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Citation: American Journal of Physics **50**, 271 (1982); doi: 10.1119/1.12866
View online: http://dx.doi.org/10.1119/1.12866
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/50/3?ver=pdfcov
Published by the American Association of Physics Teachers

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(Received 9 December 1980; accepted for publication 18 March 1981)

I. HISTORICAL NOTE

Nearly 200 years have passed since Ernst Florens Friedrich Chladni of Saxony published his treatise[1] in which he described his well-known method of using sand sprinkled on plates to show the nodal lines. In 1802 he published another work,[2] with a second edition in 1830. In it he observed that the addition of one nodal circle raised the frequency of a circular plate by about the same amount as adding two nodal diameters, a relationship that Lord Rayleigh calls Chladni's law.[3]

Chladni's lectures at European courts attracted many famous personages. Napoleon was so delighted with his demonstrations (is there a lesson in this for physics teachers?) that he financed the translation of *Die Akustik* into French, and also provided for the Institute of France a prize of 3000 francs to be awarded for a satisfactory mathematical theory of the vibrations of plates. This prize was awarded in 1815 to Mlle. Sophie Germain, who gave the correct fourth-order differential equation for plate vibrations, although her choice of boundary conditions was incorrect.[4] (Mlle. Germain apparently corresponded with Gauss using the nom de plume of "Mr. Leblanc," a not uncommon custom for women of letters in that day[5].)

In 1850 G. R. Kirchhoff published two papers on the theory of vibrating plates, which gave a more accurate treatment of the boundary conditions.[6,7] As is now well known, each normal mode of a circular plate has m nodal diameters denoted by $\cos(m\theta - \alpha) = 0$ and n concentric circles that are solutions to the equation $J_m(Kr) + \lambda J_m(iKr) = 0$. [$J_m(Kr)$ and $J_m(iKr)$ are Bessel functions; α and λ are arbitrary constants.] Rayleigh[3] pointed out that for large values of Ka (a is the radius of the plate), $Ka \simeq \frac{1}{2}\pi(m + 2n)$. Thus in a plate, f is proportional to $(m + 2n)^2$, for large f, which is a statement of Chladni's law.

A list of scientists who have employed Chladni patterns to study plate vibrations reads like a Who's Who of physics: Chladni, Savart, Strehlke, Faraday, Koenig, Debye, Young, Flügge, Wood, Andrade, etc. But the most exhaustive studies were made by Mary Désirée Waller, who wrote 31 papers on the subject plus a most remarkable book, which was published posthumously by her friends.[5] In addition to hundreds of photographs of Chladni patterns, her book includes a list of 207 references.

II. TECHNIQUES FOR OBSERVING MODES OF VIBRATION IN PLATES

Every student in a beginning physics course deserves to see a demonstration of Chladni patterns on vibrating plates (teachers: imagine that you are Chladni and some student is Napoleon). As a minimum we recommend one circular, one square, and one rectangular plate. They can be cut from a sheet of aluminium or steel about 1.5 mm thick. Painting them flat black increases visibility, especially if table salt is used in place of sand. In this section we mention several convenient ways for exciting plate vibrations, ways to observe Chladni patterns, and other ways to observe modes of vibrations in plates.

A. Driving the plate

There are a number of ways to excite the various vibrational modes in plates. Chladni stroked the edge of the plate with a violin bow.[1] Waller preferred to touch the plate with bits of dry ice,[8] a technique that is most efficient in the frequency range of 2000 to 4000 Hz. Excitation of the plate at a single frequency can be accomplished in several ways: with a loudspeaker,[9] a mechanical driver,[10] an electromagnet,[11] or a magnetostrictive rod.[12]

We have found some type of electromagnetic drive to be the most convenient means of exciting plate vibrations. Applying an alternating current to a small coil attached to the plate and inserting a cylindrical magnet into the coil works well. Alternatively, attaching a small magnet to the plate and subjecting it to the alternating field of an electromagnet (with a ferrite or powdered-iron core) produces satisfactory results. In the case of a steel plate the small magnet is unnecessary, since a magnetic field that oscillates at a frequency $f/2$ will produce a force with a frequency f.

B. Chladni patterns

Chladni generated his vibration patterns by "strewing sand" on the plate, which then collected along the nodal lines.[1] Later he noticed that fine shavings from the hair of his violin bow did not follow the sand to the nodes, but instead collected at the antinodes. Savart noted the same behavior for fine lycopodium powder.[14] This effect was explained by Faraday as being due to acoustic streaming.[15]

Waller includes a discussion of the conditions under which a given material behaves as a "nodal" versus an "antinodal" powder (see p. 112, Ref. 5). For sand, the particle diameter should exceed 100 μm to collect at the nodes and form Chladni patterns. We frequently use table salt for this purpose, although the crystals are a little larger than optimum size. On wood plates, flakes of "glitter" (such as used in Christmas decorations) work well.

C. Scanning the sound field

A small microphone in the near field of the radiated sound can be used to study the modes of vibration of a plate.[16] The plate-to-microphone spacing must be much smaller than the spacing of the nodal lines that are observed by noting the change in phase in the signal when a node is crossed. A convenient way to note changes in phase is to display the microphone output, suitably amplified and filtered, on the vertical axis of an oscilloscope versus the current in the drive coil on the horizontal axis.

D. Interferometry with laser light

The very powerful techniques of time-average holographic interferometry make it possible to study vibrations down to very small amplitudes ($\sim 10^{-7}$ m).[17] Figure 1 is a

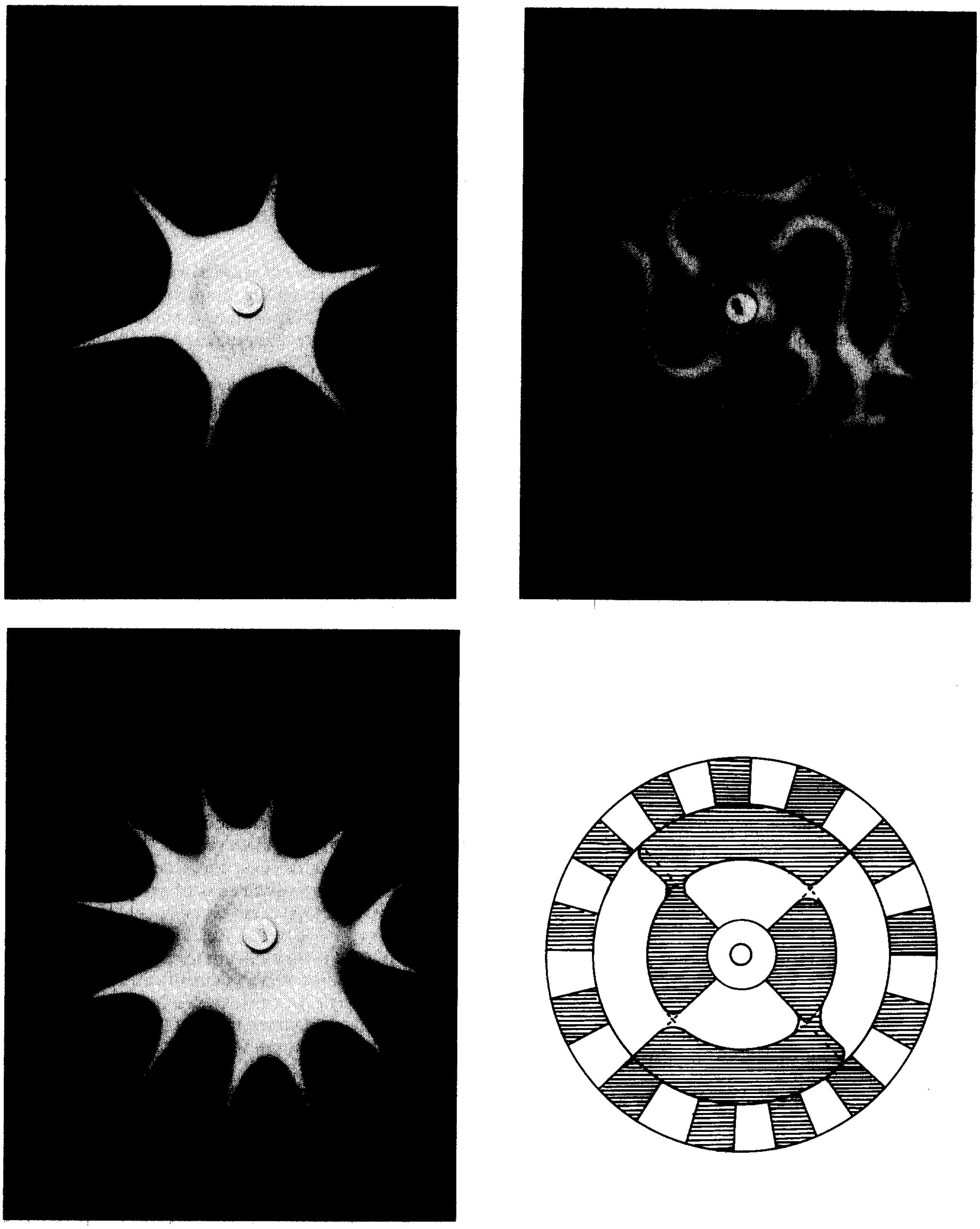

Fig. 1. Reconstructions from time-average holographic interferogram of 15-in. cymbal 18: (a) (30) mode; (b) (60) mode; (c) mixture of (13,0) and (4,2) modes; (d) schematic of the hologram in (c).

holographic reconstruction of a vibrating cymbal that shows a number of nodal circles and nodal diameters.[18]

Other interferometric means for studying modes of vibration include real-time holographic interferometry[19] and laser speckle interferometry.[20] Both of these methods have the advantage of giving continuous information about the modes, but the patterns are somewhat difficult to observe and especially to photograph.

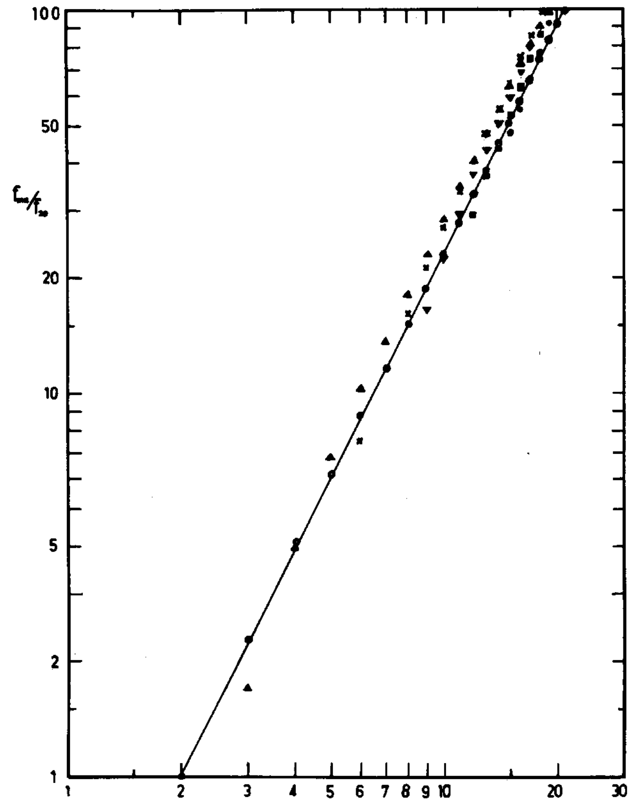

Fig. 2. Modal frequencies of a flat circular plate (data from Ref. 21) as a function of $m + 3n$ (○ $n = 0$, △ $n = 1$, × $n = 2$, ▼ $n = 3$, ■ $n = 4$).

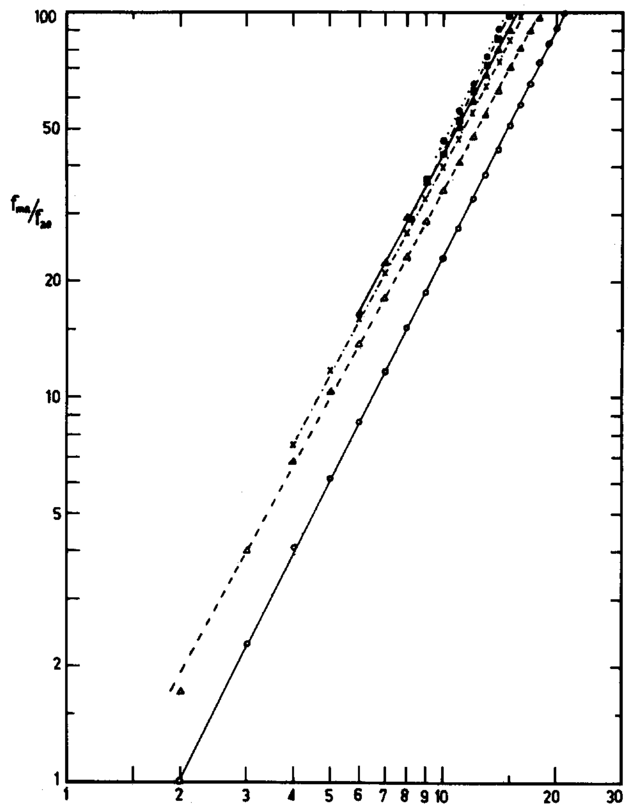

Fig. 3. Modal frequencies of a flat circular plate (data from Ref. 21) fitted to the relationship $f = c(m + 2n)^k$ (-○- $n = 0$, --△-- $n = 1$, ---×-- $n = 2$, -▲- $n = 3$, --■--$n = 4$, ··●··$n = 5$).'

III. VIBRATIONS IN CIRCULAR PLATES

A. Flat plates

Waller gives the relative frequencies of over 90 modes of vibration in a circular brass plate with a free edge.[21] She observes that these frequencies agree with Chladni's law only when m is small, which is contrary to the Kirchhoff-Rayleigh conditions. If the frequency is written as a function of $m + bn$, she states that the number b will gradually increase from 2 to 5 as m increases. She explains this discrepancy as being due to the close spacing of the nodal lines for large m. When the spacing of the nodal diameters near the center is less than the thickness, thin-plate theory can be expected to fail.

Waller's data can be fitted reasonably well to a relationship of the type $f = c(m + 3n)^k$, as shown in Fig. 2. There are two disadvantages with this, however:

(1) The exponent k varies for different values of n;

(2) The relationship disagrees with the Kirchhoff-Rayleigh condition.

The same data can also be fitted to curves of the type $f = c(m + 2n)^k$ by allowing c as well as k to vary slightly with n, as shown in Fig. 3. The agreement with Chladni's law, and with the Kirchhoff-Rayleigh condition is now reasonably good.

B. Nonflat plates

We have studied the modes of vibration in a large number of percussion musical instruments that are platelike in their behavior, such as cymbals, gongs, tamtams, and bells. It is interesting to examine some of their modal frequencies in the light of Chladni's law.

Modal frequencies of a 24-in.-diameter cymbal are shown in Fig. 4. For $n = 0$ (no circular nodes), the data follow a line with two different slopes. For $m < 6$, k_1 is determined to be 1.86. For $m > 6$, $k_2 = 1.49$.

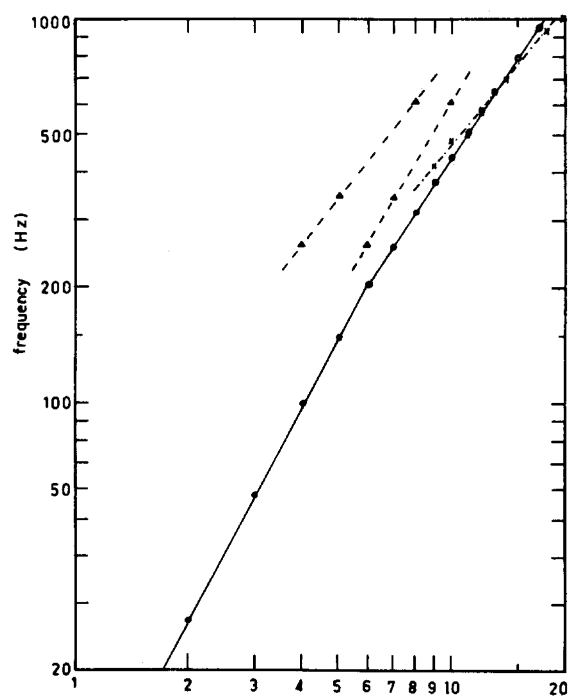

Fig. 4. Modal frequencies of a 24-in. diameter cymbal (○ $n = 0$, △ $n = 1$, × $n = 2$).

Table I. Parameters used to fit the vibration modes of cymbals having $n = 0$ to the equation $f = c(m + bn)^k$.

Cymbal	k_1	c_1(Hz)	k_2	c_2(Hz)	m_c [a]
24 in. thin	1.86	7.7	1.49	14.2	6
18 in. thin	1.75	10.1	1.56	14.1	5
	1.78	10.6	1.52	15.7	4.5
18 in. medium	1.65	13.4	1.46	18.2	4.7
	1.70	12.6	1.43	17.8	3.6
16 in. thin	1.81	10.8	1.48	18.8	5
	1.84	12.0	1.47	19.5	4
16 in. medium	1.70	13.8	1.53	18.3	5
	1.65	15.9	1.53	19.4	4.3
14 in. thick	1.47	20.6			

[a] m_c is the m value at which the slope changes from k_1 to k_2.

In a cymbal, it is difficult to observe modes with well-defined circular nodes. Often the modal frequencies can be estimated from observing modes that are combinations of two nearly-degenerate modes of different m and n [see Fig. 1(c), for example]. We have thus managed to identify three modes with $n = 1$ and six modes with $n = 2$ in this cymbal, which are also shown in Fig. 4. The $n = 2$ family fits reasonably well a line given by $f = c(m + bn)^k$ with $b = 4$, $k = 1.2$ and $c = 30$ Hz. The modes for $n = 1$ can be fitted to lines with $b = 2$, $k = 1.25$ or $b = 4$, $k = 1.7$.

Values of k and c for the $n = 0$ modes in a number of cymbals are given in Table I. In most cases a distinct change in slope (k) is noted at a particular value of m denoted as m_c. The largest k-values occur when the cymbal is large and thin; the cymbals then approach flat-plate behavior.

The vibration modes of church bells[22] and small handbells[23] can also be described in terms of the nomenclature of plate modes, which they resemble. The frequencies of the modes in the $n = 0$ group (no nodal circles) decrease from k_1 to k_2 in the same manner as the $n = 0$ group in cymbals, although the decrease in slope is more gradual in the handbells. Furthermore, the initial slope k_1 is greater in the case of handbells.

In the case of the $n = 1$ group, the initial slope k_1 is negative; that is, the frequency decreases with increasing number of nodal meridians m. This behavior can be explained by comparing the bell vibrating in one of these modes to a cylindrical shell closed at one end, which stretches as it vibrates.[23] Values of k_1 and k_2 for three bells are indicated in Table II.

IV. CONCLUSION

The vibration frequencies of flat and non-flat circular plates can be fitted to the relationship $f = c(m + bn)^k$. By proper choice of c it is possible to satisfy Chladni's law ($b = 2$, $k = 2$) in flat plates over quite a wide range of frequency. Nonflat plates require values of b and k that are slightly different from two, however.

In cymbals of different sizes, the $n = 0$ groups of modes fit curves having slopes of $k_1 = 1.65$ to 1.86 for low mode numbers and $k_2 = 1.43$ to 1.56 for high mode numbers. Rather limited data on modes having nodal circles suggest that b may vary from 2 to 4. In handbells, the modes without nodal circles have slopes which range from $k_1 = 2.32$ to 2.44 for low mode numbers and $k_2 = 1.55$ to 1.77 for high mode numbers. Thus there is some similarity with the behavior of the corresponding modes in cymbals.

Table II. Parameters used to fit the vibration modes of handbells having $n = 0$ and $n = 2$ to the equation $f = cm^k$.

Bell	n	k_1	c_1(Hz)	k_2	c_2(Hz)
C_4	0	2.44	52.3	1.55	192
C_5	0	2.37	101	1.77	266
C_6	0	2.32	193	1.74	538
C_4	1	neg.		2.01	73.9
C_5	1	neg.		1.80	214
C_6	1	neg.		1.63	492

ACKNOWLEDGMENTS

I would like to thank the Avedis Zildjian Company and Clive Hardder for the loan of cymbals, Malmark, Inc. for the loan of the handbells, and Robert Shepherd for making some of the measurements. Finally, I thank the National Science Foundation Division of International Programs for support and N. H. Fletcher and the University of New England for their hospitality during the writing of this paper.

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