2:50PM. 21st August 2024

Complex Variables MTH218

A complex number is the ~~set of negati~~

root of negative integers i.e $\sqrt{x}$

Let $\sqrt{-1} = i \Rightarrow i^2 = -1$

$$\sqrt{-2} = \sqrt{-1} \cdot \sqrt{2} = i\sqrt{2} = 1.414i$$

Let $Z = x + iy$

$$2 = 2 + 0i$$

$$1 = 1 + 0i$$

Complex Plaime

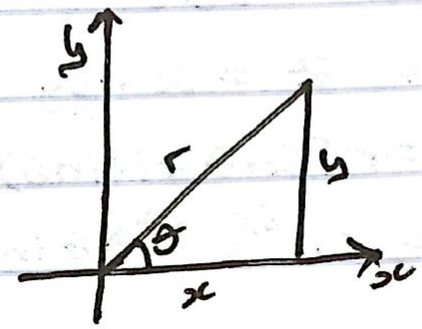


Fig 1: Argand diagram.

$\therefore r^2 = x^2 + y^2$ (Pythagoras)

$$Z = x + iy$$

$|z| = \sqrt{x^2+y^2} = r$ – modulus of a complex number

From ⊛

$$r = \sqrt{x^2+y^2}$$

Example : Find the modulus of $Z = 2+3i$

Solution: $|z| = \sqrt{2^2+3^2} = \sqrt{13}$

Operation 1 : Find the modulus of a complex number.

21st August 2024

Find the modulus of

(i) 1 (ii) $2i$

Solution

(i) $|z| = |1| = |1 + 0i| = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = \sqrt{1} = 1$

(ii) $z = 0 + 2i$

$|z| = \sqrt{0^2+2^2} = \sqrt{4} = 2$

2:58PM. Forms of Complex Numbers.

1. Cartesian Form $\Rightarrow z = x + iy$
2. Polar form $\Rightarrow z = r(\cos\theta + i\sin\theta)$
3. Euler form $\Rightarrow z = re^{ix}$ $i\theta$ θ in radians.

Polar Coordinates

(r, θ)

modulus argument

From the complex plane drawn previously

Recall: $\sin\theta = \frac{y}{r} \Leftrightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Leftrightarrow x = r\cos\theta$

But $z = x + iy$ - cartesian form

$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

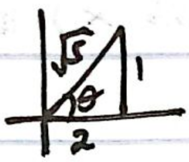
21st August 2024

But $180° = \pi$ radians

Example 2: Convert the following complex numbers to polar forms: (i) $2+i$ (ii) $2+2i$ (iii) i (iv) 2 (v) $\frac{1}{2}+i\frac{1}{2}$

Solution

(i)



No $\tan\theta = \frac{1}{2} = 0.5$

$\therefore \theta = \tan^{-1}(0.5) = 26.57°$

$z = \sqrt{5}(\cos 26.57° + i\sin 26.57°)$

(ii) $z = 2+2i \Rightarrow \theta = 45°,\ r = \sqrt{8}$

$\therefore z = \sqrt{8}(\cos 45° + i\sin 45°)$

Do 3 - 4 at home.

Example: Convert $2+2i$ to Euler form.

$180° = \pi$ radian

$45° = \frac{45\pi}{180} = \pi/4$

$\Rightarrow \underline{z = \sqrt{8}\, e^{i\pi/4}}$

Try $z = 3+3i$ on your own. to Euler form.

$1 = \frac{\mathcal{E}}{\mathcal{E}} =$ [illegible]

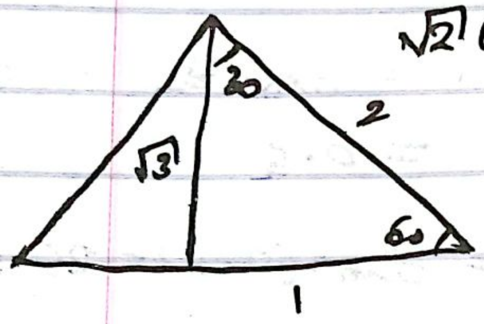
i.e $\theta = 45° = \pi/4$ radians

21st August 2024

Example 4: Convert $\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ to cartesian form.

Solution.

$$\sqrt{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$$



$$\sqrt{2}(0.866 + i0.5) = 0.866\sqrt{2} + i0.5\sqrt{2}$$

$$= 0.866 \times 1.414 + i0.5 \times 1.44$$

$\cos 30 = \frac{\sqrt{3}}{2}$

$\sin 30 = \frac{1}{2}$

hw

Convert $z = 5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\sqrt{2}}{4}\right)$ to cartesian form:

Solution

$$z = 5(\cos 135^\circ + i\sin 135^\circ)$$

$$= 5\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{5\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2}$$

Equality of Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

Then $z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$

Examples: Given that $z_1 = 3 + 2i$, $z_2 = \frac{m}{3} + i\frac{n}{4}$

21st August 2024

Then find m and n if $z_1 = z_2$

Solution.

$z_1 = z_2$ iff $3 = \frac{m}{3}$ and $2 = \frac{n}{4}$

$m = 9$ and $n = 8$.

- Find θ and r if $z_1 = 6 + 2i$ and $z_2 = r(\cos\theta + i\sin\theta)$ are equal.

$6 = r\cos\theta$

$2 = r\sin\theta$

$3 = \frac{1}{\tan\theta}$

$\theta = 18.43$

$r = \frac{6}{\cos 18.43}$

$r = 6.33$

Alternatively.

$|z_1| = \sqrt{6^2+2^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10} = r$

Thus: $r = 2\sqrt{10}$ or 6.33

$\theta = 18.43$

~~Conjugate of co~~

Conjugate of Complex Numbers.

Example: The conjugate of $z = x + iy$ is $\bar{z} = x - iy$

21st August 2024

Note $|z| = |\bar{z}|$

Find (i) $z + \bar{z}$ (ii) $z - \bar{z}$ (iii) $z \cdot \bar{z}$ (iv) $\frac{z}{\bar{z}}$

Solution.

(i) $z = x + iy \quad \bar{z} = x - iy$

$z + \bar{z} = 2x + iy - iy = 2x$

(ii) $z - \bar{z} = x + iy - (x - iy)$

$\Rightarrow x - x + iy + iy = 2iy$

(iii) $z \cdot \bar{z} = (x + iy)(x - iy)$

$= x^2 - ixy + ixy + y^2$

$= x^2 + y^2 = r^2 = |z|^2$

(iv) $\frac{z}{\bar{z}} = \frac{x+iy}{x-iy}$

$= \frac{(x+iy)(x+iy)}{(x-iy)(x+iy)}$

$= \frac{x^2 + 2(x)(iy) + i^2y^2}{x^2+y^2}$

$= \frac{x^2 + 2ixy - y^2}{x^2+y^2} = \frac{z^2}{r^2} = \frac{z^2}{|z|^2}$

Example 6:

Find $\frac{3+2i}{2-i} = \cancel{\frac{9+12i-4}{4+1}} = \cancel{\frac{15+12i}{5}}$

$= \frac{3+2i}{2-i} \times \frac{2+i}{2+i} = \frac{6+3i+4i-2}{4+1}$

21st August 2024

$$= \frac{4i + 7i}{5}$$

Put result in cartesian form or $z = a+bi$ or given form

$$z = \frac{4}{5} + \frac{7}{5}i$$