Script 3

Lemma 3.4: If A is a nonempty, finite subset of a continuum C, then A has a first and last point. We can prove that A has a first and last point through induction.

Suppose our base case $n=1$, then $A = {a_1}$ where A is a set with a single element thus that single element is both the first and last element (according to *definition 3.3*). Now for the inductive hypothesis, we can assume when $n = k$, where k is a positive integer. A has a first element a_i and the last point is a_k such that $a_1 < a_k$. Finally, our inductive step proves our assumption holds for when A contains $n=k+1$ elements. There are three possible cases :

1. $a_{k+1} < a_1 < a_k $
2. $a_1 < a_{k+1} < a_k$
3. $a_1 < a_k < a_{k+1}$

All these are true for the inductive step and maintain the existence of the first and last element, therefore any finite subset A of a continuum C, it has a first and last point.

Theorem 3.5: Suppose that A is a set of n distinct points in a continuum C, or, in other words, $A \sub C$ has cardinality n. Then symbols $a_1,...a_n$ may be assigned to each point of A so that $a_1, a_2, <... < a_n, ; ie. ; a_i < a_{i+1} ; for ; 1 \leq i \leq n-1$. Suppose we have a set X with elements ${a_1, a_2, < ... < a_n}$. We can essentially assign the elements of set X to the first point in A according to *Lemma 3.4* which says that A which is a subset of C has a first and last point. Thus as we move through set X, we can assign and remove element a_1 from set A so that each element in A will be assigned to symbols $a_1,...a_n$ in set X.

Corollary 3.7: Of three distinct points in a continuum, one must be between the two other. Consider *Theorem 3.5* where we proved each point in a subset of a continuum C to a symbol $a_1, a_2,...,a_n$ in ascending order. In this case, points assigned to index i would be less than points $i+1$. **MORE**

Exercise 3.8: **a. We define a relation $<$ on $\Z$ by $m < n $ if $n = m + c$ for some $c \in \N$. Show that, $\Z$, with the ordering $<$, satisfies *Axion 1-3*.**

Let us be reminded of *Axioms 1-3*. *Axiom 1. A continuum is a nonempty set C.* First, let us identify $\Z$ as a continuum because it's a nonempty (infinite) set of integers. For example, the set $\Z$ contains elements such as ${-1,0,1,2}$. This means $\Z$ fulfills *axiom 1*.

Axiom 2. A continuum C has an ordering $<$ We can identify m and n as distinct points in $\Z$ and c as a positive integer. For *Axiom 2* to prove true, an ordering on the set represented by $\Z \times \Z$ with elements (m,n) written as $m < n$ must satisfy these conditions according to *definition 3.1*:

Trichotomy, where for all $m,n \in \Z$ one of the following holds: $m < n, m = n, ; \textnormal{or} ; m > n $. Because we defined $m < n$ if $n = m + c$, then trichotomy is satisfied as one of the above held true. This is also because when $n=m+c$ and $c \in \N$ and $c = n-m$, $m \neq n$ because c would have to equal 0, and $m \ngeq n$ because c would have to be negative, both of which are not within the set $\N$ thus, $m < n$ is the only case and trichotomy is true in the continuum Z.

Transistivity: where for all $m,n,l \in \Z$, if $m < n$ and $n < l$ then $m < l$. Let's suppose $n = m + a$, $l = n + b$, and $l = m + c$ where $a,b,c \in \N$. From this, we can deduce $n > m$ because $a = m - n$. We can substitute $m+a$ for n to get $l = m + a + b$. If a,b are both positive integers more than 0, then l must be greater than m which proves that if $n > m$ and $l > n$, then $l > m$. Therefore, the transistivity property is true for an ordering $<$ of continuum $\Z$.

With both Trichotomy and Transistivty satisfied, we can conclude that the continuum $\Z$ has an ordering $<$.

Axiom 3. A continuum C has no first or last point

This is true, as $\Z$ has no first nor last point. This is because $\Z$ is an infinite set thus there is no $a_1 \in \Z$ and $a_i \in \Z$ where a_i is any element of $\Z$ and $a_1 < a_i$ for all i nor is there any element a_n that represents the last element of $\Z$ because $\Z$ is an infinite set with no first nor last element.

Therefore, $\Z$ with the ordering $<$, satisfies axioms 1-3.

b. Show that, for any $p = {\frac a b} \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

As we have observed in *Exercise 2.6: $[\frac a b] = [\frac {a_1} {b_1}] = (a,b)$ ~ (a_1, b_1)*. Thus, we may find an equivalent fraction for any rational number p such that $[\frac a b] = [\frac {a_1} {b_1}]$ by dividing both a,b by their greatest common divisor (GCD). We know based on the equivalence relation established in *Exercise 2.2* that $[\frac a b] = (a,b)$ and conversely $[\frac {a_1} {b_1}] = (a_1, b_1)$. Thus since $(a,b) \in p$ and (a,b) ~ (a_1, b_1), then $(a_1, b_1) \in p$. Note that by dividing b by a GCD and because p is a rational number and fractions cannot have 0 in their denominator, b must be a positive integer and thus $b > 0$.

c. We define a relation $<*mathbb{Q}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1,b_1) \in p$ be such that $0 < b_1$, and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p* <\mathbb{Q} q$ if $a_1b_2 < a_2b_1$. Show that $<_\mathbb{Q}$ is a well-defined relation on $\mathbb{Q}$.

d. Show that $\mathbb{Q}$, with the ordering $<_\mathbb{Q}$, satisfies *Axioms 1-3*.

Let us be reminded of *Axioms 1-3*. *Axiom 1. A continuum is a nonempty set C.* First, let us identify $\mathbb{Q}$ as a continuum because it's a nonempty (infinite) set of rational numbers (fractions). For example, the set $\mathbb{Q}$ contains elements such as ${(\frac 0 1), (\frac 1 1), (\frac {-1} 1), (\frac {1} 2)}$ and so on. This means $\mathbb{Q}$ fulfills *axiom 1*.

Axiom 2. A continuum C has an ordering $<$ We can identify p and q as distinct points in $\mathbb{Q}$ and define a point $x \in \mathbb{Q}$ and let $(a_3, b_3) \in x$ be such that $0 < b_3>$ and let's make x so $q = p +*mathbb{Q} x$. For Axiom 2 to prove true, an ordering on the set represented by $\mathbb{Q} \times \mathbb{Q}$ with elements p,q written as $p* <\mathbb{Q} q$ must satisfy these conditions according to *definition 3.1*:

CONTINUE HERE Trichotomy, where for all $p,q \in \mathbb{Q}$ one of the following holds: $p <*mathbb{Q} q, p* =\mathbb{Q} q, ; \textnormal{or} ; p >*mathbb{Q} q $. Because we defined $p* <\mathbb{Q} q$ if $a_1b_2 < a_2b_1$, then trichotomy is satisfied as one of the above held true. This is also because when $n=m+c$ and $c \in \N$ and $c = n-m$, $m \neq n$ because c would have to equal 0, and $m \ngeq n$ because c

would have to be negative, both of which are not within the set $\N$ thus, $m < n$ is the only case and trichotomy is true in the continuum Z.

Transistivity: where for all $m,n,l \in \Z$, if $m < n$ and $n < l$ then $m < l$. Let's suppose $n = m + a$, $l = n + b$, and $l = m + c$ where $a,b,c \in \N$. From this, we can deduce $n > m$ because $a = m - n$. We can substitute $m+a$ for n to get $l = m + a + b$. If a,b are both positive integers more than 0, then l must be greater than m which proves that if $n > m$ and $l > n$, then $l > m$. Therefore, the transistivity property is true for an ordering $<$ of continuum $\Z$.

With both Trichotomy and Transistivty satisfied, we can conclude that the continuum $\Z$ has an ordering $<$.

Axiom 3. A continuum C has no first or last point

This is true, as $\mathbb{Q}$ has no first nor last point. This is because $\mathbb{Q}$ is a dense infinite set thus there is no $a_1 \in \mathbb{Q}$ and $a_i \in \mathbb{Q}$ where a_i is any element of $\mathbb{Q}$ and $a_1 < a_i$ for all i nor is there any element a_n that represents the last element of $\mathbb{Q}$ because $\mathbb{Q}$ is an infinite set where between any two distinct rational numbers, there is another rational number. Thus $\mathbb{Q}$ has no first nor last element.

Therefore, $\Z$ with the ordering $<$, satisfies axioms 1-3.

Theorem 3.11: If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.

Let us remember *Definition 3.9: If $a, b ∊ C$ and $a < b$, then the set of points between a and b is called a region, denoted by $\underline{ab}$.* Let us mimic this region by creating a subset of C called A of points between a and b where a is the first point and b is the last point. This is possible through *Lemma 3.4* which defines the first and last points of subset A and *Corollary 3.5* which indexes each point point in the subset in ascending order. Because A can be arranged in ascending order $(a_1, a_2, a_3,...a_n)$ where $a_1 = a$ and $a_n = b$ where n is the cardinality of A, there must be an $a_i = x$ where $i \in \N$ and is between 1 and n. Therefore, x must be a point existing within the region $\underline{ab}$.

Theorem 3.13: If p is a limit point of A and $A \sub B$, then p is a limit point of B.

Let us be reminded of *Definition 3.12: a limit point is a point p of continuum C in subset A if every region R containing p has nonempty intersection with $A\backslash {p}$ which means for every region R with $p \in R$, we have $R \cap (A \backslash {p}) \neq \empty$.*

Since p is a limit point of A, then for all regions R containing p, $R \cap (A \backslash {p}) \neq \empty$*. Let $q \in A$ be an intersection point with $R \cap (A \backslash {p}) \neq \empty$, and since $A \sub B$, then $q \in B$. Thus, given an R containing p, there is a $q \in R \cap B \backslash {p}$.

Lemma: 3.15: If $\underline{ab}$ is a region in a continuum C, then, ext $\underline{ab} = {x \in C | x < a } \cup {x \in C| b < x}$

FIX Consider *Definition 3.14: If $\underline{ab}$ is a region of continuum C, then $C \backslash ({a} \cup \underline{ab} \cup {b})$. is called the exterior of $\underline{ab}$ denoted by ext $\underline{ab}$.*

We want to show ext $\underline{ab} = {x \in C | x < a } \cup {x \in C| b < x}$. We know that:

if $x \nleq a$, then $x \geq a$ if $x \ngeq b$, then $x \leq b$

We can formulate the equation: $C\backslash ({a} \cup \underline{ab} \cup {b}) = {x \in C | x < a} \cup {x \in C | b< x}$. We can simplify the LHS to $C\backslash(x\geq a \cap x \leq b)$ which equals to the RHS ${x \in C| C\backslash(x \geq a) \cup C\backslash (x \leq b)}$.

Lemma 3.16: No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.

FIX

Proof by contradiction: Consider the set $E = ext ; \underline{ab}$ where $E \cap R \neq \empty$. Let $e \in E$ where e is a limit point of $\underline{ab}$. If a and b are in the continuum C, then $e < a$ where a' is an element such that $a' < e < a$ because the continuum has no first or last point. Or $e > b$ such that there is a b' where $b < e < b'$ because the continuum has no first or last point. Because $\underline{a'a} \cap \underline{ab} \backslash e = \empty$ or $\underline{bb'} \cap \underline{ab} \backslash e = \empty$ and since $e \in \underline{ab}$, then $a < e < b$ and $a' < a$ which means that $\underline{ab} \cap \underline{a'a} \backslash {e} = \empty$. And thus $b < b'$ which means that $\underline ab \cap \underline{bb'} \backslash {e} = \empty$ or $\underline{ab} \cap E \backslash {e} = \empty$.

Theorem 3.17: If two regions have a point x in common, their intersection is the region containing x.

FIX Let $x \in \underline{ab}$ and $x \in \underline{cd}$, which means $a,b,c,d \in C$ and $a < b$ and $c < d$. We can infer that $a < x$ and $x < b$ and $c < x$ and $x < d$. If $x \in \underline{ab} \cap \underline{cd}$, then x is in a region, R, that is between the least value (whether a or c) and the greatest value b or d. Let a be the smallest, then $c < b$.

Be aware of our four cases: $a< c < b< d$, $c < a < b< d$, $a < c <d <c$, $c < a < d < b$

Because R encompasses both $\underline{ab}$ and $\underline{cd}$, then $x \in (\underline{ab} \cap \underline{cd})$.

Corollary 3.18: If n regions $R_1,..., R_n$ have a point x in common, then their intersection $R_1 \cup ... \cap R_n$ is a region containing x.

FIX

Proof by induction: If a region R, R_n x, then the intersection R

Base case $(n=1)$, R_1 contains x without intersection Inductive hypothesis, we assume $n = k$, so if $R_1... R_k$ all contains x, then $R_1 \cap R_k$ contains x. Inductive step, we can show it holds for $(n = k+1)$. Let $K = R_1 \cap R_n \cap R_n$ based on our hypothesis. We know that K is a region containing x. Thus $R_1 \cap R_2 $, $R_k \cap R_{k+1} = K \cap R_{k+1}$. The intersection of 2 regions containing x. By *theorem 3.17 *, we have $K \cap R_{k+1}$ is a region containing x.

Theorem 3.19: Let A, B be subsets of a continuum C. Then p is a limit point of $A ∪ B$ if, and only if, p is a limit point of at least one of A or B.

We can prove this this by contradiction. Let us assume that there exists a p which is a limit point of $A \cup B$ but not a limit point of either A nor B. This means p must satisfy 3 conditions:

p is a limit point of $A \cup B$: For p to be a limit point, it p is not a limit point of A:
p is not a limit pointoof B:

Corollary 3.20: Let $A_1, . . . , A_n$ be n subsets of a continuum C. Then p is a limit point of $A_1 ∪ · · · ∪ A_n$ if, and only if, p is a limit point of at least one of the sets A_k.

Theorem 3.21: If p and q are distinct points of a continuum C, then there exist disjoint regions R and S containing p and q, respectively.

Corollary 3.22. A subset of a continuum C consisting of one point has no limit points.

Theorem 3.23: A finite subset A of a continuum C has no limit points.

Corollary 3.24: If A is a finite subset of a continuum C and $x ∈ A$, then there exists a region R, containing x, such that $A ∩ R = {x}$.