Painless Stochastic Gradient: Interpolation, Line-Search, and Convergence Rates

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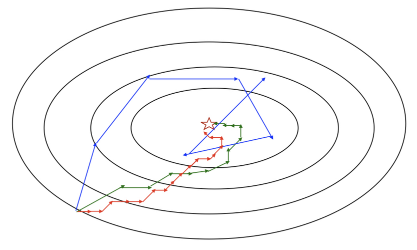
Motivation

Stochastic Gradient Descent (SGD) is the most common optimization method, but can be painful:

- Slower rate of convergence.
- Need to carefully tune the step-size.



Contributions

Strong theoretical results and good empirical performance of SGD for over-parametrized models without tuning the step-size.

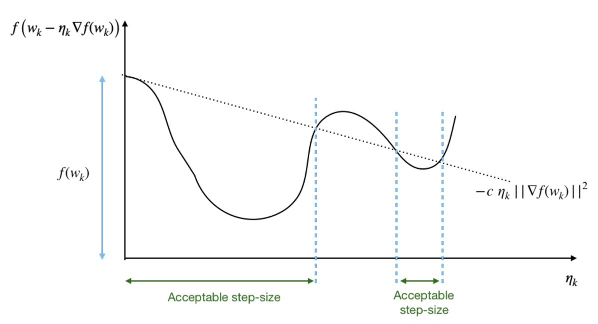
- Use line-search methods to automatically set the step-size when training over-parametrized models that can interpolate the data.
- Prove that SGD with a stochastic Armijo line-search attains the fast convergence rates of full-batch gradient descent in the interpolation setting for convex and strongly-convex functions.
- Prove that a stochastic extra-gradient method with a Lipschitz line-search attains fast convergence rates for an important class of non-convex functions and saddle-point problems satisfying interpolation.
- Compare against numerous optimization methods for standard classification tasks using both kernel methods and deep networks.

General Setup

- **Objective**: Find $w^* = \arg\min f(w) \triangleq \frac{1}{n}\sum_{i=1}^{n} f_i(w)$.
- **Technical assumptions**: Lower bounded by f^*, L-smoothness.
- **Interpolation**: If $\nabla f(w^*) = 0$, then for all f_i, $\nabla f_i(w^*) = 0$.
- **Strong growth condition (ρ-SGC)**: $\mathbb{E}_i \|\nabla f_i(w)\|^2 \leq \rho \|\nabla f(w)\|^2$

SGD + Armijo line-search

- **SGD update**: $w_{k+1} = w_k - \eta_k \nabla f_{ik}(w_k)$.
- **Stochastic Armijo condition**: Choose η_k s.t. $f_{ik}(w_k - \eta_k \nabla f_{ik}(w_k)) \leq f_{ik}(w_k) - c\,\eta_k \|\nabla f_{ik}(w_k)\|^2$.
- Use back-tracking line-search to choose a step-size that satisfies the above condition.



Lemma (Lower-bound on step-size)

The step-size η_k returned by the Armijo line-search and constrained to lie in the $(0, \eta_{max}]$ range satisfies the following inequality,

$$\eta_k \geq \min\left\{\frac{2\,(1-c)}{L_{ik}}, \eta_{max}\right\}.$$

Here L_{ik} is the Lipschitz constant of ∇f_{ik}.

Convergence of SGD with Armijo line-search

Theorem (Strongly-Convex)

Assuming (a) interpolation, (b) L_i-smoothness and (c) convexity of f_i's and (d) μ strong-convexity of f, SGD with Armijo line-search with $c = 1/2$ achieves the rate:

$$\mathbb{E}\left[\|w_T - w^*\|^2\right] \leq \max\left\{\left(1 - \frac{\bar{\mu}}{L_{max}}\right), (1 - \bar{\mu}\,\eta_{max})\right\}^T \|w_0 - w^*\|^2.$$

Here $\bar{\mu} = \sum_{i=1}^{n} \mu_i / n$ is the average strong-convexity of the finite sum and $L_{max} = \max_i L_i$ is the maximum smoothness constant in the f_i's.

Convergence of SGD with Armijo line-search

Theorem (Convex)

Assuming (a) interpolation, (b) L_i-smoothness and (c) convexity of f_i's, SGD with Armijo line-search for all $c \geq 1/2$ and iterate averaging achieves the rate:

$$\mathbb{E}\left[f(\bar{w}_T) - f(w^*)\right] \leq \frac{c \cdot \max\left\{\frac{L_{max}}{2\,(1-c)}, \frac{1}{\eta_{max}}\right\}}{(2c-1)\;T} \|w_0 - w^*\|^2.$$

Here, $\bar{w}_T = \frac{\left[\sum_{i=1}^{T} w_i\right]}{T}$ is the averaged iterate after T iterations and $L_{max} = \max_i L_i$.

Theorem (Non-convex)

Assuming (a) the SGC with constant ρ and (b) L_i-smoothness of f_i's, SGD with Armijo line-search with $c = 1/2$ and setting $\eta_{max} = \frac{3}{2\rho L}$ achieves the rate:

$$\min_{k=0,\dots,T-1} \mathbb{E}\,\|\nabla f(w_k)\|^2 \leq \frac{4\;L_{max}}{T}\left(\frac{2\rho}{3} + 1\right)(f(w_0) - f(w^*))$$

Algorithm and Practical considerations

Algorithm 1 SGD+Armijo(f, w_0, η_{max}, b, c, β, γ, opt)

- **for** $k = 0, \dots, T$ **do**
  - $i_k \leftarrow$ sample mini-batch of size b
  - $\eta \leftarrow \text{reset}(\eta, \eta_{max}, \gamma, b, k, \text{opt})/\beta$
  - **repeat**
    - $\eta \leftarrow \beta \cdot \eta$
    - $\tilde{w}_k \leftarrow w_k - \eta \nabla f_{ik}(w_k)$
  - **until** $f_{ik}(\tilde{w}_k) \leq f_{ik}(w_k) - c \cdot \eta \|\nabla f_{ik}(w_k)\|^2$
  - $w_{k+1} \leftarrow \tilde{w}_k$
- **end for**
- **return** w_{k+1}

Algorithm 2 reset(η, η_{max}, γ, b, k, opt)

- **if** k = 1 **then**
  - **return** η_{max}
- **else if** opt = 0 **then**
  - $\eta \leftarrow \eta$
- **else if** opt = 1 **then**
  - $\eta \leftarrow \eta_{max}$
- **else if** opt = 2 **then**
  - $\eta \leftarrow \eta \cdot \gamma^{b/n}$
- **end if**
- **return** η

- To allow the step-size to increase, we (i) reset the step-size to a larger value in each iteration (Algorithm 2) or (ii) use an alternative Goldstein condition.
- For faster convergence, we consider accelerated variants using momentum.

Stochastic Extra-Gradient + Lipschitz line-search

- **SEG update**: $w_k' = w_k - \eta_k \nabla f_{ik}(w_k)$, $w_{k+1} = w_k - \eta_k \nabla f_{ik}(w_k')$
- **Lipschitz line-search condition**: Choose η_k such that: $\|\nabla f_{ik}(w_k - \eta_k \nabla f_{ik}(w_k)) - \nabla f_{ik}(w_k)\| \leq c\;\|\nabla f_{ik}(w_k)\|$.
- The step-size returned by the Lipschitz line-search satisfies $\eta_k \geq \min\left\{\frac{c}{L_{ik}}, \eta_{max}\right\}$.

Theorem (Non-convex + RSI)

Assuming (a) interpolation, (b) L_i-smoothness, and (c) μ_i-RSI of f_i's, SEG with Lipschitz line-search with $c = 1/4$ and $\eta_{max} \leq \min_i 1/4\mu_i$ achieves the rate:

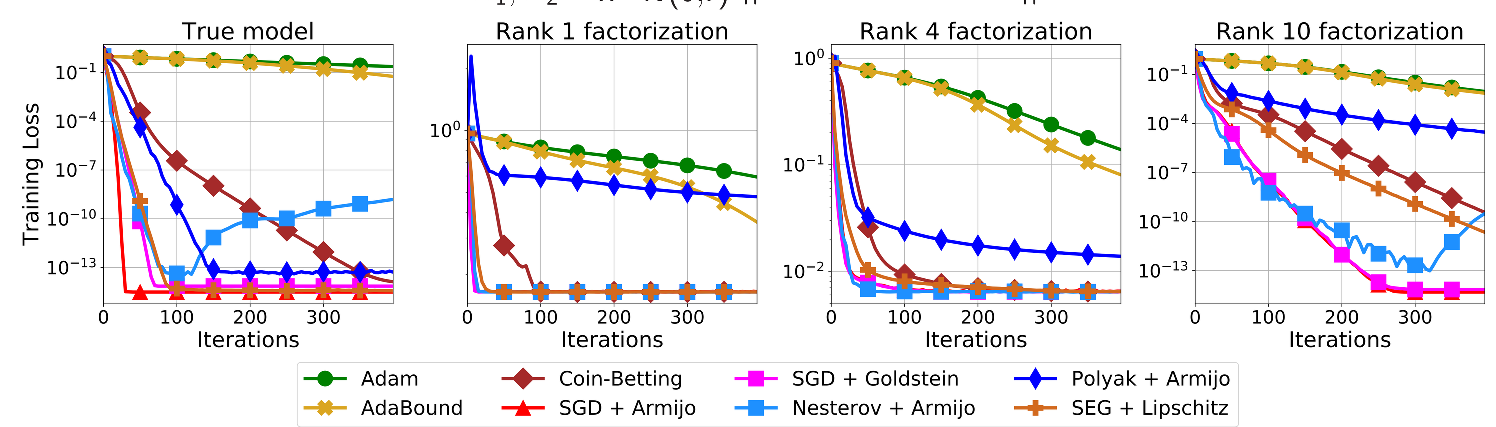
$$\mathbb{E}\left[\|w_T - \mathcal{P}_{\mathcal{X}^*}[w_T]\|^2\right] \leq \max\left\{\left(1 - \frac{\bar{\mu}}{4\;L_{max}}\right), (1 - \eta_{max}\,\bar{\mu})\right\}^T \|w_0 - \mathcal{P}_{\mathcal{X}^*}[w_0]\|^2,$$

where $\bar{\mu} = \frac{\sum_{i=1}^{n}\mu_i}{n}$ is the average RSI constant of the finite sum and $\mathcal{X}^*$ is the non-empty set of optimal solutions. $\mathcal{P}_{\mathcal{X}^*}[w]$ denotes the projection of w onto $\mathcal{X}^*$.

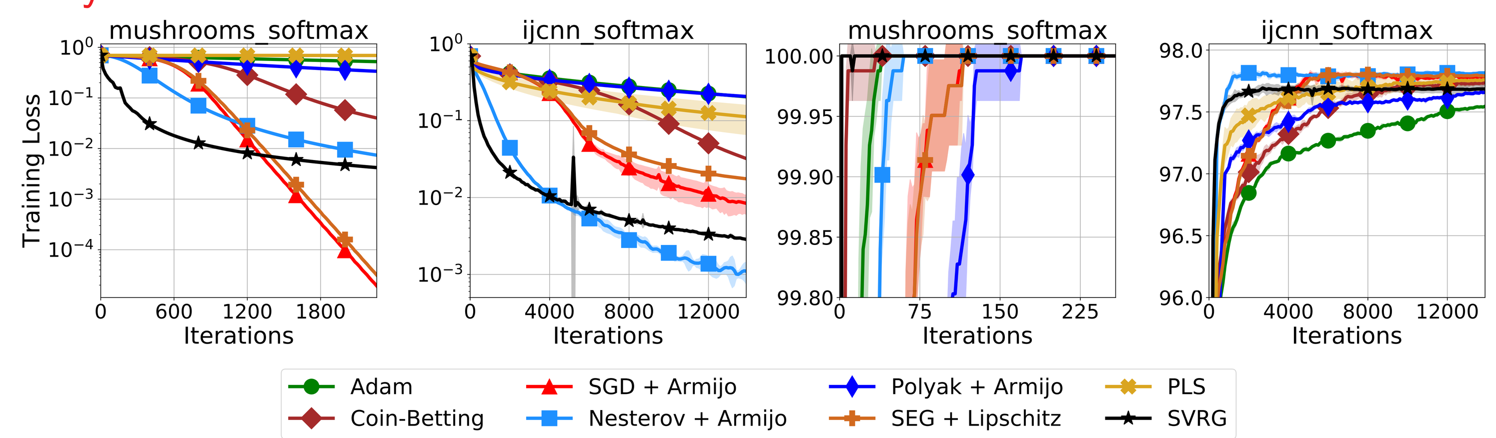
- Derive an $O(1/T)$ rate when minimizing convex functions.
- Derive linear convergence rates for strongly-convex strongly-concave and bilinear saddle point problems satisfying interpolation.

Experiments https://github.com/IssamLaradji/sls

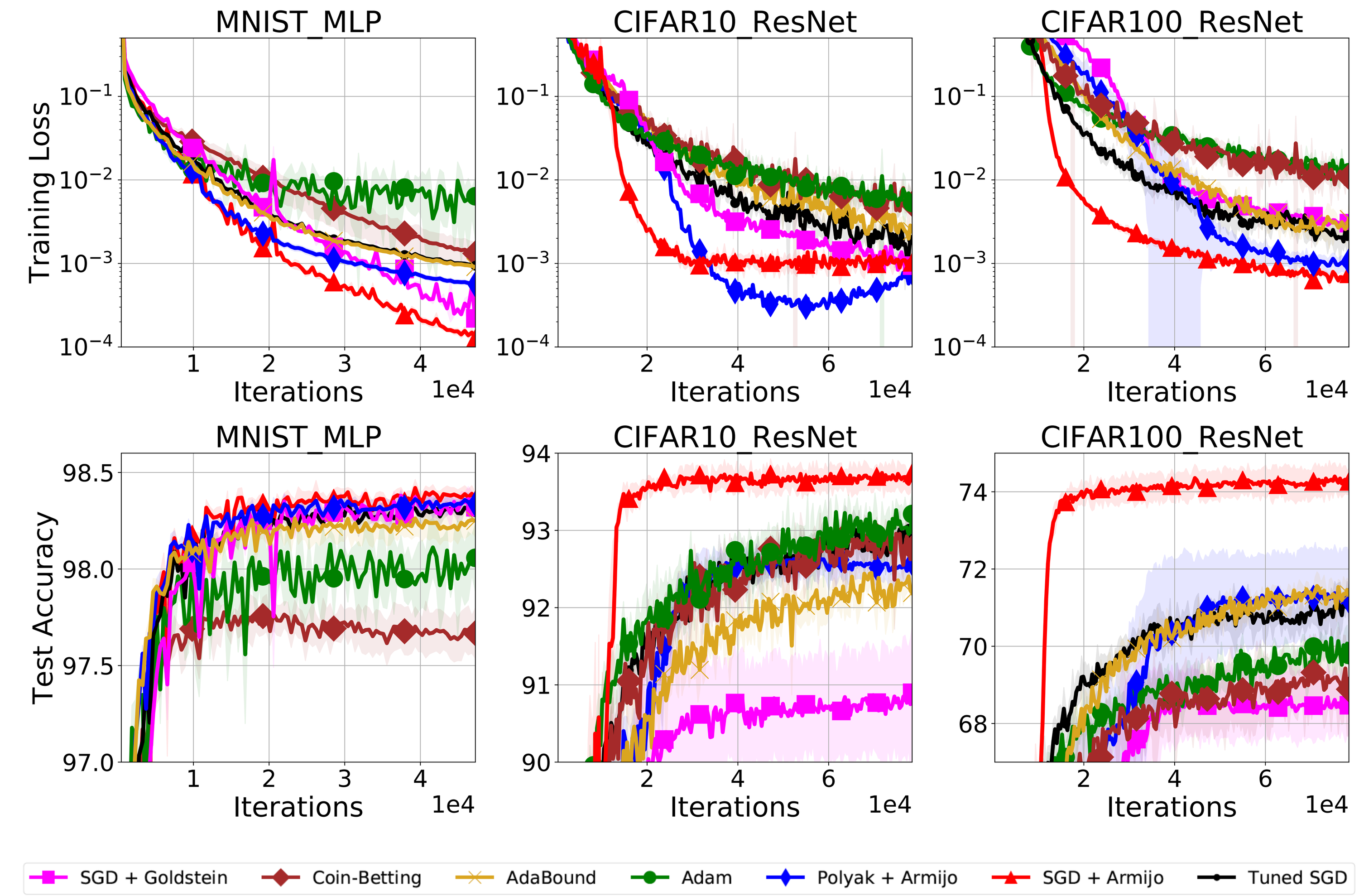
- Matrix Factorization: $\min_{W_1, W_2} \mathbb{E}_{x \sim N(0,I)} \|W_2 W_1 x - Ax\|^2$



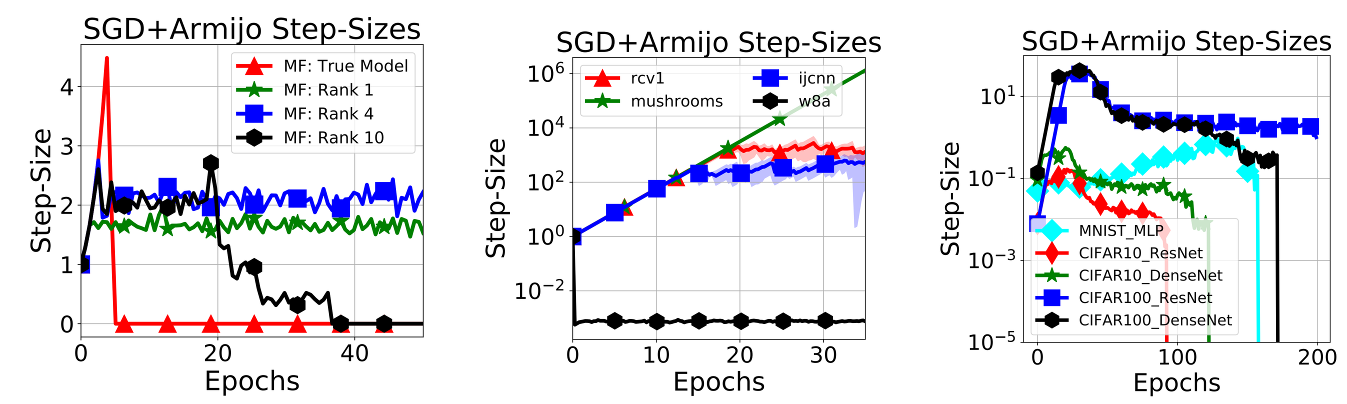
- Binary classification with RBF kernels



- Multi-class classification with deep networks



- Step-size variation for SGD with Armijo line-search



- Runtimes