MFM5052-Risk Analysis

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Lessons 04-06

Outline

1. Definitions
2. Types of Financial Risks
3. Fixed income Instruments
4. Redington's Conditions
5. Yield curve Smoothing
6. Interest rate Models

Risk Management

Risk management is the discipline that clearly shows management the risks and returns of every major strategic decision at both the institutional level and the transaction level. Moreover, the risk management discipline shows how to change strategy in order to bring the risk return trade-off into line with the best long- and short-term interests of the institution.

Above definition applies across all the following institutions

- Pension funds
- Insurance companies
- Industrial corporations
- Commercial banks
- Cooperative financial institutions like savings banks
- Securities firms
- National government treasuries
- Foundations and charities

Types of Financial Risks

- Interest Rate Risk
- Market Risk
- Credit Risk
- Operational Risk
- ....

Fixed income Instruments

- Pricing
- Yield/ARR
- Duration
- Convexity

Redington's Conditions

1. Present Value of Assets equals to the present values of liabilities
2. Duration of the assets equals the duration of the liabilities
3. Convexity of the assets is greater than the convexity of the liabilities

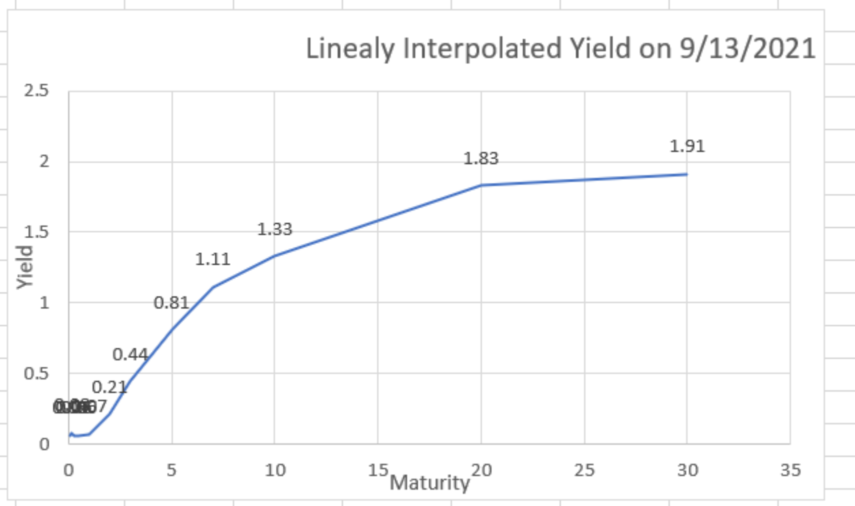
Yield curve Smoothing

- Linear Interpolation
- Cubic Splines
- Nelson-Siegel Method

Yield Data

Date	9/13/2021
Maturity	Rate
IM	0.06
2M	0.08
3M	0.06
6M	0.06
1Y	0.07
2Y	0.21
3y	0.44
5Y	0.81
7y	1.11
10Y	1.33
20Y	1.83
30Y	1.91

Linear Yield



Cubic Splines Interpolation

Definition 1

A cubic spline is a piece wise cubic function that interpolates a set of data points and guarantees smoothness at the data points.

Let us consider $n+1$ points $(x_i, y_i), \quad i = 1 \dots n+1$. Then there are $n-1$ number of interior intervals and in each interval approximate the function by a cubic polynomial

$$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \;\; x \in [x_i, x_{i+1}], \; i = 1 \dots n$$

The cubics need to match the function values y_i at the knots

$$\begin{aligned} f_1(x_1) &= y_1 \\ f_i(x_{i+1}) &= f_{i+1}(x_{i+1}) = y_{i+1}, \;\; i = 1 \dots n-1 \\ f_n(x_{n+1}) &= y_{n+1} \end{aligned}$$

Which are in total $2n$ equations.

Continuity of the derivatives

First Order Derivative

$$f_i'(x_{i+1}) = f_{i+1}'(x_{i+1}) \;\; i = 1 \dots n-1$$

$n-1$ equations
Second Order Derivative

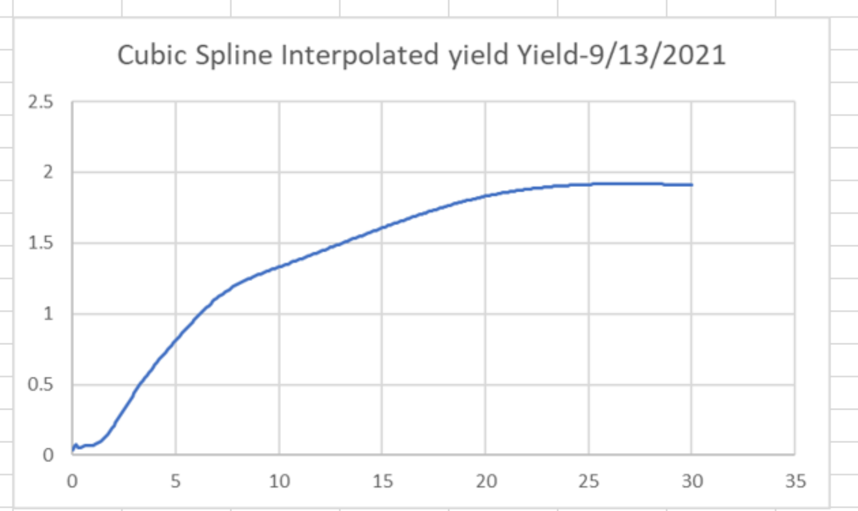
$$f"_i(x_{i+1}) = f"_{i+1}(x_{i+1}) \;\; i = 1 \dots n-1$$

$n-1$ equations For the n intervals there are $4n$ unknowns and $4n-2$ equations. We need another two equations. The natural choice is

$$f"_1(x_1) = 0, \;\; f"_n(x_{n+1}) = 0$$

We now have $4n$ equations and $4n$ unknowns.

Cubic Spline Interpolated Yield



Nelson-Siegel Model

Forward rate is given by

$$f(\tau) = \beta_0 + \beta_1 e^{-\frac{\tau}{\lambda}} + \beta_2 \frac{\tau}{\lambda} e^{-\frac{\tau}{\lambda}}$$

Where $\beta_0, \beta_1, \lambda$ are parameters with $\lambda > 0$.
Then the spot rate is given by

$$r(\tau) = \frac{1}{\tau} \int_0^{\tau} f(u) du$$

which is equal to

$$r(\tau) = \beta_0 + \beta_1 \frac{\lambda}{\tau} \left(1 - e^{-\frac{\tau}{\lambda}}\right) + \beta_2 \left(\frac{\lambda}{\tau} \left(1 - e^{-\frac{\tau}{\lambda}}\right) - e^{-\frac{\tau}{\lambda}}\right)$$

Brownian Motion

Brownian motion is described by the winner process which has the following properties

- $W_0 = 0$
- W_t has independent increments.
- $W_t - W_s \sim N(0, t-s), \; 0 \leq s \leq t$

One can easily simulate the winner process.

Vasicek Model

Mean reversion short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, t > s, \quad r(s) = r_s$$

- θ-long term mean of the short rate.
- κ - Mean reversion speed of the short rate.

Also called Ornstein-Ulembeck process and the solution is given by

$$r(T) = e^{-\kappa(T-s)}r_s + \theta\left(1 - e^{-\kappa(T-s)}\right) + \sigma \int_s^T e^{-\kappa(T-t)}dW_t$$

$$r(T) \sim N\left(e^{-\kappa(T-s)}r_s + \theta\left(1 - e^{-\kappa(T-s)}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-s)}\right)\right)$$

ZCB price under Vasicek Model

$$P(s,T) = E^Q\left(e^{-\int_s^T r_t dt}\right)$$

Following steps can be used to compute the ZCP under Vasicek Model

1. Compute expectation of $\int_s^T r_t dt$
2. Compute Variance of $\int_s^T r_t dt$
3. use the fact that

$$E^Q(e^x) = e^{E(x)+\frac{1}{2}V(x)}$$

$$E\left(\int_s^T r_t dt/F_s\right) = (r_s - \theta)B(s,T) + \theta(T - S)$$

$$Var\left(\int_s^T r_t dt/F_s\right) = \frac{\sigma^2}{\kappa^2}\left((T-s) - B(s,T) - \frac{\kappa}{2}B(s,T)^2\right)$$

where

$$B(s,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-s)}\right)$$

Then

$$P(s, T) = A(s, T)e^{-r(s)B(s,T)}$$