

Script 5

Theorem 5.1: The only subsets of a continuum C that are both open and closed are \emptyset and C .

Let's prove this by contradiction. Assume there exists a non-empty subset $A \subset C$ such that A is both open and closed and $A \neq C$.

Because A is open and closed and nonempty, its complement $C \setminus A$ is open and nonempty by *Definition 4.7* which states that a set is open if its complement is closed. We also know that because they are complements, $A \cap (C \setminus A) = \emptyset$ which means they are disjoint as they contain no similar elements. This means we have two nonempty, open sets which contradicts the definition of connected set of *Definition 4.22* because C can now be represented as the union of two non-empty disjoint open sets, thus making C disconnected which contradicts the fact that C is connected thus our assumption is false and therefore through proof by contradiction the only subsets of continuum C that are open and closed are \emptyset and C .

Corollary 5.2: Every region is infinite.

Assume, for the sake of contradiction, that there exists a finite region R denoted as \underline{ab} , where $a, b \in C$ and $a < b$. According to this assumption, the elements in the region can be arranged in a finite sequence, due to *Theorem 3.5*. This sequence is of the form $a < x_1 < x_2 < \dots < x_n < b$, with n elements in total where $n \in \mathbb{N}$.

However, this assumption leads to a contradiction. According to this sequence, there are no elements between x_1 and x_2 . This directly contradicts *Theorem 4.23*, which states that in a connected continuum (which a continuum must be by *Axiom 4*), for any two elements $x_1, x_2 \in C$ with $x_1 < x_2$, there must exist another element $z \in C$ that lies between x_1 and x_2 such that there are an infinite number of points between any two points which is in direct contrast with the finite cardinality n specified by *Theorem 3.5* for a finite region. Thus our assumption that R is a finite region is false, and every region must be infinite.

Corollary 5.3: Every point of C is a limit point of C .

Assume, for contradiction, that there exists a point $x \in C$ which is not a limit point of C . This assumption implies that there is a region R containing x , where $x \in R \subseteq C$, and R satisfies the condition: $R \cap C \setminus \{x\} = \emptyset$. In simpler terms, besides x , the region R does not contain any other points of C .

However, this assumption leads to a contradiction when we consider *Corollary 5.2*, which we previously proved. *Corollary 5.2* states that every region in C is infinite. Therefore, it's impossible for the region R to contain only the point x if it's a part of C ; R must contain infinitely many points.

Thus, our original assumption, that R contains only x and no other points from C , contradicts the fact established in *Corollary 5.2*. Therefore, $R \cap C \setminus \{x\}$ cannot be empty, implying that x must be a limit point of C . Consequently, our initial assumption is false, and it follows that every point in C is indeed a limit point of C .

Corollary 5.4: Every point of the region \underline{ab} is a limit point of \underline{ab} .

Suppose, for the sake of contradiction, that there exists a point x in the region \underline{ab} which is not a limit point of \underline{ab} . This means that we can find a region R containing x , where $x \in R \subseteq C$, and this region R satisfies the condition: $R \cap \underline{ab} \setminus \{x\} = \emptyset$. In other words, R does not contain any points from \underline{ab} other than x .

However, this assumption leads to a contradiction. Note that $R \cap \underline{ab}$ is itself a region within C . According to *Corollary 5.2*, every region within C is infinite. This implies that the region $R \cap \underline{ab}$ must contain infinitely many points.

Thus, if $R \cap \underline{ab}$ is infinite, then $R \cap \underline{ab} \setminus \{x\}$ cannot be empty. There must be other points in $R \cap \underline{ab}$ besides x . This contradicts our initial assumption that $R \cap \underline{ab}$ contains only x . Therefore, our assumption that x is not a limit point of \underline{ab} is false. It follows that every point in the region \underline{ab} must be a limit point of \underline{ab} .

Exercise 5.7: If $\sup X$ exists, then it is unique, and similarly for $\inf X$.

Assume that $\sup X$ exists and suppose for the sake of contradiction there are at least two distinct least upper bounds, call them u and u' where $u \neq u'$.

By *definition 5.6*, a supremum u must be an upper bound of X and for any upper bound u' of X , $u \leq u'$. This is the same for u' to be a supremum as well, it must be less than or equal to u . Thus they both must be upper bounds, more importantly, $u' \leq u$ and $u \leq u'$ which means $u = u'$.

However, this contradicts our assumption that there are two distinct, equal least upper bounds. Therefore, if $\sup X$ exists, then it must be unique.

This same logic can equally be applied to $\inf X$ with reverse comparative signage.

Exercise 5.8: If X has a first point L , then $\inf X$ exists and equals L . Similarly, if X has a last point U , then $\sup X$ exists and equals U .

Assume that X has a first point L such that $L < x$ where x for all $x \in X$ which means L is a lower bound of X by *definition 5.5*. To show that L is a greatest least bound of X , we must show that:

1. L is a lower bound of X and
2. If L' is any lower bound of X , then $L' \leq L$.

The first condition is met by L being the first point by *Definition 3.3* as a first point L of a set X means $L \leq x$ for all x in X . This satisfies the requirement for L to be a lower bound of X , as it is less than or equal to every element in X .

As for the second condition, Suppose there exists another lower bound L' of X . By the definition of a lower bound, $L' \leq x$ for all $x \in X$. Since L is the first point of X , it is the smallest element in X . Therefore, for any lower bound L' , it must hold that $L' \leq L$. This is because L' cannot be greater than the smallest element of X and still be a lower bound. This satisfies the condition that L is greater than or equal to every other lower bound of X . Therefore, by satisfying these two conditions, L is the greatest least bound, or the infimum, of X .

$\sup X$ can be proved by *definition 3.3* for the last point in a similar manner.

Exercise 5.9: For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of \mathbb{R} , or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$
 - i. $\sup X$ does not exist because the set of natural numbers has no upper bound as it goes on indefinitely.
 - ii. $\inf X = 1$ since 1 is the first point of the set of natural numbers, thus it is $\inf X$ by *exercise 5.8*.
2. $X = \mathbb{Q}$
 - i. neither \sup nor \inf exist for X because the set of rational numbers extend infinitely in both positive and negative directions such that there are no upper nor lower bounds.
3. $X = \{\frac{1}{n} | n \in \mathbb{N}\}$
 - i. $\sup X = 1$ because 1 is the upper bound such that there will never be a number in the set X that is greater than 1
 - ii. $\inf X$ is 0, though 0 is not in the set, it fulfills the requirements because it is a lower bound which the elements of X get close to 0 as n increases.
4. $X = \{x \in \mathbb{R} | 0 < x < 1\}$
 - i. $\sup X = 1$, even though 1 is not in the set, no element in the set X is greater than 1 such that 1 is an upper bound
 - ii. $\inf X = 0$, 0 is the greatest lower bound for X , even though it's not part of the set since all elements are greater than 0.

5. $X = \{3\} \cup \{x \in \mathbb{R} \mid -7 \leq x \leq -5\}$

i. $\sup X = 3$, since 3 is the largest number in the union of the two sets.

ii. $\inf X = -7$, because it's the smallest number in the combined set and is included in the set X .

Lemma 5.10: Suppose that X is a nonempty subset of C and $s = \sup X$ exists. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$.

Assume that X is a nonempty subset of C and that $s = \sup X$ exists. Let p be any element in C such that $p < s$. For the sake of contradiction, suppose that there does not exist an $x \in X$ such that $p < x \leq s$. This means that for all $x \in X$, $x \leq p$ or $x > s$. However, $x > s$ is impossible by *Definition 5.5* because s is the least upper bound of X .

If $x \leq p$ for all $x \in X$, then p would be an upper bound for X by *Definition 5.5*, which contradicts the fact that s is the least upper bound of X and $p < s$ by *Definition 5.6*. This contradiction implies that our assumption is false. Therefore, there must exist an $x \in X$ such that $p < x \leq s$ if $p < s$.

Theorem 5.11: Let $a < b$. The least upper bound and greatest lower bound of the region \underline{ab} are: $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.

Let $a < b$. The theorem states that the least upper bound and greatest lower bound of the region \underline{ab} are $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.

First we want to prove that $\inf \underline{ab} = a$.

To do this, we first show that a is a lower bound. By *Definition 5.5*, a lower bound for a set is an element that is less than every element in the set. Since \underline{ab} consists of points x such that $a < x < b$, it follows that for all $x \in \underline{ab}$, $a < x$. Therefore, a is a lower bound of \underline{ab} .

Next, we need to show that a is the Greatest Lower Bound which we can demonstrate by contradiction. Suppose there exists some element $l > a$ that is also a lower bound of \underline{ab} (such that a is not the greatest lower bound). By *Theorem 4.23*, the continuum is connected, meaning for any two points, there exists another point between them. Therefore, there exists a point c in the region \underline{al} such that $a < c < l$. Since c is an element of \underline{ab} and $c > a$, this contradicts the assumption that l is a lower bound of \underline{ab} . Hence, a is the greatest lower bound, or $a = \inf \underline{ab}$.

Now we want to prove that $\sup \underline{ab} = b$.

Similar for \inf , we want show b is an Upper Bound. By definition, an upper bound for a set is an element that is greater than every element in the set. Since \underline{ab} consists of points x such that $a < x < b$, it follows that for all $x \in \underline{ab}$, $x < b$. Therefore, b is an upper bound of \underline{ab} .

Next, we need to show b is the Least Upper Bound which we can demonstrate by contradiction. Suppose there exists some element $u < b$ that is also an upper bound of \underline{ab} . By the same connectedness argument of *Theorem 4.23*, there would exist a point c in the region \underline{ub} such that $u < c < b$. Since c is an element of \underline{ab} and $c < b$, this contradicts the assumption that u is an upper bound of \underline{ab} . Hence, b is the least upper bound, or $\sup \underline{ab}$.

Thus, we have proven that $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.

Theorem 5.12: Let X be a nonempty subset of C . Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of X . The same holds for $\inf X$.

Consider $s = \sup X$. Since X is nonempty, there is at least one element p in the continuum such that $p < s$. According to *Axiom 3*, which states that there is no last point and *Definition 5.6* which states that there must exist an element c such that $c > s$, we can define a region \underline{pc} that includes s . For any such region, there exists an x in X where $x \neq s$ (since $s \notin X$) and $p < x \leq s$, as stated by *Lemma 5.10*; thus for any region \underline{pc} containing x , the set $\underline{pc} \cup x \setminus \{s\}$ is nonempty, confirming that $\sup X$ is a limit point of X . The same reasoning applies to the infimum of X , showing that if the infimum exists and is not an element of X , it too is a limit point of X .

Corollary 5.13: Both a and b are limit points of the region \underline{ab} . Let $[a, b]$ denote the closure $\overline{\underline{ab}}$ of the region \underline{ab} .

We know that $a = \inf \underline{ab}$ and $b = \sup \underline{ab}$ by *Theorem 5.11*, and since $a \notin \underline{ab}$ and $b \notin \underline{ab}$, then by *Theorem 5.12*, $a, b \in \text{LP}(\underline{ab})$.

Corollary 5.14: $[a, b] = \{x \in C \mid a \leq x \leq b\}$.

The set $[a, b]$ is defined as the closure $\overline{\underline{ab}}$ as proven by *Corollary 5.13*. Thus by definition of a closure in *Definition 4.4* which says a closure is the union of the set and its limit points, $[a, b]$ must include both the elements of the region \underline{ab} such that any element x of closure $[a, b]$ must be between a and b where $a < x < b$ by *Definition 3.9* and its limit points which are a and b such that x can also equal a or b , once again as described by *Corollary 5.13*. Additionally, we know that there are no other limit points outside of the region \underline{ab} by *Lemma 3.16* which states that there is no point of the exterior of a region that can be a limit point of that region; thus, there are no other limit points outside of the closure. This means $[a, b]$ is exactly the set of all points x in C such that $a \leq x \leq b$.

Lemma 5.15: Let $X \subset C$ and define: $\Psi(X) = \{x \in C \mid x \text{ is not an upper bound of } X\}$. Then $\Psi(X)$ is open. Define: $\Omega(X) = \{x \in C \mid x \text{ is not a lower bound of } X\}$. Then $\Omega(X)$ is open.

Let $X \subset C$ and define $\Psi(X) = \{x \in C \mid x \text{ is not an upper bound of } X\}$. We want to prove that $\Psi(X)$ is open which we can accomplish by showing that the complement of $\Psi(X)$, which consists of

all upper bounds of X , is closed.

First, consider a point x in the set of limit points of the complement of $\Psi(X)$, denoted as $x \in LP(C \setminus \Psi(X))$. This indicates that x is a limit point of the set of upper bounds of X . Now, for the sake of contradiction, suppose $x \notin C \setminus \Psi(X)$. This implies that x is not an upper bound of X . Consequently, there must exist some $y \in X$ such that $x < y$, which exists because x cannot be greater than or equal to every element in X since it is not an upper bound and because the continuum is connected and has no last point.

Given that the continuum C is connected and there is no first point, there exists a point $z \in X$ such that $z < x$. This allows us to consider the region \underline{zy} , which includes x . However, the region \underline{zx} , when intersecting $C \setminus \Psi(X)$ and excluding x itself, must be empty. This emptiness arises because all points in $C \setminus \Psi(X)$ are upper bounds of X . By *Definition 5.5*, these points must be greater than y , and hence, no point in \underline{zx} (other than x) can be an upper bound of X .

Therefore, x must be in the complement of $\Psi(X)$ such that x is in the set of upper bounds of X which contradicts our original assumption. Because the complement of $\Psi(X)$ contains its limit points, then the complement of $\Psi(X)$ is closed. Following the definition of an open set, this implies that $\Psi(X)$ is open.

The same reasoning can be applied to $\Omega(X) = \{x \in C \mid x \text{ is not a lower bound of } X\}$ to show that it is open as well.

Theorem 5.16 (Least Upper Bound Property): Suppose that X is nonempty and bounded above. Then $\sup X$ exists. Similarly, if X is nonempty and bounded below, then $\inf X$ exists.

Class notes

We will prove this by contradiction. Assume for the sake of contradiction that X is nonempty and bounded above and that the $\sup X$ does not exist. We define $A = C \setminus \Psi(X)$ is the set of upper bounds of X . Because X is bounded above, then $A \neq \emptyset$, and since we know that $\sup(X)$ does not exist, then for all $y \in A$, we can say that y is not the least upper bound of X . Then, we can say there is some $z \in A$ where $z < y$. Because $z \in A$, then $z > x$ for all $x \in X$. then, for any point $a \in C$ where $a > z$, then $a > x$ for all $x \in X$.

1. For any point a in the continuum where $a > z$, then $a \in A$ because it must be in the upper bounds of X by *Definition 5.5* because it is larger than all $x \in X$. Because the continuum has no last point by *Axiom 3*, then we know that there exists a point u such that $u > y$. We define the region \underline{zu} such that $z < y < u$ thus $y \in \underline{zu}$. For all points $n \in \underline{zu}$, we know that $n > z$. By line 1, we know that for all $n \in \underline{zu}$ thus $n \in A$. Then $y \in \underline{zu}$ is a subset of A . Therefore, A is open by *Theorem 4.9*.

2. A is nonempty and open. We can say that $\Psi(X)$ is open by *Lemma 5.15*. Because X is nonempty, then we can find some $w \in X$ then there exists some $d < w$ by *Axiom 3* stating that a continuum has no first point. Therefore, d is not an upper bound of X . Now, we can state that $\Psi(X)$ is nonempty.
3. Since we know that $\Psi(X)$ is nonempty and open, $A = C \setminus \Psi(X)$ then $A \cap \Psi(X) = \emptyset$, then A and $\Psi(X)$ are disjoint. Because A and $\Psi(X)$ are empty, open, and disjoint, then $C = A \cup \Psi(X)$ and C is the union of nonempty, open, and disjoint sets thus C is disconnected by *Definition 4.22* and this is a contradiction.

Corollary 5.17: Every nonempty closed and bounded set has a first point and a last point.

We will use a proof by contradiction to demonstrate this. Let's assume there exists a set X that is nonempty, closed, and bounded, but, for the sake of contradiction, suppose X does not have a first or a last point. Given that X is bounded, by *Theorem 5.16*, it has a supremum (denoted as $\sup X$) and an infimum ($\inf X$).

First, consider the supremum of X . The last point of a set X is defined as some element $x \in X$ such that x is greater than or equal to any other element a in X . If $\sup X$ is an element of X , then it would be the largest element in X , satisfying the condition $\sup X \geq x$ for all $x \in X$. This would make $\sup X$ the last point of X , contradicting our assumption. On the other hand, if $\sup X$ is not an element of X , then by *Theorem 5.12*, $\sup X$ must be a limit point of X . However, this would imply that X does not contain all of its limit points, contradicting the assumption that X is closed.

Therefore, X must have a last point. A similar line of reasoning can be applied to show that X must also have a first point. Hence, our initial assumption leads to a contradiction, and we conclude that every nonempty, closed, and bounded set indeed has a first point and a last point.

Exercise 5.18: Is this (Corollary 5.17) true for \mathbb{Q} ?

Class notes

No, Corollary 5.17 does not hold for \mathbb{Q} (the set of rational numbers). Let's consider the set X defined as:

$$X = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

First, observe that X is bounded above and below in \mathbb{Q} . For example, $x = 5$ is an upper bound as it is greater than every element of X , and similarly, $x = -5$ is a lower bound as it is smaller than every element of X .

To establish that X does not have a first or last point, we demonstrate that for every rational number $x \in X$, there is another number $y \in X$ such that $x < y < z$, where z is the positive number (not

necessarily rational) satisfying $z^2 = 2$.

Starting with the fact that $x^2 < 2$, we follow these steps:

1. $x^2 + 2x < 2x + 2$
2. $x(x + 2) < 2x + 2$
3. $x < \frac{2x+2}{x+2}$

Therefore, if $x^2 < 2$, then $\frac{2x+2}{x+2} > x$.

Next, we want to show that $\left(\frac{2x+2}{x+2}\right)^2 < 2$ so that it belongs to X :

1. $x^2 < 2$ implies $2x^2 < 4$
2. $2x^2 + 8x + 2x^2 + 4 < 4 + 8x + 2x^2 + 4$
3. $4x^2 + 8x + 4 < 2x^2 + 8x + 8$
4. $4x^2 + 8x + 4 < 2(x^2 + 4x + 4)$
5. $\frac{4x^2+8x+4}{x^2+4x+4} < 2$

Therefore, $\left(\frac{2x+2}{x+2}\right)^2 < 2$.

Consequently, for all $x \in X$, there exists some $y \in X$ such that $y = \frac{2x+2}{x+2}$ and $x < y < z$, where $z^2 = 2$ and $z > 0$. Thus, the set X has no last point.

Similarly, for all $x \in X$, there exists some $a \in X$ such that $a = \frac{-2x+2}{x+2}$ where $x > a > b$, and b is a negative number satisfying $b^2 = 2$. Hence, the set X also has no first point.