Partial Differentiation:

If in a function of several variables, we assign definite numerical value to all but one of the variable and allow only that one to vary, the function becomes a function of one variable. Consider a function f of three variables x, y and z such that $f = f(x, y, z)$.

If y and z are held constant and only x is allowed to vary, the partial derivative with respect to x is denoted by $\frac{\partial f}{\partial x}$ and is defined as.

$$\frac{\partial f}{\partial x} = \lim_{x \to 0} \frac{f(x+\Delta x, y, z) - f(x)}{\Delta x}$$ when the limit exists.

Similarly, we define the partial derivatives of f with respect to y and z respectively as $\partial f/\partial y$ and $\partial f/\partial z$. If the limit does not exist, it means that the function does not have partial derivative with respect to that variable. Other denotations are f_x, f_y, and f_z which are partial derivatives of f with respect to x, y and z respectively i.e.

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_z = \frac{\partial f}{\partial z}$$

To find the partial derivative of f directly from definition means using the limit of the function. However, the laws of differentiation of function of one variable also apply to partial differentiation.

Example 1: Find directly from definition the partial derivative of f with respect to x at (x_0, y_0) if $f = 2x^2 - xy + y^2$

Soln: Recall that $\frac{\partial f}{\partial x} = \lim_{x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y)}{\Delta x}$

$$= \lim_{x \to 0} \frac{2(x+\Delta x)^2 - (x+\Delta x)y + y^2 - (2x^2 - xy + y^2)}{\Delta x}$$

$$= \lim_{x \to 0} \frac{\cancel{2x^2} + 4x\Delta x + 2(\Delta x)^2 - \cancel{xy} - y\Delta x + \cancel{y^2} - \cancel{2x^2} + \cancel{xy} - \cancel{y^2}}{\Delta x}$$

$$= \lim_{x \to 0} \frac{4x(\Delta x) + 2(\Delta x)^2 - y(\Delta x)}{\Delta x} = \lim_{x \to 0} \frac{\Delta x(4x + 2(\Delta x) - y)}{\Delta x} = \lim_{\Delta x \to 0} 4x + 2\Delta x - y$$

$$= 4x - y$$

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$\therefore \frac{\partial f}{\partial x} = 4x - y$

at (x_0, y_0); $\frac{\partial f}{\partial x} = 4x_0 - y_0$

Eg 2 : If $f(x,y,z) = xy + z$, find f_x, f_y, f_z

Soln : $f_x = y$, $f_y = x$, $f_z = 1$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y, z) - f(x,y)}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{(x+\Delta x)y + z - xy - z}{\Delta x}$

$= \frac{y\Delta x}{\Delta x} = y$

~~Partial derivative of higher order can be found as~~

Ex 3 : If $f(x,y) = x^2 y^3$ find f_x, f_y

Soln : $f_x = 2xy^3$, $f_y = 3x^2 y$.

Ex 4 : If $f(x,y) = x^2 + xy + y^2$, find f_x, f_y.

Soln : $f_x = 2x + y$

$f_y = \cancel{x+} 2y$.

Ex 5 : If $f(x,y) = x^3 + y^3 - 2x^2 y$, find f_x, f_y

Soln : $f_x = 3x^2 - 4xy$

$f_y = 3y^2 - 2x^2$

Ex 6 : If $f(x,y) = (2x-y)(x+3y)$ find f_x, f_y

Soln : $f_x = (2x-y)(1+0) + (2-0)(x+3y) = 2x - y + 2x + 6y = 4x + 5y$

$f_y = (2x-y)(0+3) + (x+3y)(0-1) = 6x - 3y - x - 3y = 5x - 6y$

Partial derivative of higher order can be found as :

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$, $\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right] = f_{xyz}$

We should find out that if the mixed partial derivatives f_{xy} and f_{yx} are continuous in a region then they are equal throughout the region

Continuous functions : A real valued function $f(x)$ is said to be continuous at a point a if

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$.

Ex ①

$\frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y, z) - f(x,y)}{\Delta y}$

$\lim_{\Delta y \to 0} = \frac{2x^2 + x(y+\Delta y) + (y+\Delta y)^2 - 2x^2 + xy + y^2}{\Delta y}$

$= 2x^2 + xy + x\Delta y + y^2 + 2y\Delta y + (\Delta y)^2 - 2x^2 + xy - y^2$

$= 2xy + x\Delta y + 2y\Delta y + (\Delta y)^2 = \Delta y(x - 2y$

CHAIN RULE

The total derivative of f is defined by the equation

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Whether or x, y, z are independent of each other but provided only that the partial derivatives involved are continuous. We now consider several types of dependence among x, y, z. In all the formulae to be obtained, the continuity of all derivatives on the right hand side is to be assumed.

Case 1: If x, y, z are all functions of a single variable t, then the dependent variable f may also be considered as truly a function of the dependent variable t and we may conveniently speak of x, y and z as intermediate variables. Since only one independent variable is present then $\frac{df}{dt}$ has a meaning and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

~~Example~~! Find $\frac{df}{dt}$ if $f = x^2 + y + xz$

$x = t,\ y = t^2,\ z = 2t$

Soln! $\frac{dx}{dt} = 1,\quad \frac{dy}{dt} = 2t,\quad \frac{dz}{dt} = 2$

Now $\frac{\partial f}{\partial x} = 2x + z,\quad \frac{\partial f}{\partial y} = 1,\quad \frac{\partial f}{\partial z} = x$

$$\therefore \frac{df}{dt} = (2x+z)\cdot 1 + 1\cdot 2t + x\cdot 2$$

$$= 2x + z + 2t + 2x$$

but $z = 2t$ and $x = t$

$$= 2t + 2t + 2t + 2t = 8t$$

Case II: The intermediate variables x, y and z may be functions of two or more independent variables say s and t. Then if we consider f as a function of s and t, we may investigate the partial derivatives of f with respect to t when s is held constant.

$$\left(\frac{\partial f}{\partial t}\right)_s = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial t}\right)_s + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial t}\right)_s + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s$$

The subscript indicates that s is held constant. We can also write $\left(\frac{\partial y}{\partial s}\right)$, We see that our f can be written in terms of its dependence on the variables s and t as $f(x, y, z) = f[x(s,t), y(s,t), z(s,t)]$

Hence if the subscript is understood, $= F(s,t)$

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Example: Find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ given $u=x^2+2xy-y\ln z$ and $x=s+t^2$, $y=s-t^2$, $z=2t$

$\frac{\partial x}{\partial t}=2t$, $\frac{\partial y}{\partial t}=-2t$, $\frac{\partial z}{\partial t}=2$

$\frac{\partial x}{\partial s}=1$, $\frac{\partial y}{\partial s}=1$, $\frac{\partial z}{\partial s}=0$

Soln: $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$

but $\frac{\partial u}{\partial x}=2x+2y$, $\cancel{\frac{\partial u}{\partial y}=2x-}$

$$\frac{\partial u}{\partial y}=2x-[y\cdot 0+\ln z]$$
$$=2x-\ln z$$
$$\frac{\partial u}{\partial z}=-[y\cdot\tfrac{1}{z}+\ln z\cdot 0]=\frac{-y}{z}$$

$$\therefore \frac{\partial u}{\partial s}=(2x+2y)\cdot 1+(2x-\ln z)\cdot 1+\left[\frac{-y}{z}(0)\right]$$
$=2x+2y+2x-\ln z$, but $x=s+t^2$, $y=s-t^2$, $z=2t$
$$=4(s+t^2)+2(s-t^2)-\ln 2t$$
$$=4s+4t^2+2s-2t^2-\ln 2t$$
$$=6s+2t^2-\ln 2t$$

And $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$
$$=(2x+2y)\cdot 2t+(2x-\ln z)\cdot -2t+\left(\frac{-y}{z}\right)(2)$$
$$=4xt+4yt-\cancel{4xt}+2t\ln z-\frac{2y}{z}$$
$$=\cancel{2xt}+4yt+2t\ln z-\frac{2y}{z}$$
$$=\cancel{2(s+t^2)t}+4(s-t^2)t+2t\ln 2t-\frac{\cancel{2}(s-t^2)}{\cancel{2}t}$$
$$\cancel{4st-4t^3}$$
$$=$$

Case III: If we suppose that y and z are functions of x, then f is function of the one independent variable x with y and z as intermediate variables. This, with knowledge of Case II above we can have

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

Case IV: If we suppose x and y are independent but that z is a function of both x and y ie;

$$f = f(x, y, z), \; z = z(x, y).$$

Then f can be considered as depending upon x and y directly and also intermediately on z.

$$\therefore \left(\frac{\partial f}{\partial x}\right)_y = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right)_y$$

$$\left(\frac{\partial f}{\partial y}\right)_x = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial y}\right)_x.$$

Example: If $f(x, y, z) = x^2 + xz + 2y^2$, $z = 2xy$

soln: $\frac{\partial f}{\partial x} = 2x + z + x \cdot 2y$

$$= 2x + z + 2xy = 2x + 2xy + 2xy = 2x + 4xy.$$

If we suppose that $f(x, y, z) = 0$, then

$$\left[\frac{\partial f}{\partial x}\right]_y = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right)_y = 0$$

$$\therefore \quad f_x + f_z\left(\frac{\partial z}{\partial x}\right)_y = 0$$

$$\therefore \left[\frac{\partial z}{\partial x}\right]_y = \frac{-f_x}{f_z}, \; f_z \neq 0.$$

HIGHER DERIVATIVES

Let $F = f(x, y, z(x, y))$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} + \left(\frac{\partial z}{\partial x}\frac{\partial}{\partial z}\right)f$$

Thus $\frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}$ can be regarded as an operation which follows by iteration that

$$\frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial z}\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial z}{\partial x}\frac{\partial^2 f}{\partial z \partial x} + \left[\frac{\partial z}{\partial x}\right]^2\frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial z}\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x}\frac{\partial^2 f}{\partial x \partial z} + \left[\frac{\partial z}{\partial x}\right]^2\frac{\partial^2 f}{\partial z^2}$$

Example: if $u = e^x \cos y$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Soln: $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x . 0 + \cos y . e^x = e^x \cos y.$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = e^x . -\sin y + \cos y . 0 = -e^x \sin y.$$

$$\frac{\partial^2 u}{\partial y^2} = -\left[e^x \cos y + \sin y . 0\right] = -e^x \cos y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0,$$

N/B: Any function satisfying the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called an harmonic function. The equation itself is called the Laplace equation in two variables. The symbol del ∇ is used and we write Laplace equation as:

$$\nabla^2 u = 0, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Example: if $z = x^2 \tan^{-1} y/x$, find $\frac{\partial^2 z}{\partial x \partial y}$, ~~$z = x^2$~~

Soln: $z = x^2 \tan^{-1} y/x$, ~~$u = x^2$~~

~~$f = \tan^{-1} y/x \Rightarrow f = \tan^{-1} u \Rightarrow \frac{\partial f}{\partial u} = \frac{1}{1+u^2} = \frac{1}{1+\frac{y^2}{x^2}} = \frac{x^2}{x^2+y^2}$~~

$\tan^{-1} u = \frac{1}{1+u^2}$, $u = y/x$, $\frac{du}{dx} = -\frac{y}{x^2}$

$\therefore \tan^{-1} y/x = \frac{1}{1+\frac{y^2}{x^2}} . \frac{-y}{x^2}$; $\frac{1}{\frac{x^2+y^2}{x^2}}$; $\frac{x^2}{x^2+y^2}$

$= \frac{x^2}{x^2+y^2} . \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$

$$\frac{\partial z}{\partial x} = x^2 . \frac{-y}{x^2+y^2} . x^2 + 2x . \tan^{-1} y/x$$

$$= x^2 . \frac{-y}{x^2+y^2} + 2x \tan^{-1} y/x$$

$$= 2x \tan^{-1} y/x - \frac{x^2 y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x \tan^{-1} y/x - \frac{x^2 y}{x^2+y^2}\right)$$

$$= 2x . \frac{1}{1+\frac{y^2}{x^2}} . \frac{1}{x} - \left[\frac{(x^2+y^2)(x^2) - x^2 y . 2y}{(x^2+y^2)^2}\right]$$

$$= \frac{2}{x^2+y^2} . x^2 - \frac{x^4 + x^2y^2 + 2x^2y^2}{(x^2+y^2)^2} = \frac{2x^2(x^2+y^2) - x^4 + x^2y^2 + 2x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^4 + 2x^2y^2 - x^4 + x^2y^2 + 2x^2y^2}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2} =$$

Integration by Differentiation

We state without proof the following theorem called Leibnitz Rule

Theorem: If $F(x) = \int_{\alpha(x)}^{\beta(x)} f(t,x)\,dt$ where $f(t,x)$ and $\frac{\partial F}{\partial x}$ are continuous and $\frac{\partial \beta}{\partial x}$ and $\frac{\partial \alpha}{\partial x}$ exist then.

$$\frac{dF}{dx} = \frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}\,dt + f(\beta,x)\frac{d\beta}{dx} - f(\alpha,x)\frac{d\alpha}{dx}$$

Example: Evaluate $I = \int_0^1 \frac{x^\alpha - 1}{\log_e 1}\,dx$

Soln: $\frac{dI}{d\alpha} = \frac{d}{dx}\int_0^1 \frac{x^\alpha - 1}{\log_e x}\,dx$

CHANGE OF COORDINATES

In many applied problems, it is more convenient to use other coordinate system instead of rectangular coordinates we are used to: for example it is more convenient to use polar coordinate system for problems involving a plane and in three dimension, the use of cylindrical or spherical coordinate is more convenient

Polar coordinate system:

In the polar coordinate system (r,θ) are related to the rectangular coordinates (x,y) by the equations

$x = r\cos\theta$, $y = r\sin\theta$

Here we have

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\,dr + r\sin\theta\,d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\,dr + r\cos\theta\,d\theta$$

Thus the arc element given by

$$ds^2 = dx^2 + dy^2$$
$$= (\cos\theta\,dr - r\sin\theta\,d\theta)^2 + (\sin\theta\,dr + \cos\theta\,d\theta)^2$$

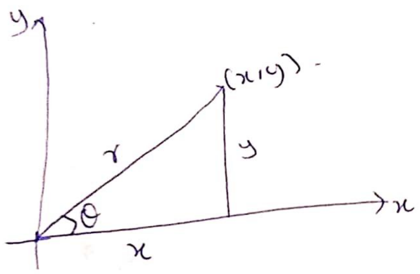
$$ds^2 = (\cos\theta\, dr - r\sin\theta\, d\theta)(\cos\theta\, dr - r\sin\theta\, d\theta) + (\sin\theta\, dr + r\cos\theta\, d\theta)(\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= \cos^2\theta\, dr^2 - r\sin\theta\cos\theta\, dr\, d\theta - r\sin\theta\cos\theta\, dr\, d\theta + r^2\sin^2\theta\, d\theta^2$$
$$+ \sin^2\theta\, dr^2 + r\sin\theta\cos\theta\, dr\, d\theta + r\sin\theta\cos\theta\, dr\, d\theta + r^2\cos^2\theta\, d\theta^2$$
$$= \cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 + \sin^2\theta\, dr^2 + r^2\cos^2\theta\, d\theta^2$$
$$= \cos^2\theta\, dr^2 + \sin^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 + r^2\cos^2\theta\, d\theta^2$$
$$= (\cos^2\theta + \sin^2\theta)dr^2 + r^2(\sin^2\theta + \cos^2\theta)d\theta^2$$
$$= dr^2 + r^2 d\theta^2.$$

$$\therefore\ ds = \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{\frac{dr^2}{d\theta^2} + \frac{r^2 d\theta^2}{d\theta^2}} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} = \sqrt{}$$

$$\text{ie } \frac{ds^2}{d\theta^2} = \frac{dr^2}{d\theta^2} + r^2\frac{d\theta^2}{d\theta^2} \Rightarrow \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$\therefore\ \frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\Rightarrow ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

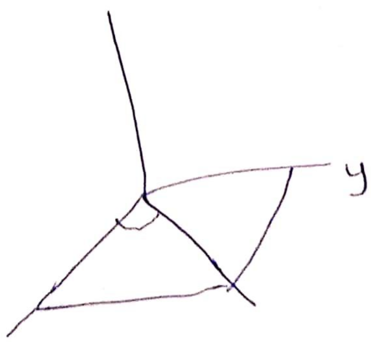


SPHERICAL AND CYLINDRICAL COORDINATE

The polar coordinate is mainly used for two dimensional plane - for the three dimensions, the two most important coordinate systems apart from the rectangular system, are the spherical and cylinderical coordinate system.

For the cylinderical coordinates, the transformation from Cartesian to cylinderical is

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$



We see that the cylinderical coordinates are just an extension of the polar coordinate into three dimension with the ~~zero~~ z for the third variable. The line element ds which is given by

$ds^2 = dx^2 + dy^2 + dz^2$ can be found

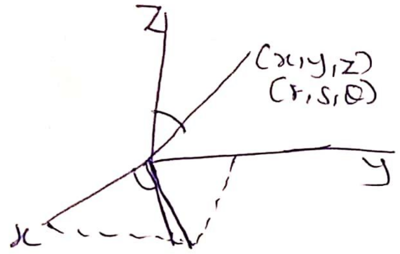
$dx = \cos\theta\, dr - r\sin\theta\, d\theta$
dy $\sin\theta\, dr + r\cos\theta\, d\theta$, $dz = dz$

$\therefore ds^2 = (\cos^2\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 + dz^2$

$= dr^2 + r^2 d\theta^2 + dz^2$

We note that there is no term cancelled out. Such coordinate system are called orthogonal. Geometrically, an orthogonal system means that the coordinate surfaces are mutually perpendicular. The three coordinate surfaces ~~are~~ given point intersect at the right angle for the spherical coordinates the transformation from cartesian is given as

$x = r\sin\theta\cos\theta \quad , y = r\sin\theta\cos\theta \quad , z = r\cos\theta$



The line element can be found as follows:

$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \partial x$

$\sin\theta\cos\theta\, dr + r[\sin\theta \cdot -\sin\theta + \cos\theta\cos\theta] d\theta +$

$\sin\theta\cos\theta\, dr + (r\sin^2\theta + r\cos^2\theta) d\theta$

$= \sin\theta\cos\theta\, dr + r\cos\theta\cos\theta\, d\theta - r\sin\theta\sin\theta$

Similarly:

$dy = \frac{dy}{dr} dr + \frac{\partial y}{\partial \theta} d\theta$

$r\cos\theta\cos\theta - r\sin\theta\sin\theta$

$= \sin\theta\cos\theta\, dr + r\cos\theta\cos\theta - r\sin\theta\sin\theta$

$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta$

$= \cos\theta\, dr + r \cdot -\sin\theta\, d\theta$

$= \cos\theta\, dr - r\sin\theta\, d\theta$

$\therefore ds^2 = dx^2 + dy^2 + dz^2$

$= (\sin\theta\cos\theta\, dr + r\cos\theta\cos\theta\, d\theta - r\sin\theta\sin\theta\, d\theta)^2 + (\sin\theta\cos\theta\, dr + r\cos\theta\cos\theta\, d\theta - r\sin\theta\sin\theta\, d\theta)^2$

$+ (\cos\theta\, dr - r\sin\theta\, d\theta)^2$

$=$

$= dr^2 + r^2 d\theta + r^2\sin^2\theta$

JACOBIANS

Problems in applied mathematics especially those involving double or triple integral lead to finding the area element $dxdy\,dz$ and volume element $dxdy$ as. We ~~state~~ without ~~proof~~ that given x and y as function of two new variables i.e,

$$x = x(s,t), \quad y = y(s,t)$$

then the area element $dxdy$ is replaced in the s,t system by the area element.

$$dA = \text{the absolute value of } \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} ds\,dt$$

The determinant is called a Jacobian or specifically the Jacobian of x, y with respect to s, t. It is often abbreviated as

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial(x,y)}{\partial(s,t)} = J(s,t).$$

Thus for the area element in the polar coordinate system can be found

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\therefore dA = J(r,\theta)\,dr\,d\theta = r\,dr\,d\theta.$$

In the same fashion, the volume of element $dv = dx\,dy\,dz$ can also be found if the coordinate element (x, y, z) are replaced by (r, θ, ϕ) then (11)

$$dv = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} dr\,d\theta\,d\phi = J(r, \theta, \phi)\, dr\,d\theta\,d\phi$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} dr\,d\theta\,d\phi$$

For the spherical coordinate we shall find that

$$J(r, \theta, \phi) = r^2 \sin\theta$$

HIGHER DERIVATIVES; LEIBNIZ FORMULA

When a function $y = f(x)$ is differentiated more than once with respect to x, the higher differential coefficient of u are written as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Where $\frac{d^ny}{dx^n}$ is the nth differential coefficient of y with respect to x.

We sometimes abbreviate them as $f'(x), f''(x), f^n(x)$ or as Dy, D^2y, D^ny where $D = \frac{d}{dx}$

Most often we are required to find the nth differential coefficient of some functions.

Example: $D^4(uv) = uD^4v + 4DuD^3v + 6D^2uD^2v + 4D^3uDv + vD^4u$.

$$= uD^4v + 4C_1DuD^3v + 4C_2D^3u + 4C_3D^3uDv + 4C_4D^4u.$$

$$= 4C_0uD^4v + 4C_1DuD^3v + 4C_2D^2u \ldots$$

$$\therefore D^4(uv) = \sum_{r=0}^{4} 4C_r D^{4-r}vD^ru.$$

Thus we have derived an important formula called the Leibniz formula for the nth differential coefficient of a function which is expressible as a product of 2 functions.

$$\therefore D^n(uv) = uD^nv + {}^nC_1D^{n-1}vDu + {}^nC_2D^{n-2}vD^2u + {}^nC_{n-1}D^{n-1}vDu + {}^nC_nvD^nu$$

$$= \sum_{r=0}^{4} {}^nC_r D^{n-r}vD^ru.$$

Example: If $y = (x^2+1)e^{2x}$, find $D^n y$.

Let $u = x^2+1$, $v = e^{2x}$

$$D^n(uv) = (x^2+1)D^n e^{2x} + {}^nC_1 D(x^2+1) D^{n-1} e^{2x} + {}^nC_2 D^2(x^2+1) D^{n-2} e^{2x} + {}^nC_3 D^3(x^2+1) D^{n-3} e^{2x} + \dots + {}^nC_n e^{2x} D^n(x^2+1).$$

$$= (x^2+1)D^n e^{2x} + 2n \cdot x D^{n-1} e^{2x} + \frac{n(n-1)}{2} 2 D^{n-2} e^{2x} + 0 + 0 + 0$$

$$= (x^2+1)2^n e^{2x} + n x 2^{n-1} e^{2x} + 2^{n-2}(n)(n-1) e^{2x}$$

$$= 2^{n-2} e^{2x}\left[4(x^2+1) + 4n + n^2 - n\right]$$

$$= 2^{n-2} e^{2x}\left[4x^2 + 1 + 4nx + n^2 - n + 4\right]$$

General Worked Examples

Prove that if f is a function of r, having 1st partial derivatives and $x = r\cos\theta$, $y = r\sin\theta$, $(r>0)$ then

$$f_x = \cos\theta \frac{\partial \phi}{\partial r} - \sin\theta \frac{\partial \phi}{\partial \theta}$$ and obtain a corresponding expression for f_y.

Prove that if $f = r^{-n}\sin n\theta$, then $\nabla^2 f = 0$, $f = f(r,\theta)$

Proof: $x = r\cos\theta$, $y = r\sin\theta$, $(r>0)$, $r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1} y/x$$

$$\therefore \frac{\partial r}{\partial x} = \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \tfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{(x^2+y^2)^{1/2}} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{1}{x^2+y^2} \cdot x^2 \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = \frac{-y}{r}\cdot\frac{1}{r} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{x^2+y^2} \cdot x^2 \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{x}{r}\cdot\frac{1}{r} = \frac{\cos\theta}{r}.$$

$$\therefore f_x = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

Also
$$f_y = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}.$$

$$f = r^{-n}\sin n\theta$$

$$\frac{\partial f}{\partial r} = -nr^{-n-1}\sin n\theta$$

$$\frac{\partial^2 f}{\partial r^2} = -n(-n-1)r^{-n-2}\sin n\theta$$
$$=(n^2+n)r^{-n-2}\sin n\theta$$

$$\frac{\partial f}{\partial \theta} = nr^{-n}\cos n\theta$$

$$\frac{\partial^2 f}{\partial \theta^2} = -n^2 r^{-n}\sin n\theta$$

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$
$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \cancel{\frac{\partial}{\partial x}}$$
$$=\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right).$$
$$= \cos\theta\frac{\partial}{\partial r}\left[\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right]$$
$$= \cos\theta\left[\cos\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial r\partial\theta} - \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}$$
$$+ \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$
$$= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

Also

$$\frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= \left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right]$$
$$= \sin\theta\frac{\partial}{\partial r}\left[\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right] + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right]$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 f}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial f}{\theta} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}\right]$$

$$+ \frac{\cos\theta}{r}\left[\sin\theta\frac{\partial^2 f}{\partial r\partial\theta} + \cos\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial\theta^2} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right]$$

$$= \sin^2\theta\frac{\partial^2 f}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta}$$

$$= \sin^2\theta\frac{\partial^2 f}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \sin^2\theta\frac{\partial^2 f}{\partial r^2} + \left[\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right]\frac{\partial f}{\partial r}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \cos^2\theta\frac{\partial^2 f}{\partial\theta^2}$$

$$= (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\left[\sin^2\theta + \cos^2\theta\right]\frac{\partial f}{\partial r} + \frac{1}{r^2}\left[\sin^2\theta + \cos^2\theta\right]\frac{\partial^2 f}{\partial\theta^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

Substituting for $\frac{\partial^2 f}{\partial r^2}$, $\frac{\partial f}{\partial r}$ and $\frac{\partial^2 f}{\partial\theta^2}$

$$\nabla^2 f = (n^2+n)r^{-n-2}\sin n\theta + \frac{1}{r}\cdot -n r^{-n-1}\sin n\theta - \frac{1}{r^2}n^2 r^{-n}\sin n\theta$$

$$= \cancel{n^2 r^{-n-2}\sin n\theta} + \cancel{n r^{-n-2}\sin n\theta} - \cancel{n r^{-n-2}\sin n\theta} - \cancel{n^2 r^{-n-2}\sin n\theta} = 0.$$

$\therefore \nabla^2 f = 0$ proved.

If $u = e^x(x\cos y - y\sin y)$. prove that

(i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(ii) $e^{-u}\left(\frac{\partial^2 e^u}{\partial x^2} + \frac{\partial^2 e^u}{\partial y^2}\right) = e^{2x}\left[(x+1)^2 + y^2\right]$

Soln: $u = e^x(x\cos y - y\sin y)$.

$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$

$= u + 2e^x\cos y$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y)$

$= e^x(-x\cos y - 2\cos y + y\sin y)$

$= -e^x(x\cos y - y\sin y) - e^x \cdot 2\cos y$

$= -u - 2e^x\cos y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u + 2e^x\cos y - u - 2e^x\cos y = 0$ proved.

ii). $\frac{\partial e^u}{\partial x} = e^u\frac{du}{dx}$

$\frac{d^2 e^u}{dx^2} = \frac{d}{dx}\left(e^u\frac{du}{dx}\right)$

$= e^u\left(\frac{du}{dx}\right)^2 + e^u\left(\frac{d^2u}{dx^2}\right)$

Also $\frac{de^u}{dy} = e^u\frac{du}{dy}$

$\frac{d^2e^u}{dy^2} = \frac{d}{dy}\left(e^u\frac{du}{dy}\right)$

$$= e^u\left(\frac{du}{dy}\right)^2 + e^u\frac{d^2u}{dy^2}$$

$$\therefore \frac{d^2e^u}{dx^2}+\frac{d^2e^u}{dy^2} = e^u\left(\frac{du}{dx}\right)^2 + e^u\left(\frac{d^2u}{dx^2}\right) + e^u\left(\frac{du}{dy}\right)^2 + e^u\left(\frac{d^2u}{dy^2}\right)$$

$$\Rightarrow \left(\frac{d^2e^u}{dx^2}+\frac{d^2e^u}{dy^2}\right) = e^u\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right] + e^u\underbrace{\left[\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right]}_{\Rightarrow 0}$$

$$\Rightarrow \left[\frac{d^2e^u}{dx^2}+\frac{d^2e^u}{dy^2}\right]\frac{1}{e^u} = \frac{e^u}{e^u}\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right]$$

$$\Rightarrow e^{-u}\left[\frac{d^2e^u}{dx^2}+\frac{d^2e^u}{dy^2}\right] = \left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2$$

$$= \left[e^x(x\cos y - y\sin y + \cos y)\right]^2 + \left[e^x(-x\sin y - \sin y - y\cos y)\right]^2$$

$$= e^{2x}(x\cos y - y\sin y + \cos y)^2 + e^{2x}(-x\sin y - \sin y - y\cos y)^2$$

$$= e^{2x}\left[(x\cos y - y\sin y + \cos y)(x\cos y - y\sin y + \cos y) + (-x\sin y - \sin y - y\cos y)(-x\sin y - \sin y - y\cos y)\right]$$

$$= e^{2x}\left[x^2\cos^2 y - xy\sin y\cos y + x\cos^2 y - xy\sin y\cos y + y^2\sin^2 y - y\sin y\cos y + x\cos^2 y - y\sin y\cos y + \cos^2 y + x^2\sin^2 y + x\sin^2 y + xy\sin y\cos y + x\sin^2 y + \sin^2 y + y\sin y\cos y + xy\sin y\cos y + y\sin y\cos y + y^2\cos^2 y\right]$$

$$= e^{2x}\left[x^2\sin^2 y + y^2\cos^2 y + 2xy\sin y\cos y + 2y\sin y\cos y + 2x\sin^2 y + \sin^2 y + x^2\cos^2 y + y\sin^2 y - 2xy\sin y\cos y - 2y\sin y\cos y + 2x\cos^2 y + \cos^2 y\right]$$

$$= e^{2x}\left[x^2(\sin^2 y + \cos^2 y) + y^2(\sin^2 y + \cos^2 y) + 2x(\sin^2 y + \cos^2 y) + (\sin^2 y + \cos^2 y)\right]$$

$$= e^{2x}\left[x^2 + y^2 + 2x + 1\right] = e^{2x}\left(x^2 + 2x + 1 + y^2\right]$$

$$= e^{2x}\left[(x+1)^2 + y^2\right]$$

Proved.

TAYLOR'S SERIES

TAYLOR'S SERIES EXPANSION OF FUNCTIONS

Consider a convergence series $\sum_{n=0}^{\infty} A_n(x-x_0)^n$ which converges in non-zero interval about $x=x_0$ and let $f(x)=\sum A_n(x-x_0)^n$

$\therefore f(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots + \cdots$

if it is possible to differentiate $f(x)$ K-times at the point x_0, then we have

$$f(x_0) = A_0$$

$$f'(x_0) = A_1$$

$$f''(x_0) = 2A_2 + 3\cdot 2 A_3(x-x_0) + 4\cdot 3 A_4(x-x_0)^2 + \cdots$$

$$f''(x_0) = 2A_2 = 2!A_2$$

$$f'''(x_0) = 3\cdot 2\cdot 1 A_3 + 4\cdot 3\cdot 2\cdot A_4(x-x_0) + 5\cdot 4\cdot 3 A_5(x-x_0)^2 + \cdots$$

$$f'''(x_0) = 3\cdot 2\cdot 1 A_3$$

$$\therefore f^K(x_0) = K!\, A_K$$

$$\therefore A_K = \frac{f^K(x_0)}{K!}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)(x-x_0)^n}{n!}$$

We now state the Taylor's theorem

TAYLOR'S THEOREM

If $f(x)$ is a continuous, single valued function of x with continuous derivative $f'(x), f''(x)\cdots$ up to and including $f^n(x)$ in a given interval $a \leq x \leq b$ and if $f(x)$ exists in $a < x < b$ then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$$

$$f(a+h) = f(a) + h\frac{\partial f(a)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(a)}{\partial x^2} + \cdots + \frac{h^n}{n!}\frac{\partial^n f(a)}{\partial x^n} + \cdots$$

TAYLOR'S SERIES EXPANSION FOR FUNCTION OF TWO VARIABLES

Let $f(x,y)$ be a function of two variables, x, y and let ht be an increment in x and Kt an increment in y, in which case x, y, h and K are held temporarily constant. Thus $f(x,y)$ is now function of one variable t and $f(t)$ which can be ~~expand~~ expanded in Taylor's expansion at $t=0$

$$\therefore f(x) = \sum_{K=0}^{n} f^{K}(0)t^{K} + f^{n+1}(0)t^{n+1}$$

But $f(t) = f(x+ht, y+Kt)$.

$$\frac{df}{dt} = h\frac{\partial f}{\partial x} + K\frac{\partial f}{\partial y} = \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]f$$

By iteration

$$\frac{d^{n}f}{dx^{n}} = \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]^{n} f(x+ht, y+Kt).$$

At $t=0$

$$\frac{d^{n}f(0)}{dx^{n}} = \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]^{n} f(x,y)$$

$$E_n(T) = \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]^{n+1} f(x+h, y+K)$$

$$\therefore f(t) = f(x+ht, y+Kt)$$

$$\therefore f(x) = \sum_{K=0}^{n} \frac{\left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]}{n!} f(x+h)t^{k} + E_n$$

If we expan $f(t)$ to the second order terms then we have

$$E_n(x+h, y+h) = f(x,y) + \left(h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right) f(x,y) + \left[h\frac{\partial}{\partial x} + K\frac{\partial}{\partial y}\right]^{2} f(x,y) + E_n$$

$$= f(x,y) + h\frac{\partial f}{\partial x} + K\frac{\partial f}{\partial y} + \frac{1}{2}h^{2}\frac{\partial^{2} f}{\partial x^{2}} + hK\frac{\partial^{2} f}{\partial x \partial y} + \frac{K^{2}}{2}\frac{\partial^{2} f}{\partial y^{2}} + E_n$$

If this is evaluated at the point (a,b) then we have

$$f(a+h, b+K) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + K\frac{\partial f}{\partial y}(a,b) + \frac{h^{2}}{2}\frac{\partial^{2} f}{\partial x^{2}}(a,b) + hK\frac{\partial^{2} f}{\partial x \partial y}(a,b) + \frac{K^{2}}{2}\frac{\partial^{2} f}{\partial y^{2}}(a,b) + E_n$$

This is the Taylor series expansion of $f(x,y)$ for a function $f(x,y)$ defined in a region R in the ~~xy~~ xy plane and all it's partial derivatives of orders up to and including the $(n+1)$th are continuous in R.

Alternatively, we can let

$$x = a+h \Rightarrow h = x-a$$

$$y = b+K \Rightarrow K = y-b$$

$\therefore f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y} + (x-a)^2\frac{\partial^2 f}{\partial x^2} + (x-a)(y-b)\frac{\partial f}{\partial y}(a,b)$ (3)

$+ (y-b)^2\frac{\partial^2 f}{\partial y^2} + R_3$

Example I: Expand the function $f(x,y) = \sin xy$ about the point $(1, \pi/3)$ neglecting terms of degree three and higher

soln:

$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-a)\frac{\partial f}{\partial y}(a,b) + \frac{1}{2}(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b)$

$+ \frac{1}{2}(y-b)^2\frac{\partial^2 f}{\partial y^2} + R_3$.

$(a,b) = (1, \pi/3)$

$f(x,y) = \sin xy$

$\therefore f(a,b) = \sin \pi/3 = \sqrt{3}/2$

$\sin u \Rightarrow \frac{du}{} = \cos u$

$u = xy \quad \frac{du}{dx} = y$

$\frac{du}{dy} = x \qquad \Rightarrow \frac{\partial f}{\partial x} = \cos u \cdot y = y\cos xy$

$\frac{\partial f}{\partial x} = y\cos xy$

$\frac{\partial f}{\partial x}(a,b) = \pi/3 \cos \pi/3 = \pi/3 \cdot \frac{1}{2} = \pi/6$

$\frac{\partial f}{\partial y} = x\cos xy$

$\frac{\partial f}{\partial y}(a,b) = \cos \pi/3 = \frac{1}{3}$

$\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$

$\frac{\partial^2 f}{\partial x^2}(a,b) = -\left(\frac{\pi}{3}\right)^2 \sin \pi/3 = \frac{2}{9} \cdot \frac{\sqrt{3}}{2} = \frac{-\sqrt{3}\pi}{18}$

$\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$

$\frac{\partial^2 f}{\partial y^2}(a,b) = -\sin \pi/3 = -\frac{\sqrt{3}}{2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cos xy\right)$

$= -xy \sin xy + \cos xy$

$= \cos xy - xy \sin xy$.

$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \cos \pi/3 - \pi/3 \sin \pi/3$

$= \frac{1}{2} - \pi/3 \sqrt{3}/2$

$= \frac{1}{2} - \frac{\sqrt{3}\pi}{6} = \frac{3 - \sqrt{3}\pi}{6}$

$\therefore f(x,y) = \sin(x,y) = \frac{\sqrt{3}}{2} + \frac{\pi}{6}(x-1) + \frac{1}{3}(y-1) - \frac{1}{2}\frac{\sqrt{3}\pi}{18}(x-1)^2 + \frac{3-\sqrt{3}\pi}{6}(x-1)(y-\pi/3)$

$- \frac{1}{2}\frac{\sqrt{3}}{2}(y-\pi/3)^2$

2. Expand $f(x,y)=\sin x\cos y$ in a two variable power series at $(0,0)$ ignoring terms higher than degree 3. (4)

Soln: $f(x,y)=f(0,0)+\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2+\frac{\partial^2 f}{\partial x\partial y}(0,0)xy$
$+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2+R_3$

$f(x,y)=\sin x\cos y$

$\frac{\partial f}{\partial x}=\cos x\cos y$, $\frac{\partial f}{\partial x}(0,0)=1$

$\frac{\partial^2 f}{\partial x^2}=-\sin x\cos y$, $\frac{\partial^2 f}{\partial x^2}(0,0)=0$

$\frac{\partial f}{\partial y}=-\sin x\sin y$, $\frac{\partial f}{\partial y}(0,0)=0.$

$\frac{\partial^2 f}{\partial y^2}=-\sin x\cos y$, $\frac{\partial^2 f}{\partial y^2}(0,0)=0.$

$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left[-\sin x\sin y\right]$
$=-\cos x\sin y$

$\therefore \frac{\partial^2 f}{\partial x\partial y}(0,0)=0.$

$\therefore f(x,y)=0+x+0+0+0+0+R_3$
$=x+R_3$

3) Using Taylor's series for a function of two variables expand $f(x,y)=e^{xy}$ to three terms about the point $x=2, y=3$

Soln: $f(x,y)=e^{xy}$ $\Rightarrow f(2,3)=e^6$

$\frac{\partial f}{\partial x}=ye^{xy}$, $\frac{\partial f}{\partial x}(2,3)=3e^6$

$\frac{\partial f}{\partial y}=xe^{xy}$, $\frac{\partial f}{\partial y}(2,3)=2e^6$

$\frac{\partial^2 f}{\partial x^2}=y^2e^{xy}$, $\frac{\partial^2 f}{\partial x^2}(2,3)=9e^6$

$\frac{\partial^2 f}{\partial y^2}=x^2e^{xy}$, $\Rightarrow \frac{\partial^2 f}{\partial y^2}(2,3)=4e^6$

$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(xe^{xy}\right)$
$=xye^{xy}+e^{xy}$

$\therefore \frac{\partial^2 f}{\partial x\partial y}(2,3)=6e^6+e^6=7e^6$

$\therefore f(x,y)=e^6+3(x-2)e^6+2(y-3)e^6+\frac{9}{2}(x-2)^2e^6+7(x-2)(y-3)e^6+\frac{4}{2}(y-3)^2e^6+R_3$

$=\left[1+3(x-2)+2(y-3)+\frac{9}{2}(x-2)^2+7(x-2)(y-3)+2(y-3)^2\right]e^6+R_3$

APPLICATION OF TAYLOR'S THEOREM

Maximum and Minimum

Consider $f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2} + R_3$

$\Delta f(x+h, y+k) - f(x,y) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2} + R_3$

A function $f(x,y)$ is said to have relative maximum value at a Point $P(x_0, y_0)$ if $\Delta f(x_0+h, y_0+k) - f(x_0, y_0) < 0$

Similarly the function $f(x,y)$ is said to have a relative minimum at $P(x_0, y_0)$ if $\Delta f(x_0+h, y_0+k) - f(x_0, y_0) > 0$

Where h and k are sufficiently small permissible increment in x and y respectively. If a maximum and minimum occurs at (x_0, y_0) the curves lying in the plane $x = x_0$, $y = y_0$ must also have maximal or minimal at (x_0, y_0).

Consequently the tangents to each of these curves at (x_0, y_0) must be parallel to the Ox and Oy axes respectively. This implies that a necessary condition for f to have a relative maximum or minimum is

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

At all maximal and minimal

If h and k are sufficiently small, the terms involving degree two and higher will be sufficiently small and

$$\Delta f(x_0+h, y_0+k) - f(x_0, y_0) > 0$$

$$= h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0)$$

In this case the sign of $\Delta f(x_0, y_0)$ will be the same as the sign of $h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0)$ when this quantity is not zero. And the sign of the quantity will change as the sign of h and k change unless $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (x_0, y_0).

As pointed out earlier, if a maximum or minimum occurs at (x_0, y_0) the $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, $\frac{\partial f}{\partial y}(x_0, y_0) = 0$.

The solution of these equations gives the coordinates of points of possible maximum and minimum and also points called saddle points which we shall define later.

In general, we speak of the solutions of the above equations as giving the stationary or critical point of $f(x,y)$. In deciding the nature of the stationary point we now consider sign of f at (x_0,y_0).

From the Taylor's expansion, the sign of $f(x_0,y_0)$ is dependent on the values of h and k. At the stationary points

$$\Delta f = \frac{1}{2}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]$$

If we let

$$\left.\begin{aligned} A &= \frac{\partial^2 f}{\partial x^2} \\ B &= \frac{\partial^2 f}{\partial x\partial y} \\ C &= \frac{\partial^2 f}{\partial y^2} \end{aligned}\right\} \text{ at } (x_0,y_0) \text{ then}$$

$$\Delta f = \frac{1}{2}\left[Ah^2 + 2hkB + k^2C\right] = \frac{1}{2}A\left[\left(h^2+\frac{B}{A}\right)^2 + k^2\frac{C}{A} - \frac{k^2B^2}{A^2}\right]$$

$$= \frac{1}{2}A\left[\left(h^2+\frac{B}{A}\right)^2 - k^2\left(AC-B^2\right)\right]$$

Now if $A>0$ and $AC-B^2>0$, then $\Delta f>0$. This implies that the point (x_0,y_0) is a relative minimum. If $A<0$ and $AC-B^2>0$, then Δf depends on the values of h and k. In this case the stationary point is called a saddle point. Such a point is neither a maximum nor a minimum. $AC-B^2=0$, a more refined test will be required to determine the nature of a given stationary point.

We thus summarise as follows.

(i) At any stationary point,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ at } (x_0,y_0).$$

The solution of these equations give the critical values the coordinates of the stationary points

ii) For relative maximum to exist at (x_0,y_0).

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

iii) A relative minimum exists if

$$\frac{\partial^2 f}{\partial x^2} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

iv). A saddle point exists if

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2.$$

v) If $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$ more investigation is necessary to determine the nature of the stationary point.

Example: Find the stationary points of the surface $z = x^3 + xy + y^2$ (7)

Determine their nature

Soln: $z = x^3 + xy + y^2$

$$\frac{\partial z}{\partial x} = 3x^2 + y$$

$$\frac{\partial z}{\partial y} = x + 2y$$

At stationary points, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$.

$\therefore$ $3x^2 + y = 0$ ... (1)

$x + 2y = 0$ ... (2)

$\therefore$ $x = -2y$

Substituting for x in eqn (1)

$\Rightarrow$ $3(-2y)^2 + y = 0$

ie $3(4y^2) + y = 0$

$\therefore$ $12y^2 + y = 0$ — quadratic eqn

$\Rightarrow$ $y(12y+1) = 0$

$\therefore$ $y = 0$ or $y = -\frac{1}{12}$

Also $x = -2y$

when $y = 0$, $x = 0$

and $y = \frac{-1}{12}$

$\Rightarrow$ $x = -\cancel{2}\,\frac{-1}{\cancel{12}_{6}} = \frac{1}{6}$

$\therefore$ Stationary points are $(0,0)$ and $\left(\frac{1}{6}, \frac{-1}{12}\right)$.

To determine their nature:

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1$$

The point (0,0).

$$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} = 6x \times 2 = 6(0)\times 2 = 0$$

$$\therefore \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 1^2 = 1$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} < \left[\frac{\partial^2 z}{\partial x \partial y}\right]^2.$$

$\therefore$ (0,0) is a saddle point.

The point $\left(\frac{1}{6}, \frac{-1}{12}\right)$

$$\therefore \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} = \left(6\cdot\frac{1}{6}\right)\cdot(2) = 2$$

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 1^2 = 1$$

$$\therefore \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 6\cdot\frac{1}{6} = 1 > 0$$

$\therefore \left(\frac{1}{6}, -\frac{1}{12}\right)$ is minimum point.

Example!

Show that $f(x,y) = x^3 + y^3 - 2(x^2+y^2) + 3xy$ has stationary ~~Point~~ values at (0,0) and $\left[\frac{1}{3}, \frac{+1}{3}\right]$ and investigate their nature

Soln:

$$f(x,y) = x^3 + y^3 - 2(x^2+y^2) + 3xy$$

$$\frac{\partial f}{\partial x} = 3x^2 - 4x + 3y$$

$$\frac{\partial f}{\partial y} = 3y^2 - 4y + 3x$$

If (0,0), $\left(\frac{1}{3}, \frac{+1}{3}\right)$ are stationary values then

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial f}{\partial x}\left(\frac{1}{3}, \frac{+1}{3}\right) = \frac{\partial f}{\partial y}\left(\frac{1}{3}, \frac{+1}{3}\right) = 0$$

$\Rightarrow$ At $\left(\frac{1}{3}, \frac{1}{3}\right)$

$\frac{\partial f}{\partial x} = 3\left(\frac{1}{3}\right)^2 - 4\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)$ $\qquad \frac{\partial f}{\partial y} = 3\left(\frac{1}{3}\right)^2 - 4\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)$

$= 3\left(\frac{1}{9}\right) - 4\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)$ $\qquad = 0$.

$= \frac{1}{3} - \frac{4}{3} + 1 = 0$.

$\frac{\partial^2 f}{\partial x^2} = 6x - 4$ $\quad$ at $(0,0) \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6(0) - 4 = -4$

$\frac{\partial^2 f}{\partial y^2} = 6y - 4$ $\quad$ & $\frac{\partial^2 f}{\partial y^2} = 6(0) - 4 = -4$

$\frac{\partial^2 f}{\partial x \partial y} = 3$ $\quad$ at $(0,0)$, $\frac{\partial^2 f}{\partial x \partial y} = 3 \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 9$.

$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = -4 \cdot -4 = 16$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

At $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\frac{\partial^2 f}{\partial x^2} = 6\left(\frac{1}{3}\right) - 4 = 2 - 4 = -2$ & $\frac{\partial^2 f}{\partial y^2} = 6\left(\frac{1}{3}\right) - 4 = -2$.

$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = -2 \cdot -2 = 4$

& $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 3^2 = 9$

$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$\therefore \left(\frac{1}{3}, \frac{1}{3}\right)$ is a saddle point

Example: Find the stationary points of the function and determine their nature $f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$

Soln: $f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$

$\frac{\partial f}{\partial x} = 4x^3 + 8xy^2 - 4x$

$\frac{\partial f}{\partial y} = 8x^2y + 4y$.

At stationary Points $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. (10)

$$\frac{\partial f}{\partial x} = 4x^3 + 8xy^2 - 4x = 0$$
$$= 4x(x^2 + 2y^2 - 1) = 0$$

$\Rightarrow 4x = 0$ or $x^2 + 2y^2 - 1 = 0 \quad \ldots (1)$

$\Rightarrow x = 0$ or

$$\frac{\partial f}{\partial y} = 8x^2y + 4y = 0$$
$$= 4y(2x^2 + 1) = 0$$

$\Rightarrow 4y = 0$ or $2x^2 + 1 = 0 \quad \ldots (2)$

$y = 0$

In equation (2), since x and y are real, $y = 0$ substuting in (1)

We have $x^2 - 1 = 0 \Rightarrow x = \pm 1$

Therefore the stationary points are $(0,0), (-1,0), (1,0)$.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 8y^2 - 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = 16xy$$

$$\frac{\partial^2 f}{\partial y^2} = 8x^2 + 4$$

To test their nature

point $(0,0)$

$$\frac{\partial^2 f}{\partial x^2} = -4 < 0, \quad \frac{\partial^2 f}{\partial y^2} = 4$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = -4 \cdot 4 = -16$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$\therefore (0,0)$ is saddle point

At point $(-1,0)$.

(11)

$$\frac{\partial^2 f}{\partial x^2} = 12(x^2) + \cancel{8y^2} - 4 \cancel{- 8((-1)^2) + 4 = 8+4}$$

$$= 12(-1)^2 + 8(0) - 4 = 12 - 4 = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 8x^2 + 4 = 8(-1)^2 + 4 = 8 + 4 = 12$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = 96$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (16xy)^2 = [16(-1)(0)]^2 = (0)^2 = 0$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$\therefore$ ~~$\Rightarrow$~~ $(1,0)$ is a minimum point

point $(1,0)$

$$\frac{\partial^2 f}{\partial x^2} = 12(1)^2 + 8(0) - 4 = 8 > 0$$

$$\frac{\partial^2 f}{\partial y^2} = 8(1)^2 + 4 = 12.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6(1)(0) = 0$$

$$\therefore \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

$\therefore$ $(1,0)$ is a minimum point

LANGRANGIAN MULTIPLIER

Situations may occur in which a function f to be maximized or minimized depends upon variables which are not dependent but are interrelated by one or more constraint conditions. For example we can be expected to find the stationary points of $f(x, y, z)$ where x, y, z are related by an equation $(x, y, z) =$ constant. In some problems it could be possible to use the given equation and the constraint to eliminate one variable. In which case, we now have equation in two variables which we can easily find its maximum or minimum or saddle point. For example in finding the stationary points of $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 =$ constant c, we can eliminate z by the substitution $z^2 = c - x^2 - y^2$. In which case, we now find the maximum point of $f(x, y, z) = x^2y^2(c - x^2 - y^2)$

However, not all problems could easily be treated explicit in this manner. Langrange developed a method which introduces the use of a multiplier called Langrangian multiplier or Undetermined multipliers.

Let's assume we are seeking the maximum or minimum point of $f(x, y, z)$. Subject to the constraint $g(x, y, z) = \phi$, We find the differential of f and g. Since $\phi =$ constant, we get $dg = 0$. We put $df = 0$ because we want to find maximum or minimum points of f.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$$

We then multiply dg equation by λ (this is the determined multiplier) and add it to the df equation. Hence we have

$$\left[\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right]dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left[\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right]dz$$

We choose such that

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad \ldots \quad (1)$$

This implies also that

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad \ldots \quad (2)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \quad \ldots \quad (3)$$

We solve these three equations together with $g(x,y,z) =$ constant to determine the values of x, y, z and λ.

Example I : Find the volume of the largest Parallelepiped (that is box with edges parallel to the axes inscribed in the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Let the point (x,y,z) be the corner in the first octant where the box touches the ellipsoid. Then (x,y,z) satisfies the ellipsoid equation and the volume of the box is $8xyz$ (since there are 8 octants).

Hence we are to maximize $f(x,y,z) = 8xyz$ subject to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$df = 8yz\,dx + 8xz\,dy + 8xy\,dz$$

$$dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz$$

We now solve

$$8yz + \lambda \frac{2x}{a^2} = 0 \quad \ldots \quad (1)$$

$$8xz + \lambda \frac{2y}{b^2} = 0 \quad \ldots \quad (2)$$

$$8xy + \lambda \frac{2z}{c^2} = 0 \quad \ldots \quad (3)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \cdots \quad (4)$$ (4)

multiply eqns (1), (2) and (3) by x, y and z respectively and add the three equations

$$8xyz + 2\lambda \frac{x^2}{a^2} = 0$$

$$8xyz + 2\lambda \frac{y^2}{b^2} = 0$$

$$8xyz + 2\lambda \frac{z^2}{c^2} = 0$$

$$\Rightarrow 8xyz + 2\lambda \frac{x^2}{a^2} + 8xyz + 2\lambda \frac{y^2}{b^2} + 8xyz + 2\lambda \frac{z^2}{c^2} = 0$$

$$= 24xyz + 2\lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0,$$

$$= 12xyz + \lambda = 0$$

$$\lambda = -12xyz$$

Using this in eqn (1)

$$8yz - \frac{24x^2}{a^2}yz = 0$$

$x \neq 0$, $y \neq 0$, $z \neq 0$ since the edges of the box is not at the origin.

$$\therefore 8 - \frac{24x^2}{a^2} = 0$$

$$x^2 = \frac{a^2}{3} \quad \Rightarrow \quad x = \frac{a}{\sqrt{3}}$$

Similarly, we can solve for

$$y^2 = \frac{b^2}{3} \quad \Rightarrow \quad y = \frac{b}{\sqrt{3}}$$

$$z^2 = \frac{c^2}{3}, \quad z = \frac{c}{\sqrt{3}}$$

$\therefore$ The maximum Volume $= 8xyz$

$$= \frac{8abc}{3\sqrt{3}}$$

There are times when we are faced with more than one constraint; this is overcome by introducing more than one Langrangian constants. For example, if we are to maximize $f(x,y,z)$ subject to the constraints $g_1 =$ constant and $g_2(x,y,z) =$ constant, we introduce λ_1 and λ_2 and solve the equations

$$\frac{\partial f}{\partial x} + \lambda_1 \frac{\partial g_1}{\partial x} + \lambda_2 \frac{\partial g_2}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda_1 \frac{\partial g_1}{\partial y} + \lambda_2 \frac{\partial g_2}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda_1 \frac{\partial g_1}{\partial z} + \lambda_2 \frac{\partial g_2}{\partial z} = 0$$

$g_1(x,y,z) =$ constant

$g_2(x,y,z) =$ constant

Eg. find the minimum distance from the origin to the Inter Section $xy = 12$ with $x + 2z = 0$.

The distance from the origin is given by $\sqrt{x^2+y^2+z^2}$ so we are to minimize $f(x,y,z) = x^2+y^2+z^2$ subject to $xy = 12$, $x+2z=0$

$df = 2x\,dx + 2y\,dy + 2z\,dz$

$dg_2 = dx + 2dz$

Introducing λ_1 and λ_2 we have

$2x + \lambda_1 y + \lambda_2 = 0$ ——— (1)

$2y + \lambda_1 x + \cancel{\lambda_2} = 0$ ——— (2)

$2z + 2\lambda_2 = 0$ ——— (3)

Also $xy = 12$ ——— (4)

$x + 2z = 0$ ——— (5)

From (5) $x = -2z$

(3) $\lambda_2 = -z$

(2) $\lambda_1 = -\frac{2y}{x}$

Eliminating λ_1 and λ_2 in eqn (1)

$$\Rightarrow 2x - \frac{2y^2}{x} - z = 0$$

$$\therefore 2x^2 - 2y^2 - xz = 0$$

But $z = -\frac{x}{2}$

$$\Rightarrow 2x^2 - 2y^2 + \frac{x^2}{2}$$

$$4x^2 - 4y^2 + x^2$$

$$= 5x^2 - 4y^2 = 0$$

Also $y = \frac{12}{x}$

$$\therefore 5x^2 - 4\left(\frac{12}{x}\right)^2 = 0$$

$$= 5x^2 - 4 \cdot \frac{(12)^2}{x^2} = 0$$

$$\therefore 5x^4 - 576 = 0$$

$$x^4 - \frac{576}{5}$$

$$x = -2\sqrt[4]{\frac{36}{5}}$$

$$z = -\frac{x}{2} = \pm 4\sqrt{\frac{36}{5}}$$

$$y = \frac{12}{x} = \frac{12}{2}\left(\frac{36}{5}\right).$$

$$= 6\left(\sqrt[4]{\frac{36}{5}}\right)^{-1} = 6\sqrt[4]{\frac{36}{5}}.$$

$\therefore$ The minimum distance

$$d = \sqrt{x^2+y^2+z^2}$$

$$d^2 = \sqrt[4]{\frac{36}{5}} + 36\sqrt{\frac{5}{36}} + \sqrt{\frac{36}{5}}$$

$$d^2 = \sqrt[4]{\frac{36}{5}} + 36\sqrt{\frac{5}{36}} + \sqrt{\frac{36}{5}}$$

$$= 5\sqrt{\frac{36}{5}} + 36\sqrt{\frac{5}{36}}$$

$$= 30\sqrt{\frac{1}{5}} + \frac{36}{6}\sqrt{5}$$

$$= \frac{30\sqrt{5}}{5} + 6\sqrt{5} \Rightarrow d^2 = \sqrt{12\sqrt{5}}$$

2) Find the shortest distance from the origin to $x^2 - y^2 = 1$ (17)

The distance from the origin to any point is given as $d^2 = x^2 + y^2$.

Hence we are to minimize the function

$f(x,y) = x^2 + y^2$ subject to constraint $x^2 - y^2 = 1$

Soln:

$df = 2x\,dx + 2y\,dy$

$g = x^2 - y^2$

$dg = 2x\,dx - 2y\,dy$

Introducing the Langrangian multiplier

$\therefore 2x + 2x\lambda = 0$

$\Rightarrow 2x(1+\lambda) = 0$ ——— (1)

$2y - 2y\lambda = 0$

$\Rightarrow 2y(1-\lambda) = 0$ ——— (2)

But $x^2 - y^2 = 1$ ——— (3)

From (1) $x = 0$ or $\lambda = -1$

In (3) Since x and y are real, $x \neq 0$

$\therefore \lambda = -1$

In (2) we have $y = 0$

From (3) if $y = 0$, $x = \pm 1$

Therefore the stationary points are $(-1, 0), (1, 0)$

Now $d^2 = x^2 + y^2 = 1 + 0 = 1$

$\therefore d = 1$.

3) The temperature in a rectangular plate bounded by $x=0$, $y=0$, $x=3$ and $y=5$ is $T = xy^2 - x^2y + 1000$

Find the hottest and coldest points of the plate. We first set the partial derivatives to zero to find the critical points.

Soln: $\dfrac{\partial T}{\partial x} = y^2 - 2xy = 0$

$\Rightarrow y = 0$

$\frac{\partial T}{\partial y} = 2xy - x^2 = 0$

$\rightarrow x = 0$

Hence the solution $x = 0, y = 0$

$\therefore T = 100$

This is at one corner of the plate

We next try to see the temperature of the other corners.

These corners are $(3,0), (3,5), (0,5)$.

At $(3,0)$, $T = 100$

At $(0,5)$, $T = 100$

At $(3,5)$, $T = 3(5)^2 - 3^2(5) + 100$
$= 3(25) - 9(5) + 100 = 130$

We then also consider the entire boundary of the plate

Along $x = 0$, $T = 0$

Along $y = 0$, $T = 0$

Along $x = 3$, $T = 3y^2 - 9y + 100$

We find whether it is a maximum or minimum

$T = 3y^2 - 9y + 100$

$\frac{\partial T}{\partial y} = 6y - 9 = 0$

$\therefore 6y - 9 = 0$

$y = 9/6 = 3/2$.

$\therefore$ At point $\left(3, \frac{3}{2}\right)$

$T = 3\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) + 100$
$= 3\left(\frac{9}{4}\right) - 9\left(\frac{3}{2}\right) + 100 = 93\frac{1}{4}$

Along $y = 5$

~~$T = 3(5)^2 -$~~ $T = 5^2(x) - 5x^2 + 100$

$T = 25x - 5x^2 + 100$

$\frac{\partial T}{\partial x} = 25 - 10x = 0$

$\therefore x = 25/10 = 5/2$.

At Point $(5/2, 5)$.

$T = \frac{5}{2} \cdot 25 - \frac{25}{4} + 100$

$= 131\frac{1}{4}$

Comparing all the results we see that the hottest point is $\left(\frac{5}{2}, 5\right)$ and temperature is

$T = 131\frac{1}{4}$

The coldest point is $\left(3, \frac{3}{2}\right)$ with $T = 93\frac{1}{4}$.