Script 5

Theorem 5.1: The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.

Let's prove this by contradiction. Assume there exists a non-empty subset $A \subset C$ such that A is both open and closed and $A \neq C$.

Because A is open and closed and nonempty, its complement $C \backslash A$ is open and nonempty by *Definition 4.7* which states that a set is open if its complement is closed. We also know that because they are complements, $A \cap (C \backslash A) = \emptyset$ which means they are disjoint as they contain no similar elements. This means we have two nonempty, open sets which contradicts the definition of connected set of *Definition 4.22* because C can now be represented as the union of two non-empty disjoint open sets, thus making C disconnected which contradicts the fact that C is connected thus our assumption is false and therefore through proof by contradiction the only subsets of continuum C that are open and closed are $\emptyset$ and C.

Corollary 5.2: Every region is infinite.

Assume, for the sake of contradiction, that there exists a finite region R denoted as $\underline{ab}$, where $a, b \in C$ and $a < b$. According to this assumption, the elements in the region can be arranged in a finite sequence, due to *Theorem 3.5*. This sequence is of the form $a < x_1 < x_2 < \ldots < x_n < b$, with n elements in total where $n \in \mathbb{N}$.

However, this assumption leads to a contradiction. According to this sequence, there are no elements between x_1 and x_2. This directly contradicts *Theorem 4.23*, which states that in a connected continuum (which a continuum must be by *Axiom 4*), for any two elements $x_1, x_2 \in C$ with $x_1 < x_2$, there must exist another element $z \in C$ that lies between x_1 and x_2 such that there are an infinite number of points between any two points which is in direct contrast with the cardinality n specified by *Theorem 3.5* for a finite region. Thus our assumption that R is a finite region is false, and every region must be infinite.

Corollary 5.3: Every point of C is a limit point of C.

Assume, for contradiction, that there exists a point $x \in C$ which is not a limit point of C. This assumption implies that there is a region R containing x, where $x \in R \subseteq C$, and R satisfies the condition: $R \cap C \setminus \{x\} = \emptyset$. In simpler terms, besides x, the region R does not contain any other points of C.

However, this assumption leads to a contradiction when we consider *Corollary 5.2*, which we previously proved. *Corollary 5.2* states that every region in C is infinite. Therefore, it's impossible for the region R to contain only the point x if it's a part of C; R must contain infinitely many points.

Thus, our original assumption, that R contains only x and no other points from C, contradicts the fact established in *Corollary 5.2*. Therefore, $R \cap C \setminus \{x\}$ cannot be empty, implying that x must be a limit point of C. Consequently, our initial assumption is false, and it follows that every point in C is indeed a limit point of C.

Corollary 5.4: Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.

Suppose, for the sake of contradiction, that there exists a point x in the region $\underline{ab}$ which is not a limit point of $\underline{ab}$. This means that we can find a region R containing x, where $x \in R \subseteq C$, and this region R satisfies the condition: $R \cap \underline{ab} \setminus \{x\} = \emptyset$. In other words, R does not contain any points from $\underline{ab}$ other than x.

However, this assumption leads to a contradiction. Note that $R \cap \underline{ab}$ is itself a region within C. According to *Corollary 5.2*, every region within C is infinite. This implies that the region $R \cap \underline{ab}$ must contain infinitely many points.

Thus, if $R \cap \underline{ab}$ is infinite, then $R \cap \underline{ab} \setminus \{x\}$ cannot be empty. There must be other points in $R \cap \underline{ab}$ besides x. This contradicts our initial assumption that $R \cap \underline{ab}$ contains only x. Therefore, our assumption that x is not a limit point of $\underline{ab}$ is false. It follows that every point in the region $\underline{ab}$ must be a limit point of $\underline{ab}$.
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Exercise 5.7: If sup X exists, then it is unique, and similarly for inf X.

Assume that sup X exists and suppose for the sake of contradiction there are at least two distinct least upper bounds, call them u and u' where $u \neq u'$.

By *definition 5.6*, a supremum u must be an upper bound of X and for any upper bound u' of X, $u \leq u'$. This is the same for u' to be a supremum as well, it must be less than or equal to u. Thus they both must be upper bounds, more importantly, $u' \leq u$ and $u \leq u$ which means $u = u'$. However, this contradicts our assumption that there are two distinct, equal least upper bounds. Therefore, if sup X exists, then it must be unique.

This same logic can equally be applied to inf X with reverse comparative signage.

Exercise 5.8: If X has a first point L, then inf X exists and equals L. Similarly, if X has a last point U, then sup X exists and equals U.

Assume that X has a first point L such that $L < x$ where x for all $x \in X$ which means L is a lower bound of X by *definition 5.5*. To show that L is a greatest least bound of X, we must show that:

1. L is a lower bound of X and
2. If L' is any lower bound of X, then $L' \leq L$.

The first condition is met by L being the first point by *Definition 3.3* as a first point L of a set X means $L \leq x$ for all x in X. This satisfies the requirement for L to be a lower bound of X, as it is less than or equal to every element in X.

As for the second condition, Suppose there exists another lower bound L' of X. By the definition of a lower bound, $L' \leq x$ for all $x \in X$. Since L is the first point of X, it is the smallest element in X. Therefore, for any lower bound L', it must hold that $L' \leq L$. This is because L' cannot be greater than the smallest element of X and still be a lower bound. This satisfies the condition that L is greater than or equal to every other lower bound of X. Therefore, by satisfying these two conditions, L is the greatest least bound, or the infimum, of X.

Sup X can be proved by *definition 3.3* for the last point in a similar manner.

Exercise 5.9: For this exercise, we assume that $C = \mathbb{R}$. Find sup X and inf X for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$
    i. sup X does not exist because the set of natural numbers has no upper bound as it goes on indefinitely.
    ii. inf $X = 1$ since 1 is the first point of the set of natural numbers, thus it is inf X by *exercise 5.8*.
2. $X = \mathbb{Q}$
    i. neither sup nor inf exist for X because the set of rational numbers extend infinitely in both positive and negative directions such that there are no upper nor lower bounds.
3. $X = \{\frac{1}{n} | n \in N\}$
    i. sup X = 1 because 1 is the upper bound such that there will never be a number in the set X that is greater than 1
    ii. inf X is 0, though 0 is not in the set, it fulfills the requirements because it is a lower bound which the elements of X get close to 0 as n increases.
4. $X = \{x \in \mathbb{R} | 0 < x < 1\}$
    i. sup X = 1, even though 1 is not in the set, no element in the set X is greater than 1 such that 1 is an upper bound
    ii. inf $X = 0, 0$ is the greatest lower bound for X, even though it's not part of the set since all elements are greater than 0.

5. $X = \{3\} \cup \{x \in \mathbb{R} | -7 \leq x \leq -5\}$
   i. sup $X = 3$, since 3 is the largest number in the union of the two sets.
   ii. inf $X = -7$, because it's the smallest number in the combined set and is included in the set X.

Lemma 5.10: Suppose that X is a nonempty subset of C and $s =$ sup X exists. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$.

Assume that X is a nonempty subset of C and that $s = \sup X$ exists. Let p be any element in C such that $p < s$. For the sake of contradiction, suppose that there does not exist an $x \in X$ such that $p < x \leq s$. This means that for all $x \in X$, $x \leq p$ or $x > s$. However, $x > s$ is impossible by *Definition 5.5* because s is the least upper bound of X.

If $x \leq p$ for all $x \in X$, then p would be an upper bound for X by *Definition 5.5*, which contradicts the fact that s is the least upper bound of X and $p < s$ by *Definition 5.6*. This contradiction implies that our assumption is false. Therefore, there must exist an $x \in X$ such that $p < x \leq s$ if $p < s$

Theorem 5.11: Let $a < b$. The least upper bound and greatest lower bound of the region $\underline{ab}$ are: sup $\underline{ab}$ = b and inf $\underline{ab}$ = a.

Let $a < b$. The theorem states that the least upper bound and greatest lower bound of the region $\underline{ab}$ are $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.

First we want to prove that $\inf \underline{ab} = a$.

To do this, we first show that a is a lower bound. By *Definition 5.5*, a lower bound for a set is an element that is less than every element in the set. Since $\underline{ab}$ consists of points x such that $a < x < b$, it follows that for all $x \in \underline{ab}$, $a < x$. Therefore, a is a lower bound of $\underline{ab}$.

Next, we need to show that a is the Greatest Lower Bound which we can demonstrate by contradiction. Suppose there exists some element $l > a$ that is also a lower bound of $\underline{ab}$ (such that a is not the greatest lower bound). By *Theorem 4.23*, the continuum is connected, meaning for any two points, there exists another point between them. Therefore, there exists a point c in the region $\underline{al}$ such that $a < c < l$. Since c is an element of $\underline{ab}$ and $c > a$, this contradicts the assumption that l is a lower bound of $\underline{ab}$. Hence, a is the greatest lower bound, or $a = \inf \underline{ab}$.

Now we want to prove that $\sup \underline{ab} = b$.

Similar for inf, we want show b is an Upper Bound. By definition, an upper bound for a set is an element that is greater than every element in the set. Since $\underline{ab}$ consists of points x such that $a < x < b$, it follows that for all $x \in \underline{ab}$, $x < b$. Therefore, b is an upper bound of $\underline{ab}$.

Next, we need to show b is the Least Upper Bound which we can demonstrate by contradiction. Suppose there exists some element $u < b$ that is also an upper bound of $\underline{ab}$. By the same connectedness argument of *Theorem 4.23*, there would exist a point c in the region $\underline{ub}$ such that $u < c < b$. Since c is an element of $\underline{ab}$ and $c < b$, this contradicts the assumption that u is an upper bound of $\underline{ab}$. Hence, b is the least upper bound, or $\sup \underline{ab}$.

Thus, we have proven that $\sup \underline{ab} = b$ and $\inf \underline{ab} = a.\backslash$

Theorem 5.12: Let X be a nonempty subset of C. Suppose that sup X exists and sup $X \notin X$. Then sup X is a limit point of X. The same holds for inf X.

Consider $s = \sup X$. Since X is nonempty, there is at least one element p in the continuum such that $p < s$. According to *Axiom 3*, which states that there is no last point and *Definition 5.6* which states that there must exist an element c such that $c > s$, we can define a region $\underline{pc}$ that includes s. For any such region, there exists an x in X where $x \neq s$ (since $s \notin X$) and $p < x \leq s$, as stated by *Lemma 5.10*; thus for any region $\underline{pc}$ containing x, the set $\underline{pc} \cup x \backslash \{s\}$ is nonempty, confirming that $\sup X$ is a limit point of X. The same reasoning applies to the infimum of X, showing that if the infimum exists and is not an element of X, it too is a limit point of X.

Corollary 5.13: Both a and b are limit points of the region $\underline{ab}$. Let $[a, b]$ denote the closure $\overline{ab}$ of the region $\underline{ab}$.

We know that $a =$ inf $\underline{ab}$ and $b =$ sup $\underline{ab}$ by *Theorem 5.11*, and since $a \notin \underline{ab}$ and $b \notin \underline{ab}$, then by *Theorem 5.12*, $a, b \in$ LP$(\underline{ab})$.

Corollary 5.14: $[a, b] = x \in C | a \leq x \leq b$**.**

The set $[a, b]$ is defined as the closure $\overline{ab}$ as proven by *Corollary 5.13*. Thus by definition of a closure in *Definition 4.4* which says a closure is the union of the set and its limit points, $[a, b]$ must include both the elements of the region $\underline{ab}$ such that any element x of closure $[a, b]$ must be between a and b where $a < x < b$ by *Definition 3.9* and its limit points which are a and b such that x can also equal a or b, once again as described by *Corollary 5.13*. Additionally, we know that there are no other limit points outside of the region $\underline{ab}$ by *Lemma 3.16* which states that there is no point of the exterior of a region that can be a limit point of that region; thus, there are no other limit points outside of the closure. This means $[a, b]$ is exactly the set of all points x in C such that $a \leq x \leq b$.

Lemma 5.15: Let $X \subset C$ and define: $\Psi(X) = \{x \in C | x \text{ is not an upper bound of } X\}$. Then $\Psi(X)$ is open. Define: $\Omega(X) = \{x \in C | x \text{ is not a lower bound of } X\}$.Then $\Omega(X)$ is open.

Let $X \subset C$ and define $\Psi(X) = \{x \in C | x \text{ is not an upper bound of } X\}$. We want to prove that $\Psi(X)$ is open which we can accomplish by showing that the complement of $\Psi(X)$, which consists of all upper bounds of X, is closed.

First, consider a point x in the set of limit points of the complement of $\Psi(X)$, denoted as $x \in LP(C \backslash \Psi(X))$. This indicates that x is a limit point of the set of upper bounds of X. Now, for the sake of contradiction, suppose $x \notin C \backslash \Psi(X)$. This implies that x is not an upper bound of X. Consequently, there must exist some $y \in X$ such that $x < y$, which exists because x cannot be greater than or equal to every element in X since it is not an upper bound and because the continuum is connected and has no last point.

Given that the continuum C is connected and there is no first point, there exists a point $z \in X$ such that $z < x$. This allows us to consider the region $\underline{zy}$, which includes x. However, the region $\underline{zx}$, when intersecting $C \backslash \Psi(X)$ and excluding x itself, must be empty. This emptiness arises because all points in $C \backslash \Psi(X)$ are upper bounds of X. By *Definition 5.5*, these points must be greater than y, and hence, no point in $\underline{zx}$ (other than x) can be an upper bound of X.

Therefore, x must be in the complement of $\Psi(X)$ such that x is in the set of upper bounds of X which contradicts our original assumption. Because the complement of $\Psi(X)$ contains its limit points, then the complement of $\Psi(X)$ is closed. Following the definition of an open set, this implies that $\Psi(X)$ is open.

The same reasoning can be applied to $\Omega(X) = \{x \in C | x \text{ is not a lower bound of } X\}$ to show that it is open as well.

Theorem 5.16 (Least Upper Bound Property): Suppose that X is nonempty and bounded above. Then sup X exists. Similarly, if X is nonempty and bounded below, then inf X exists.
Class notes

We will prove this by contradiction. Assume for the sake of contradiction that X is nonempty and bounded above and that the sup$X$ does not exist. We define $A = C \backslash \Psi(X)$ is the set of upper bounds of X. Because X is bounded above, then $A \neq \emptyset$, and since we know that sup(X) does not exist, then for all $y \in A$, we can say that y is not the least upper bound of X. Then, we can say there is some $z \in A$ where $z < y$. Because $z \in A$, then $z > x$ for all $x \in X$. then, for any point $a \in C$ where $a > z$, then $a > x$ for all $x \in X$.

1. For any point a in the continuum where $a > z$, then $a \in A$ because it must be in the upper bounds of X by *Definition 5.5* because it is larger than all $x \in X$. Because the continuum has no last point by *Axiom 3*, then we know that there exists a point u such that $u > y$. We define the region $\underline{zu}$ such that $z < y < u$ thus $y \in \underline{[zu]}$. For all points $n \in \underline{zu}$, we know that $n > z$. By line 1, we know that for all $n \in \underline{zu}$ thus $n \in A$. Then $y \in \underline{zu}$ is a subset of A. Therfore, A is open by *Theorem 4.9*.

2. A is nonempty and open. We can say that $\Psi(X)$ is open by *Lemma 5.15*. Because X is nonempty, then we can find some $w \in X$ then there exists some $d < w$ by *Axiom 3* stating that a continuum has no first point. Therefore, d is not an upper bound of X. Now, we can state that $\Psi(X)$ is nonempty.
3. Since we know that $\Psi(X)$ is nonempty and open, $A = C \backslash \Psi(X)$ then $A \cap \Psi(X) = \emptyset$, then A and $\Psi(X)$ are disjoint. Because A and $\Psi(X)$ are empty, open, and disjoint, then $C = A \cap \Psi(X)$ and C is the union of nonempty, open, and disjoint sets thus C is disconnected by *Definition 4.22* and this is a contradiction.

Corollary 5.17: Every nonempty closed and bounded set has a first point and a last point.

We will use a proof by contradiction to demonstrate this. Let's assume there exists a set X that is nonempty, closed, and bounded, but, for the sake of contradiction, suppose X does not have a first or a last point. Given that X is bounded, by *Theorem 5.16*, it has a supremum (denoted as sup$X$) and an infimum (inf$X$).

First, consider the supremum of X. The last point of a set X is defined as some element $x \in X$ such that x is greater than or equal to any other element a in X. If sup$X$ is an element of X, then it would be the largest element in X, satisfying the condition sup$X \geq x$ for all $x \in X$. This would make sup X the last point of X, contradicting our assumption. On the other hand, if sup$X$ is not an element of X, then by *Theorem 5.12*, sup$X$ must be a limit point of X. However, this would imply that X does not contain all of its limit points, contradicting the assumption that X is closed.

Therefore, X must have a last point. A similar line of reasoning can be applied to show that X must also have a first point. Hence, our initial assumption leads to a contradiction, and we conclude that every nonempty, closed, and bounded set indeed has a first point and a last point.

Exercise 5.18: Is this (Corollary 5.17) true for $\mathbb{Q}$?
Class notes

No, Corollary 5.17 does not hold for $\mathbb{Q}$ (the set of rational numbers). Let's consider the set X defined as:

$$X = \{x \in \mathbb{Q} | x^2 < 2\}$$

First, observe that X is bounded above and below in $\mathbb{Q}$. For example, $x = 5$ is an upper bound as it is greater than every element of X, and similarly, $x = -5$ is a lower bound as it is smaller than every element of X.

To establish that X does not have a first or last point, we demonstrate that for every rational number $x \in X$, there is another number $y \in X$ such that $x < y < z$, where z is the positive number (not necessarily rational) satisfying $z^2 = 2$.

Starting with the fact that $x^2 < 2$, we follow these steps:

1. $x^2 + 2x < 2x + 2$
2. $x(x + 2) < 2x + 2$
3. $x < \frac{2x+2}{x+2}$

Therefore, if $x^2 < 2$, then $\frac{2x+2}{x+2} > x$.

Next, we want to show that $\left(\frac{2x+2}{x+2}\right)^2 < 2$ so that it belongs to X:

1. $x^2 < 2$ implies $2x^2 < 4$
2. $2x^2 + 8x + 2x^2 + 4 < 4 + 8x + 2x^2 + 4$
3. $4x^2 + 8x + 4 < 2x^2 + 8x + 8$
4. $4x^2 + 8x + 4 < 2(x^2 + 4x + 4)$
5. $\frac{4x^2+8x+4}{x^2+4x+4} < 2$

Therefore, $\left(\frac{2x+2}{x+2}\right)^2 < 2$.

Consequently, for all $x \in X$, there exists some $y \in X$ such that $y = \frac{2x+2}{x+2}$ and $x < y < z$, where $z^2 = 2$ and $z > 0$. Thus, the set X has no last point.

Similarly, for all $x \in X$, there exists some $a \in X$ such that $a = \frac{-2x+2}{x+2}$ where $x > a > b$, and b is a negative number satisfying $b^2 = 2$. Hence, the set X also has no first point.