{x_2}{12} = \dfrac{x_3}{6} = k; \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}18\\12\\6\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

i.e Dividing by $\underline{6}$.

The corresponding eigenvectors are

$\lambda = 2 \Rightarrow \begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad \lambda = 3 \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \lambda = 5 \Rightarrow \begin{pmatrix}3\\2\\1\end{pmatrix}$

QUADRATIC FORMS

Consider $A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ square matrix. Then the quadratic form of matrix A is given as X^TAX

where $X =$ column matrix $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ and $X^T = (x_1 \ \ x_2)$

So that $X^TAX = (x_1 \ \ x_2)\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$

multiplying we get

$\Rightarrow X^TAX = (a_{11}x_1 + a_{21}x_2 \quad a_{12}x_1 + a_{22}x_2)\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}$

again gives

$X^TAX = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_{21}x_1x_2$

$= a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2$

Observe that $(a_{12}+a_{21})$ gives the coefficient of X_1X_2. The original matrix can be gotten from the quadratic form as $A = \frac{1}{2}[A + A^T]$.

Example:

(I) Express $\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = A$ in Quadratic form.

here $a_{11}X_1^2 + (a_{12}+a_{21})X_1X_2 + a_{22}X_2^2 \Rightarrow$

$$1X_1^2 + (3+4)X_1X_2 + 6X_2^2 \Rightarrow \underline{X_1^2 + 7X_1X_2 + 6X_2^2}$$

Observe that to take this back to the former, we don't (will encounter) difficulty trying to say which of a_{12}/a_{21} is $\underline{4 \text{ or } 3}$

That is:

(II) Find the Matrix with the Quadratic form

$$X_1^2 + 7X_1X_2 + 6X_2^2 \Rightarrow a_{11}X_1^2 + (a_{12}+a_{21})X_1X_2 + a_{22}X_2^2$$

$\therefore a_{11} = 1,\ a_{22} = 6 \quad (a_{12}+a_{21}) = 7 \ \therefore a_{12} = a_{21} = \frac{7}{2} = 3.5$

i.e. $(3.5 + 3.5) = 7$

$\therefore A = \begin{bmatrix} 1 & 3.5 \\ 3.5 & 6 \end{bmatrix}$ Hint: divide the value of $(a_{12}+a_{21})$ by $\underline{2}$ to get each a_{12} & a_{21}.

3X3

Consider $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and $X^T = [X_1\ X_2\ X_3]$ the

In [illegible] as of the 2X2 matrix

Quadratic form of $Q = X^TAX$.

$= a$ [illegible] $+ (a_{12}+a_{21})X_1X_2 + (a_{23}+a_{32})$ [illegible]

[illegible] $+ (a_{13}+a_{31})X_1X_3$

Ex

Find [illegible] quadratic form of $A = \begin{bmatrix} 1 & [illegible] & [illegible] \\ [illegible] & [illegible] & 4 \\ 3 & 6 & 5 \end{bmatrix}$

a_{11} [illegible] $a_{33} = 5$

$= 1X$ [illegible] $+ (\ldots+2)X_1X_2 + (4+\ldots)X_2X_3 +$

$(-1+3)X_1X_3$

$=$ [illegible] $+ \ldots X_1X_2 + 10X_2X_3 + 2X_1X_2$

Rev[illegible] [illegible] from the [illegible]

[illegible]

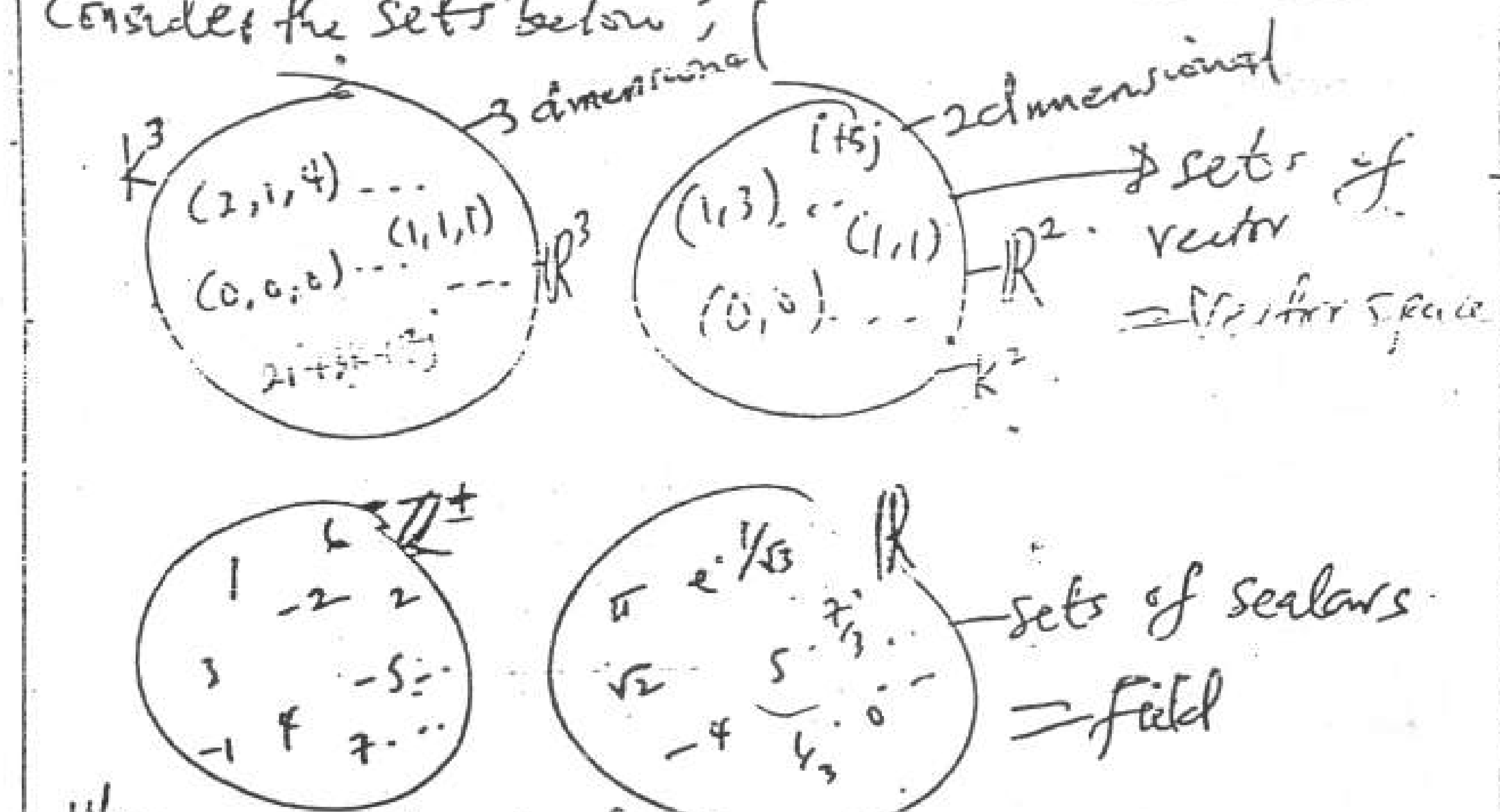
$\Rightarrow$ [illegible]

$=$ [illegible]

$=$ [illegible]

SUMMARY (III) VECTOR SPACES (V) (21)

Consider the sets below;



We now define vector space as; A set whose elements (vectors) are defined over a field K (F or $\mathbb{R}$). Vector space is always defined over a field. (i.e vectors consists of numbers (with direction))

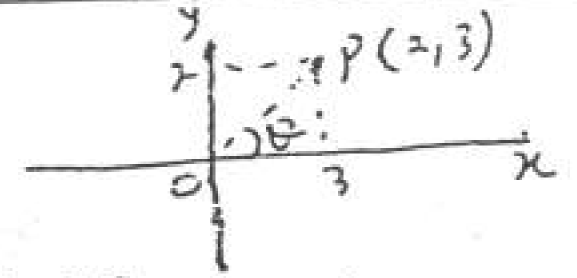
Hence:

We get vectors from vector space $u, v, w \in V$ and

Scalars from a field $a, b, c, \alpha, \beta, \gamma \in F$ or K

Also Note that a vector space can be represented as $\mathbb{R}^2, \mathbb{R}^3 \ldots \mathbb{R}^n$ or K^n or F^n (n-dimensional)

Because the elements of such set consist of two or more scalars $\Rightarrow$ (direction) i.e



$2 \in \mathbb{R} \to$ scalar, $3 \in \mathbb{R} \to$ scalar

but $(2,3) \notin \mathbb{R}$, $(2,3) \in V$ (vector).

DEFINITION:

Let V be a non-empty set with two operations;

(i) Vector Addition (+); i.e $u+v$, $u, v \in V$

(ii) Scalar multiplication (·); $a \cdot v$ or αu, $\alpha \in F$, $u \in V$

Then V is called a vector space if the following holds;

PROPERTIES OF A VECTOR SPACE

Addition:

(i) $(u+v)+w = u+(v+w)$, $u, v, w \in V$

(ii) $\exists\ 0 \in V$, $\ni 0+u = u+0 = u$, $\forall u \in V$.

(iii) $\exists -u \in V$ if $u \in V \ni u+(-u) = -u+v = 0$

(iv) $u+v = v+u$, $\forall u, v \in V$

Multiplication:

(i) $k(u+v) = ku+kv$, $k \in F$, $u, v \in V$

(ii) $(\alpha+\beta)u = \alpha u + \beta u$, $\alpha, \beta \in F$ and $u \in V$

(iii) $(\alpha\beta)u = \alpha(\beta u)$, $\alpha, \beta \in F$, $u \in V$

(iv) $\exists\ 1 \in F$ (Unit scalar) $\ni 1.u = u.1 = u$.

Also if $0 \in V$ then 0 is called the zero vector eg vector of the form $(0,0)$, $(0,0,0)$ etc. but $0 \in F$ denote a zero scalar.

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THEOREMS; Let V be a vector space over a field K;

(i) for any $\alpha \in K$ and $0 \in V$, $k0 = 0$

(ii) for $0 \in K$ and $u \in V$, $0u = 0$

(iii) if $\alpha u = 0$, where $\alpha \in K$ and $u \in V$, either $\alpha = 0$ or $u = 0$

(iv) for any $\alpha \in K$ and $u \in V$ $(-\alpha)u = \alpha(-u) = -\alpha u$

Examples of Vector Spaces

(i) Space K^n (e.g an n-dimensional Co-ordinate system)

i.e a $u \in K^n$ Consists of n-tuples of scalars in K.

K^n is a space since $(a_1, a_2, \ldots a_n) + (b_1 + b_2, \ldots b_n)$

$= (a_1 + b_1, a_2 + b_2, \ldots \ldots a_n + b_n)$

and

$k(a_1, a_2, \ldots a_n) = (ka_1, ka_2, \ldots ka_n)$.

and also $0 = (0, 0, \ldots 0) \in K^n$ and

$-(a_1, a_2 \ldots a_n) = (-a_1, -a_2, \ldots -a_n) \in K^n$.

(ii) Polynomial space $P(t)$ or $P(x)$.

This denote the set of all polynomial of the form

$P(t) = a_0 + a_1t^1 + a_2t^2 + \ldots + a_nt^n$, $n = (1, 2, \ldots)$

(iii) Matrix Space (M_{mn})

Where M_{mn} denotes the set of all $m \times n$ matrices with entries in a field K.

(iv) Polynomial Space $P_n(t)$ or $P_n(x)$;

$P_n(t)$ denotes the set of all polynomials $P(t)$ over a field K, where the degree of $P(t) \leq n$

i.e if $P(t) = a_0 + a_1t^1 + a_2t^2 + \ldots + a_rt^r$ then $r \leq n$

So that $P_2(t) \rightarrow$ denotes a polynomial of degree 2 (quadratic). $\Rightarrow a_0 + a_1t + a_2t^2$

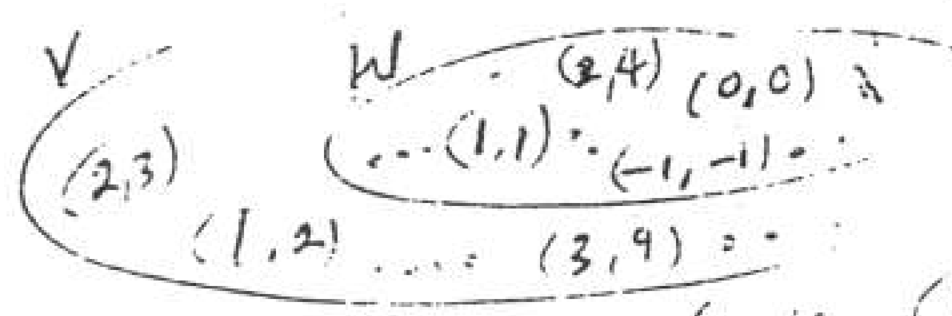
SUBFIELD

Suppose F is an extension of the field K, then K can be viewed as a subfield of the field F.

SUBSPACE

Let V be a vector space over a field K and let W be a subset of V. Then W is a subspace of V if W itself is a vector space over K with respect to vector $(+)$ and scalar $(\cdot)$.

Consider



Here $W \subset V$ Subspace of Space V.

So that W must also satisfy the (8) properties of a space that V satisfies.

Generally: if W is a subspace of the vector space V, the same axioms automatically holds in W.

THEOREM (i) ON SUBSPACE:

Suppose W is a subset of a vector space V, then W is a subspace of V if the ff. holds:

(i) The zero vector is in W i.e $0 \in W$

(ii) For all $(\forall)$ $u, v \in W$, $k \in F$ or $k \in K$.

(i) $u+v \in W$ (ii) $ku \in W$. (vector (+) & Scalar (·)).

Example (i)

Let $V = R^3$ consist of vectors in R^3 (3-dimensional) which is defined as (a,b,c).

Define a space $W = \{(a,b,c) : a \neq b \neq c\}$. (such that)

This implies that any space that consist of the vector of the form (a,a,a) cannot be a subspace of the vector space W. (trivial)

Note that 0 and W are subspace of W (itself).

Example (ii)

If $V = R^3$, which of these is not a subspace of V? (a) $W_1 = \{(a,b,c) : a \geq 0\}$

(b) $W_2 = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$

(c) $W_3 = \{(a,b,c) : a = b = c\}$

Solution

We need to check if they satisfy vector addition & Scalar Multiplication.

(a) The first entry of W_1 is always positive i.e ≥ 0

Suppose $a > 0$, Consider $-k \in F$ so that if $u \in W$

$-ku = -k(a,b,c) = (-ka, -kb, -kc)$ which gives a negative entry in the first position).

Hence (a) $\rightarrow$ does not obey Scalar (·).

(b) W_2 consists of vector whose entries are $a^2+b^2+c^2 \leq 1$

i.e $1^2+0^2+0^2$ also $0^2+1^2+0^2 \leq 1$ so that $(1,0,0) \in W$ and $(0,1,0) \in W$.

Set $u = (1,0,0)$ & $v = (0,1,0)$ we see that

$u+v = (1,0,0) + (0,1,0) = (1,1,0) \notin W$

because $1^2+1^2+0^2 > 1$ i.e $2 > 1$.

Hence W dis-obeys vector addition.

(c) W_3 consist of vectors whose entries are equal

Consider $\pm ku$, where $\pm k \in F$ and $u \in W$.

$\therefore \pm ku = \pm k(a,a,a) = (\pm ka, \pm ka, \pm ka)$ whose entries are also equal so $\pm ku \in W$.

Also consider $u = (a,a,a)$ and $v = (b,b,b)$

then $u+v = (a+b, a+b, a+b) \in W$ whose entries are also equal $\therefore W_3 \subset V$ and $W_1, W_2 \notin V$.

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Example (3)

Let $V = P(t)$ be the space of real polynomials. Which of these is not a subspace of V.

(i) $W_1 = P(t)$ which consists of polynomials with Integral Coefficients.

(ii) $W_2 = P(t)$, polynomials with degree ≥ 6 and the zero polynomial

(iii) $W_3 = P(t)$, polynomials with even powers.

Solution:

We only need to check if they satisfies Vector (+) and (·) scalar (multiplication).

(i) Not a subspace because it does not satisfies $\alpha u \in W_1$, $\alpha \in F$ and $u \in W$. i.e let $3 + 5t + 2t^2 \in W$ and let $1/7 \in F$ then $\frac{1}{7}(3 + 5t + 2t^2) =$

$$= \frac{3}{7} + \frac{5}{7}t + \frac{2}{7}t^2 \notin W \text{ but } 1/7 \in F \text{ hence}$$

Suppose $\alpha = 1/7$, $u = (3 + 5t + 2t^2)$ then $\alpha u \notin W$ because $3/7, 5/7, 2/7$ are not Integral Coefficients.

(ii) $P(t)$ with degree ≥ 6, consider $\alpha \in K$ and $u = at^6 + bt^7 \in W_2$ we see that αu

$\Rightarrow \alpha(at^6 + bt^7) = \alpha at^6 + \alpha bt^7 \in W_2$ since it is still polynomial of degree ≥ 6

also define $v = ct^7 + dt^8$ Consider $u + v =$

$= (at^6 + bt^7) + (ct^7 + dt^8) = at^6 + (b+c)t^7 + dt^8$

$\in W_2$ (A polynomial of degree ≥ 6)

$\therefore W_2$ is a subspace of V.

(iii) $P(t)$ = polynomials with even powers for

Suppose $u \in W_3$ then $u = a_2t^2 + a_4t^4 + \dots + a_{2n}t^{2n}$

Note that Multiplying u by a scalar $\alpha \in F$ does not change the degree of polynomial of u. and adding another vector v of even powers ($p(t)$) to u also does not affect the powers of the polynomial (u).

Hence W_3 is a Subspace. So that

$W_1 \notin V$, $W_2 \subset V$ and $W_3 \subset V$.

LINEAR COMBINATION

Let V be a V·S (vector space) over a field K, then the vector $v \in V$ is a linear combination of $u_1, u_2, \dots, u_n$ if $\exists$ (there exist) scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ $\ni$ (such that) $\Longrightarrow$

$V = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$. Alternatively V is a linear combination of $u_1, u_2 \dots u_n$ if there is a solution to the system of equ;

$$V = x_1 u_1 + x_2 u_2 + \dots + x_n u_n$$

Otherwise (i.e if the system has no solution) then V is not a linear combination of the U's.

Example: ①

Express $V = (7, -2)$ as a linear combination of $u_1 = (3, -5)$ and $u_2 = (-1, 4)$.

Write $V = u_1 x_1 + u_2 x_2 \Rightarrow \begin{pmatrix} 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} x_1 + \begin{pmatrix} -1 \\ 4 \end{pmatrix} x_2$

$\Rightarrow \begin{aligned} 3x_1 - x_2 &= 7 \\ -5x_1 + 4x_2 &= -2 \end{aligned}$ Using Calculator MODE → 5 (EQU).

you get $x = \frac{26}{7}$ & $y = \frac{29}{7}$.

So that $V = \frac{26}{7} u_1 + \frac{29}{7} u_2$ Hence V is a linear Comb. of u_1 & u_2.

Example ⑪

Express $V = (1, -2, 5)$ in R^3 as a Linear Comb. of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$ & $u_3 = (2, -1, 1)$

$\therefore \quad V = u_1 x_1 + u_2 x_2 + u_3 x_3$

$\therefore \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

So that
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= -2 \\ x_1 + 3x_2 + x_3 &= 5 \end{aligned}$$
Using Calculator you get $x_1 = -6$, $x_2 = 3$, $x_3 = 2$

So that $V = -6u_1 + 3u_2 + 2u_3$

Example (iii)

Express the polynomial $V = t^2 + 4t - 3$ in $P(t)$ as a linear Combination of the polynomials $p_1 = t^2 - 2t - 5$, $p_2 = 2t^2 - 3$, $p_3 = t + 1$ i.e

$V = p_1 x_1 + p_2 x_2 + p_3 x_3$.

$p_1 = 1t^2 - 2t - 5$
$p_2 = 2t^2 + 0t - 3$
$p_3 = 0t^2 + t + 1$

So that

$$\begin{pmatrix} 1\cancel{t^2} \\ 4\cancel{t} \\ -3 \end{pmatrix} = x_1 \begin{pmatrix} 1\cancel{t^2} \\ -2\cancel{t} \\ -5 \end{pmatrix} + x_2 \begin{pmatrix} 2\cancel{t^2} \\ 0\cancel{t} \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 0\cancel{t^2} \\ 1\cancel{t} \\ 1 \end{pmatrix}$$

$\therefore \begin{aligned} x_1 + 2x_2 + 0x_3 &= 1 \\ -2x_1 + 0x_2 + x_3 &= 4 \\ -5x_1 - 3x_2 + x_3 &= -3 \end{aligned}$ Using Calculator to solve the system gives $x_1 = 11/3$, $x_2 = \frac{-4}{3}$, $x_3 = \frac{34}{3}$

So that $V = \frac{11}{3} p_1 - \frac{4}{3} p_2 + \frac{34}{3} p_3$

Now Consider Example (iv)

$V = (0, 0)$, $u_1 = (1, 2)$ & $u_2 = (1, -3)$

then $V = X_1 U_1 + X_2 U_2$

$$\begin{pmatrix}0\\0\end{pmatrix} = X_1\begin{pmatrix}1\\2\end{pmatrix} + X_2\begin{pmatrix}1\\-3\end{pmatrix} \Rightarrow \begin{matrix} X_1 + X_2 = 0 \\ 2X_1 - 3X_2 = 0\end{matrix}$$

Using Calculator gives $x = 0, y = 0$ which implies that the system has no solution hence V is not a linear Combination of U_1 & U_2.

Q When does a system have a solution?

Unique Solution	Infinite Solution
when Calculator gives a unique value.	when Calculator gives math-Error ⇒ the system has many solutions which are infinite in no.

See page (15) Solvability.

LINEAR DEPENDENCE & INDEPENDENCE

Let V be a V-space over F, we say that the vectors $v_1, v_2, \ldots v_n$ are linearly dependent if ∃ (there exist) scalars $\alpha_1, \alpha_2 \ldots \alpha_n \in F$ ∋ (such that) $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

Otherwise they are said to be linearly Independent. or specifically

If $X_1V_1 + X_2V_2 + \ldots + X_nV_n = 0$ has a solution

Examples: (1)

Which of the following pair of vectors are linearly dependent.

(i) $U = (1,2)$, $V = (1,-3)$

(ii) $U = (1,-3)$, $V = (-2,6)$

(iii) $U = (1,2,-3)$, $V = (4,5,-6)$

Solution

(i) $UX_1 + VX_2 = 0$ → must be equal to zero for linear dependency.

$$\therefore \begin{pmatrix}1\\2\end{pmatrix}X_1 + \begin{pmatrix}1\\-3\end{pmatrix}X_2 = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{matrix} X_1 + X_2 = 0 \\ 2X_1 - 3X_2 = 0\end{matrix}$$

Using Calculator $x = 0, y = 0$

So that the system has no solution. Hence (i) the vectors are linearly Independent.

(ii) $U = (1,-3)$, $V = (-2,6)$ So that $UX_1 + VX_2 = 0$

$$\begin{matrix} X_1 - 2X_2 = 0 \\ -3X_1 + 6X_2 = 0\end{matrix}$$

Using Calculator gives Math-Error So that the System have Infinite Solution

Hence (ii) the vectors are linearly dependent. To get one of the solution to above system you can get one of the solution to above by choosing one equation i.e $X_1 - 2X_2 = 0$

So that $X_1 = 2X_2$ ∴ Solution $= \begin{pmatrix}X_1\\X_2\end{pmatrix} = \begin{pmatrix}2X_2\\X_2\end{pmatrix}$

$$\begin{pmatrix}2X_2\\X_2\end{pmatrix} = \begin{pmatrix}2K\\K\end{pmatrix} = K\begin{pmatrix}2\\1\end{pmatrix}$$ (an Integral Multiple of the Solution)

Note $K \in \mathbb{Z}$

Remarks ②; Suppose v is a non-zero vector. Then v itself is linearly dependent because $kv = 0$, $v \neq 0$ implies $k = 0$.

Remarks ③: Suppose two of the vectors $(v_1, v_2, \ldots, v_m)$ are equal or one is a scalar multiple of the other say $v_1 = kv_2$. Then the vectors must be linearly dependent because $v - kv_2 + 0v_3 + \cdots + 0v_m = 0$.

Remark ④: The vectors v_1 and v_2 are linearly dependent iff one is a multiple of the other i.e $v_1 = kv_2$.

Remarks ⑤: If the set $(v_1, v_2, \ldots v_m)$ is linearly independent then the rearrangement of the vectors $(v_{i_1}, v_{i_2}, \ldots v_{i_m})$ is also linearly independent.

Remarks ⑥: If a set S of vectors is linearly independent, then any subset of S is linearly independent.

OR if S contains a linearly dependent subset, then S is linearly dependent.

(BASIS OF A VECTOR SPACE)

Definition: The set $S = \{v_1, v_2, \ldots v_n\}$ is a basis of V iff (if and only if)

... V and

e.g ① Standard basis in R^2 is $\{e_1, e_2\}$. Vectors $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

(ii) Standard basis of $M_2(R)$ or $M_2(F)$ i.e 2×2 matrices having real entries or entries in a field $\{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$= \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right]$$

(iii) For vector space K^n (Standard/Usual basis $e_1 = (1, 0, 0, \ldots 0)$, $e_2 = (0, 1, 0, 0, \ldots 0) \ldots e_n = (0, 0, \ldots 1)$) are bases of K^n. Since they form a linearly independent set i.e they form a matrix in echelon form so that $u = (a_1, a_2, \ldots a_n)$ in K^n can be written uniquely as a linear combination of the above e_is i.e

$$V = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \text{ where } a_1, a_2 \ldots \in F.$$

THEOREM ③ BASIS

① Let V be a vector space and suppose $S = \{u_1, u_2, \ldots u_n\} \subseteq V$. Then S is a basis of V iff every element $v \in V$ can be written uniquely as a linear elements. Then $m=n$.

Examples: which of the following vectors form a basis of $\mathbb{R}^3$. ⓐ $(1,1,1),(1,0,1)$. (It suffices to show linear Independency)

ⓑ $(1,2,3),(1,3,5),(1,0,1),(2,3,0)$

ⓒ $(1,1,1),(1,2,3),(2,-1,1)$ ⓓ $(1,1,2),(1,2,5)(5,3,4)$.

Since $\mathbb{R}^3$ is 3-dimensional then the basis of $\mathbb{R}^3$ must contain exactly 3-elements Hence ⓐ & ⓑ Cannot form a basis for $\mathbb{R}^3$.

ⓒ We check for linear independency (See def. of basis) i.e

$$\begin{pmatrix}1\\1\\1\end{pmatrix}x_1+\begin{pmatrix}1\\2\\3\end{pmatrix}x_2+\begin{pmatrix}2\\-1\\1\end{pmatrix}x_3=0 \Rightarrow \begin{matrix}x_1+x_2+2x_3=0\\x_1+2x_2-x_3=0\\x_1+3x_2+x_3=0\end{matrix}$$

Using Calculator gives

$x_1=0, x_2=0, x_3=0$ i.e it has no solution

Hence these vectors are linearly Independent So that they must form a basis for $\mathbb{R}^3$.

ⓓ the same way as in ⓒ above check.

$$\begin{pmatrix}1\\1\\2\end{pmatrix}x_1+\begin{pmatrix}1\\2\\5\end{pmatrix}x_2+\begin{pmatrix}5\\3\\4\end{pmatrix}x_3=\begin{pmatrix}0\\0\\6\end{pmatrix}$$ Using Calculator gives

Math-Error which implies Infinite Solution Hence the vectors in (d) are linearly dependent, so do not form a basis of $\mathbb{R}^3$.

DIMENSION OF A VECTOR SPACE

The dimension of a vector space V, is the number of vectors in a basis of V. e.g

(i) $\dim(F^n)=n$ So that $\dim(\mathbb{R})=1$, $\dim(\mathbb{R}^2)=2$.

(ii) $\dim(M_{m\times n})(F)=m\times n$ i.e an $m\times n$ matrix over a field F. e.g $\dim(M_{2\times3}(F))=2\times3=6$.

(iii) If $A\in M_{m\times n}(\mathbb{R})$. Then the nullity of A equal the dimension of null space of Matrix A.

i.e $V\in V$, $A\in M_{m,n}(\mathbb{R})$.

$\therefore$ $Null(V)=\{v\in V: vA=0\}$.

If $A\in M_{mn}(\mathbb{R})$ then the span of the rows of A is the row space of A and the rank of the Matrix A is the dimension of the rowspace A.

SYLVESTER'S LAW

Suppose $A\in M_{mn}(F)$. Then the;

$$\text{rank } k(A)+\text{nullity}(A)=n$$

THEOREM ④ BASIS & DIMENSION

① A Vector Space V is said to be of finite dimension or n dimensional written as $\dim(V)=n$

② If V has a basis with n-elements; then all basis of V have the same no of elements.

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(b) The vector space {0} is defined to have the dimension 0 i.e $dim(0) = 0$.

(c) Suppose a vector space, does not have a finite basis. Then its of infinite dimension.

Lemma:

Suppose $\{v_1, v_2 \ldots v_n\}$ spans V and suppose $\{u_1, u_2 \ldots u_m\}$ is linearly independent then $m \leq n$ and V is spanned by the set of the form $\{u_1, u_2, \ldots u_m, u_{i1}, u_{i2}, \ldots u_{in-m}\}$.

Theorem (2) Let V be a vector space of finite dimension n. Then;

(i) Any n+1 or more vectors in V are linearly independent.

(ii) Any linearly independent set $S = [v_1, v_2 \ldots v_n]$ of V with n elements is a basis of V.

(iii) Any ~~linearly independent~~ spanning set $T = [v_1, v_2 \ldots v_n]$ of V with n elements is a basis of V.

Theorem (3) Suppose S spans a vector space V. then

(i) Any max no of linearly independent vectors in form a basis of V.

(ii) Suppose one deletes from S every vector that is a linear combination of preceding

(30) forms a basis of V

(iii) Let V be a vector space of finite dimension and let $S = \{u_1, u_2 \ldots u_n\}$ be a set of linearly independent vectors in V. Then S is part of a basis of V. i.e S may be extended to a basis of V.

Theorem (4) Dimension.

Let W be a subspace of an n-dimensional vector space V. then $dim(W) \leq n$ s.t. that $dim(W) = n$ if $W = V$. E.g.

Let W be a subspace of $\mathbb{R}^3$ with dimension = 3. then dim of W can only (exactly) be 0, 1, 2 or 3. if

(a) $dim(W) = 0$, then W is a point.

(b) $dim(W) = 1$, then W is a line through the origin

(c) $dim(W) = 2$, then W is a plane through the origin

(d) $dim(W) = 3$, then W is the entire space $\mathbb{R}^3$.

Note:

(i) the polynomial set $S = (1, t, t^2, \ldots t^n)$ the dimension of S $\Rightarrow dim(S) = n+1$. i.e starting from $0 \to n$

(ii) The rank of a matrix A. $Max(\mp)$ is also the dimension of A.

i.e $dim\ A = rank(A)$

LINEAR TRANSFORMATION / MAP SUMMARY (IV)

Let U & V be vector spaces over a Field F. We say that a mapping $T: U \longrightarrow V$ (T maps from U to V) is a linear transformation if it satisfies;

(1) $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in V$

(2). $T(\alpha u) = \alpha T(u) \quad \forall\, u \in V$ and $\alpha \in F$.

Linear transformation from U to V is denoted by $L(U,V)$.

THEOREM (I)

(I) Let $T_1, T_2 \in L(U,V)$ Then (i) $T_1 + T_2 \in L(U,V)$
(ii) $\alpha T_1 \in L(U,V)$

(II) Let U, V, W be vector spaces over a field F. Suppose $T_1 \in L(U,V)$ and $T_2 \in L(V,W)$ so that $T_2 \circ T_1 : U \longrightarrow W$ (composition).

then $T_2 \circ T_1 \in L(U,W)$.

Definition:

Suppose $T \in L(U,V)$, we define the null space a Kernel of T as; $Ker(T) \equiv Null(T) = \{v \in U : T(u) = 0\}$

Also Range of T as: $range(T) = \{T(u) : u \in U\} \subseteq V$

THEOREM (II)

(I) Let $T \in L(U,V)$, be invertible in the inverses $T^{-1} : V \longrightarrow U$ Then $T^{-1} \in L(V,U)$ → Note the order of U & V

That is $\begin{pmatrix} T: U \longrightarrow V & \text{and} \\ U \longleftarrow V : T^{-1} \end{pmatrix}$.

(II) Let $T \in L(U,V)$, then

(i) $Ker(T)$ is a subspace of U.

(ii) $Range(T)$ is a subspace of V.

Example (1).

Show if the function $f(x,y) = [x-y, x]$ is a linear transformation or not on R^2

Solution:

We need to show if f satisfies the two conditions above: Note that is is 2-dimensional i.e $\dim(R^2) = 2$.

$\therefore$ select $u = (x_1, x_2)$ & $v = (y_1, y_2)$

So that $f(u) = f(x_1, x_2) = (x_1 - x_2, x_1)$
$\quad\quad\quad\quad\quad\quad\quad(x \;\; y) \quad\quad (x-y, x)$ from the definition of the function

and $f(v) = f(y_1, y_2) = (y_1 - y_2, y_1)$

(1) Consider $u + v = (x_1, x_2) + (y_1, y_2)$
$= (x_1 + y_1, x_2 + y_2)$

$f(u+v) = f(x_1 + y_1, x_2 + y_2) =$
$= [(x_1 + y_1) - (x_2 + y_2), (x_1 + y_1)]$
$= (x_1 + y_1 - x_2 - y_2, x_1 + y_1]$
$= ((x_1 - x_2) + (y_1 - y_2), x_1 + y_1]$

BY SIR WILLIAMS

$\therefore = [(x_1-x_2)+(y_1-y_2), x_1+y_1]$

$= (x_1-x_2, x_1) + (y_1-y_2, y_1)$

$f(u+v) = \underline{f(u) + f(v)}$ Condition (I) satisfied

(II) let $\alpha \in F$ Consider $\alpha u = \alpha(x_1, x_2)$

$= (\alpha x_1, \alpha x_2)$

$\therefore f(\alpha u) = f(\alpha x_1, \alpha x_2) = (\alpha x_1 - \alpha x_2, \alpha x_1)$

$= (\alpha(x_1-x_2), \alpha x_1)$

$= \alpha(x_1-x_2, x_1)$

Hence $f=(x-y, x)$ $\quad = \underline{\alpha f(u)}$ — (II) satisfied

is a linear transformation.

Example (II)

Say if $f(x,y,z) = [x, x+y, z-x, yz]$ is a linear transformation or not on $\mathbb{R}^3$ i.e 3-dim.

Select $u=(x_1, x_2, x_3)$ So that

$f(u) = f(x_1, x_2, x_3) = [x_1, x_1+x_2, x_3-x_1, x_2 x_3]$

i.e replace x with x_1, y with x_2 & z with x_3

(II) let $\alpha \in F$ Consider $\alpha u = \alpha(x_1, x_2, x_3)$

Consider $\alpha u = (\alpha x_1, \alpha x_2, \alpha x_3)$

$f(\alpha u) = f(\alpha x_1, \alpha x_2, \alpha x_3)$

$\Rightarrow [\alpha x_1, \alpha x_1 + \alpha x_2, \alpha x_3 - \alpha x_1, \alpha x_2 \cdot \alpha x_3]$

$= [\alpha(x_1), \alpha(x_1+x_2), \alpha(x_3-x_1), \alpha(\alpha x_2 x_3)]$

$= \alpha[x_1, x_1+x_2, x_3-x_1, \alpha x_2 x_3]$

$\neq \alpha f(u)$ because one of α is still inside the bracket of which it doesn't suppose to be. This was caused by the two variables (yz) multiplying each other in the original function i.e $\alpha y \cdot \alpha z = \alpha(\alpha yz)$ (which doubled the α in the above)

Hence functions that involves products or squares aren't a transformation e.g

$f(x,y) = [x^2, x+y]$, $f(x,y) = [xy, x+y, x-y, x]$

$f(x,y,z) = [x, x-z, xy]$ are not a linear T.

So example (II) the function is not a linear transformation.

SHORTCUT (But might cut you short, it sometim as looks like a hit & miss)

To test for linear transformation; test condition (II) first i.e $f(\alpha u) = \alpha f(u)$ owing to its simplicity, if the function fails to be a transformation then no need checking for (I) $f(u+v)$ but if the function satisfies (II) then one doesn't have a long road because solving always results (cumbersome).

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DEPARTMENT OF MATHEMATICS
FACULTY OF PHYSICAL SCIENCES
UNIVERSITY OF BENIN, BENIN CITY, NIGERIA.

FIRST SEMESTER B.Sc. EXAMINATIONS 2014/2015 SESSION

COURSE TITLE: MTH 230 (Linear Algebra)

TIME ALLOWED: 1½ Hours.

INSTRUCTIONS: (i) Write and circle your attendance list serial number on the objective answer paper. (ii) Attempt all questions by SHADING (using HB pencil) the letter box that corresponds to the correct option. Information about your Mat.No., Name, Course code, Faculty code and Departmental code must be clearly written and CORRECTLY SHADED. YOU MUST SUBMIT YOUR QUESTION PAPER ALONG WITH YOUR ANSWER SHEET.

1. Given a matrix $A = (a_{ij})_{44} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$. Find the determinant of (a) $a_{11}a_{23}a_{14}a_{33}$ (b) $a_{11}a_{22}a_{33}a_{44}$ (c) $a_{11}a_{22}a_{23}a_{44}$ (d) $a_{11}a_{12}a_{22}a_{44}$ (e) none of the above

2. Given a matrix $A = \begin{bmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{bmatrix}$ Find x if det(A) = -35 (a) 2 (b) 0 (c) 5 (d) 3 (e) None of the above

₦1,500
3,000
2,500
8,000

3. Given a matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{bmatrix}$ Find the product $A^T I$, where I is a unit matrix. (a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 8 \\ 3 & 7 & 0 \\ 5 & 4 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 5 & 8 \\ 4 & 3 & 2 \end{bmatrix}$ (e) none of the above

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4. Find the Eigen values of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$ (a) 0, 2 and 3 (b) 3, 2 and 4 (c) 1, 2 and 3 (d) -1, 1 and 3

5. Given a matrix $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 4 & -2 \\ 2 & 10 & -5 \end{bmatrix}$. Find the cofactor A_{22} (a) 5 (b) 6 (c) -6 (d) -8 (e) none of the above

6. Given a matrix $A = \begin{bmatrix} 1 & 1 & -k \\ k & -3 & 11 \\ 2 & 4 & -8 \end{bmatrix}$. If det $A = 0$ find the values of k (a) 1 and 0.5 (b) 1 and -0.5 (c) 0 and 0.6 (d) 1 and -0.6

7. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 4 & 6 \end{bmatrix}$. Find the cofactors of A_{32} and A_{33} (a) 8 and -10 (b) 6 and 15 (c) -5 and 8 (d) -8 and 10

8. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{bmatrix}$ (a) $A^{-1} = \frac{1}{30}\begin{bmatrix} 8 & -58 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{bmatrix}$ (b) $A^{-1} = \frac{1}{38}\begin{bmatrix} 8 & -58 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{bmatrix}$ (c) $A^{-1} = \frac{1}{38}\begin{bmatrix} 8 & -58 & 38 \\ 6 & -4 & 0 \\ -5 & -35 & -19 \end{bmatrix}$ (d) $A^{-1} = \frac{1}{40}\begin{bmatrix} 8 & -58 & 38 \\ 6 & -4 & 0 \\ -5 & -35 & -19 \end{bmatrix}$

$B := t, u,$ B' $B'(u,7,7)$ 5 $\{2,4,\}$ $\{2,4,5\}$

9. Which of the following is/are not true? (a) $A\bigcup(B\bigcap C)=(A\bigcup B)\bigcap(A\bigcup C)$ (b) $A-B=A\bigcap B'$ (c) $(A\bigcap B)'=A'\bigcup B'$ (d) If U is the universal set then $B'=\{x: x\in B, x\notin U\}$ (e) None of the above

10. Consider the set $A=\{1,2,3,5,7\}$, $B=\{0,3,6,7,9\}$ and $C=\{4,5,6,8\}$. Determine $(A-B)\bigcup(B-A)$ (a) (1, 2, 7, 1) (b) (0, 6, 9, 1) (c) (ϕ) (d) (0, 1, 2, 5, 6, 9) (e) None of the above

11. A relation from a set E to a set F is a subset of (a) $E\bigcap F$ (b) $E\bigcup F$ (c) E x F (d) E F (e) None of the above

12. Let A = {a, c} and B = { a, b, c, f }. What is n(A x B) . (a) 16 (b) 6 (c) 8 (d) 4 (e) None of the above

13. Which of the following is not a linear transformation from IR^3 to IR^3? (a) T (x, y, z) = (x, 2y, 3x – y) (b) T (x, y, z) = (0, 0, 0) (c) T (x, y, z) = (x, 2y, 5z) (d) T (x, y, z) = (1, x, z) (e) None of the above

14. Which of the following is not true? (a) If $T: U\to V$ is any linear transformation from U to V, then T (x.y) = T (x) T (y) for all vectors. (b) The set A of all linear transformations of a vector space into itself is also a vector space (c) The set T of all linear transformations of a vector space into itself. (d) The set A of all linear transformations of a vector space into itself is a ring with respect to addition and multiplication. (e) None of the above

15. $T: R^3 \to R^3$ such that $T(x,y,z)=(-x, y-z, x-1)$ is not a linear map because it is: (a) Not additive (b) not well defined (c) Neither homogeneous nor additive (d) Not closed with respect to x, y and z (e) None of the above

16. Given a 2 x 3 column vector $A=\begin{bmatrix}0 & 1\\ -2 & 2\\ 1 & 0\end{bmatrix}$, then the linear transformation $T: IR^2\to IR^3$ is defined by

(a) $T(x,y)=(-2x+z, x+2y)$ (b) $T(x,y)=(x+2y, 2x+y, 0)$ (c) $T(x,y)=(y, -2x+2y, x)$ (d) $T(x,y)=(-x+3y)$ (e) None of the above

17. Suppose $f: IR^n \to IR^m$ is linear, then (a) The kernel of f is a subspace of IR^n (b) The range of f is a subspace of IR^m (c) f (u + v) = f (u) + f (v) for all u, v, $\in IR^n$ (d) f (ku) = k f (u) for all u $\in IR^n$ and k, a scalar (e) None of the above

18. Which of the following function is not a linear transformation (a) $f: IR^3\to IR^2$ such that f (x, y, z) = (x, -y, -x) (b) $f: IR^3\to IR^3$ such that f (x, y, z) = (x, y, z)+(0, -1, 0) (c) $h: IR^2\to IR^2$ such that h (x, y) = (2x, y-x) (d) $t: IR^3\to IR^3$ such that t(x, y, z) = (x+y, y- z, x) (e) None of the above

19. Which of the following statements is incorrect? (a) The empty set is a subspace of every vector space (b) Every subspace of a vector space is also a vector space (c) Every vector space is an additive abelian group (d) Every vector space is also a subspace (e) None of the above

20. Which of the following is a vector space? (a) The set V of all m x n matrices with real entries together with the operation of matrix addition and scalar multiplication. (b) The points on a plane V through the origin with addition and scalar multiplication (c) The points on a line passing through the origin in IR^2 with addition and scalar multiplication (d) All of the above (e) None of the above

21. Which of these vector space is finite dimensional even though it does not have a linearly independent set and therefore no basis? (a) The n-dimensional vector space (b) The zero vector space (c) The one dimensional vector space (d) The infinite dimensional space (e) None of the above

22. Let V be a vector space, then (i) The set IR of real numbers is an element of V (ii) The set S of all linear combinations of the subspaces of V is also a subspace of V (iii) V is an additive group that is also commutative: (a) (i) and (iii) only (b) (ii) only (c) (ii) and (iii) only (d) (i), (ii) and (iii) (e) None of the above

23. Which of these is not a vector space? (a) The set of all vectors b for which a given system Ax = b has a solution (b) The set of vectors with positive entries (c) The vector V consisting of the single object zero with standard operations of addition and scalar multiplication. (d) The set V = IR^n (e) None of the above

24. Let V be a vector space such that one of its basis has m-elements and another has n-elements, then (a) m = n + 1 (b) m = 2n (c) m // n (d) m + 1 = n (e) None of the above $m = n$

25. Find scalars x and y such that the vector is a linear combination of u = (1, 2, -1) and v = (6, 4, 2) (a) x = 3, y = 2 (b) x = -2, y = -3 (c) x = -3, y = -2 (d) x = 1, y = 5 (e) None of the above

26. Suppose $S=\{x_1, x_2, \ldots, x_n\}$ is a finite set of vectors and there exists scalars $k_1, k_2, \ldots, k_n$ not all zero such that $k_1x_1 + k_2x_2 + \ldots + k_nx_n = 0$, then the set S is called? (a) linearly dependent (b) linear combination (c) linear independence (d) linear span (e) None of the above

27. A bijective map is one that is: (a) one to one (b) onto (c) injective and surjective (d) Not dependent

28. Which of following set is a subspace of IR^3? (a) $V_1=\{(x,y,z): x,y,z,\in IR \text{ and } x+y=1\}$ (b) $V_2=\{(r, r+2, 0): r,\in IR\}$ (c) $V_3=\{(x,y,z): x,y,z,\in IR \text{ and } x+2y+z=0\}$ (d) $V_4=\{(x,y,z): x,y,z,\in IR \text{ and } x\cdot x=z\cdot z\}$ (e) None of the above

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29. Obtain scalars a, b, c in V = $ax_1 + bx_2 + cx_3$ such that the vector V = (5, 2, 4) in $\mathbb{R}^3$ is written as a linear combination of the vectors x_1 = (1, 2, 3), x_2 = (2, 3, 7) and x_3 = (3, 5, 6). (a) a = 2, b = 4, c = 3 (b) a = -2, b = -4, c = 3 (c) a = 2, b = -4, c = -3 (d) a = 0, b = 0, c = 0 (e) None of the above

30. Express U = (6, 3, 4) as a linear combination of u_1 = (1, 1, 2), u_2 = (1, -1, -3), and u_3 = (-1, 1, 2) in $\mathbb{R}^3$

(a) $U = \frac{-9}{2}u_1 + 8u_2 + \frac{13}{2}u_3$ (b) $U = \frac{9}{2}u_1 + 2u_2 + \frac{1}{2}u_3$ (c) $U = 6u_1 - u_2 + u_3$ (d) $U = \frac{9}{2}u_1 - 8u_2 + \frac{13}{4}u_3$

31. Which of the following statements is/are not true? (a) If the set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors is linearly independent, then any re-arrangement of the vectors is also linearly independent. (b) If v_1 is a non-zero vector, the v_1 by itself is linearly dependent (c) The vectors U = (1, 1, 0), V = (1, 3, 2) and W = (4, 9, 5) are linearly dependent (d) The vectors U_1 = (1, 1, 1) and U_2 = (1, 2, -3, 0) are orthogonal (e) All of the above

32. Which of the following statements is most correct? (a) For every vector space there exists a basis (b) The dimension of a vector space is uniquely defined (c) The vector space R^3 has ((1, 0, 0), (0, 1, 0), (0, 0, 1)) as its basis (d) All of the above (e) None of the above

33. To show that a set is a basis for a given vector space, the set must: (i) span the vector space (ii) be linearly dependent (iii) be a closed set (iv) be linearly independent
(a) (i) and (ii) (b) (iii) only (c) (i) and (iv) (d) (i), (ii) and (iii) (e) None of the above

34. Which of the following is a basis in R^3? (a) ((0, 0, 1), (1, 2, 5), (0, 1, 3)) (b) ((1, 2, 3), (2, 3, 4), (3, 4, 5)) (c) Both (a) and (c) (d) ((1, 0, 0), (0, 1, 0), (1, 1, 0)) (e) None of the above

35. Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & -1 \end{bmatrix}$ (a) $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -2$

(b) $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -2$ (c) $\lambda_1 = \lambda_2 = \lambda_3 = 2$ (d) $\lambda_1 = \lambda_2 = \lambda_3 = -2$ (e) none of the above

36. Let $M = S + A$ and $M^T = S^T + A^T$. Where S is a symmetric matrix and A is an antisymmetric matrix, hence find an expression for S and A. (a) $S = 2(M + M^T)$, $A = \frac{1}{2}(M - M^T)$ (b) $S = \frac{1}{2}(M + M^T)$ and $A = 2(M + M^T)$ (c) $S = \frac{1}{2}(M + M^T)$ and $A = \frac{1}{2}(M - M^T)$ (d) $S = 2(M - M^T)$ and $A = 3(M + M^T)$ (e) None of the above

37. Find the cofactor matrix A_{ij} of the matrix $A = \begin{bmatrix} 1 & 7 & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{bmatrix}$. (a) $\begin{bmatrix} 2 & 0 & 36 \\ -33 & 2 & 19 \\ 30 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 52 & -32 \\ -33 & -1 & 8 \\ 36 & -28 & 32 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & 30 & -32 \\ 4 & 1 & 54 \\ 0 & 7 & -11 \end{bmatrix}$ (d) $\begin{bmatrix} -12 & 1 & 12 \\ 10 & 35 & 16 \\ 36 & 0 & 34 \end{bmatrix}$ (e) None of the above

38. Suppose the set of equations $\begin{matrix} x + 2y + z = 4 \\ 3x - 4y - 2z = 2 \\ 5x + 3y + 5z = -1 \end{matrix}$ is of the form $AX = b$. If $A^{-1} = -\frac{1}{35}\begin{bmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{bmatrix}$,

find x, y and z (a) 4,3 and -4 (b) 2, 3 and -4 (c) 0, 0 and -3 (d) 2, 3 and 4 (e) None of the above

39. If $A = \begin{bmatrix} 1 & x & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{bmatrix}$ Find x if det $A = 192$ (a) -4 (b) 2 (c) 7 (d) 0 (e) none of the above

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{bmatrix}9 & \sqrt{3}\\ \sqrt{3} & 11\end{bmatrix}$ (a) 2, 5 and $\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (b) 2, 3 and $\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}$

(c) -2, 2 and $\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (d) 2, 3 and $\begin{bmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix}$ (e) None of the above

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Suppose $A=\begin{bmatrix}1 & -1 & 2\\ -1 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2\\ -5 & 7\\ -1 & 3\end{bmatrix}$. Find $\det(AB)$ (a) 174 (b) 192 (c) 34 (d) 0 (e) None of the above

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42. Suppose $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 1 & 5\\ 6 & 0 & 2\end{bmatrix}$. Find A^{-1}. (a) $\frac{1}{28}\begin{bmatrix}2 & -4 & 7\\ -22 & -16 & 17\\ -6 & 12 & -7\end{bmatrix}$ (b) $\frac{1}{16}\begin{bmatrix}2 & 6 & -2\\ -22 & -16 & 7\\ -6 & 12 & 8\end{bmatrix}$

(c) $\frac{1}{28}\begin{bmatrix}2 & -2 & 7\\ 22 & -4 & 3\\ 4 & 12 & -7\end{bmatrix}$ (d) $\frac{1}{8}\begin{bmatrix}4 & 27 & 7\\ 22 & 16 & 8\\ -6 & 12 & -7\end{bmatrix}$ (e) None of the above

43. Suppose $A=\begin{bmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{bmatrix}$. Find A^{-1}. (a) $\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$ (b) $\frac{1}{\cos\beta-\sin\beta}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$ (d) $\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & -\cos\beta\end{bmatrix}$ (e) none of the above

44. Let $A=\begin{bmatrix}1 & 2 & 3 & 5\\ 2 & 3 & 1 & -1\\ 3 & 6 & 9 & 15\\ 1 & 3 & 2 & 7\end{bmatrix}$. Find $\det(A)$ (a) 5 (b) 3 (c) 0 (d) 0.5 (e) none of the above

45. Let $A=\begin{bmatrix}0 & 4 & 0 & 5\\ 1 & 1 & 2 & 1\\ 0 & 0 & 1 & 0\\ 2 & 3 & 4 & 1\end{bmatrix}$. Find $\det(A)$ (a) 18 (b) 4 (c) 2 (d) 9 (e) 10

46. Suppose $A=\begin{bmatrix}1 & -1 & 2\\ -1 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2\\ -5 & 7\\ -1 & 3\end{bmatrix}$ Find $\det(BA)$ (a) 12 (b) 0 (c) 10 (d) 6 (e) None of the above

47. Given the matrix $\sigma=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. Find the eigenvalues and eigenvectors (a) 1, −1 and $\begin{pmatrix}1\\ i\end{pmatrix}$ (b) −2, 1 and $\begin{pmatrix}1\\ -i\end{pmatrix}$

(c) −3, 2 and $\begin{pmatrix}1\\ i\end{pmatrix}$ (d) 2, −2 and $\begin{pmatrix}1\\ -i\end{pmatrix}$ (e) none of the above.

48. Find an expression for the Eigenvalue of a 2x2 matrix (a) $\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0$

(b) $\lambda^2-(a_{11}-a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0$ (c) $\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}+a_{12}a_{21}=0$

(d) $\lambda^2-(a_{11}+a_{22})\lambda-a_{11}a_{22}-a_{12}a_{21}=0$ (e) None of the above

49. Find the Rank of the matrix $A=\begin{bmatrix}2 & 4 & 3\\ 1 & 2 & 0\\ 0 & 0 & 2\\ 6 & 2 & 5\end{bmatrix}$ (a) 4 (b) 3 (c) 2 (d) 1 (e) none of the above

50. Given matrices $A=\begin{bmatrix}2 & 1 & 3\\ 0 & 4 & 5\\ 2 & 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 2\\ 1 & 4 & y\end{bmatrix}$. Find the value of y if $\det(AB)=48$ (a) −12 (b) 8 (c) 12 (d) ...

26/04/2015

MTH 230: Linear Algebra [Paper Type B]

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Option C

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN, BENIN CITY
FIRST SEMESTER B.Sc. EXAMINATIONS 2013/2014 SESSION

COURSE CODE: MTH230 COURSE TITLE: LINEAR ALGEBRA

INSTRUCTIONS: (i) Write and circle your attendance list serial number on the objective answer paper. (ii) Attempt all questions by SHADING (using HB pencil) the letter box that corresponds to the correct option. Information about your Mat. No., Name, Course code, Faculty code and Departmental code must be clearly written and CORRECTLY SHADED. YOU MUST SUBMIT YOUR QUESTION PAPER ALONG WITH YOUR ANSWER SHEET.

Time Allowed: 1½hrs

NAME______________________ MAT. NO.______________ DEPT______________

Suppose $A=\begin{bmatrix}1 & -1 & 2\\ -1 & 3 & 1\end{bmatrix}$ And $B=\begin{bmatrix}4 & 2\\ -5 & 7\\ -1 & 3\end{bmatrix}$ Find $\det(BA)$ (a) 12 (b) 0 (c) 10 (d) 6 (e) none of the above

2×3 3×2

Suppose $A=\begin{bmatrix}1 & -1 & 2\\ -1 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2\\ -5 & 7\\ -1 & 3\end{bmatrix}$ Find $\det(AB)$ (a) 174 (b) 192 (c) 34 (d) 0 (e) none of the above

If V is a vector space of dimension n then: (a) n + 1 or more vectors in V are linearly dependent (b) n + 1 or more vectors in V are linearly independent (c) n or less vectors in V are linearly dependent (d) n or less vectors in V are linearly independent (e) None of the above

Suppose $x_1^2 + 6x_1x_2 + x_2^2$ is expressed in a form $q(x) = x^T Ax$ Find the matrix A (a) $\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ (c) $\begin{bmatrix}0 & 3\\ 3 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (e) none of the above

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{bmatrix}9 & \sqrt{3}\\ \sqrt{3} & 11\end{bmatrix}$ (a) 2,5 and $\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (b) 2, 3 and $\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (c) -2, 2 and $\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (d) 2, 3 and $\begin{bmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{bmatrix}$ (e) none of the above

$3^{1/5} \times 3^{1/5}$ $\sqrt{3}$ $3^{1/2}$ $\frac{1}{3}$

(d) i) S is linearly independent ii) S spans V (e) None of the above

Let V be a vector space such that one basis has m elements and another basis has n elements. (a) Then $m \neq n$ (b) $m = n + 1$ (c) $m = 2n$ (d) $m + 1 = n$ (e) None of the above $m = n$

Given a matrix $A=\begin{bmatrix}1 & x & 1\\ 3 & -4 & -2\\ 5 & 3 & 5\end{bmatrix}$ Find x if det(A)=-35 (a) 2 (b) 0 (c) 5 (d) 3 (e) none of the above

Let W be a subspace of an n-dimensional vector space: (a) Then dim W ≤ n. In particular, if dim W = n + 1, then W = V (b) Then dim W ≤ N. In particular, if dim W = n, then W = V + 1 (c) Then dim W ≥ n. In particular, if W = n + 1, then W = V (d) Then dim V ≥ n. In particular, if dim W = n, then W = V (e) None of the above then dim W≤n in particular if dim W = n, W = V

Find the eigenvalues of the matrix $A=\begin{bmatrix}1 & 0 & -\sqrt{3}\\ 0 & 2 & 0\\ -\sqrt{3} & 0 & -1\end{bmatrix}$ (a) $\lambda_1=\lambda_2=2$ and $\lambda_3=-2$ (b) $\lambda_1=1$, $\lambda_2=2$ and $\lambda_3=-1$ (c) $\lambda_1=\lambda_2=\lambda_3=2$ (d) $\lambda_1=\lambda_2=\lambda_3=-2$ (e) none of the above

$\lambda_1 = 2$
$\lambda_2 = 1$
$\lambda_3 = -1$

$-\lambda^2 +$

$\lambda_1 = 2$ $\lambda_2 = 1$ $\lambda_3 = -1$

Option C

1. Consider a vector space V over a field F. A vector v ∈ V is a linear combination of vectors $u_1, u_2, \ldots, u_m$ in F if there exist scalar $a_1, a_2, \ldots, a_m$ in V such that (a) $v = a_1u_1 + a_2u_2 + \ldots + a_mu_m$ (b) $u = a_1v_1 + a_2v_2 + \ldots + a_mv_m$ (c) $w = a_1y_1 + a_2y_2 + \ldots + a_my_m$ (d) All of the above (e) None of the above

2. Given a matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{bmatrix}$ Find the product $A^T I$, where I is a unit matrix.

(a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 8 \\ 3 & 7 & 0 \\ 5 & 4 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 5 & 8 \\ 4 & 3 & 2 \end{bmatrix}$ (e) none of the above

3. Let $M = S + A$ and $M^T = S^T + A^T$. Where S is a symmetric matrix and A is an antisymmetric matrix, hence find an expression for S and A. (a) $S = 2(M + M^T)$, $A = \frac{1}{2}(M - M^T)$ (b) $S = \frac{1}{2}(M + M^T)$ and $A = 2(M + M^T)$ (c) $S = \frac{1}{2}(M + M^T)$ and, $A = \frac{1}{2}(M - M^T)$ (d) $S = 2(M - M^T)$ and $A = 3(M + M^T)$ (e) none of the above

4. Use the law of set to simplify $(B \cap C) \cup (B \cap C')$ (a) B' (b) C (c) B (d) C' (e) None of the above

5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 1 & -1 \\ 3 & 6 & 9 & 15 \\ 1 & 3 & 2 & 7 \end{bmatrix}$ Find $\det(A)$ (a) 5 (b) 3 (c) 0 (d) 0.5 (e) none of the above

3 6 9 15

6. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{bmatrix}$ Find A^{-1} (a) $\frac{1}{28}\begin{bmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{bmatrix}$ (b) $\frac{1}{16}\begin{bmatrix} 2 & 6 & -2 \\ 22 & -16 & 7 \\ -6 & 12 & 8 \end{bmatrix}$ (c) $\frac{1}{28}\begin{bmatrix} 2 & -2 & 7 \\ 22 & -4 & 8 \\ 4 & 12 & -7 \end{bmatrix}$ (d) $\frac{1}{18}\begin{bmatrix} 4 & 27 & 7 \\ 22 & 16 & 8 \\ -6 & 12 & -7 \end{bmatrix}$ (e) None of the above

7. Suppose V is a vector space over a field F and W is a subspace of V. (a) Then W is a subspace of V if W is itself a vector space over F with respect to the operations of scalar addition and vector multiplication on V. (b) Then W is a subspace of V if W is itself a vector space over F with respect to the operations of vector addition or scalar multiplication on V. (c) Then W is a subspace of V if W is itself a vector space over F with respect to the operations of scalar multiplication only. (d) All of the above (e) None of the above

8. Let $A = \begin{bmatrix} 0 & 4 & 0 & 5 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ Find $\det(A)$ (a) 18 (b) 4 (c) 2 (d) 9 (e) 10

9. What values of scalars m, n, p in m u + n v + p w = 0 would make the vectors u = (1, 2, 3), v = (2, 5, 7), w = (1, 3, 5) linearly independent? (a) m = 1, n = 0, z = 0 (b) m = 0, n = 1, p = 0 (c) m = 0, n = 0, p = 1 (d) m = 2, n = 4, p = 3 (e) None of the above

10. Find the cofactor matrix A_{ij} of the matrix $A = \begin{bmatrix} 1 & 7 & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{bmatrix}$ (a) $\begin{bmatrix} 2 & 0 & 36 \\ -33 & 2 & 19 \\ 30 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 52 & -32 \\ -33 & -1 & 8 \\ 36 & -28 & 32 \end{bmatrix}$

$B \cap C = B - C = B$

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OPTION A

20. Which of the following is/are not correct (a) If $A \cap B = \phi$, then A and B are disjoint (b) A - B = A ∩ B'

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (d) All of the above (e) None of the above

21. A square matrix A is Skew Symmetric if (a) A= A^T (b) A = - A^T (c) A = A^{-1} (d) A = - A^{-1} e) none of the above.

22. Find $|A|$ if A = $\begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix}$ (a) 150 (b) 20 (c) 100 (d) 120 (e) none of the above.

23. Which of the following set operations in not correct? (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $(A \cup B)' = A' \cap B'$ (c) $(A \cap B)' = A' \cup B'$ (d) If U is the universal set, then $A' = \{x \text{ such at } x \notin, x \in U\}$

24. A relation from a set A to a set B is a subset of (a) $A \cup B$ (b) A x B (c) $A \cap B$ (d) A – B (e) None of the above

25. If A = $\begin{bmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ Find AB (a) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 \\ -2 & -3 \\ 10 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 1 \\ 1 & 1 \\ 12 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 8 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}$

26. Find scalars K_1 and K_2 such that the vector W = (9, 2, 7) is a linear combination of U = (1, 2, -1) and V = (6, 4, 2) in R^3 (a) $K_1 = 3$, $K_2 = 2$ (b) $K_1 = -3$, $K_2 = -2$ (c) $K_1 = -3$, $K_2 = 2$ (d) $K_1 = 3$, $K_2 = -2$ (e) None of the above

27. Let $\theta : X \to Y$ be an onto map, the Y is called (a) Range (b) domain (c) open set (d) All of the above (e) None of the above

28. A matrix is singular if (a) $|A| = 0$ (b) $|A| < 0$ (c) $|A| = 2$ (d) $|A| > 0$ (e) none of the above.

29. If A = $\begin{bmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{bmatrix}$ find the value of x if $|A| = -35$ (a) 0 (b) 2 (c) 3 (d) 5 (e) None of the above.

30. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors. If there exist scalar $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all equal to zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \ldots, \lambda_n v_n = 0$. Then the set S is called (a) linearly independent (b) independent set (c) dependent set (d) Linearly dependent (e) None of the above

31. Determine the Adjoint of A if A = $\begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{bmatrix}$ (a) $\begin{bmatrix} -21 & 0 & 7 \\ 7 & 11 & 17 \\ 14 & -22 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -21 & 2 & 3 \\ 7 & 11 & 5 \\ 6 & 13 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 32 & 1 & 8 \\ 7 & 10 & 5 \\ 14 & 16 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 7 & 1 & 0 \\ 3 & 11 & -6 \\ 4 & -10 & -1 \end{bmatrix}$ (e) none of the above.

2013

OPTION A

20. Which of the following is/are not correct (a) If $A \cap B = \phi$, then A and B are disjoint (b) $A - B = A \square B'$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (d) All of the above (e) None of the above

21. A square matrix A is Skew Symmetric if (a) $A = A^T$ (b) $A = -A^T$ (c) $A = A^{-1}$ (d) $A = -A^{-1}$ c) none of the above.

22. Find $|A|$ if $A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix}$ (a) 150 (b) 20 (c) 100 (d) 120 (e) none of the above.

23. Which of the following set operations in not correct? (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $(A \cup B)' = A' \cap B'$ (c) $(A \cap B)' = A' \cup B'$ (d) If U is the universal set, then $A' = \{x \text{ such at } x \notin, x \in U\}$

24. A relation from a set A to a set B is a subset of (a) $A \cup B$ (b) A x B (c) $A \cap B$ (d) A − B (e) None of the above

25. If $A = \begin{bmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ Find AB (a) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 \\ -2 & -3 \\ 10 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 1 \\ 1 & 1 \\ 12 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 8 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}$

26. Find scalars K_1 and K_2 such that the vector W = (9, 2, 7) is a linear combination of U = (1, 2, -1) and V = (6, 4, 2) in $\mathbb{R}^3$ (a) $K_1 = 3$, $K_2 = 2$ (b) $K_1 = -3$, $K_2 = -2$ (c) $K_1 = -3$, $K_2 = 2$ (d) $K_1 = 3$, $K_2 = -2$ (e) None of the above

27. Let $\theta : X \to Y$ be an onto map, the Y is called (a) Range (b) domain (c) open set (d) All of the above (e) None of the above

28. A matrix is singular if (a) $|A| = 0$ (b) $|A| < 0$ (c) $|A| = 2$ (d) $|A| > 0$ (e) none of the above.

29. If $A = \begin{bmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{bmatrix}$ find the value of x if $|A| = -35$ (a) 0 (b) 2 (c) 3 (d) 5 (e) None of the above.

30. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors. If there exist scalar $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all equal to zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \ldots, \lambda_n v_n = 0$. Then the set S is called (a) linearly independent (b) independent set (c) dependent set (d) Linearly dependent (e) None of the above

31. Determine the Adjoint of A if $A = \begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{bmatrix}$ (a) $\begin{bmatrix} -21 & 0 & 7 \\ 7 & 11 & 17 \\ 14 & -22 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -21 & 2 & 3 \\ 7 & 11 & 5 \\ 6 & 13 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 32 & 1 & 8 \\ 7 & 10 & 5 \\ 14 & 16 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 7 & 1 & 0 \\ 3 & 11 & -6 \\ 4 & -10 & -1 \end{bmatrix}$ (e) none of the above.

2013

OPTION A

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN, BENIN CITY
FIRST SEMESTER B.Sc. 2012/2013 EXAMINATION

COURSE CODE: MTH230 (Linear Algebra)

INSTRUCTION: (i) Write and circle your attendance list serial number on the objective answer paper (ii) Attempt all questions by SHADING (using HB pencil) the letter box that corresponds to the correct option. Information about your Mat. No., Name, Course code, Faculty code and Departmental code must be clearly written and CORRECTLY SHADED. YOU MUST SUBMIT YOUR QUESTION PAPER ALONG WITH YOUR ANSWER SHEET. Time allowed: $1\frac{1}{2}$ hrs

1. In any mapping $\theta : X \to Y$, if every element y ∈ ~~Y is~~ an image of an x ∈ X then θ is an (a) onto mapping (b) into mapping (c) onto and into mapping (d) All of the above (e) None of the above

2. If the image set of a map $\theta : X \to Y$ is the same as the codomain Y, then the mapping is said to be (a) injective (b) surjective (c) bijective (d) All of the above (e) None of the above

3. If $V_1, V_2, \ldots, V_n$ are vectors in a vector space V. Then the set W of all linearly combinations of $V_1, V_2, \ldots, V_n$ is (a) a subspace of V (b) linearly dependent (c) linearly independent (d) All of the above (e) None of the above

4. Find the determinant of A = $\begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 2 & 8 & 6 \end{pmatrix}$ (a) 10 (b) -12 (c) 40 (d) 30 (e) none of the above

5. Let V be a vector space containing the vectors $\{V_1, V_2, \ldots, V_n\}$. Then the dimension of V is (a) infinite (b) finite (c) countably infinite (d) Zero

6. Which of the following is wrong? (a) The set A of all linear transformations of a vector space V into itself forms a ring with respect to addition and scalar multiplication (b) The set m of all non-singular linear transformation of a vector space V into itself form a group (c) The set A of all linear transformations of a vector space V into itself is itself a vector space (d) All of the above (e) None of the above

7. Find the eigenvalues of A if A = $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ (a) 7 and 3 (b) 2 and 3 (c) 1 and 3 (d) 0 and 3 (e) none of the above

8. Let $F : V \to W$ be a function from the vector space V into the vector space W. Then F is a linear transformation if (a) $F(u+v) = F(u) + F(v)\ \forall\ u, v \in V$ (b) $F(\alpha u) = \alpha F(u)\ \forall\ \alpha \in R, u \in V$ (c) property (a) and (b) holds (d) the domain is equal to the codomain (e) None of the above

9. Consider the F: $R^2 \to R^3$ defined by $F(v) = (x, x+y, x-y) \forall V \in R^2, V \in (x, y)$ if $V_1 = (x_1, y_1)$ and $V_2 = (x_2, y_2)$, then $F(v_1 + v_2)$ is a (a) non-linear transformation (b) quasi–linear transformation (c) linear transformation (d) All of the above

10. If this equation $x_1^2 + 4x_1x_2 - 3x_1x_3 + x_2^2 + 4x_2x_3 - x_3^2$ Could be expressed in the form $x^T Ax$

Find A (a) $\begin{bmatrix} 2 & 0 & 1 \\ 7 & 3 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1.5 \\ 2 & 1 & 2 \\ -1.5 & 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 3 & 0 \\ 5 & 1 & -9 \\ 1 & 0 & -1 \end{bmatrix}$ (e) none of the above.

OPTION A

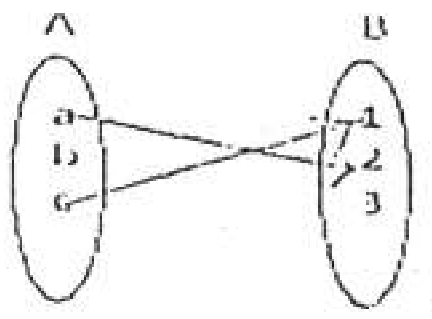
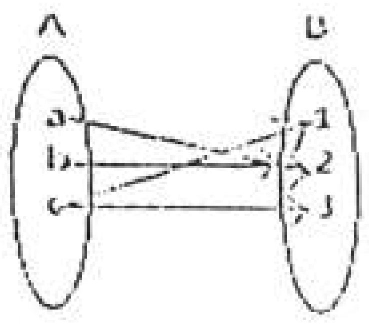
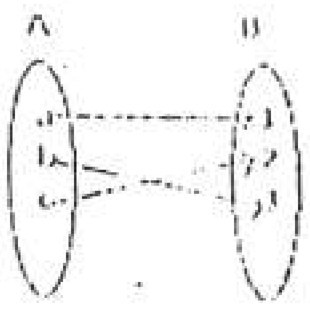
1. If W = $\{V_1, V_2, \ldots, V_n\}$ are vectors in a vector space V, and if every vectors in V is expressible as a linear combination of $V_1, V_2, \ldots, V_n$. Then (a)V space W(b) W span V(c)V span V (d) all of the above (e) None of the above

2. Determine A^{-1} if A = $\begin{bmatrix}2&-1&3\\1&3&-1\\2&-2&5\end{bmatrix}$ (a) $\frac{1}{9}\begin{bmatrix}13&1&-8\\-7&4&-5\\-8&-2&7\end{bmatrix}$ (b) $\frac{1}{20}\begin{bmatrix}13&-1&-8\\-7&4&5\\-8&2&7\end{bmatrix}$ (c) $\frac{1}{6}\begin{bmatrix}5&-1&8\\6&-4&6\\-8&2&7\end{bmatrix}$

(d) $\frac{1}{7}\begin{bmatrix}13&-1&-8\\-7&4&8\\-8&2&7\end{bmatrix}$ (e) None of the above.

3. A mapping that is both injective and surjective is called (a) injective and surjective (b) surjection (c) injection (d) bijective (e) None of the above

4. An injective mapping is also called (a)bijective (b) one – to – one (c) surjective (d) one – to – two(e)None of the above

5. Which of the following is a map (a)  (b)  (c)  (d) all of the above
(e) None of the above

6. Determine the inverse of A if A = $\begin{bmatrix}2&7&4\\3&1&6\\5&0&8\end{bmatrix}$ (a) $\frac{1}{28}\begin{bmatrix}8&56&38\\-6&-4&0\\-5&-35&-19\end{bmatrix}$ (b) $\frac{1}{38}\begin{bmatrix}8&-56&38\\6&-4&0\\-5&35&-19\end{bmatrix}$

(c) $\frac{1}{28}\begin{bmatrix}2&-4&7\\22&-16&7\\-6&12&-7\end{bmatrix}$ (d) $\frac{1}{40}\begin{bmatrix}3&-5&7\\23&15&-8\\10&-3&2\end{bmatrix}$ (e)none of the above

7. Let V be any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ a finite set of vectors in V. Then S is called a basis for V if

(a) S is linearly dependent and S span V (b) S is linearly independent and S do not span V (c)S is linearly independent and S span V (d) S is linearly dependent and S do not span V (e) None of the above

8. If A = $\begin{bmatrix}2&1&3\\0&4&5\\2&1&1\end{bmatrix}$ and B = $\begin{bmatrix}1&1&3\\0&1&2\\1&4&y\end{bmatrix}$. Find the value of y if $|AB| = 48$ (a)-12(b) 8(c) 12 (d) 6 (e) none of the above

9. Find the eigenvalues of the matrix A = $\begin{bmatrix}1&-1&0\\1&2&1\\-2&1&-1\end{bmatrix}$ (a)-1, 1, and 2 (b) 3, 2, and 6 (c) -1, 2, and -2 (d) -1, 5, and -2

(e)none of the above

2013

OPTION A

20. Which of the following is/are not correct (a) If $A \cap B = \phi$, then A and B are disjoint (b) $A - B = A \cap B'$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (d) All of the above (e) None of the above

21. A square matrix A is Skew Symmetric if (a) $A = A^T$ (b) $A = -A^T$ (c) $A = A^{-1}$ (d) $A = -A^{-1}$ (e) none of the above

22. Find $|A|$ if $A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix}$ (a) 150 (b) 20 (c) 100 (d) 120 (e) none of the above.

23. Which of the following set operations in not correct? (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $(A \cup B)' = A' \cap B'$ (c) $(A \cap B)' = A' \cup B'$ (d) If U is the universal set, then $A' = \{x \text{ such at } x \notin A, x \in U\}$

24. A relation from a set A to a set B is a subset of (a) $A \cup B$ (b) A x B (c) $A \cap B$ (d) A − B (e) None of the above

25. If $A = \begin{bmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ Find AB (a) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 \\ -2 & -3 \\ 10 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 1 \\ 1 & 1 \\ 12 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 8 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}$

26. Find scalars K_1 and K_2 such that the vector W = (9, 2, 7) is a linear combination of U = (1, 2, -1) and V = (6, 4, 2) in R^3 (a) $K_1 = 3$, $K_2 = 2$ (b) $K_1 = -3$, $K_2 = -2$ (c) $K_1 = -3$, $K_2 = 2$ (d) $K_1 = 3$, $K_2 = -2$ (e) None of the above

27. Let $\theta : X \to Y$ be an onto map, the Y is called (a) Range (b) domain (c) open set (d) all of the above (e) None of the above

28. A matrix is singular if (a) $|A| = 0$ (b) $|A| < 0$ (c) $|A| = 2$ (d) $|A| > 0$ (e) none of the above.

29. If $A = \begin{bmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{bmatrix}$ find the value of x if $|A| = -35$ (a) 0 (b) 2 (c) 3 (d) 5 (e) None of the above.

30. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors. If there exist scalar $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all equal to zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \ldots, \lambda_n v_n = 0$. Then the set S is called (a) linearly independent (b) independent set (c) dependent set (d) Linearly dependent (e) None of the above

31. Determine the Adjoint of A if $A = \begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{bmatrix}$ (a) $\begin{bmatrix} 21 & 0 & 7 \\ 7 & 11 & 17 \\ 14 & -22 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -21 & 2 & 3 \\ 7 & 11 & 5 \\ 6 & 13 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 32 & 1 & 8 \\ 7 & 10 & 5 \\ 14 & 16 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 7 & 1 & 0 \\ 3 & 11 & -6 \\ 4 & -10 & -1 \end{bmatrix}$ (e) none of the above.

2013

OPTION A

20. Which of the following is/are not correct (a) If $A \cap B = \phi$, then A and B are disjoint (b) A − B = A ∩ B'

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (d) All of the above (e) None of the above

21. A square matrix A is Skew Symmetric if (a) A = A^T (b) A = $-A^T$ (c) A = A^{-1} (d) A = $-A^{-1}$ e) none of the above

22. Find $|A|$ if A = $\begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix}$ (a) 150 (b) 20 (c) 100 (d) 120 (e) none of the above.

23. Which of the following set operations in not correct? (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $(A \cup B)' = A' \cap B'$ (c) $(A \cap B)' = A' \cup B'$ (d) If U is the universal set, then $A' = \{x \text{ such that } x \notin A, x \in U\}$

24. A relation from a set A to a set B is a subset of (a) $A \cup B$ (b) A x B (c) $A \cap B$ (d) A − B (e) None of the above

25. If A = $\begin{bmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ Find AB (a) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 \\ -2 & -3 \\ 10 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 1 \\ 1 & 1 \\ 12 & -5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 8 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}$

26. Find scalars K_1 and K_2 such that the vector W = (9, 2, 7) is a linear combination of U = (1, 2, -1) and V = (6, 4, ... R^3 (a) $K_1 = 3$, $K_2 = 2$ (b) $K_1 = -3$, $K_2 = -2$ (c) $K_1 = -3$, $K_2 = 2$ (d) $K_1 = 3$, $K_2 = -2$ (e) None of the above

27. Let $\theta : X \to Y$ be an onto map, the Y is called (a) Range (b) domain (c) open set (d) All of the above (e) None of the above

28. A matrix is singular if (a) $|A| = 0$ (b) $|A| < 0$ (c) $|A| = 2$ (d) $|A| > 0$ (e) none of the above.

29. If A = $\begin{bmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{bmatrix}$ find the value of x if $|A| = -35$ (a) 0 (b) 2 (c) 3 (d) 5 (e) None of the above.

30. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors. If there exist scalar $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all equal to zero su... $\lambda_1 v_1 + \lambda_2 v_2 + \ldots, \lambda_n v_n = 0$. Then the set S is called (a) linearly independent (b) independent set (c) dependent set (d) Linearly dependent (e) None of the above

31. Determine the Adjoint of A if A = $\begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{bmatrix}$. (a) $\begin{bmatrix} -21 & 0 & 7 \\ 7 & 11 & 1 \\ 14 & & \end{bmatrix}$ $\begin{bmatrix} & & 3 \\ 7 & 11 & 5 \\ 6 & 13 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 32 & 1 & 8 \\ 7 & 10 & 5 \\ 14 & 16 & - \end{bmatrix}$

(d) $\begin{bmatrix} 7 & 1 & 0 \\ 3 & 11 & -6 \\ 4 & -10 & -1 \end{bmatrix}$ (e) none of the above.

2017

OPTION A

DEPARTMENT OF MATHEMATICS
FACULTY OF PHYSICAL SCIENCES
UNIVERSITY OF BENIN, BENIN CITY

FIRST SEMESTER EXAMINATIONS 2016/2017 SESSION

COURSE TITLE: Linear Algebra (MTH 230) TIME ALLOWED: 2 Hours

INSTRUCTIONS: (i) Write and circle your attendance list serial number on the objective answer paper. (ii) Attempt all questions by SHADING (using HB pencil) the letter box that corresponds to the correct option. Information about your Mat. No., Name, Course code, Faculty code and Departmental code must be clearly written and CORRECTLY SHADED. YOU MUST SUBMIT YOUR QUESTION PAPER ALONG WITH YOUR ANSWER SHEET.

1. If $A+I_3=\begin{bmatrix}2&3&6\\4&0&3\\4&2&1\end{bmatrix}$, Evaluate $(A-I_3)(A-I_3)$ (a) $\begin{bmatrix}2&3&6\\4&0&3\\4&2&1\end{bmatrix}$ (b) $\begin{bmatrix}39&30&27\\20&21&27\\20&24&30\end{bmatrix}$ (c) $\begin{bmatrix}36&12&15\\12&18&21\\12&10&29\end{bmatrix}$ (d) $\begin{bmatrix}40&18&27\\20&18&27\\20&14&31\end{bmatrix}$ (e) None of the above

2. Find the values of p and q so that $\begin{pmatrix}p&6\\5&q\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}7\\3\end{pmatrix}$ (a) $[p=6, q=-\frac{1}{2}]$ (b) $[p=3, q=1]$ (c) $[p=\frac{1}{2}, q=-7]$ (d) $[p=7, q=-\frac{1}{2}]$ (e) None of the above

3. Consider the square matrix $A=\begin{pmatrix}3&2&4\\1&5&3\\-1&8&2\end{pmatrix}$, then what can be said about A? (a) A is invertible (b) A is non-singular matrix (c) A is symmetric (d) A is a singular matrix (e) None of the above

4. Use Crammer's ruler to solve the following linear system of equations: $\begin{matrix}x+x+z=6\\x-y+z=2\\2x+y+z=7\end{matrix}$

(a) 2, 1, 4 (b) 3, 2, 1 (c) 1, 2, 3 (d) 0, 1, 2 (e) None of the above

5. Solve the following system of linear equations: $\begin{matrix}-x+3y-2z=5\\4x-y-3z=-8\\2x+2y-5z=7\end{matrix}$ (a) $\frac{-55}{11},\frac{-56}{11},\frac{87}{11}$

(b) $-10,\frac{-79}{11},\frac{-91}{11}$ (c) $\frac{55}{11},\frac{56}{11},\frac{-87}{11}$ (d) $1,\frac{79}{11},\frac{91}{11}$ (e) None of the above.

6. Let $F:V\to U$ be a non-singular linear map. Then the image of any linearly independent set under F is (a) independent (b) linearly dependent (c) dependent (d) linearly independent (e) None of the above

7. Consider the sets $A=\{1,5,9\}$ and $B=\{1,2,3,4,5\}$. Which of the following best describes the relation between A and B. (a) $A\not\subset B$ (b) $A\subseteq B$ (c) $A\subset B$ (d) $A\in B$ (e) None of the above

8. Determine which of the sets is a null set: (i) $X=\{x: x^2=9,\ 2x=4\}$ (ii) $Y=\{x: x\neq x\}$ (iii) $Z=\{x: x+8=8\}$ (a) i and iii (b) i and ii (c) i only (d) iii only (e) None of the above

9. Given the set $A=\{\{1,2,3\},\{4,5\},\{6,7,8\}\}$ which of the following is correct? (a) $1\in A$ (b) $\{1,2,3\}\subseteq A$ (c) $\{6,7,8\}\in A$ (d) $\phi\in A$ (e) None of the above

10. Determine the power set $p(A)$ of $A=\{a,b,c,d\}$ (a) $p(A)$ has 4 elements (b) $p(A)$ has 8 elements (c) $p(A)$ has 12 elements (d) $p(A)$ has 16 elements (e) None of the above

11. Consider the vector space $V=p(t)$ of polynomials over the real field IR, and let $H:V\to V$ be the third derivative operator, that $H[f(t)]=d^3f/dt^3$, then (a) kernel H = {polynomial of degree ≤ 1} $=p_1(t)$, and image $H=V$ (b) kernel H = {polynomial of degree ≤ 2} $=p_2(t)$, and image $H=V$ (c) kernel H = {polynomial of degree ≤ 3} $=p_3(t)$, and image $V=H$ (d) kernel H = {polynomial of degree ≤ 4} $=p_4(t)$, and image $V=H$ (e) None of the above

12. Let $F:IR^4\to IR^4$ be the linear mapping defined by $F(x,y,z,t)=(x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$. Find a basis and dimension of the

image of F. (a) (1, 2, 3) forms a basis of image of F, and dim(Image F) = 3 (b) (1, 2, 3, 4) forms a basis of image of F, and dim(Image F) = 3 (c) (1, 2, 3) and (0, 1, 1) forms a basis of image of F, and dim(Image F) = 2 (d) All of the above (e) None of the above

13. Let V and U be vector spaces over a field K. Then the collection of all linear mappings from V into U with the operations of addition and scalar multiplication forms: (a) a vector space over K (b) a subspace over K (c) a set of linear maps over K (d) a closed space over K (e) None of the above

14. Consider the mapping $F: IR^2 \to IR^2$ defined by $F(x,y) = (3y, 2x)$. Let S be the unit circle in IR^2, that is the solution set of $x^2 + y^2 = 1$. Find $F(S)$ (a) $(3y, 2x) = (a, b)$ (b) $\frac{a^2}{9} + \frac{b^2}{4} = 1$ (c) $\frac{a^2}{4} + \frac{b^2}{9} = 1$ (d) $\frac{a^2}{3} + \frac{b^2}{2} = 1$ (e) None of the above

15. Let $F: IR \to IR$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto, hence, f has an inverse mapping f^{-1}. Find a formular for f^{-1} (a) $f^{-1}(x) = \frac{1}{2}(x - 3)$ (b) $f^{-1}(x) = \frac{1}{2x - 3}$ (c) $f^{-1}(x) = \frac{1}{2}(x + 3)$ (d) $f^{-1}(x) = \frac{1}{4}(2x + 1)$ (e) None of the above

16. Solve the following system of equations for x_1 and x_2 in terms of x_3, $\begin{pmatrix} x_1 + x_2 + x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \end{pmatrix}$
(a) $x_1 = 6 - 2x_3$, $x_2 = x_3 - 6$ (b) $x_1 = 6 + 3x_3$, $x_2 = 6x_3 + 2$ (c) $x_1 = 3x_3 - 6$, $x_2 = 2x_3$
(d) $x_1 = 6x_3 - 2$, $x_2 = 6 - x_3$ (e) None of the above

17. Determine the eigenvalues of $A = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$ (a) $\lambda = 1, 2$ (b) $\lambda = 1, -2$ (c) $\lambda = -1, -2$
(d) $\lambda = -1, 2$ (e) None of the above

18. If $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find a matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ such that $AX = B$ (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(b) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^t$ (c) $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix}$ (e) None of the above

19. Given that $F(t) = 2t^2 + 4t + 6$. Find $F(A)$. If $A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ (a) $\begin{pmatrix} 132 & 100 \\ 60 & 52 \end{pmatrix}$ (b) $\begin{pmatrix} 60 & 132 \\ 52 & 100 \end{pmatrix}$
(c) $\begin{pmatrix} 52 & 100 \\ 60 & 132 \end{pmatrix}$ (d) $\begin{pmatrix} 52 & 60 \\ 100 & 132 \end{pmatrix}$ (e) None of the above

20. If $E = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & x \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{pmatrix}$ and $\det(E) = -546$. Find the value of x. (a) 6 (b) 4 (c) 12 (d) 1
(e) None of the above

21. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$ (a) $\begin{pmatrix} 7.5 & -1.5 & -4.75 \\ -4 & 1 & 2.5 \\ -2.5 & 0.5 & 1.75 \end{pmatrix}$
(b) $\begin{pmatrix} 7.5 & -4 & 2.5 \\ -1.5 & 1 & 0.5 \\ -4.75 & 2.5 & 1.75 \end{pmatrix}$ (c) $\begin{pmatrix} 30 & -16 & -10 \\ -6 & 4 & 2 \\ -19 & 10 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 0.8 & 0.4 & -0.2 \\ -0.16 & 0.1 & 0.06 \\ -0.53 & 0.28 & 0.19 \end{pmatrix}$ (e) None of the above

22. Determine the value of k for which the following set of homogenous equations has non-trivial solutions $\begin{matrix} 4x_1 + 3x_2 - x_3 = 0 \\ 7x_1 - x_2 - 3x_3 = 0 \\ 3x_1 - 4x_2 + kx_3 = 0 \end{matrix}$ (a) 0.5 (b) 4 (c) -2 (d) 2 (e) None of the above

23. If $AX = \lambda X$, where $A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. Determine the eigenvalues of the matrix A. (a) (-2, 1, 0)
(b) (1, 2, 4) (c) (-2, 1, 1) (d) (0, 1, 1) (e) None of the above

OPTION A

24. If $A=\begin{pmatrix}1 & 0 & 0\\ 2 & x & 0\\ 3 & 4 & x+2\end{pmatrix}$ and $det\ A = 15$. Find the eigenvalues of A if x is non-negative (a) 0, 5, 7 (b) $\frac{3}{5}, \frac{3}{7}, 1$ (c) 1, 3, 5 (d) 4, 5, 7 (e) None of the above

25. If $A=\begin{pmatrix}1 & 4 & 3\\ 6 & 2 & 5\\ 1 & 7 & 0\end{pmatrix}$ determine $Adj\ A$. (a) $\begin{pmatrix}-35 & 5 & 40\\ 21 & -3 & -3\\ 14 & 13 & -22\end{pmatrix}$ (b) $\begin{pmatrix}-35 & 20 & 120\\ 126 & -6 & -15\\ 14 & 91 & 0\end{pmatrix}$ (c) $\begin{pmatrix}-35 & 12 & 14\\ 20 & -6 & 9\\ 120 & -15 & 0\end{pmatrix}$ (d) $\begin{pmatrix}-35 & 21 & 14\\ -5 & -3 & -13\\ 40 & -3 & -22\end{pmatrix}$ (e) None of the above

26. Determine the value of k such that the system has unique solution. $\begin{aligned}x+y+kz&=2\\ 3x+4y+2z&=k\\ 2x+3y-z&=1\end{aligned}$ (a) 3 (b) 2 (c) -2 (d) 0 (e) None of the above

27. Find a and b, given that $\begin{pmatrix}4 & 2a & 3\\ -1 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 5\\ 3 & 0\\ -6 & 3b\end{pmatrix}=\begin{pmatrix}-2 & 2\\ 5 & -5\end{pmatrix}$ (a) $[a=2, b=2]$ (b) $[a=5, b=0]$ (c) $[a=1, b=2]$ (d) $[a=2, b=-2]$ (e) None of the above

28. Given that $\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}3\\ k\end{pmatrix}$. Find the two possible values of k. (a) ½, -3 (b) -6, 36 (c) 2, 3 (d) 2, 30 (e) None of the above

29. If $P=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$, $QP=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$. Find Q. (a) $\begin{pmatrix}\frac{2}{3} & 1\\ 0 & 2\end{pmatrix}$ (b) $\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$ (c) $\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ 0 & 2\end{pmatrix}$ (d) $\begin{pmatrix}-\frac{2}{3} & -1\\ 0 & 2\end{pmatrix}$ (e) None of the above

30. If $A=\begin{pmatrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$. Compute $3A^3+2A^2-5I$ (a) $\begin{pmatrix}-63 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 32\end{pmatrix}$ (b) $\begin{pmatrix}-68 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & 27\end{pmatrix}$ (c) $\begin{pmatrix}-68 & 0 & 27\\ 0 & -5 & 0\\ 0 & 0 & 0\end{pmatrix}$ (d) $\begin{pmatrix}63 & 0 & 32\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ (e) None of the above

31. Find the number of elements in the power set of the set $B=\{\text{positive divisors of }12\}$ (a) 64 (b) 32 (c) 16 (d) 4 (e) None of the above

32. Express $V=(3,7,-4)$ in IR^3 as a linear combination of the vectors $u_1=(1,2,3)$, $u_2=(2,3,7)$, $u_3=(3,5,7)$ (a) $V=4u_1+2u_2-6u_3$ (b) $V=2u_1-4u_2+3u_3$ (c) $V=u_1-u_2+u_3$ (d) $V=u_1+3u_2-2u_3$ (e) None of the above

33. Express the polynomial $v=3t^2+5t-5$ as a linear combinator of the polynomials $p_1=t^2+2t+1$, $p_2=2t^2+5t+4$, $p_3=t^2+3t+6$ (a) $v=p_1+p_2-p_3$ (b) $v=3p_1+3p_2-4p_3$ (c) $v=3p_1+p_2-2p_3$ (d) $v=2p_1+2p_2-3p_3$ (e) None of the above

34. Suppose $\{v_1, v_2, \ldots, v_n\}$ spans a vector space v, and suppose $\{w_1, w_2, \ldots, w_m\}$ is linearly independent. Then (a) $m=n$ (b) $m\geq n$ (c) $m\neq n$ (d) $m\leq n$ (e) None of the above

35. A set $S=\{u_1, u_2, \ldots, u_n\}$ of vectors is basis of V if it has the following properties: (a) S is linearly independent, and S spans V (b) S is linearly dependent, and S spans V (c) V is linearly independent, and V spans S. (d) V is linearly dependent, and V spans S (e) None of the above

36. Let V be a vector space of finite dimension n. Then which of the following statement(s) is/are correct? (i) Any $n+1$ or more vectors in V are linearly independent. (ii) Any linearly independent set $S=\{u_1, u_2, \ldots, u_n\}$ with n elements is a basis of V (iii) Any spanning set $T=\{v_1, v_2, \ldots, v_n\}$ of V with n elements is a basis of V. (a) i and ii (b) ii and iii (c) i and iii (d) i, ii and iii (e) None of the above

37. Let W be a subspace of an n-dimensional vector space V. Then ……(a) $\dim W=n+1$ (b) $\dim W\neq n$ (c) $\dim W\leq n$ (d) If $\dim W=n$, then $W\subseteq V$ (e) None of the above

38. Suppose W_1 and W_2 are finite dimensional subspaces of a vector space V. Then $W_1 + W_2$ has finite dimension given by: (a) $\dim\ (W_1 + W_2) = \dim W_1 + \dim W_2$ (b) $\dim\ (W_1 + W_2) = \dim (W_1 \cup W_2)$ (c) $\dim\ (W_1 + W_2) = \dim (W_1 \cap W_2)$ ($\cancel{d}$) $\dim\ (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim\ (W_1 \cap W_2)$ (e) None of the above

39. Express M as a linear combination of the matrices A, B, C where $M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$. (a) $M = 2A + 3B - C$ (b) $M = A + B - C$ (c) $M = 2A - 3B - 2C$ (d) $M = A - 3B + C$ (e) None of the above

40. Let V be the vector space of functions $f : IR \to IR$, which of the following is not a subspace of V. (a) $W = \{f(x) : f(1) = 0\}$ (b) $W = \{f(x) : f(3) = f(1)\}$ (c) $W = \{f(t) : f(-x) = -f(x)\}$ ($\cancel{d}$) All of the above (e) None of the above

41. Find the Rank of the matrix $K = \begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 2 & 5 & 3 & 10 & 7 \\ 3 & 5 & 7 & 10 & 4 \end{pmatrix}$ (a) 4 ($\cancel{b}$) 3 (c) 6 (d) 5 (e) None of the above

42. Find AB if $A = \begin{pmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$ (a) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 3 \\ 21 & 4 & 6 \end{pmatrix}$ ($\cancel{c}$) $\begin{pmatrix} 7 & 8 \\ -2 & -5 \\ 10 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4 & 7 \\ 0 & 6 & -2 \\ 21 & 14 & 10 \end{pmatrix}$ (e) None of the above

43. Express $v = (1, -2, 5)$ in IR^3 as a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (2,-1,1)$ (a) $v = -4u_1 + 7u_2 + 8u_3$ (b) $v = 6u_1 - 3u_2 - 2u_3$ (c) $v = -8u_1 + 7u_2 - 6u_3$ ($\cancel{d}$) $v = -6u_1 + 3u_2 + 2u_3$ (e) None of the above

44. Find a basis of the subspace W of the vector space $V = IR^3$, where $W = \{(a,b,c) : a + b + c = 0\}$ (a) $\dim W = 2$ (b) $\dim W = 3$ (c) $\dim W = 4$ (d) $\dim W = 5$ ($\cancel{e}$) None of the above

45. Find the conditions on a, b, c so that $v = (a,b,c)$ in IR^3 belongs to $W = span(u_1, u_2, u_3)$ where $u_1 = (1,2,0)$, $u_2 = (-1,1,2)$, $u_3 = (3,0,-4)$ (a) $v = (a,b,c)$ belongs to W if and only if the system is inconsistent ($\cancel{b}$) $v = (a,b,c)$ belongs to W if and only if the system is consistent (c) $v = (a,b,c)$ belongs to W if the system is linearly dependent (d) None of the above (e) All of the above

46. Let V be a vector space of finite dimension n. Then which of the following statements is/are correct? (i) Any $n+1$ or more vectors must be linearly independent (ii) Any linearly independent set $S = \{u_1, u_2, \ldots, u_n\}$ with n elements is not a basis of V (iii) Any spanning set $T = \{v_1, v_2, \ldots, v_n\}$ of V with n elements is a basis of V. (a) i (b) ii ($\cancel{c}$) iii (d) i, ii & iii (e) None of the above

47. Given that $A = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$, calculate $A \bullet A^T$ (a) $\begin{pmatrix} -56 & 62 \\ 60 & 114 \end{pmatrix}$ (b) $\begin{pmatrix} 56 & 114 \\ 62 & 62 \end{pmatrix}$ (c) $\begin{pmatrix} 62 & 56 \\ 114 & 62 \end{pmatrix}$ ($\cancel{d}$) $\begin{pmatrix} 56 & 62 \\ 62 & 114 \end{pmatrix}$ (e) None of the above

48. Express the polynomial $v = t^2 + 4t - 3$ in $p(t)$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 1$ (a) $v = -3p_1 + 2p_2 + 4p_3$ (b) $v = -5p_1 + 8p_2 + 7p_3$ (c) $v = 8p_1 - 4p_2 - 7p_3$ (d) $v = -9p_1 + 4p_2 + 3p_3$ ($\cancel{e}$) None of the above

49. A square matrix $A = [a_{ij}]$ such that $a_{ij} = 0$ when $i > j$ and $a_{ij} = 0$ whenever $i < j$ is called ---- (a) Canonical matrix ($\cancel{b}$) Diagonal matrix (c) Upper triangular matrix (d) Lower triangular matrix (E) None of the above

50. Which of the following statements is incorrect? (a) If A is a square matrix, then A is non-singular (b) If A is non-singular matrix, then A^{-1} is unique ($\cancel{c}$) If A^{-1} is the inverse of an $n \times n$ matrix A, then A^{-1} is $n \times m$ (d) A square matrix A of order n has an inverse if and only if $\det A \neq 0$ (e) None of the above

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN, BENIN CITY.
CONTINUOUS ASSESMENT TEST FOR MTH230 (LINEAR ALGEBRA) 2016/2017 SESSION
DATE: Friday 13th 2016, Time allowed: 45 minutes

1. Express $v = (1,-2,5)$ in R^3 as a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$, $u_3 = (2,-1,1)$ (a) $v = -6u_1 + 3u_2 + 2u_3$ (b) $v = 6u_1 - 3u_2 - 2u_3$ (c) $v = -8u_1 + 7u_2 - 6u_3$ (d) $v = -4u_1 + 7u_2 + 8u_3$
2. Express the polynomial $v = t^2 + 4t - 3$ in $P(t)$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 1$ (a) $v = 8p_1 - 4p_2 - 7p_3$ (b) $v = -3p_1 + 2p_2 + 4p_3$ (d) $v = -5p_1 + 8p_2 + 7p_3$ (d) $v = -9p_1 + 4p_2 + 3p_3$ (e) None of the above
3. Let $V = R^3$. Which of the following space is a subspace of W?
(a) $W = \{(a,b,c): a \geq 0\}$ (b) $W = \{(a,b,c): a^2 + b^2 + c^2 \leq 1\}$ (c) All of the above (d) None of the above
4. Let $V = P(t)$, the vector space of real polynomials. Which of the following is not a subspace of V?
(a) W consists of all polynomials with integral coefficients (b) W consists of all polynomials with degree ≥ 6 and the zero polynomial (c) W consists of all polynomials with only even powers of t (d) None of the above
5. Which of the following pairs of vectors, u and v, are linearly dependent?
(a) $u = (1,2), v = (3,-5)$ (b) $u = (1,-3), v = (-2,6)$ (c) $u = (1,2,-3), v = (4,5,-6)$ (d) All of the above (e) None of the above
6. Which of the following pairs of polynomials (and matrices), u and v, are linearly dependent?
(i) $u = 2t^2 + 4t - 3, v = 4t^2 + 8t - 6$ (ii) $u = 2t^2 - 3t + 4, v = 4t^2 - 3t + 2$
(iii) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}, v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$ (iv) $u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, v = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$
(a) i & iii (b) ii & iv (c) i & iv (d) i, ii, iii & iv (e) None of the above
7. Suppose the vectors u, v, w are linearly independent. Are the vectors $u + v, u - v, u - 2u + w$ linearly dependent or independent? (a) Linearly dependent (b) Linearly independent (c) Dependent(d) All of the above (e) None of the above
8. Which of the following vectors forms a basis of R^3? (a) $(1,1,1), (1,0,1)$ (b) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$ (c) $(1,1,1), (1,2,3), (2,-1,1)$ (d) None of the above
9. Find a basis of the subspace W of the space $V = R^3$ where $W = \{(a,b,c): a + b + c = 0\}$
(a) $\dim W = 3$ (b) $\dim W = 4$ (c) $\dim W = 2$ (d) $\dim W = 5$ (e) None of the above
10. Given that $B = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ determine $B \cdot B^T$ (a) $\begin{pmatrix} -56 & 62 \\ 60 & 114 \end{pmatrix}$ (b) $\begin{pmatrix} 56 & 114 \\ 62 & 62 \end{pmatrix}$ (c) $\begin{pmatrix} 56 & 62 \\ 62 & 144 \end{pmatrix}$
(d) None of the above
11. Which of the following is wrong? (a) if A is a square matrix, then A is non-singular (b) if A^{-1} is the inverse of an $n \times n$ matrix A, then A^{-1} is $n \times m$ (c) if A is non-singular matrix, then A^{-1} is unique (d) A square matrix of order n has an inverse if and only if $\det A \neq 0$ (e) if A is an $n \times n$ matrix then $A^{-1} = \frac{1}{\det A} adjA$, provided $\det A \neq 0$
12. If A and B are symmetric matrices of the same order, then $AB - BA$ is a (a) Null matrix (b) symmetric matrix (c) skew-symmetric matrix (d) All of the above (e) None of the above
13. A square matrix $A = [a_{ij}]$ such that $a_{ij} = 0$ whenever $i > j$ and $a_{ij} = 0$ whenever $i < j$ is called (a) Upper triangular matrix (b) Lower triangular matric (c) Diagonal matrix (d) Canonical matrix (e) None of the above
14. The main diagonal of an $m \times n$ skew-symmetric matrix consists of only the element (a) 0 (b) 1 (c) $^1/_2$ (d) 2 (e) None of the above
15. Evaluate the determinants $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$ (a) 20 (b) 0 (c) 27 (d) 25 (e) None of the above

LINEAR TRANSFORMATION / MAP SUMMARY (IV)

Let U & V be vector spaces over a Field F. We Say that a Mapping $T: U \longrightarrow V$ (T maps from U to V) is a linear transformation if it satisfies;

① $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in V$

② $T(\alpha u) = \alpha T(u) \quad \forall\, u \in V,\ \alpha \in F$

Linear transformation from U to V is denoted by $\mathcal{L}(U,V)$.

THEOREM ①

(i) Let $T_1, T_2 \in \mathcal{L}(U,V)$ Then (i) $T_1 + T_2 \in \mathcal{L}(U,V)$ (ii) $\alpha T_1 \in \mathcal{L}(U,V)$

(ii) Let U, V, W be vector spaces over a field F. Suppose $T_1 \in \mathcal{L}(U,V)$ and $T_2 \in \mathcal{L}(V,W)$. So that $T_2 \circ T_1 : U \longrightarrow W$ (composition). then $T_2 \circ T_1 \in \mathcal{L}(U,W)$.

Definition;

suppose $T \in \mathcal{L}(U,V)$, we define the null space a Kernel of T as; $Ker(T) \equiv Null(T) = \{v \in U : T(v) = 0\}$

Also Range of T as: $range(T) = \{T(u) : u \in U\} \subset V$

THEOREM (II)

(i) Let $T \in \mathcal{L}(U,V)$, be invertible in the Inverses $T^{-1} : V \longrightarrow U$ Then $T^{-1} \in \mathcal{L}(V,U)$ → Note the order of U & V

That is $\begin{pmatrix} T : U \longrightarrow V \text{ and} \\ U \longleftarrow V : T^{-1} \end{pmatrix}$

(ii) Let $T \in \mathcal{L}(U,V)$, then

(i) $Ker(T)$ is a subspace of U

(ii) $Range(T)$ is a Subspace of V

Example ①

Show if the function $f(x,y) = [x-y, x]$ is a linear transformation or not on $\mathbb{R}^2$

Solution:

We need to show if f satisfies the two Conditions above: Note that is is 2 dimensional i.e $\dim(\mathbb{R}^2) = 2$.

select $u = (x_1, x_2)$ & $v = (y_1, y_2)$

So that $f(u) = f(\underset{(x}{x_1}, \underset{y)}{x_2}) = \underset{(x-y,\ x)}{(x_1 - x_2, x_1)}$ from the definition of the function

and $f(v) = f(y_1, y_2) = (y_1 - y_2, y_1)$

① Consider $u + v = (x_1, x_2) + (y_1, y_2)$

$= (x_1 + y_1, x_2 + y_2)$

$f(u+v) = f(x_1 + y_1, x_2 + y_2) =$

$= [(x_1 + y_1) - (x_2 + y_2),\ (x_1 + y_1)]$

$= [x_1 + y_1 - x_2 - y_2,\ x_1 + y_1]$

$= ((x_1 - x_2) + (y_1 - y_2),\ x_1 + y_1]$

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$\therefore = [(x_1-x_2)+(y_1-y_2),\ x_1+y_1]$

$= (x_1-x_2, x_1) + (y_1-y_2, y_1)$

$f(u+v) = f(u) + f(v)$ Condition (i) satisfied

(ii) let $\alpha \in F$ Consider $\alpha u = \alpha(x_1, x_2)$

$= (\alpha x_1, \alpha x_2)$

$\therefore f(\alpha u) = f(\alpha x_1, \alpha x_2) = (\alpha x_1 - \alpha x_2, \alpha x_1)$

$= (\alpha(x_1-x_2), \alpha x_1)$

$= \alpha(x_1-x_2, x_1)$

Hence $f=(x-y, x)$ $= \alpha f(u)$ — (ii) satisfied

Is a linear transformation.

Example (ii)

Say if $f(x,y,z) = [x, x+y, z-x, yz]$ is a linear transformation or not on $\mathbb{R}^3$ i.e 3-dim.

Select $u=(x_1, x_2, x_3)$ so that

$f(u) = f(x_1, x_2, x_3) = [x_1, x_1+x_2, x_3-x_1, x_2x_3]$

i.e replace x with x_1, y with x_2 & z with x_3

(ii) let $\alpha \in F$ Consider $\alpha u = \alpha(x_1, x_2, x_3)$

Consider $\alpha u = (\alpha x_1, \alpha x_2, \alpha x_3)$

$f(\alpha u) = f(\alpha x_1, \alpha x_2, \alpha x_3)$

$\Rightarrow [\alpha x_1, \alpha x_1+\alpha x_2, \alpha x_3-\alpha x_1, \alpha x_2 \cdot \alpha x_3]$

$= [\alpha(x_1), \alpha(x_1+x_2), \alpha(x_3-x_1), \alpha(\alpha x_2 x_3)]$

$= \alpha[x_1, x_1+x_2, x_3-x_1, \alpha x_2 x_3]$

$\neq \alpha f(u)$ because one of α is still inside the bracket of which it doesn't suppose to be. This was caused by the two variables (yz) multiplying each other in the original function

i.e $\alpha y \cdot \alpha z = \alpha(\alpha yz)$ (which doubled the α in the above)

Hence, functions that involves products or squares aren't a transformation e.g

$f(x,y) = [x^2, x+y]$, $f(x,y) = [xy, x+y, x-y, x]$

$f(x,y,z) = [x, x-z, xy]$ — are not a linear T.

So example (ii) the function is not a linear transformation.

SHORTCUT (But might cut you short, it sometimes looks like a hit & miss)

To test for linear transformation; Test condition (ii) first i.e $f(\alpha u) = \alpha f(u)$ owing to its simplicity, if the function fails to be a transformation then no need checking for (i) $f(u+v)$ but if the function satisfies (ii), then you are doomed for a long haul because solving for (i) is always tedious, cumbersome.

(1) $\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$ See page (8) det of a triangular matrix is the product of its diagonal elements

Ans = $a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44}$ (B)

(2) If $\det(A) = -35$ i.e $\begin{vmatrix} 1 & x & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{vmatrix} = -35$

So that;

$1(-20+6) - x(15+10) + 1(9+20) = -35$

$\therefore -14 - 25x + 29 = -35$

$-25x = -50 \quad \therefore x = 2$ = Ans (A)

(3) $A^T I = A^T$ See page (7) i.e for an identity matrix I, $A \cdot I = I \cdot A = A$ $\forall$ (for all matrix).

$A = \cancel{\begin{pmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 0 & 6 \end{pmatrix}} \sim \begin{pmatrix} 2 & 1 & 8 \\ 3 & 7 & 0 \\ 5 & 4 & 6 \end{pmatrix} = A^T$ and $A^T \cdot I = A^T$

Ans = (B)

(4) If $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ then $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & 0-\lambda \end{vmatrix}$

See eigenvalue page (17)

$\Rightarrow 2-\lambda[(4-\lambda)(-\lambda)+2] - 0 + 1[-2 + (4-\lambda)] = 0$

$(2-\lambda)[\lambda^2 - 4\lambda + 2] + [-2 + 4 - \lambda] = 0$

$(2-\lambda)[\lambda^2 - 4\lambda + 2 \quad +1] = 0$

$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \quad \therefore (2-\lambda)(\lambda-1)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3 \rightarrow$ eigenvalues. Ans (C)

(5) $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ $\therefore$ the cofactor of a_{22}

is $a_{22}^C = + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = + \begin{vmatrix} 2 & -2 \\ 2 & -5 \end{vmatrix}$

$= -10 - (-4) = -6$

Ans = (C)

(6) $\det(A) = 0$ then $\begin{vmatrix} 1 & 1 & -k \\ k & -3 & 11 \\ 2 & 4 & -8 \end{vmatrix} = 0$

$\Rightarrow 1(24-44) - 1(-8k-22) - k(4k+6) = 0$

$-20 + 8k + 22 - 4k^2 - 6k = 0$

$\therefore \cancel{4}k^2 - \cancel{2}k - \cancel{2} = 0$

$\therefore 2k^2 - k - 1 = 0$

Use Calculator, $k = 1$ and -0.5 or
$k = 1$ and $-1/2$ Ans = (B)

(7) $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 11 & 4 & 0 \end{bmatrix}$ a_{32} is entry 4 & 8 a_{33} is entry 0 see question (5) above

$a_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix}$ & $a_{33} = +\begin{vmatrix} 2 & 3 \\ \end{vmatrix}$ (Note: close the row and column)

(8) $A = \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{pmatrix}$ See page (10) Use the shortcut to get A^c (Cofactor of A).

Write Matrix A as

$$\begin{array}{ccccc} \cancel{2} & \cancel{7} & \cancel{4} & \cancel{2} & \cancel{7} \\ 3 & 1 & 6 & 3 & 1 \\ 5 & 0 & 8 & 5 & 0 \\ 2 & 7 & 4 & 2 & 7 \\ 3 & 1 & 6 & 3 & 1 \end{array}$$

$\therefore A^c = \begin{pmatrix} 8-0 & 30-24 & 0-5 \\ 0-56 & 16-20 & 35-0 \\ 42-4 & 12-12 & 2-21 \end{pmatrix} = \begin{pmatrix} 8 & 6 & -5 \\ -56 & -4 & 35 \\ 38 & 0 & -19 \end{pmatrix}$

$\therefore A^{cT} = \begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix} = A^{adj}$

$|A| = \begin{vmatrix} 2 & 7 & 4 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{vmatrix} = 2(8-0) - 7(24-30) + 4(-5)$
$= 38$

$\therefore A^{-1} = \frac{1}{|A|} A^{adj} = \frac{1}{38}\begin{pmatrix} 8 & -56 & 38 \\ 6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix}$ Ans = None

(9) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ — distributive ✓
$A - B = A \setminus B = A \cap B'$ — difference ✓
$(A \cap B)' = A' \cup B'$ — Demorgan's Law
But B' means $t \notin B$ and $t \in U$ (Universal set)
$\therefore$ option (D) is not correct

(10) $A = \{1, 2, 3, 5, 7\}$, $B = \{0, 3, 6, 7, 9\}$, $C = \{4, 5, 6, 8\}$
$A - B$ means removing elements of B from A.
$\therefore A - B = \{1, 2, \cancel{3}, 5, \cancel{7}\} = \{1, 2, 5\}$
$B - A = \{0, \cancel{3}, 6, \cancel{7}, 9\} = \{0, 6, 9\}$.
$\therefore (A-B) \cup (B-A) = \{0, 1, 2, 5, 6, 9\}$ Ans = (D).

(11) A relation from one set A to another set B is a subset of their cartesian product $A \times B$.
i.e $ArB \subseteq A \times B$ Ans = (C)

(12) $A = \{a, c\}$, $B = \{a, b, e, f\}$
$A \times B \rightarrow$ set of all ordered pairs
$A \times B = \{(a,a), (a,b), (a,e), (a,f), (c,a), (c,b), (c,e), (c,f)\}$
$\Rightarrow n(A \times B) = 8$

Shortcut
$n(A \times B) = n(A) \times n(B)$
$= 2 \times 4 = 8$

where $n(A)$ denotes the no of element in the set.

(13) See page (32)
its easy to test for Condition (ii) $f(\alpha u) = \alpha f(u)$.
Shortcut
(A) & (C) are linear transformation because of their linearity i.e
$\cdots) = (x, 2y, 3x - y)$ select $u = (x_1, x_2, x_3)$

and (C) $T(x,y,z) = (x, 2y, 5z)$ these two are obvious enough. see page (32)

but (D) doesn't look clear enough and (B)

(D) $T(x,y,z) = (1, x, z)$ select $u = (x_1, x_2, x_3)$

Consider $\alpha u = \alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3)$

$\therefore T(\alpha u) = T(\alpha x_1, \alpha x_2, \alpha x_3) = [1, \alpha x_1, \alpha x_3]$

$x \quad y \quad z \quad = (1, \alpha x_1, \alpha x_3)$

Note we can't factorize α out of the bracket to get $\alpha f(u)$, and $f(u) \neq (1, \alpha x_1, \alpha x_3)$

Hence (D) is not a linear Tr.

Ans = (D).

(14) Ans = (A) see page (31)

(15) Test for $T(0,0,0) = 0$

$T(x,y,z) = (-x, y-z, x-1)$

$\therefore T(0,0,0) = (0, 0, -1) \neq 0$ Ans = (D)

(16) $A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix}$; $T: \mathbb{R}^2 \to \mathbb{R}^3$ (see matrix multiplication)

$= \begin{pmatrix} 0 & 1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 0x + 1 \cdot y \\ -2x + 2y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} y \\ -2x + 2y \\ x \end{pmatrix}$

$= (y, -2x + 2y, x)$ Ans = (C)

(17) for linearity $f(u+v) = f(u) + f(v)$. Ans = (C)

(18) its clear that (A) (C) & (D) should be a Linear T.

Check (B) $f(x,y,z) = (x,y,z) + (0,-1,0)$

select $u = (x_1, x_2, x_3)$ & $v = (y_1, y_2, y_3)$

so that $f(u) = f(x_1, x_2, x_3) = (x_1, x_2, x_3) + (0, -1, 0)$

and $f(v) = f(y_1, y_2, y_3) = (y_1, y_2, y_3) + (0, -1, 0)$

Consider $u + v = ((x_1 + y_1), (x_2 + y_2), (x_3 + y_3))$

$f(u+v) = f((x_1+y_1), (x_2+y_2), (x_3+y_3))$

$= [(x_1+y_1), (x_2+y_2), (x_3+y_3)] + [0, -1, 0]$

$= [x_1, x_2, x_3] + [y_1, y_2, y_3] + [0, -1, 0]$

$= (x_1, x_2, x_3] + f(y_1, y_2, y_3)$

$= \underline{(x_1, x_2, x_3) + f(v)}$. but $f(u) \neq (x_1, x_2, x_3)$

Hence B is not a linear Transformation

(19) (E) see vector spaces page (21)

(20) (A) i.e $M_{m \times n}$ is a vector space

$\mathbb{R}^3$ & $\mathbb{R}^2$ are vector spaces.

All of the above.

(21) Only a zero vector has no basis Ans = (B)

because e.g $V_3 = \{(x, y, z) ; x, y, z \in \mathbb{R}$ and $x+2y+z\}$

Compare and $V_3 = \{(x, y, z) ; x, y, z, \in, \mathbb{R}$ and $x+2y+z\}$

those commas does not suppose to be there.

That is

If x is an element of $\mathbb{R}$ we write $x \in \mathbb{R}$ not ~~$, x, \in, R$~~.

(29) $V = ax_1 + bx_2 + cx_3$ [Note: typographical error $V = ax_1 + bx_2 + cx_2$ error]

$$\begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + c\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$$

$a + 2b + 3c = 5$
$2a + 3b + 5c = 2$
$3a + 7b + 6c = 4$

Using Calculator to punch we get

$a = \frac{-63}{4}, b = 13/4, c = \frac{19}{4}$ Ans = (E)

(30) Re-writing gives

$$x_1\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$ Using Calculator.

See linear combination page (25)

$x_1 = 9/2$, $x_2 = 2$, $x_3 = 1/2$ Ans = (B)

(31) Ans (B) See page (28). Remark (2)

(32) All are most correct Ans = (D)

(33) (c) Spans the vector space and (ii) linearly independent Ans = (C) (see basis).

(34) It suffice to find the vectors that are linearly independent. Hence use calculator to evaluate ;

0	1	0 = 0	1	2	3 = 0	1	0	1 = 0
0	2	1 = 0	2	3	4 = 0	0	1	1 = 0
1	5	3 = 0 ,	3	4	5 = 0	0	0	0 = 0

(A) $x_1 = 0, x_2 = 0, x_3 = 0$ No solution (Hence linear independency)

(B) Math-Error Infinite Soln (linear dependency)

(D) Math-Error Infinite Soln. (linear dependency)

Ans = (A) because a basis must be linearly Independent.

(35) $$\begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-\lambda & 0 & -\sqrt{3} \\ 0 & 2-\lambda & 0 \\ -\sqrt{3} & 0 & -1-\lambda \end{pmatrix}$$ characteristics matrix

$\Rightarrow |A - \lambda I| = 0$ + − +

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -\sqrt{3} \\ 0 & 2-\lambda & 0 \\ -\sqrt{3} & 0 & -1-\lambda \end{vmatrix} = 0$$

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$(1-\lambda)[(2-\lambda)(-1-\lambda)+0] - \cancel{0} + (-\sqrt{3})[0 - (+\sqrt{3})(2-\lambda)] = 0$

$\therefore (1-\lambda)(2-\lambda)(-1)(1+\lambda) - \sqrt{3}(\sqrt{3}(2-\lambda)) = 0$

$-(2-\lambda)(1-\lambda)(1+\lambda) - 3(2-\lambda) = 0$

Factorizing

$-(2-\lambda)[(1-\lambda^2)+3] = 0$

$\therefore -(2-\lambda)(4-\lambda^2) = 0 \Rightarrow -(2-\lambda)(2^2-\lambda^2) = 0$

$\Rightarrow -(2-\lambda)(2-\lambda)(2+\lambda) = 0 \quad \therefore \lambda = 2, 2, -2$

order of the answer does not matter

Ans = (A).

(36) $M = S + A$, $M^T = S^T + A^T$ for symmetric matrix $A = A^T$, $S = S^T$

But S is symmetric so that $S^T = S$ — For skew-symmetric M. $S^T = -S$, $A^T = -A$.

and A is antisymmetric so that $A^T = -A$.

$\therefore M = S + A$ and $M^T = S + (-A) \Rightarrow M^T = S - A$

$$\begin{array}{ll} & M = S + \cancel{A} \\ (+) & M^T = S - \cancel{A} \\ \hline \text{add} & M + M^T = 2S \end{array}$$

$S = \frac{1}{2}(M + M^T)$

$$\begin{array}{ll} M = \cancel{S} + A & \\ M^T = \cancel{S} - A & (-) \text{ subtract} \\ \hline M - M^T = A - (-A) & \end{array}$$

$\therefore 2A = M - M^T$

$\therefore A = \frac{1}{2}(M - M^T)$

(37) Cofactor of $\begin{pmatrix} 1 & 7 & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{pmatrix}$ shortcut.

$$\equiv \begin{pmatrix} \cancel{1} & \cancel{7} & \cancel{5} & \cancel{1} & \cancel{7} \\ -4 & 4 & 8 & -4 & 4 \\ 2 & 6 & 9 & 2 & 6 \\ 1 & 7 & 5 & 1 & 7 \\ -4 & 4 & 8 & -4 & 4 \end{pmatrix}$$

Cofactor $= \begin{pmatrix} -12 & 52 & -32 \\ -33 & -1 & 8 \\ 36 & -28 & 32 \end{pmatrix}$ Ans = (B).

(38) See Inverse Method page (11)

To solve $AX = B$, $X = A^{-1}B$ and $A^{-1} = \frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$

The inverse is already given

all we will do is to multiply $A^{-1}B$

$$\therefore X = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

$$= -\frac{1}{35}\begin{pmatrix} -56-14+0 \\ -100+0-5 \\ 116+14+10 \end{pmatrix} = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ +140 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-70}{-35} \\ \frac{-105}{-35} \\ \frac{+140}{-35} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$ Ans = (B)

pg (59)

(39) $\det(A) = 192$ i.e.
$$\begin{array}{ccc} + & - & + \\ \hline 1 & x & 5 \\ -4 & 4 & 8 \\ 2 & 6 & 9 \end{array} = 192$$

$\therefore 1(36-48) - x(-36-16) + 5(-24-8) = 192$

$-12 - x(-52) + 5(-32) = 192$

$\therefore 52x = 364$

$\therefore x = 7$ Ans = (C)

(40) $A = \begin{bmatrix} 9 & \sqrt{3} \\ \sqrt{3} & 11 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 9-\lambda & \sqrt{3} \\ \sqrt{3} & 11-\lambda \end{vmatrix} = 0 \Rightarrow (9-\lambda)(11-\lambda) - (\sqrt{3})(\sqrt{3}) = 0$

$\Rightarrow (9-\lambda)(11-\lambda) - 3 = 0$

$\therefore 99 - 9\lambda - 11\lambda + \lambda^2 - 3 = 0$

$\therefore \lambda^2 - 20\lambda + 96 = 0$

$\therefore (\lambda - 8)(\lambda - 12) = 0 \quad \therefore \lambda = 8, 12$ (or use Calculator)

No need solving further the eigenvalues do not tally with the options Ans = (E).

(41) $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 \\ -5 & 7 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4+5-2 & 2-7+6 \\ -4-15-1 & -2+21+3 \end{pmatrix}$

$= \begin{pmatrix} 7 & 1 \\ -20 & 22 \end{pmatrix}$

$\therefore \det(AB) = \begin{vmatrix} 7 & 1 \\ -20 & 22 \end{vmatrix} = 154 - (-20)$

$= 144$ Ans = (A)

(42) $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ ... adj

$\therefore A^c$ (cofactor) = Shortcut
$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 4 & 1 & 5 & 4 & 1 \\ 6 & 0 & 2 & 6 & 0 \\ 1 & 2 & 3 & 1 & 2 \\ 4 & 1 & 5 & 4 & 1 \end{array}$$

To give
$$A^c = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{pmatrix}$$

Note: that $\begin{matrix} 1 & 5 \\ 0 & 2 \end{matrix} \equiv \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix}$

$= (1)(2) - (0)(5)$

$= 2 - 0 = 2$

$\therefore A^{CT} = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} = A^{adj}$

And
$$|A| = \begin{array}{ccc} + & - & + \\ \hline 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{array} = 1(2+0) - 2(8-30) + 3(0-6)$$

$= 2 + 44 - 18 = 28$

$\therefore A^{-1} = \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$ Ans = (A)

(43) See Inverse of 2×2 Matrix i.e for a Matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$A = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$. So that $|A| = \begin{vmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{vmatrix}$

$= \cos^2\beta + \sin^2\beta = 1$

$\therefore A^{-1} = \frac{1}{1}\begin{pmatrix} \cos\beta & -\sin\beta \\ +\sin\beta & \cos\beta \end{pmatrix}$ Ans = (A)

(44) $A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 1 & -1 \\ 3 & 6 & 9 & 15 \\ 1 & 3 & 2 & 7 \end{pmatrix}$ Note that $R_3 = 3R_1$, i.e. Row 3 is a scalar multiple of Row 1.

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 1 & -1 \\ 3(1) & 3(2) & 3(3) & 3(5) \\ 1 & 3 & 2 & 7 \end{pmatrix} = 3\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 1 & -1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 2 & 7 \end{pmatrix}$$

Hence the det. = 0 see property (3, page (3))

(45) $A = \begin{pmatrix} 0 & 4 & 0 & 5 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ Row 3 contains more zeros than Row 1

$\therefore R_3 \leftrightarrow R_1$.

but the sign of the det. will change.

$$\approx -\begin{pmatrix} \overset{+}{0} & \overset{-}{0} & \overset{+}{1} & \overset{-}{0} \\ 1 & 1 & 2 & 1 \\ 0 & 4 & 0 & 5 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

So we expand entry a_{13}

$$(-)(+1)\begin{vmatrix} \overset{+}{1} & \overset{-}{1} & \overset{+}{1} \\ 0 & 4 & 5 \\ 2 & 3 & 1 \end{vmatrix}$$

$$\Rightarrow -\left(1(4-15) - 1(0-10) + 1(0-8)\right)$$

$$= -(-11+10-8) = \underline{+9} \quad \text{Ans} = (D)$$

(46) $B = \begin{pmatrix} 4 & 2 \\ -5 & 7 \\ -1 & 3 \end{pmatrix} \cdot A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 4-2 & -4+6 & 8+2 \\ -5-7 & 5+21 & -10+7 \\ -1-3 & 1+9 & -2+3 \end{pmatrix}$

$\therefore B\cdot A = \begin{pmatrix} 2 & 2 & 10 \\ -12 & 26 & -3 \\ -4 & 10 & 1 \end{pmatrix} \quad \therefore \det(B\cdot A) = \begin{vmatrix} 2 & +2 & 10 \\ -12 & 26 & -3 \\ -4 & 10 & 1 \end{vmatrix}$

$$\Rightarrow 2(26+30) - 2(-12-12) + 10(-120+104)$$

$$= 2(56) - 2(-24) + 10(-16)$$

$$= 112 + 48 - 160 = \underline{0} \quad \text{Ans} = (B)$$

(47) $\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{vmatrix} = 0 \quad |A - \lambda I|$

$\therefore \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + i^2 = 0 \quad \therefore \lambda^2 + (-1) = 0$

$\therefore \lambda^2 = 1 \therefore \lambda = \pm 1, \quad \underline{\lambda = 1, -1}$

(a) the eigenvalues are $\Longrightarrow$

(b) Eigenvectors of $\lambda = 1$ Substitute into

$(A - \lambda I)X = 0 \quad \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

For $\lambda = 1$.

$\Rightarrow \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ so that $\begin{array}{l} -x_1 - ix_2 = 0 \\ ix_1 - x_2 = 0 \end{array}$

Pick one equation $-x_1 - ix_2 = 0$

$\therefore x_1 = -ix_2$

$\therefore$ Solution $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ix_1 \\ x_2 \end{pmatrix} = \underline{K\begin{pmatrix} -i \\ 1 \end{pmatrix}}$

when $\lambda = -1$

$$\Rightarrow \begin{pmatrix} 1-\lambda & -i \\ i & 1-\lambda \end{pmatrix} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} x_1 - ix_2 = 0 \\ ix_1 + x_2 = 0 \end{matrix}$$

choose one equation $x_1 - ix_2 = 0 \Rightarrow x_1 = ix_2$

$\therefore$ Solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} i \\ 1 \end{pmatrix} = k\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\therefore$ the eigenvalues and their corresponding eigenvectors are $\lambda_1, k\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\lambda_2, \begin{pmatrix} i \\ 1 \end{pmatrix}$ i.e two eigenvalues gives two eigenvectors

for $\underline{\lambda_1 = 1}$. for $\underline{\lambda_2 = -1}$

Ans = (E)

(48) For a 2X2 Matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \quad \left(\text{i.e } |A-\lambda I| = 0\right)$$

$\therefore$ Expanding

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$= a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

$$= \underline{\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0}$$

Ans = (A)

(49) See page (14) Rank of a Matrix
(we reduced/decompose the Matrix to triangular or Row lechelon form.

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$$A = \begin{pmatrix} 2 & 4 & 1 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 2 & 5 \end{pmatrix}$$

We will want to change the pivot $a_{11} = 2$ to (1) by interchanging column (1) with column (3) i.e $C_1 \leftrightarrow C_3$

$$\simeq \begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & -2 & -1 & 0 \\ 2 & 0 & 0 & 2 \\ 2 & 6 & 3 & 5 \end{pmatrix} \begin{matrix} \\ R_2 \\ R_3 \\ R_4 \end{matrix}$$

So that $a_{11} = 1 =$ pivot .

So that ; $\Rightarrow$

Measure for $R_2 = \dfrac{a_{21}}{a_{11}} = \dfrac{1}{1}$ | $R_2 = R_2 - \dfrac{1}{1}R_1$

Measure for $R_3 = \dfrac{a_{31}}{a_{11}} = \dfrac{2}{1}$ | $R_3 = R_3 - \dfrac{2}{1}R_1$

Measure for $R_4 = \dfrac{a_{41}}{a_{11}} = \dfrac{2}{1}$ | $R_4 = R_4 - \dfrac{2}{1}R_1$

$$\sim \begin{pmatrix} 1 & 4 & 2 & 3 \\ 1-\frac{1}{1}(1) & -2-\frac{1}{1}(4) & -1-\frac{1}{1}(2) & 0-\frac{1}{1}(3) \\ 2-\frac{2}{1}(1) & 0-\frac{2}{1}(4) & 0-\frac{2}{1}(2) & 2-\frac{2}{1}(3) \\ 2-\frac{2}{1}(1) & 6-\frac{2}{1}(4) & 3-\frac{2}{1}(2) & 5-\frac{2}{1}(3) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & -6 & -3 & -3 \\ 0 & -8 & -4 & -4 \\ 0 & -2 & -1 & -1 \end{pmatrix}$$

Now because of the difficulty of -ve sign.
Multiply R_2, R_3 & R_4 by (-1) ..

$$\sim\begin{pmatrix}1&4&2&3\\0&6&3&3\\0&8&4&4\\0&2&1&1\end{pmatrix}$$ Now R_2 has a common factor (3), R_3 has a common factor (4)

$\therefore R_2 = \frac{1}{3}R_2$, $R_3 = \frac{1}{4}R_3$

$$\sim\begin{pmatrix}1&4&2&3\\0&2&1&1\\0&2&1&1\\0&2&1&1\end{pmatrix}$$ Since some rows are equal the just subtract some from another.

i.e R_3 becomes $R_3 = R_3 - R_2$,
R_4 becomes $R_4 = R_4 - R_2$

$$\sim\begin{pmatrix}1&4&2&3\\0&2&1&1\\0-0&2-2&1-1&1-1\\0-0&2-2&1-1&1-1\end{pmatrix}\sim\begin{pmatrix}1&4&2&3\\0&2&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

Note that the matrix is now in triangular form. Zero rows

$\therefore$ Rank = No of non-zero rows = R_1 & R_2

Hence Rank = $\underline{2}$. Ans = (C).

(5c)

$$A \cdot B = \begin{pmatrix}2&1&3\\0&4&5\end{pmatrix}\cdot\begin{pmatrix}1&1&3\\0&1&2\end{pmatrix}$$ See Matrix Multiplication

$$\begin{pmatrix}2+0+3 & 2+1+12 & 6+2+3y\\0+0+5 & 0+4+20 & 0+8+5y\\2+0+1 & 2+1+4 & 6+2+y\end{pmatrix} = \begin{pmatrix}5&15&8+3y\\5&24&8+5y\\3&7&8+y\end{pmatrix}$$

$$\therefore \det(AB) = |A\cdot B| = \begin{vmatrix}\overset{+}{5} & \overset{-}{15} & \overset{+}{8+3y}\\5&24&8+5y\\3&7&8+y\end{vmatrix} = 48 \text{ (given)}$$

Expanding

$\Rightarrow$

$$5\left[24(8+y)-7(8+5y)\right]-15\left(5(8+y)-3(8+5y)\right)+(8+3y)(35-72)$$

$$5(192+24y-56-35y)-15(40+5y-24-15y)+(8+3y)(-37)$$

$$5(136-11y)-15(16-10y)+(8+3y)(-37)=48$$

$$680-55y-240+150y-296-111y=48$$

$$440+95y-296-111y=48$$

$$\therefore 144-16y=48$$

$$\therefore 16y=144-48$$

$$\therefore 16y=96$$

$$\therefore y=\underline{\underline{6}} \quad \text{Ans} - (D)$$

(1) $\det(BA) = 0$ See 2015 Question (46) Ans = (B)

(2) $\det(AB) = 174$ See 2015, Question (41) Ans = (A)

(3) $n+1$ or more are linearly Independent
See page (30) Theorem (2)(i) Ans = (B)

(4) $X_1^2 + 6X_1X_2 + X_2^2$. This is equivalent to;
$a_{11}X_1^2 + (a_{11} + a_{21})X_1X_2 + a_{22}X_2^2$ (see quadratic forms page (19)/(20))

$\therefore a_{11} = 1, a_{22} = 1, a_{12} = a_{21} = \frac{6}{2} = 3$

$\therefore$ The Matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ Ans = (A)

(5) Ans = (E) See 2015, Question (40)

(6) $m = n$, See question (24); 2015

(7) $\det(A) = -35$, Ans = (A) See 2015, Question (2)

(8) If $\dim(W) = \dim(V)$; where $W \subseteq V$
So that $\dim(W) = n$ if $\dim(V) = n$ then $W = V$.
See page (30) Theorem (4) dimension Ans = (E)
It should be; Let W be a subspace of an n-dimensional vector space V,
Then $\dim(W) \leq n$. In particular if $\dim(W) = n$, then $W = V$.

(9) Ans = (A), $\lambda_1, \lambda_2 = 2, \lambda_3 = -2$ See 2015 Question (35)

(10) Incorrect statement, Ans = (E) None
Because vectors $u_1, u_2 \ldots u_m \in V$ not $\notin F$ and the Scalars $a_1 a_2 \ldots a_m \in F$ not $\notin V$
See page (24) linear Combination

(11) See 2015 Question (3) Ans = (B)

(12) See 2015 Question (36) Ans = (C)

(13) $(B \cap C) \cup (B \cap C')$ let $Z = (B \cap C)$
$\therefore Z \cup (B \cap C')$ Using distributive law
i.e $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\Rightarrow (Z \cup B) \cap (Z \cup C')$

$\Rightarrow = (B \cap C) \cup B \cap (B \cap C) \cup C'$ (Distributing both)

$= [(B \cup B) \cap (C \cup B)] \cap [(B \cup C') \cap (C \cup C')]$

But $B \cup B = B$ and $C \cup C' = \mathcal{U}$ (universal set)

$\Rightarrow [B \cap (C \cup B)] \cap [(B \cup C') \cap \mathcal{U}]$.

Note: For any given set Y, $Y \cap \mathcal{U} = Y$.
and since $B \subseteq (B \cup C)$ then $B \cap (B \cup C) = B$
and $B \subseteq (B \cup C')$ then $B \cap (B \cup C') = B$.
So that $[B \cap (B \cup C)] \cap [(B \cup C') \cap \mathcal{U}]$

$\Rightarrow (B) \cap [(B \cup C')]$

$\Rightarrow B \cap (B \cup C')$

$= B$ Ans = (C)

(14) See 2015 Question (44) Ans = (C)

(15) See 2015 Question (42) Ans = (A)

(16) W is a subspace of V if W itself is a vector space over a field F. w.r.t;
(i) Scalar Multiplication $(\cdot)$ and
(ii) Vector addition $(+)$.

Scalar $(\cdot)$ and vector $(+)$ but in the option they used (or). Ans = (E)

(17) See 2015 Question (45) Ans = (D)

(18) For linear dependency $mu + nv + pw = 0$
i.e $m(1 \ldots) \ldots (2 \ldots) + n(\ldots$

$\Rightarrow$ $m + 2n + p = 0$
$2m + 5n + 3p = 0$
$3n + 7n + 5p = 0$

$\therefore$ The vectors will be linearly Independent if the above equation has no solution.

i.e $m = n = p = 0$ Ans = (E).

(19) See 2015 Question (37), Ans = (B)

(20) Ans = (C) See page (30) Theorem (3)(1)

(21) $A: \mathbb{R}^4 \to \mathbb{R}^3$

$$\therefore A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{pmatrix} = \text{Image of } A.$$

$\therefore$ The basis of ~~Image~~ (A) & dimension can be gotten, by decomposing Image (A) into Echelon form.

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \quad \begin{array}{l} R_2 = R_2 - \frac{2}{1}R_1 \\ R_3 = R_3 - \frac{3}{1}R_1 \\ R_4 = R_4 - \frac{1}{1}R_1 \end{array} \sim \begin{pmatrix} 1 & 1 & 3 \\ 2 - \frac{2}{1}(1) & 3 - \frac{2}{1}(1) & 8 - \frac{2}{1}(3) \\ 3 - \frac{3}{1}(1) & 5 - \frac{3}{1}(1) & 13 - \frac{3}{1}(3) \\ 1 - \frac{1}{1}(1) & -2 - \frac{1}{1}(1) & -3 - \frac{1}{1}(3) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & \ldots \\ \ldots \end{pmatrix} \begin{array}{l} \text{pivot} \\ R_2 \\ R_3 \end{array} \quad \begin{array}{l} R_3 = R_3 - \frac{2}{1}(R_2) \\ R_4 = R_4 - (-3)R_2 = R_4 + 3R_2 \end{array}$$

$$\therefore \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0-\frac{2}{1}(0) & 2-\frac{2}{1}(1) & 4-\frac{2}{1}(2) \\ 0+\frac{3}{1}(0) & -3+\frac{3}{1}(1) & -6+\frac{3}{1}(2) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \rightarrow \text{Non zero Rows} \\ \\ R_3 \\ R_4 \end{matrix} \text{ Zero rows}$$

- Dimension = Rank = 2 (No of non-zero Rows)
Basis = R_1 & R_2 = (1, 1, 3) & (0, 1, 2)

Ans = (D)

(22) If zero be one of the vectors $v_1, v_2 \ldots v_m \in V$
∴ Then they must be linearly dependent because of the following;

$$1 \cdot v_1 + 0v_2 + 0v_3 + \ldots + 0v_m = 0 \quad (v_1 \neq 0)$$

i.e at least there should be a scalar $a \neq 0$ Ans = (E).

Note: that in the option they combined Upper-Case letters $1V_1 + 0V_2 + \ldots + 0V_m$
Note that V — space v — vector so that $v \in V$. (lower case v to rep. vectors)

(23) Option (A) ∵ $A = \{x \mid \cos x = \sqrt{3}/2, 0^\circ < x < 360^\circ\}$
read as → The set A having the element x such that cos x = $\sqrt{3}/2$ between 0° and 360°
So A is finite because only two values between 0 & 360° can give $\sqrt{3}/2$ which are, $\cos 30^\circ = \sqrt{3}/2$ and $\cos(360-30) = \cos 330 = \sqrt{3}/2$

option (B) is impossible
because $\cos x = 2$ satisfies no $\mathbb{R}$ (real line)
option (C) Integer multiple of 10 are Infinite.

(24) See 2015, Question (33)
If S is linearly Independent and S spans V Ans = None.

(25) See 2015 Question (36). (S – Symmetric, A – Anti symmetric)
If $Z = S + A$
∴ $S = \frac{1}{2}(Z + Z^T)$ and $A = \frac{1}{2}(Z - Z^T)$

$$S = \frac{1}{2}\left(\begin{bmatrix} 2 & 8 & -2 \\ 0 & 3 & 14 \\ 0 & 0 & 9 \end{bmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 8 & 3 & 0 \\ -2 & 14 & 9 \end{pmatrix}\right)$$

$\frac{1}{2}(Z \quad + \quad Z^T)$

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$$= \frac{1}{2}\begin{pmatrix} 4 & 8 & -2 \\ 8 & 6 & 14 \\ -2 & 14 & 18 \end{pmatrix} \quad \text{and } A = \frac{1}{2}(Z - Z^T)$$

$$\therefore A = \frac{1}{2}\left(\begin{pmatrix} 2 & 8 & -2 \\ 0 & 3 & 14 \\ 0 & 0 & 9 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 8 & 3 & 0 \\ -2 & 14 & 9 \end{pmatrix}\right)$$

$$\quad\quad Z \quad\quad - \quad\quad Z^T$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 8 & -2 \\ -8 & 0 & 14 \\ 2 & -14 & 0 \end{pmatrix}$$

$$\therefore S = \frac{1}{2}\begin{pmatrix} 4 & 8 & -2 \\ 8 & 6 & 14 \\ -2 & 14 & 18 \end{pmatrix} \ \& \ A = \frac{1}{2}\begin{pmatrix} 0 & 8 & -2 \\ -8 & 0 & 14 \\ 2 & -14 & 0 \end{pmatrix}$$

Ans = E

(26) $xu_1 + yu_2 + zu_3 = V$

$$x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} + z\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \Rightarrow \begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 5z &= 7 \\ 3x + 7y + 6z &= -4 \end{aligned}$$

Using Calculator gives

$x = 2, y = -4, z = 3$ Ans = B

(27) All are not correct Ans = A

$\because$ scalar $\alpha \notin V$ Instead $\alpha \in F$ and $u, v \in V$.

(28) See 2015 Question (49) Ans = C

(29) $M = \{x / x \in N : x \text{ is even}\}$, $\alpha(n) = 2n$, $n \in N$

$f : \alpha \to M$, observe that the set M ~~and~~ ~~...~~ contains only even positive Integers Hence f is both one-to-one and onto.

i.e $2(1) = 2 \in M$ for $n = 1, 2, 3 \ldots\ldots$

$2(2) = 4 \in M$

$2(3) = 6 \in M$ and for no $m, n \in \alpha(n)$

$\vdots$ is $f(m) = f(n)$

$2(n) = 2n \in M$ i.e $2(2) \neq 2(3)$.

$\therefore$ Ans = B

(30) $K = \begin{pmatrix} 2 & 3 & 5 & 2 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{pmatrix}$. $Q \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$

$$KQ = \begin{pmatrix} 2+3+0+2 & -2+0+0+2 \\ 1-2+0+2 & -1+0+0+2 \\ 6+5+0+1 & -6+0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 1 & 1 \\ 12 & -5 \end{pmatrix} \quad \text{Ans} = E$$

(31) See 2016 Question (39), Ans = C

(32) Ans = B See Page 27 Theorem Li Spans

(33) See 2015, Question (1) Ans = (B)

(34) $\alpha P_1 + \beta P_2 + \delta P_3 = V \Rightarrow$

$$\alpha\begin{pmatrix}\cancel{t^2}\\ 2\cancel{t}\\ 1\end{pmatrix} + \beta\begin{pmatrix}2\cancel{t^2}\\ 5\cancel{t}\\ 4\end{pmatrix} + \delta\begin{pmatrix}\cancel{t^2}\\ 2\cancel{t}\\ 6\end{pmatrix} = \begin{pmatrix}3\cancel{t^2}\\ 5\cancel{t}\\ -5\end{pmatrix}$$

$(P_1) \quad (P_2) \quad (P_3) \quad (V)$

$$\equiv \quad \alpha + 2\beta + \delta = 3$$
$$2\alpha + 5\beta + 2\delta = 5$$
$$\alpha + 4\beta + 6\delta = -5$$

Using Calculator gives $\alpha = \frac{31}{5}$; $\beta = -1$, $\delta = \frac{-6}{5}$

Ans = (E)

(35) See 2015, Question (50) Ans = (D)

(36) See 2015, Question (43) Ans = (A)

(37) Ans = (C) See page (28) Remark (3)

(38) See 2016 Question (47) Ans = (E)

(39) $M = \begin{pmatrix}4 & 7\\ 7 & 9\end{pmatrix}$, $A = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $B = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $C = \begin{pmatrix}1 & 1\\ 4 & 5\end{pmatrix}$

By Inspection $\alpha\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \gamma\begin{pmatrix}1\\1\\4\\5\end{pmatrix} = \begin{pmatrix}4\\7\\7\\9\end{pmatrix}$

Which gives

(i) $\alpha + \beta + \gamma = 4$
(ii) $\alpha + 2\beta + \gamma = 7$
(iii) $\alpha + 3\beta + 4\gamma = 7$
(iv) $\alpha + 4\beta + 5\gamma = 9$

Picking any three i.e (i) $\alpha + \beta + \gamma = 4$
(ii) $\alpha + 2\beta + \gamma = 7$
(iii) $\alpha + 3\beta + 4\gamma = 7$

Using Calculator gives $\alpha = 2$, $\beta = 3$ & $\gamma = -1$

$\therefore$ $M = 2A + 3B - C$ Ans = (A)

(40) $A = [1, 2, 3, 6, 8]$
$B = [2, 5, 6, 7, 9]$
$C = [4, 5, 6, 8]$.

where $A - B = A \cap B^c$ means difference of the set A w.r.t B.

$A - B = [A - (A \cap B)] = [1, \cancel{2}, 3, \cancel{6}, 8] - [\cancel{2}, \cancel{6}]$
$= [1, 3, 8]$

and $B - A = [B - (B \cap A)] = [\cancel{2}, 5, \cancel{6}, 7, 9] - [\cancel{2}, \cancel{6}]$
$= [5, 7, 9]$

$\therefore (A - B) \cup (B - A)$
$= [1, 3, 8] \cup [5, 7, 9]$
$= [1, 3, 5, 7, 8, 9]$ Ans = (C)

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20/2/2013

(1) Ans = (A), i.e Range(f) = Y (Co-domain)

(2) Onto (surjective) ans = (B) i.e all elements of the codomain is an Image of an element the domain.

(3) Ans = (A) i.e $W \subseteq V$, where V is a Vector Space

(4) $A = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 2 & 8 & 6 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 2 & 8 & 6 \end{vmatrix}$

$= 3(42-72) - 2(24-18) + 5(32-14)$

$= -\cancel{90} - 12 + \cancel{90}$

$= \underline{\underline{-12}}$ Ans = (B)

(5) The dimension of V is finite and dim (v) = finite number Ans = (B)

i.e $\mathbb{R}^n$ or $\mathbb{K}^n$.

(6) (E) None see 2015 Question (14)

(7) $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$ i.e $|A - \lambda I| = 0$

$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0 \therefore 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\therefore \lambda^2 - 5\lambda + 6 = 0 \therefore \underline{\underline{\lambda = 2, 3}}$ Ans = (B)

(8) Ans = (C) both are Correct i.e (A) & (B)

see page (31) & (32) Linear Transformation

See page (31) & (32) $u + v = ((x_1+y_1), (x_2+y_2))$

So that $f(u+v) = f((x_1+y_1), (x_2+y_2))$

recall that $f(x, y) = (x, x+y, x-y)$

$\mathbb{R}^2 \quad = \quad \mathbb{R}^3$

$\therefore f((x_1+y_1), (x_2+y_2)) =$

$[(x_1+y_1), (x_1+y_1)+(x_2+y_2), (x_1+\cancel{y_1}) - (\cancel{x_2}+y_2)]$

$= [x_1+y_1, (x_1+x_2)+(y_1+y_2), (x_1-x_2)+(y_1-y_2)]$.

$= [x_1, x_1+x_2, x_1-x_2] + [y_1, y_1+y_2, y_1-y_2]$

$= \underline{f(u) + f(v).}$ Ans = (C)

(10) $x_1^2 + 4x_1x_2 - 3x_1x_3 + x_2^2 + 4x_2x_3 - x_3^2$ is equivalent to

$a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + (a_{13}+a_{31})x_1x_3$

$+ a_{22}x_2^2 + (a_{23}+a_{32})x_2x_3 + a_{33}x_3^2$

$\therefore a_{11} = 1, a_{22} = 1, a_{33} = -1$

$a_{12} = a_{21} = 4/2 = 2, \quad a_{13} = a_{31} = \frac{-3}{2} = -1.5$

and $a_{23} = a_{32} = 4/2 = 2$ Ans = (B)

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & -1.5 \\ 2 & 1 & 2 \\ -1.5 & 2 & -1 \end{pmatrix}$ See Quadratic Forms Page (i) page 49

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(11) (B) = Ans see page (27) Span (Definition)

(12) $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{pmatrix}$, $|A| = \begin{vmatrix} \overset{+}{2} & \overset{-}{-1} & \overset{+}{3} \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{vmatrix}$

$|A| = 2(15-2) + 1(5+2) + 3(-2-6)$

$= 26 + 7 - 24 = 9$ $\therefore |A| = 9$

A^{c} = Cofactor $\begin{pmatrix} 2 & -1 & 3 & 2 & -1 \\ 1 & 3 & -1 & 1 & 3 \\ 2 & -2 & 5 & 2 & -2 \\ 2 & -1 & 3 & 2 & -1 \\ 1 & 3 & -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 13 & -7 & -8 \\ -1 & 4 & 2 \\ -8 & 5 & 7 \end{pmatrix}$ → Note the pattern see page (10)

$A^{cT} = \begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix} = A^{adj}$

$\therefore A^{-1}$ (Inverse) $= \frac{1}{|A|} \cdot A^{adj} = \frac{1}{9} \begin{pmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{pmatrix}$

Ans = (E)

(12) Ans (D) A bijective function is a function which is both (injective) one to one and (surjective) onto.

(14) (B) = Ans one to one = Injective page (49)

(15) (D) All.

(16) See 2015 Question (8) Ans = (B)

i.e $A^{-1} = \begin{pmatrix} 8 & -36 & 38 \\ -6 & -4 & 0 \\ -5 & 35 & -19 \end{pmatrix} \cdot \frac{1}{38}$ But in 2015 it was option (E)

(17) Ans (C) see spans page (27) & Bases page (28).

(18) See 2015 Question (50) Ans = (D).

(19) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix} \Rightarrow \begin{vmatrix} \overset{+}{1-\lambda} & \overset{-}{-1} & \overset{+}{0} \\ 1 & 2-\lambda & 1 \\ -2 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow$

$(1-\lambda)[(2-\lambda)(-1-\lambda)-1] + 1[(-1-\lambda)+2] + 0 = 0$

$(1-\lambda)[(2-\lambda)(-1)(1+\lambda)-1] + (1-\lambda) = 0$

$(1-\lambda)[(-1)(2-\lambda)(1+\lambda)-1] + (1-\lambda) = 0$

$(1-\lambda)((-1)[2+2\lambda-\lambda-\lambda^2]-1) + (1-\lambda) = 0$

$(1-\lambda)(-2-2\lambda+\lambda+\lambda^2-1) + (1-\lambda) = 0$

$(1-\lambda)(\lambda^2-\lambda-3) + (1-\lambda) = 0$

factorising $(1-\lambda)$. $\therefore (1-\lambda)[\lambda^2-\lambda-3+1] = 0$

$\therefore (1-\lambda)[\lambda^2-\lambda-2] = 0$

$(1-\lambda)(\lambda-2)(\lambda+1) = 0$

$\therefore \lambda = 1, 2, -1$ Ans = (A) (order does not matter)

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(20) All are correct Ans = (E)

See 2015 Question (9)

(21) Ans = (B) i.e $A^T = -A$ or $A = -A^T$

See types of matrix page (8)

(22) $|A| = \begin{vmatrix} \overset{+}{5} & \overset{-}{2} & \overset{+}{1} \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{vmatrix} = 5(42-12) - 2(0-24) + 1(0-48)$

$= 150 + 48 - 48$

$= 150$ Ans = (A)

(23) (A) — distributive

(B) = Dé Morgan's law

(C) = Dé Morgan's law

(D) if U = Universal set $A' \Rightarrow x \in U, x \notin A$. (implies that)

option/Question not properly stated for (D).

Ans. = (D).

(24) $ArB \subseteq A \times B$ Ans = (B) A relation is a subset of cartesian product.

(25) $A = \begin{pmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2+3+0+3 & -2+0+0+3 \\ 1-2+0+2 & -1+0+0+2 \\ 6+5+0+1 & -6+0+0+1 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 1 & 1 \\ 12 & -5 \end{pmatrix}$

(26) i.e $W = k_1U + k_2V$ or $k_1U + k_2V = W$

$\therefore k_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix} \equiv \begin{matrix} k_1 + 6k_2 = 9 \\ 2k_1 + 4k_2 = 2 \\ -k_1 + 2k_2 = 7 \end{matrix}$

(This is two variable 3 equ.)

Choose only two equations i.e $k_1 + 6k_2 = 9$, $2k_1 + 4k_2 = 2$

Use calculator and punch

1 6 = 9

2 4 = 2

Ans $k_1 = -3$ & $k_2 = 2$

Ans = (C)

(27) if $\theta: X \to Y$ if θ is a function from X to Y then Y is called the co-domain so that if the function is onto (surjective) then Y (co-domain) = Range of θ.

Ans = (A).

(28) if $|A| = 0$ i.e $\det(A) = 0$. Ans = (A)

See page (8) types of Matrix

(29) See 2015 Question (2) Ans = (B)

(30) Ans = (D) See page (26)

(31) $A = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 0 & 3 \end{pmatrix}$. Adjoint of $A = A^{adj}$

$= A^{CT}$

$\therefore A^c = \begin{pmatrix} 5 & 2 & 1 & 5 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 4 & 6 & 3 & 4 & 6 \\ 5 & 2 & 1 & 5 & 2 \\ 3 & 1 & 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 3-24 & 16-9 & 18-4 \\ 6-6 & 15-4 & 8-30 \\ 8-1 & 3-20 & 5-6 \end{pmatrix}$

$= \begin{pmatrix} -21 & 7 & 14 \\ 0 & 11 & -22 \\ 7 & -17 & -1 \end{pmatrix} \therefore A^{cT} = \begin{pmatrix} -21 & 0 & 7 \\ 7 & 11 & -17 \\ 14 & -22 & -1 \end{pmatrix}$

$= \underline{A^{adj}}$ Ans = (E)

(32) See page (13) on application of Gaussian Elimination Method:

$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} \equiv \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2 & -1 & -1 & 11 \\ 3 & 2 & 1 & -5 \end{array}\right)$

$R_2 = R_2 - \frac{2}{1}R_1$, $R_3 = R_3 - \frac{3}{1}R_1$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 2-\frac{2}{1}(1) & -1-\frac{2}{1}(2) & -1-\frac{2}{1}(-3) & 11-\frac{2}{1}(3) \\ 3-\frac{3}{1}(1) & 2-\frac{3}{1}(2) & 1-\frac{3}{1}(-3) & -5-\frac{3}{1}(3) \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 5 & 5 \\ & & & -14 \end{array}\right)$

$\therefore R_2 = -\frac{1}{5}R_2$ and $R_3 = -\frac{1}{2}R_3 \sim$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -5 & 7 \end{array}\right) \therefore R_3 = R_3 - \frac{2}{1}R_2$

next pivot

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0-\frac{2}{1}(0) & 2-\frac{2}{1}(1) & -5-\frac{2}{1}(-1) & 7-\frac{2}{1}(-1) \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & 9 \end{array}\right)$ then

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 9 \end{pmatrix} \Rightarrow \begin{aligned} X_1+2X_2-3X_3 &= 3 \\ X_2 - X_3 &= -1 \\ -3X_3 &= 9 \end{aligned}$

from equ (3) $X_3 = \underline{-3}$ $\therefore$

from equ (2) $X_2 - X_3 = -1 \therefore X_2 - (-3) = -1$
$X_2 = \underline{-4}$

from equ (1) $X_1 + 2X_2 - 3X_3 = 3$

$\therefore X_1 + 2(-4) - 3(-3) = 3 \therefore X_1 = 2$

$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$ Ans = (A)

Since the exam is obj. Use Calculator to solve the system

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Shortcut Don't Use [illegible]

Use Calculator to punch

(Mode → 5 − EQU → 2.)

1	2	−3	= 3
2	−1	−1	= 11
3	2	1	= −5
a	b	c	d

Ans = (A)

(33) Image are the Mapped elements in the Codomain Ans = $\{a, b, c, e\}$ = (C).
Note (d) is not mapped.

(34) $x_1^2 + 2x_1x_2 - 3x_2^2$ $\quad a_{11} = 1, a_{22} = -3$

$a_{21} = a_{12} = 2/2 = 1$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

Ans = (A)

page (19) See quadratic forms

(35) Zero vector has no basis Ans = (D)
See 2015 Question (21)

(36) $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$. $A^c =$

$$\begin{pmatrix} 2 & 3 & 5 & 2 & 3 \\ 4 & 1 & 6 & 4 & 1 \\ 1 & 4 & 0 & 1 & 4 \\ 2 & 3 & 5 & 2 & 3 \\ 4 & 1 & 6 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0-24 & 6-0 & 16-1 \\ 20-0 & 0-5 & 3-8 \\ \end{pmatrix}$$

page (52)

$$= \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix} = A^c \quad \therefore A^{CT} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

and $A^{CT} = A^{adj}$ = Adjoint $= A^{adj}$

Ans = (B)

(37) Ans (D) See examples of Vector Space page (22)

(38) See 2015, Question (49) Ans = (D)

(39) $\lambda_1 V_1 + \lambda_2 V_2 + \dots + \lambda_n V_n = 0 \Rightarrow$ Implies $\lambda_1, \lambda_2 \dots \lambda_n = 0$

Note: this mean's the system of equation formed by this vectors has no solution
i.e $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \dots = \lambda_n = 0$
Hence the vectors Must be linearly Independent

Ans = (C)

See page (26)

(40) Symmetric Matrix is a matrix ∋
$A = A^T$ (See page (5))

Ans = (C) (Types of Matrix)

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2016/2017 option (A) where $I_3 = 3\times3$ Identity Matrix

(1) $A+I_3 = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 0 & 3 \\ 4 & 2 & 1 \end{pmatrix}$ i.e $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 0 & 3 \\ 4 & 2 & 1 \end{pmatrix} - I_3 = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 0 & 3 \\ 4 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & 3 & 6 \\ 4 & -1 & 3 \\ 4 & 2 & 0 \end{pmatrix}$ So that $(A-I_3) = \begin{pmatrix} 1 & 3 & 6 \\ 4 & -1 & 3 \\ 4 & 2 & 0 \end{pmatrix} - I_3$

$\therefore (A-I_3) = \begin{pmatrix} 1 & 3 & 6 \\ 4 & -1 & 3 \\ 4 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 3 & 6 \\ 4 & -2 & 3 \\ 4 & 2 & -1 \end{pmatrix} = (A-I_3)$.

$\therefore (A-I_3)(A-I_3) = \begin{pmatrix} 0 & 3 & 6 \\ 4 & -2 & 3 \\ 4 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 & 6 \\ 4 & -2 & 3 \\ 4 & 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0+12+24 & 0-6+12 & 0+9-6 \\ 0-8+12 & 12+4+6 & 24-6-3 \\ 0+8-4 & 12-4-2 & 24+6+1 \end{pmatrix} = \begin{pmatrix} 36 & 6 & 3 \\ 4 & 22 & 15 \\ 4 & 6 & 31 \end{pmatrix}$

Ans = (E)

(2) $\begin{pmatrix} p & 6 \\ 5 & q \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\therefore \begin{pmatrix} 2p+6 \\ 10+q \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ $\therefore 2p+6=7$ | $10+q=3$ $\therefore q=-7$ $(p=\frac{1}{2}, q=-7)$ Ans = (C)

(3) $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ Hint: Note that An Invertible Matrix is a non-singular matrix i.e a matrix that its inverse exists. i.e $|A| \neq 0$

So that option (A) means option (B)

Hence we test for Singularity option (D) & Symmetry option (C)

Recall that if A is Singular then $\det(A) = |A| = 0$ see page (8).

$\begin{vmatrix} \overset{+}{3} & \overset{-}{2} & \overset{+}{4} \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{vmatrix} = 3(10-24)-2(2+3)+4(8+5)$

$= 3(-14)-2(5)+4(13)$

$= -42-10+52 = -52+52$

$= \underline{0}$

Hence A is Singular Ans = (D)

Also

Note that for a matrix to be symmetric i.e $A = A^T$ (say a 3×3 matrix) entry $a_{21} = a_{12}$ and $a_{31} = a_{13}$.

i.e let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & ? \\ a_{31} & ? & a_{33} \end{pmatrix}$ $\therefore A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & ? \\ a_{13} & ? & a_{33} \end{pmatrix}$

So that comparing $a_{21} = a_{12}$ and $a_{31} = a_{13}$

So that

$A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ and $A^T = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 5 & 8 \\ 4 & 3 & 2 \end{pmatrix}$ See page (8)

See that $A \neq A^T$ Hence A is not Symmetric (D)

(4) Since the exam is obj don't use any rule for system's of equation its a waste of time.

i.e Use calculator and punch Mode $\rightarrow$ 5(EQN) $\rightarrow$ 2(3x3) $\Rightarrow$

$$\begin{array}{cccc} a & b & c & d \\ 1 & 1 & 1 & =6 \\ 1 & -1 & 1 & =2 \\ 2 & 1 & 1 & =7 \end{array}$$

Ans = 1, 2, 3 = (C) for Crammer's rule see page (4)

(5) punch

$$\begin{array}{cccc} -1 & 3 & -2 & =5 \\ 4 & -1 & -3 & =-8 \\ 2 & 2 & -5 & =7 \end{array}$$

Ans $= -10, \frac{-79}{11}, \frac{-91}{11}$ = (B)

(6) Ans = (D)

linearly Independent.

(7) $A=[1,5,9]$, $B=[1,2,3,4,5]$ Here, observe that $9 \in A$ but $9 \notin B$ Hence $A \not\subseteq B$ (A is not a sub-set of B)

Also: $2,3,4 \in B$ but $2,3,4 \notin A$

So that $B \not\subseteq A$ (B is not a subset of A)

Ans = (A)

(8) Null set means ~~a~~ set that is empty. "$\emptyset$"

Consider $X = \{x : x^2=9, 2x=4\}$ i.e X has the element x such that $x^2=9$ and $2x=4$

If $2x=4$ then $x=2$, so that $x^2=9 \Rightarrow 2^2=9$ a contradiction since $2^2 \neq 9$. Hence $X=\emptyset$ (empty set)

Consider $Y=\{x : x \neq x\}$. Y has the element x such that $x \neq x \Rightarrow$ Y has no element hence $Y = \emptyset$ (empty set)

Consider $Z = \{x : x+5=8\}$ Z has the element x such that $x+5=8$ (only one element (number) satisfies this)

i.e $\exists$ (there exist) c such that $c+5=8$.

Hence Z has not empty i.e $Z=\{c\}$ has one element $\{c\}$ $\therefore$

Ans = (i) & (ii) Ans = (B)

(9) Analysis:

(A) $1 \notin A$. the element 1 is not an element of A because A contains elements that are in groups.

(B) If $\{1,2,3\} \subseteq A$ [i.e if $\{1,2,3\}$ is a subset of A] then 1, 2, and $3 \in A$, a contradiction since the elements of A are in groups

(C) Ans = (C) because the set $\{6,15\} \in A$.

Note: that (D) is wrong $\emptyset \notin A$ but although $\emptyset \subseteq A$

Also Note that $\emptyset$ (empty set) would have been an element of A if A was defined to be;

$A = [(1,2,3), (4,5), (6,7,8), \emptyset]$.

Hence Ans $=$ C

(10) The Power set of a set A, denoted as $P(A)$ is all the possible subsets of the set A.

Note the number of these subsets denoted as

$n(P(A)) = 2^m$ (where m is the number of elements in the set A.

$\therefore$ Since $A = [a,b,c,d]$ — i.e Contains 4 elements

$\therefore n(P(A)) = 2^4 = \underline{16}$ (i.e 16 possible subsets)

Ans $=$ D

(11) $V = P(t)$ polynomials over a field $\mathbb{R}$ (Real nos).

$H: V \to V$, $H[f(t)] = d^3f/dt^3$

but Kernel of $H = \{x \in V : f(x) = 0\}$ See page (37) Definition of Kernel.

So that $\text{kernel}(H) = \left[P_n(t) \in V : \frac{d^3 P_n(t)}{dt^3} = 0\right]$

i.e the kernel of the mapping/function H.

is the polynomial in the domain (V) such that the function $\frac{d^3 f}{dt^3} = 0$. And

This is only possible when the polynomial $P_n(t)$ is of degree less than or equal to 2.

i.e $\frac{d^3}{dt^3}(at^2) = 0$ i.e $\frac{d}{dt}(at^2) = 2at$, $\frac{d^2}{dt^2}(at^2) = 2a$

i.e kernel$(H) = P_2(t)$.

$\therefore$ kernel$(H) = (P_n(t)$ of degree $\leq 2)$ Ans $=$ B

(12) $f: \mathbb{R}^4 \to \mathbb{R}^4$ a linear mapping defined by;

$f(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$

So that $\Rightarrow$

$$\begin{pmatrix} x-y+z+t \\ 2x-2y+3z+4t \\ 3x-3y+4z+5t \end{pmatrix} \approx \begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$\therefore$ The Image

$$= \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

pivot of operation.

To find basis & dimension of a Matrix write/transform the Matrix into Row-Echelon form. See page (16).

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix} \sim \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 = R_3 - R_1 \\ R_4 = R_4 - R_1 \end{array} \sim \begin{pmatrix} 1 & 2 & 3 \\ -1+1 & -2+2 & -3+3 \\ 1-1 & 3-2 & 4-3 \\ 1-1 & 4-2 & 5-3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

Note: R_2 Contains only zero $\therefore$ Interchanging $R_2 \leftrightarrow R_3$ & $R_3 \leftrightarrow R_4$.

$$\sim \begin{array}{l} \\ \text{pivot} \\ a_{32} \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we only have a_{32} (entry) left to be turned to zero $\therefore$ $R_3 = R_3 - 2R_2$.

BY SIR WILLIAMS

18. $A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B$ So that $I = A^{-1}B$

(Since $A^{-1}A = I$, and $I.X = X$]
see page (11) Inverse method

Then

$|A| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$ So that $A^{-1} = \frac{1}{|A|}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ (See Inverse of 2x2 page (9))

$\therefore \Rightarrow X = A^{-1}.B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2-1 & 2-1 \\ -3+2 & -3+2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = X$ Ans = (E)

(19) $f(t) = 2t^2 + 4t + 6$
then $f(A) = 2A^2 + 4A + 6$

(Now since $A = 2\times2$ matrix it is convinient to change $6 = 6I$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ i.e 2×2 Identity matrix)

$f(A) = 2A^2 + 4A + 6I$

$A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ So that $A^2 = A \cdot A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 4+15 & 6+18 \\ 10+30 & 15+36 \end{pmatrix} = \begin{pmatrix} 19 & 24 \\ 40 & 51 \end{pmatrix}$

$\therefore 2A^2 + 4A + 6I$

$\Rightarrow 2\begin{pmatrix} 19 & 24 \\ 40 & 51 \end{pmatrix} + 4\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 38 & 48 \\ 80 & 102 \end{pmatrix} + \begin{pmatrix} 8 & 12 \\ 20 & 24 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 52 & 60 \\ 100 & 132 \end{pmatrix}$

Ans = (D)

(20) $E = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & X \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{pmatrix}$ and $\det(E) = -546$.

Using Laplace Expansion see page (1) for (4X4)

$|E| = \begin{vmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{0} & \overset{-}{3} \\ 2 & 7 & 0 & X \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix} = \Bigg(1\begin{vmatrix} 7 & 0 & X \\ 6 & 3 & 0 \\ 3 & 1 & -5 \end{vmatrix} - \cancel{0|\ | + 0|\ |}$ (vanishes)

$-3\begin{vmatrix} 2 & 7 & 0 \\ 0 & 6 & 3 \\ 7 & 3 & 1 \end{vmatrix}\Bigg) = -546$

Now

$\begin{vmatrix} \overset{+}{7} & \overset{-}{0} & \overset{+}{X} \\ 6 & 3 & 0 \\ 3 & 1 & -5 \end{vmatrix} = 7(-15+0) - \cancel{0(-30+0)} + X(6-9)$
$= (-105 - 3X)$ and

$\begin{vmatrix} \overset{+}{2} & \overset{-}{7} & \overset{+}{0} \\ 0 & 6 & 3 \\ 7 & 3 & 1 \end{vmatrix} = 2(6-9) - 7(0-21) + \cancel{0(0-42)}$
$= -6 + 147 = 141$

$\Rightarrow \left(1(-105 - 3X) - 3(141)\right) = -546$

$\Rightarrow -105 - 3X - 423 = -546$

$-3X - 528 = -546$

$-3X = -18$

$\therefore X = 6$ Ans = (A)

BY SIR WILLIAMS

(21) $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$ first get cofactor. Short cut =

$$A^c = \begin{pmatrix} 2 & 1 & 4 & 2 & 1 \\ 3 & 5 & 1 & 3 & 5 \\ 2 & 0 & 6 & 2 & 0 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 5 & 1 & 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 30-0 & 2-18 & 0-10 \\ 0-6 & 12-8 & 2-0 \\ 1-20 & -12-2 & 10-3 \end{pmatrix}$$

$$= \begin{pmatrix} 30 & -16 & -10 \\ -6 & 4 & 2 \\ -19 & 10 & 7 \end{pmatrix} \therefore A^{CT} = \begin{pmatrix} 30 & -6 & -19 \\ -16 & 4 & 10 \\ -10 & 2 & 7 \end{pmatrix} = A^{adj} \text{ (Adjoint)}$$

$$\therefore |A| = \begin{vmatrix} \overset{+}{2} & \overset{-}{1} & \overset{+}{4} \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{vmatrix} = 2(30) - 1(16) + 4(-10)$$
$$= 60 - 16 - 40$$
$$= 60 - 56 = \underline{4}$$

$\therefore$ Inverse $A^{-1} = \dfrac{1}{|A|} A^{adj}$

$$= \frac{1}{4}\begin{pmatrix} 30 & -6 & -19 \\ -16 & 4 & 10 \\ -10 & 2 & 7 \end{pmatrix} = \begin{pmatrix} \frac{30}{4} & \frac{-6}{4} & \frac{-19}{4} \\ \frac{-16}{4} & 4/4 & 10/4 \\ -10/4 & 2/4 & 7/4 \end{pmatrix}$$

$$= \begin{pmatrix} 7.5 & -1.5 & -4.75 \\ -4 & 1 & 2.5 \\ -2.5 & 0.5 & 1.75 \end{pmatrix} \text{ Ans} = A^{-1}$$

(22) Hint: for a ... equations syst... For trivial & non-trivial, write the augmented matrix of such system. i.e if $AX = B$ then write $[A \,\vdots\, B]$ and put $C = [A \,\vdots\, B]$ and reduce the matrix to Row-echelon form. Then;

if $\text{Rank}(A) = \text{Rank}(C) \Rightarrow \text{Rank}(A) <$ no of unknowns

i.e Non-trivial if (i.e if $\text{Rank}(A) < 3$)

And Trivial (Unique) if $\text{Rank}(A) = 3$ (no of unknowns)

Summary

Trivial (Unique) Soln	Non trivial Infinite Solution
if Rank A = 3	If Rank(A) < 3 & (Rank(A) = Rank(C)) Rank(A) = Rank(C) < 3.

Solution:

$$\begin{aligned} 4x_1 + 3x_2 - x_3 &= 0 \\ 7x_1 - x_2 - 3x_3 &= 0 \\ 3x_1 - 4x_2 + kx_3 &= 0 \end{aligned} \Rightarrow \overset{A}{\begin{pmatrix} 4 & 3 & -1 \\ 7 & -1 & -3 \\ 3 & -4 & k \end{pmatrix}} \overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overset{B}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$[A \,\vdots\, B] = \begin{pmatrix} 4 & 3 & -1 & \vdots & 0 \\ 7 & -1 & -3 & \vdots & 0 \\ 3 & -4 & k & \vdots & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \quad (7/4) \\ R_3 \quad (3/4) \end{array}$$

So that

$R_2 = R_2 - \frac{7}{4}R_1$ and

$R_3 = R_3 - \frac{3}{4}R_1$ (see Elementary operations page 11)

$$\begin{pmatrix} 4 & 3 & -1 & 0 \\ 7 & -1 & -3 & 0 \\ 3 & -4 & k & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 7-\frac{7}{4}(4) & -1-\frac{7}{4}(3) & -3-\frac{7}{4}(-1) & 0-\frac{7}{4}(0) \\ 3-\frac{3}{4}(4) & -4-\frac{3}{4}(3) & k-\frac{3}{4}(-1) & 0-\frac{3}{4}(0) \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0 & -25/4 & k+\frac{3}{4} & 0 \end{pmatrix} \therefore R_3 = R_3 - R_2$$

$$\sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0-0 & \cancel{-\frac{25}{4}-\left(-\frac{25}{4}\right)} & k+\frac{3}{4}-\left(-\frac{5}{4}\right) & 0-0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0 & 0 & k+\frac{8}{4} & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0 & 0 & k+2 & 0 \end{pmatrix}$$

Row ~~Echelon~~ Form.

Now

$$A_{RE} = \begin{pmatrix} 4 & 3 & -1 \\ 0 & -25/4 & -5/4 \\ 0 & 0 & k+2 \end{pmatrix} \text{ And } C = \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0 & 0 & k+2 & 0 \end{pmatrix}$$

Now Rank$(A) = 3$ And Rank$(C) = 3$ = no of Unknown Variables (X_1, X_2, X_3).

For Non trivial solution Rank(A) = Rank$(C) \leq 3$

for this to happen $k+2=0$ i.e $k = -2$

So that $A = \begin{pmatrix} 4 & 3 & -1 \\ 0 & -25/4 & -5/4 \\ 0 & 0 & 0 \end{pmatrix}$ And $C = \begin{pmatrix} 4 & 3 & -1 & 0 \\ 0 & -25/4 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ So that Rank(A) = Rank(C) $= 2 < 3$

Ans is when $k = -2$.

Ans = (C)

See rank of a Matrix page (14).

(23) $AX = \lambda X$ so that $AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0$ i.e

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \Rightarrow \begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0 \quad |A-\lambda I| = 0$$

See eigenvalues. page (17) Expanding

$$\Rightarrow 2-\lambda\left[(3-\lambda)(2-\lambda)-2\right] - 2\left[(2-\lambda)-1\right] - 2\left[2-(3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[6-3\lambda-2\lambda+\lambda^2-2\right] - 2(1-\lambda) - 2(2-3+\lambda] = 0$$

$$(2-\lambda)\left[\lambda^2-5\lambda+4\right] - 2(1-\lambda) - 2(-1+\lambda) = 0$$

$$(2-\lambda)((\lambda-1)(\lambda-4)) - 2(1-\lambda) - 2(\lambda-1) = 0$$

$$(2-\lambda)(\lambda-1)(\lambda-4) - 2(-(\lambda-1)) - 2(\lambda-1) = 0$$

$$(2-\lambda)(\lambda-1)(\lambda-4) + 2(\lambda-1) - 2(\lambda-1) = 0$$

Factorize out $(\lambda-1)$ in each term

$$(\lambda-1)\left[(2-\lambda)(\lambda-4) + \cancel{2} - \cancel{2}\right] = 0$$

$$(\lambda-1)(2-\lambda)(\lambda-4) = 0 \quad \therefore \lambda = 1, 2, 4$$

Ans = (B)

(24) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & x & 0 \\ 3 & 4 & x+2 \end{pmatrix}$ and $\det(A) = 15$ Note: This is an ~~upper~~ triangular Matrix (Lower triangular).

so that $\det(A) = a_{11} \cdot a_{22} \cdot a_{33} = 1 \cdot x \cdot (x+2) = 15$

i.e $x(x+2) = 15$ so that $x^2+2x-15=0$

then $(x-3)(x+5)=0$ $x=3$ or $x=-5$

Since x is non-negative then $x=3$, $x \neq -5$

then A becomes $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 3+2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

And Eigenvalues of A

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0 \Rightarrow$ Recall lower triangular Matrix $\det = a_{11} \cdot a_{22} \cdot a_{33}$

$\Rightarrow (1-\lambda)(3-\lambda)(5-\lambda) = 0$ $\therefore \lambda = 1, 3, 5$ Ans = C

(25) $A = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ 1 & 7 & 0 \end{pmatrix}$ $\therefore A^c$ Cofactor $=$

$\begin{matrix} \cancel{1} & \cancel{4} & \cancel{3} & \cancel{1} & \cancel{4} \\ 6 & 2 & 5 & 6 & 2 \\ 1 & 7 & 0 & 1 & 7 \\ 1 & 4 & 3 & 1 & 4 \\ 6 & 2 & 5 & 6 & 2 \end{matrix}$

$= \begin{pmatrix} 0-35 & 5-0 & 42-2 \\ 21-0 & 0-3 & 4-7 \\ 20-6 & 18-5 & 2-24 \end{pmatrix}$

$= \begin{pmatrix} -35 & 5 & 40 \\ 21 & -3 & -3 \\ 14 & 13 & -22 \end{pmatrix} = A^c$

Adjoint(A) $= A^{adj}$

$= A^{cT}$ (Transpose of Co-factor)

$\therefore$ $($ -35 21 14 $)$ A ... (E)

for a unique solution if (Use Calculator to check for uniqueness.)

$x+y+kz=2$
$3x+4y+2z=k$
$2x+3y-z=1$

Consider this as a system i.e

Use Calculator to punch:

1	1	k= 3	=	2
3	4	2	=	3 =k
2	3	-1	=	1

(Math Error — means infinite solution)

1	1	k= 2	=	2
3	4	2	=	k= 2
2	3	-1	=	1

($x=12, y=-8, z=-1$ (Unique Solution)

1	1	k= -2	=	2
3	4	2	=	k=-2
2	3	-1	=	1

($x=0, y=0, z=-1$ Unique Solution)

1	1	k= 0	=	2
3	4	2	=	k= 0
2	3	-1	=	1

($x=6, y=-4, z=-1$ Unique Solution.)

$\therefore$ Ans = B, C, or D (two or more answers).

(27) $\begin{pmatrix} 4 & 2a & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 0 \\ -6 & 3b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$ (See matrix multiplication page 6)

$= \begin{pmatrix} 4+6a-18 & 20+0+9b \\ -1+6+0 & -5+0+0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$

$\therefore 4+6a-18=-2$ — (i)

$\therefore 6a-14=-2$

$\therefore 6$ ...

and $2a+9b=2$ $\therefore 9b=-18$ $\therefore b=\underline{-2}$

Hence $[a,b]=\underline{[2,-2]}$ Ans $=$ D

(28) $\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}=\begin{pmatrix}3\\ k\end{pmatrix}$ ~~find k~~. Observe that;

this forms a system $\begin{matrix}5x_1+2x_2=3\\ 2x_1+4x_2=k\end{matrix}$ We can only find the possible values of k, if some information about x_1 & x_2 (i.e the unknowns) were given i.e find the values of k for which the system has; (i) Unique Solution (ii) Infinite Solution (iii) No solution. Hence k can take any value. (Since there is no restriction to which the value of k can be)

Hence Ans $=$ E.

(29) $P=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$; $QP=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$ let $R=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$

So that $QP=R$ (multiply both sides by P^{-1})

$(QP)P^{-1}=RP^{-1}$ $\therefore Q(PP^{-1})=RP^{-1}$ $\therefore QI=RP^{-1}$

$\therefore Q=RP^{-1}$ (where $P.P^{-1}=I$ and $IQ=QI=Q$) (where $I=$ Identity, $P^{-1}=$ Inverse of P.)

$|P|=\begin{vmatrix}3 & 0\\ 0 & 1\end{vmatrix}=3-0=3$

So that $P^{-1}=\frac{1}{|P|}\begin{pmatrix}1 & -0\\ -0 & 3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$\therefore Q=RP^{-1}=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$=\frac{1}{3}\begin{pmatrix}2+0 & 0+3\\ 0+0 & 0+6\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2 & 3\\ 0 & 6\end{pmatrix}=\begin{pmatrix}2/3 & 3/3\\ 0/3 & 6/3\end{pmatrix}$

Ans $=Q=\begin{pmatrix}2/3 & 3/3\\ 0/3 & 6/3\end{pmatrix}=\underline{\begin{pmatrix}2/3 & 1\\ 0 & 2\end{pmatrix}}$ Ans $=$ A

(30) $A=\begin{pmatrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$, Now $A^2=A\cdot A=\begin{pmatrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$

$=\begin{pmatrix}9+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+4\end{pmatrix}=\underline{\begin{pmatrix}9 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{pmatrix}}$ And

$A^3=A\cdot A\cdot A=A\cdot A^2=\begin{pmatrix}-3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}\cdot\begin{pmatrix}9 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{pmatrix}$

$=\begin{pmatrix}-27+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+8\end{pmatrix}=\underline{\begin{pmatrix}-27 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 8\end{pmatrix}}$

So that $3A^3+2A^2-5I$

$=3\begin{pmatrix}-27 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 8\end{pmatrix}+2\begin{pmatrix}9 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{pmatrix}-5\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$=\begin{pmatrix}-81 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 24\end{pmatrix}+\begin{pmatrix}18 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 8\end{pmatrix}+\begin{pmatrix}-5 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & -5\end{pmatrix}$

$$\doteq \begin{pmatrix} -65 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 27 \end{pmatrix}, \quad Ans = B$$

(31) See Definition of powerset Question (10).

$B = \{$positive divisors of $12\}$. $\therefore B = \{1,2,3,4,6,12\}$.

$\therefore n(P(B)) = 2^m$ where (m = no of elements in B)

$\therefore = 2^6 = \underline{64}$ Ans = (A)

(32) See page (25)

$$\begin{array}{cccc} u_1 & u_2 & u_3 & V \end{array}$$

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}2\\3\\7\end{pmatrix} + x_3\begin{pmatrix}3\\5\\7\end{pmatrix} = \begin{pmatrix}3\\7\\-4\end{pmatrix}$$

Use Calculator to evaluate;

$$\begin{pmatrix} a & b & c & d \\ 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 7 & -4 \end{pmatrix}$$

Ans = $x_1 = 1$, $x_2 = -5$, $x_3 = 4$

$V = \underline{u_1 - 5u_2 + 4u_3}$ Ans = (E)

(33) See page (25) linear Combination of Polynomials

write;

$$\begin{array}{cccc} (P_1) & (P_2) & (P_3) & (V) \end{array}$$

$$x_1\begin{pmatrix}t^2\\2t\\1\end{pmatrix} + x_2\begin{pmatrix}2t^2\\5t\\4\end{pmatrix} + x_3\begin{pmatrix}t^2\\3t\\6\end{pmatrix} = \begin{pmatrix}3t^2\\5t\\-5\end{pmatrix} \Rightarrow$$

$$x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\5\end{pmatrix} + x_3\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$$ use Calculator to evaluate

So that; $V = \underline{3P_1 + P_2 - 2P_3}$ $V = 3P_1 + P_2 - 2P_3 = (C)$

Ans = (C)

(34) See Lemma on page (30)

Ans = $m \le n$ = (D)

(35) Ans = (A) See Definition of Basis page (28)

(36) See page (30) Theorem (2) (i), (ii) (iii)

Ans = (D)

(37) if $W \subseteq V$ and $\dim(V) = n$ then $\dim(W) \le n$

Because if $\dim(W) = n$ then $W = V$.

Ans = (C) See page (30) Theorem (4)

(38) THEOREM

SUMS & DIRECT SUMS

Let U & W be a subspace of V(vector space), such that U & W are finite dimensional subspaces of V.

$U+W$ is finite dimensional X

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Ans = (D) See Sums & Direct sum (Linear Algebra SCHAUM'S);

(40) Observe that all these functions maps from $\mathbb{R}'$(field) to $\mathbb{R}'$(field) because $\mathbb{R}$ is a field.

... of W in (A) (B) and (C)

are all scalars (α, say) and $\alpha \notin V$. $\therefore W \not\subset V$

Ans = (B).

(39) i.e. $M = \alpha A + \beta B + \gamma C$, Observe that the Matrix C was not given, Hence Ans = (E) None.

Consider the case where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$ (Assuming a typographical error)

See 2014 Question (39) Ans = $\underline{2A + 3B - C} = M$

Ans = (A)

(41) pivot
$$\begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 2 & 5 & 3 & 10 & 7 \\ 3 & 5 & 7 & 10 & 4 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$
Reduce to row-echelon form (See page (12), (13), (14) & (15))

$R_2 = R - \frac{2}{1}(R_1)$, $R_3 = R_3 - \frac{3}{1}(R_1)$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 2-\frac{2}{1}(1) & 5-\frac{2}{1}(2) & 3-\frac{2}{1}(2) & 10-\frac{2}{1}(4) & 7-\frac{2}{1}(2) \\ 3-\frac{3}{1}(1) & 5-\frac{3}{1}(2) & 7-\frac{3}{1}(2) & 10-\frac{3}{1}(4) & 4-\frac{3}{1}(2) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & -1 & 1 & -2 & -2 \end{pmatrix}$$
Now to turn (a_{32} to 0). put $R_3 = R_3 + R_2$

a_{32} (i.e adding the 2nd row to the 3rd row turns a_{32} to 0 i.e $1 + (-1) = 0$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 0 & 1 & -1 & 2 & 3 \\ 0+0 & \cancel{-1+1} & \cancel{1+(-1)} & -2+2 & -2+3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & 4 & 2 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Observe that R_3 (row 3) still contains a number that is not zero) Hence Rank = $\underline{3}$ (No of non-zero rows). Ans = (B)

(42) A·B
$$\begin{pmatrix} 2 & 3 & 5 & 3 \\ 1 & -2 & -3 & 2 \\ 6 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow$$

$$= \begin{pmatrix} 2+0+5+0 & 0+3+5+0 \\ 1+0-3+0 & 0-2-3+0 \\ 6+0+4+0 & 0+5+4+0 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ -2 & -5 \\ 10 & 9 \end{pmatrix}$$ Ans = (C)

(43) See page (25) Ans. $v = -6u_1 + 3u_2 + 2u_3$

Ans = (D)

(44) Basis was asked but answers are in dimension. Ans = (E)

(45) Note: if $v = (a, b, c)$ belongs to $W = \text{Span}(u_1, u_2, u_3)$ then this implies that v can be written as a linear combination of the "u's" in W). Hence; $\exists$ (there exist) scalars $\alpha, \beta, \gamma \in F$ $\ni$ $\alpha u_1 + \beta u_2 + \gamma u_3 = V$. Hence; V belongs to W iff the system is Consistent (has a Solution).

See linear Combination & Spans: page (24)
Ans = (B).

(46) See Question (36) (here) Ans = (C)
Observe that (i) is not Correct it should be ;
Any n+1 or more vectors are linearly Independent.
See page (30) also Theorem (2)

(47) $A = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ ∴ $A^T = \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix}$

$A \cdot A^T = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 16+4+36 & 4+16+42 \\ 4+16+42 & 1+64+49 \end{pmatrix}$

$= \begin{pmatrix} 56 & 62 \\ 62 & 114 \end{pmatrix}$ Ans = (D)

(48) See page (25) write (transform as ;)

$x_1 \overset{P_1}{\begin{pmatrix} 1t^2 \\ -2t \\ +5 \end{pmatrix}} + x_2 \overset{P_2}{\begin{pmatrix} 2t^2 \\ -3t \\ +0 \end{pmatrix}} + x_3 \overset{P_3}{\begin{pmatrix} 0t^2 \\ 1t \\ +1 \end{pmatrix}} = \overset{V}{\begin{pmatrix} 1t^2 \\ +4t \\ -3 \end{pmatrix}}$ (Remove all the variable t's)

$x_1 \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ Using Calculator gives

$x_1 = \frac{-17}{11}, x_2 = \frac{14}{11}, x_3 = \frac{52}{11}$

So that $x_1 = \frac{-17}{11}$, $x_2 = \frac{14}{11}$ $x_3 = 52/11$ ⇒

$V = \frac{-17}{11} P_1 + \frac{14}{11} P_2 + \frac{52}{11} P_3$ Ans = (E)

(49) $A = [a_{ij}]$ $a_{ij} = 0$ when $i > j$ and
$a_{ij} = 0$ when $i < j$

i.e $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ 1<2, 1<3, 2<3 So that $a_{12} = a_{13} = a_{23} = 0$.

2>1, 3>1, 3>2 So that $a_{21} = a_{31} = a_{32} = 0$ gives a diagonal matrix. Ans = (B)

i.e $\equiv \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

(50) (A) Yes a square matrix can either be singular or non-singular.

(B) Yes if A is non-singular that means $|A| \neq 0$ thence $A^{-1} = \frac{1}{|A|} A^{adj}$ (which is Unique).

(C) An Aberration because a 2×2 Matrix can never have a (2×3)/(2 by 3) inverse So an (n×n) Matrix will have an inverse of order (n×n).

(D) Yes if $\det(A) \neq 0$ then A has an Inverse i.e A is Invertible.

∴ (C) ...

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(1) See 2017 Question (43) Ans = A

(2) See 2017 Question (45) Ans = (E)

(3) Ans = (D) i.e (A) & (B) are not subspace of V
See page (23) Example (II)

(4) Ans = (A) See page (24) Example (3)

(5) for linear dependency $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$

(A) $\alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}3\\-5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ Solving $\alpha_1 + 3\alpha_2 = 0$, $2\alpha_1 - 5\alpha_2 = 0$

you get $\alpha_1 = 0, \alpha_2 = 0$

Hence not linearly dependent.

(B) linearly dependent. Ans = (B)

(C) Not linearly dependent See page (26) & (27) Example (i)

(6) (i) $u = \begin{pmatrix}2\\4\\-3\end{pmatrix}, v = \begin{pmatrix}4\\8\\-6\end{pmatrix} \Rightarrow x_1\begin{pmatrix}2\\4\\-3\end{pmatrix} + x_2\begin{pmatrix}4\\8\\-6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

choose only 2 equations (because no of unknowns is two)

$2x_1 + 4x_2 = 0$
$4x_1 + 8x_2 = 0$ ∴ ans $(x_1 = 2, x_2 = -1)$

(Observe that Calculator gives math Error)

i.e Infinite Solution.

Hence the vectors in (i) are linearly dependent.

(ii) $u = \begin{pmatrix}2\\-3\\4\end{pmatrix}, \begin{pmatrix}4\\-3\\2\end{pmatrix}$ choose 2 equ. $2 \quad 4 = 0$, $-3 \quad -3 = 0$

Calculator gives $x = 0, y = 0$

Hence they are linearly independent.

Alternatively:

See Remark (4) page (28)

for v_1 & v_2 (vectors) to be linearly dependent one must be a multiple of the other,

i.e in (i) above $v = 2u = 2\begin{pmatrix}2\\4\\-3\end{pmatrix} = \begin{pmatrix}4\\8\\-6\end{pmatrix}$

(ii) one is not a multiple of the other.

Now

in (iii) $v = -4u$ i.e $v = \begin{bmatrix}-4 & -12 & 16\\ 20 & 0 & 4\end{bmatrix} = -4\begin{pmatrix}1 & 3 & -4\\ 5 & 0 & -1\end{pmatrix}$

Hence they are linearly dependent, so that $v = -4u$. $1v + 4u = 0$.

(iv) but the vectors here are not multiple of the other. Hence Ans = (i) & (iii) = (A)

(7) Suppose $L = u + v$, $M = 2u - v$, $N = u - 2v + w$

Consider $L = u + v + 0w$
$M = u - v + 0w$ so that
$N = u - 2v + w$

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$$x_1\begin{pmatrix}4\\4\\0\end{pmatrix}+x_2\begin{pmatrix}4\\-4\\0\end{pmatrix}+x_3\begin{pmatrix}4\\-24\\40\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1\begin{pmatrix}1\\1\\0\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{array}{l} x_1+x_2+x_3=0\\ x_1-x_2-2x_3=0\\ 0x_1+0x_2+x_3=0\end{array}$$

Using Calculator to Evaluate.
you get $x_1=0, x_2=0, X_3=0$ i.e no Solution
Hence they are linearly Independent.

Ans = (B)

(8) See page (29) Example Ans = (C)

(9) See 2017 Question (44) Ans = (E)

(10) See 2017 Question (47) Ans = (E)

(11) Ans = (B) See 2017 Question (50)

(12) To check for this we construct any two arbitrary symmetric matrices i.e a matrix that $A^T=A$ where A = Square Matrix

Now:
consider $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ so that $A^T=\begin{pmatrix}1&2\\2&1\end{pmatrix}=A$.

and $B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ So that $B^T=\begin{pmatrix}1&0\\0&1\end{pmatrix}=B$

Now Consider $A\cdot B=\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1+0&0+2\\2+0&0+1\end{pmatrix}$

$=\begin{pmatrix}1&2\\2&1\end{pmatrix}$.

and $B\cdot A=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&2\\2&1\end{pmatrix}=\begin{pmatrix}1+0&2+0\\0+2&0+1\end{pmatrix}$

$=\begin{pmatrix}1&2\\2&1\end{pmatrix}$

$AB-BA=\begin{pmatrix}1&2\\2&1\end{pmatrix}-\begin{pmatrix}1&2\\2&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ = Null matrix

Ans = (A).

Observe that an Identity matrix is a symmetric matrix as in (B) above $B=I_{2\times2}$ so that

$A\cdot B=A\cdot I=A$ & $B\cdot A=I\cdot A=A$

So that $A\cdot B-B\cdot A=A-A=\underline{\underline{0}}$ (Null)

(13) See 2017 Question (49)

(14) Ans = (A) i.e diagonal elements see page (8) Skew-symmetric

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