Theorem 4.2: The sets $\emptyset$ and C are closed.
A closed set is a set that contains all of its limit points. Remember for there to be a limit point p of $\emptyset$, then for every region R where $p \in R$ this must hold true: $R \cap (\emptyset \backslash \{p\}) \neq \emptyset$. However, $\emptyset$ does not contain any elements in its set, therefore $R \cap \emptyset$ will always be the empty set therefore there are no limit points, and since $\emptyset$ has no limit points, then the set trivially contains all of its limit points.

For the set C which contains all elements of the continuum C, recall that limit points p must be in C, thus C contains all its limit points based on the definition of a limit point.

Theorem 4.3: A subset C containing a finite number of points is closed.
According to *theorem 3.23: a finite subset A of a continuum C has no limit points*. Similar to the empty set, when a set contains no limit points, then trivially the set contains all its limit points and is closed.

Theorem 4.5: $X \subset C$ is closed if and only if $X = \overline{X}$.
The closure of X is $\overline{X} = X \cup LP(X)$ which is the union of all elements in X and the limit points of x.

First we prove that if $X \subset C$ is closed then $X = \overline{X}$. Suppose we have a set $A = LP(X)$. If X is closed, then that means all limit points $LP(X) \in X$ such that $A \subset X$. Because X contains all elements in A, we understand that $X \cup A = X$ because there are no elements in A that are not in X. Now we have proved $\overline{X} = X \cup LP(X) = X$.

Now we have to prove that if $X = \overline{X}$ then $X \subset C$ is closed. We know that $\overline{X} = X \cup LP(X)$, so if $X = \overline{X}$, that means $X = X \cup LP(X)$. Because the set X contains all its limit points, then the set X must be closed.

Theorem 4.6: Let $X \subset C$. Then $\overline{X}$ is closed. (equivalently, $\overline{X} = \overline{\overline{X}}$)
To prove $\overline{X}$ is closed, we need to show $\overline{X}$ contains all of its limit points.
We want to show $y \in \overline{X}$, so let $y \in LP(\overline{X})$ so $y \in LP(X \cup LP(X))$ which means based on *theorem 3.19* :
$y \in LP(X)$ which means $y \in \overline{X}$ based on the definition of closure: $\overline{X} = X \cup LP(X)$.

or

$y \in LP(LP(x))$ where we let R_1 is a region that contains y where $y \in R$ and we want to show $R_1 \cap (X \backslash \{y\} \neq \emptyset)$. We know that $R_1 \cap (X \backslash \{y\} \neq \emptyset)$ for all regions R that $y \in R$. Let $Z \in R_1 \cap (LP(X) \backslash \{y\})$ where $z \in LP(X)$which means tha $z \in R_1$ and $z \in LP(X) \backslash \{y\}$. We can then say that $R_1 \cap (X \backslash \{y\} \neq \emptyset)$. Now, let $a \in R_1 \cap (X \backslash \{y\})$.

case 1: $a \neq y$ which means $R_1 \cap x \backslash \{y\} \neq \emptyset$.

case 2: $a = y$ which means $y \in R_1 \cap (x\backslash\{z\}) and by * theorem 3.21 *,$y\in R_2,z \in R_3$, and$ R_2 \cap R_3 = \empty$. We can surmise that$z \in R_1 \cap R_3$such that$(R_1 \cap R_3) \cap (x \backslash {z} \neq \empty)$. Let$b \in (R_1 \cap R_3) \cap (x \backslash {z}$which means$b \in R_1 \cap R_3andb\in Xandb \in R_1 \cap (x \backslash {y}) such that $y \in LP(X)$ which means $y \in \overline{X}$ and because all $LP(\overline{X} \subset X) then$\overline{X}$ is closed.