Quantum measurement theory

Projection & sampling noise

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Qubit example

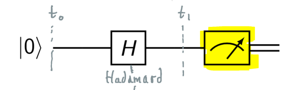
Pauli Operators

$$\hat{I} \to \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & 1 & 0 \\ |1\rangle & 0 & 1 \end{array}, \quad \hat{X} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{Y} \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{Z} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hat{Z} = 1\,|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1|$

$\hat{Z}|0\rangle = (+1)|0\rangle$

$\hat{Z}|1\rangle = (-1)|1\rangle$



$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

$\langle \hat{X} \rangle = \langle \psi | \hat{X} | \psi \rangle = 0$

$\langle \hat{Y} \rangle = \langle \psi | \hat{Y} | \psi \rangle = 0$

$\langle \hat{Z} \rangle = \langle \psi | \hat{Z} | \psi \rangle = 1$

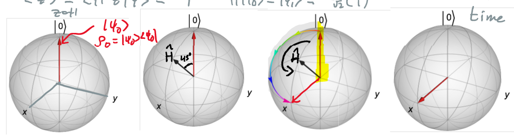
$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{X} + \hat{Z})$

$\hat{H}|\psi_0\rangle = |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

time t_1

$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\rho_1 := |\psi_1\rangle\langle\psi_1| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(\hat{I} + \hat{Z})$



Deterministic state evolution

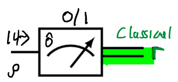
Measurement: probabilistic

The standard (von Neumann) measurement of a quantum system.

von Neumann measurement is efficient, strong, and projective

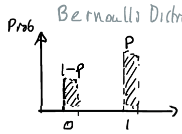
Q: Is the qubit in the 1 state or not?

result: $\begin{cases} r = 0 \to |0\rangle \\ r = 1 \to |1\rangle \end{cases}$ classical variable



$\hat{O} = \sum_{r \in \{0,1\}} r\,|r\rangle\langle r|$ $\qquad r \in \{0, 1\}$

Bernoulli Distribution



$\hat{\Pi}_r = |r\rangle\langle r|$

$\Pi_0 = |0\rangle\langle 0|$

$\Pi_1 = |1\rangle\langle 1|$

$I = \Pi_0 + \Pi_1$ $\qquad$ resolution of the identity

$\hat{O} = 0\,|0\rangle\langle 0| + 1\,|1\rangle\langle 1| = |1\rangle\langle 1| = \frac{I - Z}{2}$

result $r=0$: $\Pr(r=0) = |\langle 0|\psi\rangle|^2 = \mathrm{Tr}(|0\rangle\langle 0|\,\rho) = \mathrm{Tr}(\hat{\Pi}_0 \rho) =$

result $r=1$: $\Pr(r=1) = |\langle 1|\psi\rangle|^2 = \mathrm{Tr}(|1\rangle\langle 1|\,\rho) = \mathrm{Tr}(\hat{\Pi}_1 \rho) =$

Projection noise

$\underbrace{\mathbb{E}[r]}_{\text{classical}}$ classical expect. classical

$\mathbb{E}[r] = \sum_r r\,\mathbb{P}(r) = \sum_r r\,\mathrm{Tr}(|r\rangle\langle r|\,\rho)$

$= \mathrm{Tr}\left(\sum_r r\,|r\rangle\langle r|\,\rho\right)$

$= \mathrm{Tr}(\hat{O}\rho)$

$= \underbrace{\langle\psi|\hat{O}|\psi\rangle}_{\text{Quantum}}$ $\qquad$ for pure state.

$\mathbb{E}[r^2] = \sum_r r^2\,\mathbb{P}(r) = \mathrm{Tr}(\hat{O}^2\rho)$

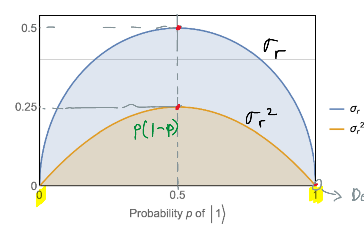
$\mathbb{V}[r] = \mathbb{E}[(r - \mathbb{E}[r])^2] = \mathbb{E}[r^2] - \mathbb{E}[r]^2$

For $\hat{O} = |1\rangle\langle 1|$

$\mathbb{E}[r] = \mathrm{Tr}(|1\rangle\langle 1|\,\rho) = p$ $\qquad$ Prob to be excited

$\mathbb{E}[r^2] = \mathrm{Tr}[|1\rangle\langle 1|\,\rho) = p$

$\mathbb{V}[r] = p - p^2 = p(1-p)$ $\qquad =: \sigma_r^2$



Random coin toss

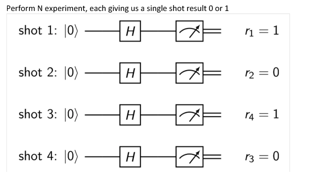
Bernoulli distribution

Parameters determined by quantum algorithm

Projection noise and sampling error

Let's turn to the example of finite number of shots we execute for our experiment.

Perform N experiment, each giving us a single shot result 0 or 1



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Sample: $\{r_1, r_2, r_3, \ldots r_N\}$

$r_n \in \{0, 1\}$ $\qquad n \in \{1, 2, \ldots N\}$

$r_n \sim$ i.i.d. $B(p = \mathrm{Tr}(\hat{O}\rho))$

For the k-th circuit $\rho \to \mathrm{Tr}(|1\rangle\langle 1|\rho) = p$

$N = 3$ For example with 3 samples 2^3 possible sequences

000	100
001	101
010	110
011	111

$\{r_1, r_2, r_3\}$

2^N result sequences

Sample mean

$$S = \frac{1}{N}\sum_{n=1}^{N} r_n$$

r_n ← random variable

S ↑ random variable

Recall