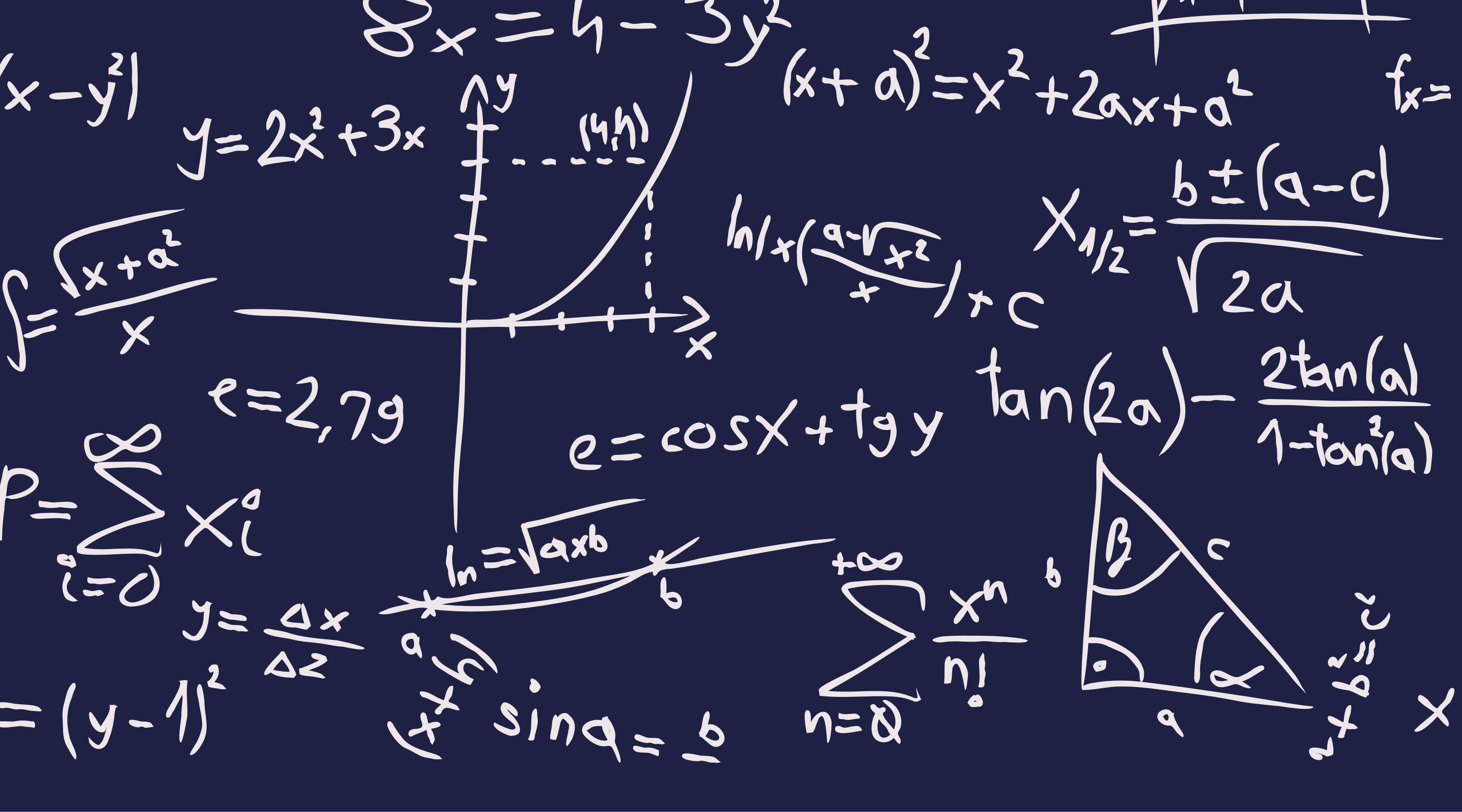
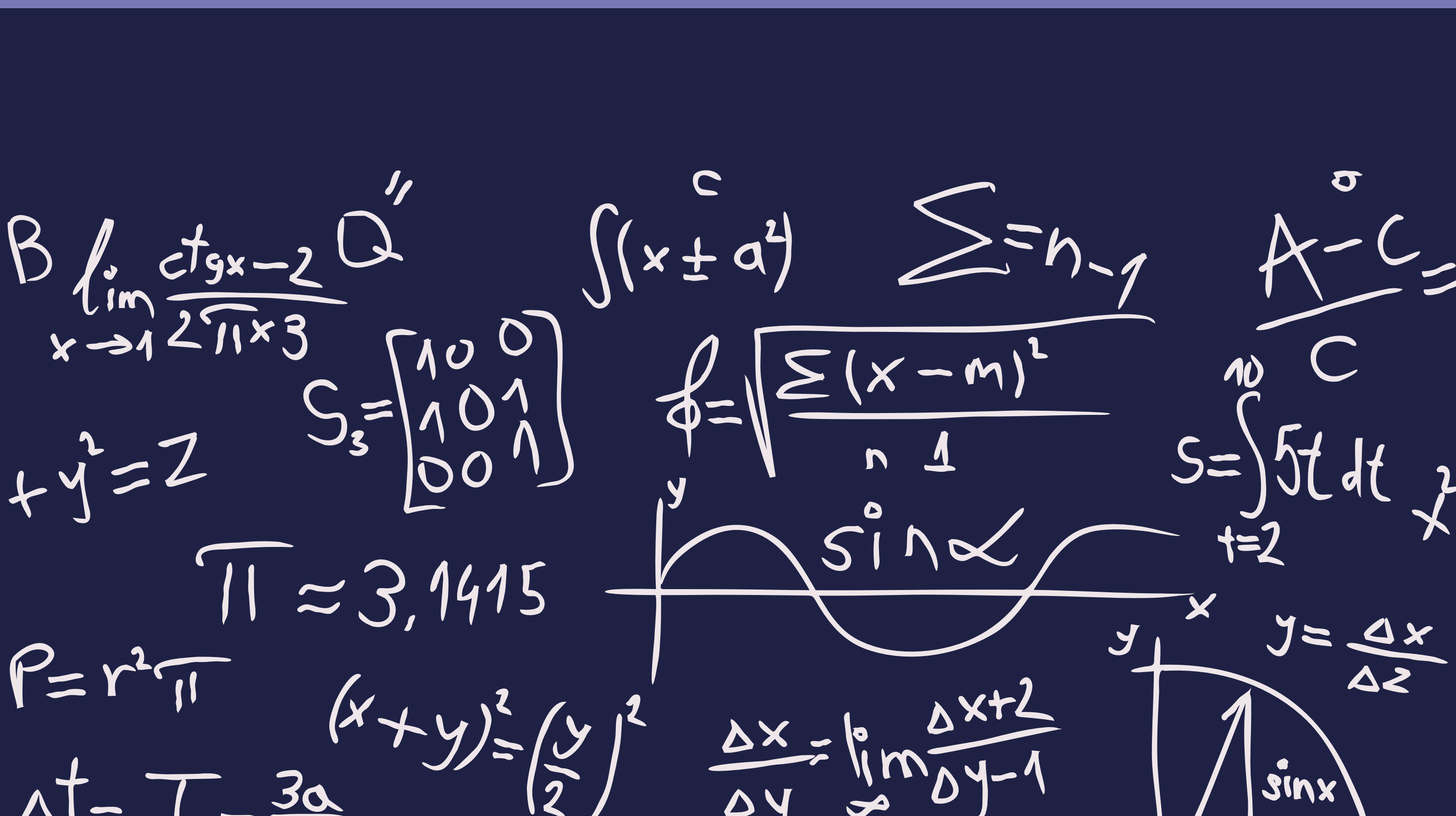


Álgebra Lineal 2

Una aventura en las matemáticas

Rudik Roberto Rompich



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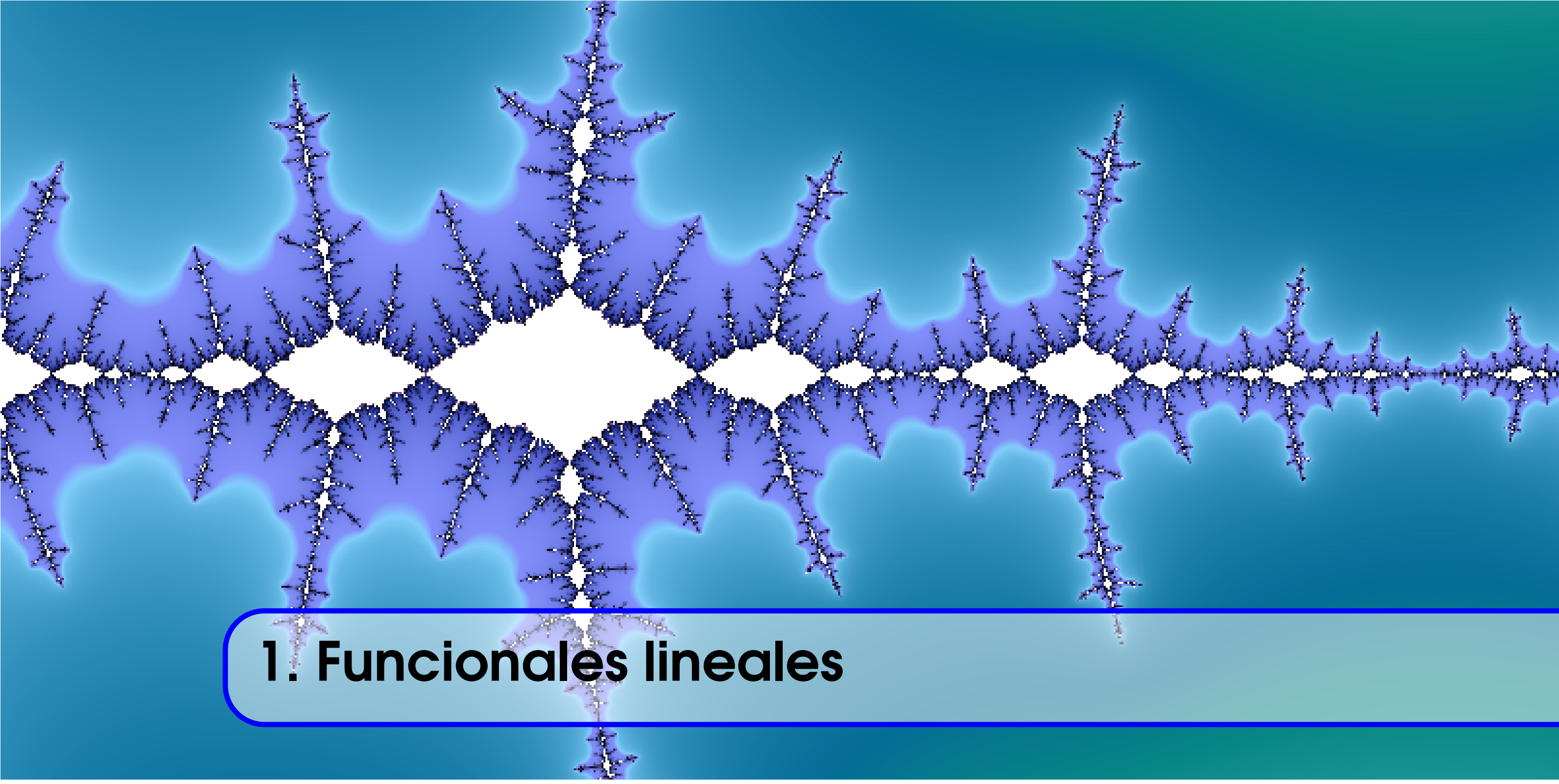
Funcionales lineales



1. Funcionales lineales

1.1 Ejercicios y Teoremas

Definición 1.1.1 Sea V un espacio vectorial sobre $\mathbb{F}$. Entonces, una transformación lineal

$$f : V \mapsto \mathbb{F}$$

es un funcional lineal sobre V.

Ejercicio 1.1 Sean $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Definimos:

$$\Phi : \mathbb{R}^n \mapsto \mathbb{R} \ni$$

$$\Phi(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

$\Rightarrow \Phi$ es funcional lineal. ■

Nota 1.1.

$$f : \mathbb{R}^2 \mapsto \mathbb{R} \ni$$

$$\Phi(x, y) = 2x - y$$

Ejercicio 1.2 Sea $C[0,1]$ un conjunto de funciones continuas en [2,1] y considere:

$$T : C[0,1] \mapsto \mathbb{R} \ni$$

$$T(g) = \int_0^1 g(x)dx$$

Nótese si $f, g \in [0,1]$ y $\alpha \in R \Rightarrow T(\alpha f + g) = \int_0^1 [\alpha f + g](x)dx = \int_0^1 [(\alpha f)(x) + g(x)]dx = \alpha \int_0^1 f(x)dx + \int_0^1 f(x)dx + \int_0^1 g(x)dx = \alpha T[f] + T[g] \Rightarrow$ es lineal $\Rightarrow$ T es funcional lineal. ■

Ejercicio 1.3 Sea

$$d : \mathbb{R}^{nxn} \mapsto \mathbb{R} \ni$$

$$d(A) = \text{determinante de A}$$

Recordar que:

$$det(A+B) \neq det(A) + det(B)$$

$$det(\alpha A) \neq \alpha det(A)$$

$$d(A) \text{ no es funcional lineal}$$

■

Ejercicio 1.4 Sea

$$T : \mathbb{R}^{nxn} \mapsto \mathbb{R} \ni$$

$$T(A) = \text{ traza de A}$$

$$\text{Si } A = [a_{ij}] \Rightarrow Tr(A) = \sum_{i=1}^{n} a_{ij}$$

$$\Rightarrow Tr(A) \text{ es funcional lineal.}$$

■

Ejercicio 1.5 Sea V el espacio de todas las funciones sobre $\mathbb{R}$. Definimos

$$C_t : V \mapsto \mathbb{R} \ni$$

$$C_t(f) = f(t) \text{ , donde t es un número fijo.}$$

Nótese que:

1. Sea $f, g \in V \Rightarrow L_t[f+g] = (f+g)(t) = L_t(f) + L_t(g)$
2. Sea $\alpha \in \mathbb{R} \mapsto L_t(\alpha f) = (\alpha f)(t) = \alpha f(t) = \alpha L_t(f) \Rightarrow$ Es funcional lineal.

■

Nota 1.2. *Considere el funcional lineal*

$$f : \mathbb{R}^n \mapsto \mathbb{R} \ni$$

$$f(x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

$$\alpha_1 \in \mathbb{R}$$

(fijos) Sea $B = \{e_1, \ldots, e_n\}$ la base usual de $\mathbb{R}^n$ y sea $B' = \{1\}$ la base usual de $\mathbb{R}$.

$$f(1,0,\ldots,0) = \alpha_1(1) + \alpha_2(0) + \ldots + \alpha_1(0)$$

$$= \alpha_1$$

$$f(0,1,0,\ldots,0) = \alpha_2$$

$$\vdots$$

$$\Rightarrow [f]_B^{B'} = [\alpha_1\, \alpha_2 \ldots \alpha_n]$$

Nota 1.3. *Si f son funcionales lineales.*

$$f \in f[V,\mathbb{F}] \text{ si dim(V)=n}$$

$$\Rightarrow dim(f[V,\mathbb{F}]) = n \cdot 1 = n$$

Definición 1.1.2 — V^*. Al espacio de funciones lineales de V es $\mathbb{V}$ es $\mathbb{F}$ se le llama al espacio dual de V.

Nota 1.4. *Si $dim(V) = n \Rightarrow dim(V^*) = n$*

1.1.1 Teorema

Teorema 1.1.1 Sea V un espacio vectorial finito dimensional y $B = \langle v_1, ..., v_n \rangle$ una base ordenada de V. Entonces, existe una base: $B^* = \{\Phi_1, ..., \Phi_n\}$ de V^*, tal que:

$$\Phi_i(V_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

$$\Phi_i(V_j) = \delta_{ij} \xleftarrow{\text{Delta de Kronecker}}$$

1.1.2 Ejercicio

Ejercicio 1.6 Considere la base de $\mathbb{R}^2$,

$$B = \{(2,1),(3,1)\}$$

entonces, encuentre una base para para $(\mathbb{R}^{\nvDash})^* \xleftarrow{\mathscr{L}[\mathbb{R}^2,\mathbb{R}^2]}$ ■

Solución:

$$B^* = \{\phi_1, \phi_2\} \qquad \text{es tal que:} \tag{1.1}$$

Debemos encontrar $\alpha_1, \alpha_2, \beta_1, \beta_2$

$$\phi_1(x,y) = \alpha_1 x + \alpha_2 y \tag{1.2}$$
$$\phi_2(x,y) = \beta_1 x + \beta_2 y \tag{1.3}$$

Encontramos $\alpha_1 y \alpha_2$:

$$\phi_1(v_1) = \phi(2,1) = 2\alpha_1 + \alpha_1 = 1 \qquad \delta_{11} \tag{1.4}$$
$$\phi_1(v_2) = \phi(3,1) = 3\alpha_1 + \alpha_2 = 0 \qquad \delta_{11} \tag{1.5}$$
$$\Longrightarrow \alpha_1 = -1, \alpha = 3 \tag{1.6}$$
$$\Longrightarrow \phi_1(x,y) = -x + 3y \tag{1.7}$$

Encontramos B_1, B_2:

$$\phi_2(v_1) = \phi_2(2,1) = 2\beta_1 + \beta_2 = 0 \tag{1.8}$$
$$\phi_2(v_2) = \phi_2(3,1) = 3\beta_1 + \beta_2 = 1 \tag{1.9}$$
$$\Longrightarrow \beta_1 = 1, \beta_2 = -2 \tag{1.10}$$
$$\Longrightarrow \phi_2(x,y) = x - 2y \tag{1.11}$$

$\Longrightarrow$ La base dual de B, denotada por B^* (i.e. la base del espacio dual $V^{)}*$, es $B^* = \{-x+3y, -x+2y\}$

1.1.3 Ejercicio

Ejercicio 1.7 Dada la base de $\mathbb{R}^3$:

$$B = \{(1,-1,3),(0,1,-1),(0,3,-2)\},$$

encuentre la base dual de B^* (i.e. la basa para $\mathscr{L}[\mathbb{R}^3,\mathbb{R}]$) ■

$$\phi_1(x,y,z) = \alpha_1 x + \alpha_2 y + \alpha_3 z \tag{1.1}$$
$$\phi_2(x,y,z) = \beta_1 x + \beta_2 y + \beta_3 z \tag{1.2}$$
$$\phi_3(x,y,z) = \gamma_1 x + \gamma_2 y + \gamma_3 z \tag{1.3}$$

Encontramos $\alpha_1, \alpha_2, \alpha_3$:

$$\phi_1(v_1) = \phi_1(1,-1,3) = \alpha_1 - \alpha_2 + 2\alpha_3 = 1 \tag{1.4}$$
$$\phi_2(v_2) = \phi_1(0,1,-1) = 0\alpha_1 + \alpha_2 - 1\alpha_3 = 0 \tag{1.5}$$
$$\phi_3(v_3) = \phi_1(0,3,-2) = 0\alpha_1 + 3\alpha_2 - 2\alpha_3 = 0 \tag{1.6}$$
$$\implies \alpha_1 = 1, \alpha_2 = \alpha_3 = 0 \implies \phi_1(x,y,z) = x \tag{1.7}$$

Encontramos $\beta_1, \beta_2, \beta_3$:

$$\phi_2(v_1) = \phi_2(1,-1,3) = 1\beta - 1\beta + 2\beta = 0 \tag{1.8}$$
$$\phi_2(v_2) = \phi_2(0,1,-1) = 0\beta + 1\beta - 1\beta = 1 \tag{1.9}$$
$$\phi_2(v_3) = \phi_2(0,3,-2) = 0\beta + 3\beta - 2\beta = 0 \tag{1.10}$$
$$\implies \beta_1 = 7, \beta_2 = -2, \beta_3 = -3 \implies \phi_2(x,y,z) = 7x - 2y - 3z \tag{1.11}$$

Encontramos $\gamma_1, \gamma_2, \gamma_3$

$$\phi_3(v_1) = \phi_3(1,-1,3) = 1\gamma - 1\gamma + 2\gamma = 0 \tag{1.12}$$
$$\phi_3(v_2) = \phi_3(0,1,-1) = 0\gamma + 1\gamma - 1\gamma = 1 \tag{1.13}$$
$$\phi_3(v_3) = \phi_3(0,3,-2) = 0\gamma + 3\gamma - 2\gamma = 0 \tag{1.14}$$
$$\implies \gamma_1 = 2, \gamma_2\gamma_3 = 1 \implies \phi_3(x,y,z) = -2x + y + z \tag{1.15}$$

Por lo tanto:

$$B^* = \{x, 7x - 2y - 3z, -2x + y + z\} \tag{1.16}$$

Por otra parte, es necesario probar que B^* es linealmente independiente

$$\text{Considere: } \upsilon_1\phi_1 + \upsilon_2\phi_2 + \ldots + \upsilon_n\phi_n = 0 \tag{1.17}$$
$$\text{A probar: } \upsilon_1 = \upsilon_2 = \ldots = \upsilon_n = 0 \tag{1.18}$$

Sea $v_i \in B$, $1 \leq i \leq n$

$$\implies (\upsilon_1\phi_1 + \ldots + \upsilon_1\phi_i + \ldots + \upsilon_n\phi_n)(v_i) = 0(v_i) \tag{1.19}$$
$$\implies \upsilon_1\phi_1(v_i) + \ldots + \upsilon_1\phi_i(v_i) + \ldots + \upsilon_n\phi_n)(v_i) = 0 \tag{1.20}$$
$$\implies 0 + 1 + 0 \qquad = 0 \tag{1.21}$$
$$\implies \upsilon_1 = \upsilon_2 = \ldots = \upsilon_n = 0 \implies \{\phi_1, \ldots, \phi_n\} \text{ es linealmente independiente} \tag{1.22}$$
$$\tag{1.23}$$

1.1.4 Teorema

Teorema 1.1.2 Sea V un espacio vectorial finito dimensiona y sea $B = \{x_1, \ldots, x_2\}$ una base ordenada para V. Entonces existe una base $B^* = \{\phi_1, \ldots, \phi_n\}$ para $V^* \ni$

$$\phi_i(x_j) = \delta_i j$$

Además,

(i) $\forall \phi \in V^*$ se tiene:

$$\phi = \phi(x_1)\phi_1 + \phi(x_2)\phi_2 + \ldots + \phi(x_n)\phi_n \qquad (1.1)\ (1.2)$$

(ii) $\forall x \in V$ se tiene que:

$$x = \phi_1(x)x_1 + \phi_2(x)x_2 + \ldots + \phi_n(x)x_n \qquad (1.3)\ (1.4)\ (1.5)$$

Demostración.

A probar: B^* es linealmente independiente. Considere:

$$\begin{aligned} \alpha_1\phi_1 + \ldots + \alpha_n\phi_n &= 0_n && (1.1)\ (1.2) \\ \text{funcional lineal} &= \text{funcional lineal} && (1.3) \end{aligned}$$

Aplicando a $x_i(i = 1, \ldots, n)$

$$\begin{aligned} &(\alpha_1\phi_1 + \ldots + \alpha_i\phi_i + \ldots + \alpha_n\phi_n)(x_j) && (1.4)\ (1.5) \\ \Longrightarrow\ &\alpha_1\phi_1(x_i) + \ldots + \alpha_i\phi_i(x_i) + \ldots + \alpha_n\phi_n(x_i) = 0 && (1.6) \\ &\Longrightarrow\ \alpha_i = 0 && (1.7) \\ &\Longrightarrow\ B^* \text{ es linealmente independiente.} && (1.8) \end{aligned}$$

(2)B^* genera a V^*

Sea $\phi \in v^*$. Entonces, sean:

$$\phi(x_1) = \mathscr{L}[_]1, \phi(x_2) = \mathscr{L}[_]2, \ldots, \phi(x_n) = \mathscr{L}[_]n \qquad (1.9)$$

Por otro lado, hagamos:

$$\begin{aligned} \sigma = \mathscr{L}[_]1\phi_1 + \mathscr{L}[_]2\phi_2 + \ldots + \mathscr{L}[_]n\phi_n && (1.10) \\ \Longrightarrow\ \sigma(x_1) &= (\mathscr{L}[_]i\phi_1 + \ldots + \mathscr{L}[_]n\phi_n)(x_1) && (1.11) \\ &= \mathscr{L}[_]1\phi_1(x_1) + \mathscr{L}[_]2\phi_2(x_1) + \ldots + \lambda_n\phi_n(x_1) && (1.12) \\ &= \mathscr{L}[_]1 + 0 + 0 && (1.13) \end{aligned}$$

De la misma forma:

$$\sigma(x_i) = \mathscr{L}[_]i \tag{1.14}$$

$$\implies \phi(x_i) = \sigma(x_i) \tag{1.15}$$

$$\implies \phi = \sigma = \mathscr{L}[_]1\phi_1 + \mathscr{L}[_]2\phi_2 + ... + \mathscr{L}[_]n\phi_n \tag{1.16}$$

(ii) Sea $x \in V \implies x = \alpha_1 x_1 + ... + \alpha_n x_n$, donde $\alpha_1, ..., \alpha_n \in \mathbb{F}$

$$\implies \phi_1(x) = \phi_1(\alpha_1 x_1 + ... + \alpha_n x_n) \tag{1.17}$$

$$= \alpha_1 \phi_1(x_1) + \alpha_2 \phi_1(x_2) + ... + \alpha_n \phi_1(x_n) \tag{1.18}$$

$$\implies \phi_1(x) = \alpha_1 \tag{1.19}$$

En general, $\phi_i(x) = \alpha_i, i = 1, ..., n$

$$\implies x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \tag{1.20}$$

$$\implies x = \phi_1(x)x_1 + \phi_2(x)x_2 + \phi_n(x)x_n \tag{1.21}$$

Sea $x \in V \implies x = \phi_1(x)x_1 + ... + \phi_n(x)x_n$

Si $\phi \in V^*$, entonces:

$$\phi_(x) = \phi[\phi_1(x)x_1 + ... + \phi_n(x)x_n] \tag{1.22}$$

$$= \phi_1(x) * \phi(x_1) + ... + \phi_1(x)\phi(x_1) \tag{1.23}$$

$$= \phi(x_1)\phi_1(x) + ... + \phi(x_2)\phi(x) \tag{1.24}$$

$$= [\phi(x_1)\phi_1 + \phi(x_2)\phi_2 + ... + \phi(x_n)\phi_n](x) \tag{1.25}$$

$$\implies \phi = \phi(x_1)\phi_1 + ... + \phi(x_2)\phi_n \tag{1.26}$$

■

1.1.5 Ejercicio

Ejercicio 1.8

Sea:

$$\mathbb{R}_1[x] = \{a + bx : a, b \in \mathbb{R}\} \tag{1.1}$$

Sean:

$$\phi_1 : V \mapsto \mathbb{R} \text{ y } \phi_2 : V \mapsto \mathbb{R} \ni \tag{1.2}$$

$$\phi_1(f(x)) = \int_0^1 f(x)\,dx \,,\ \phi_2(f(x)) = \int_0^2 f(x)\,dx \tag{1.3}$$

$$\text{Encuentre una base } B = \{f1, f2\} \text{ cuya base dual es: } B^* = \{\phi_1, \phi_2\} \tag{1.4}$$

■

Demostración.

Sean:

$$f_1 = a + bx \tag{1.1}$$
$$f_2 = c + dx \tag{1.2}$$
$$\phi_1(f_1) = 1 = \int_0^1 a + bx\,dx = ax + \frac{1}{2}bx^2|_0^1 = 1 \tag{1.3}$$
$$\phi_2(f_1) = 0 = \int_0^2 a + bx\,dx = ax + \frac{1}{2}bx^2|_0^2 = 0 \tag{1.4}$$
$$\tag{1.5}$$

$$a + \frac{1}{2}b = 1 \tag{1.1}$$
$$2a + 2b = 0 \tag{1.2}$$
$$\implies a = 2, b = -2 \tag{1.3}$$
$$\implies f_1 = 2 - 2x \tag{1.4}$$
$$\tag{1.5}$$

$$\phi_1(f_1) = 0 = \int_0^1 c + dx\,dx = c + \frac{1}{2}d \quad (1.1)$$

$$\phi_2(f_2) = 1 = \int_0^2 c + dx\,dx = 2c + 2d \quad (1.2)$$

$$\implies c + \frac{1}{2}d = 0 \quad (1.3)$$

$$2c + 2d = 1 \quad (1.4)$$

$$\implies c = \frac{-1}{2}, d = 1 \quad (1.5)$$

$$\implies f_2 = \frac{-1}{2} + x \quad (1.6)$$

$$\therefore B = \{1 - 1x, \frac{-1}{2} + x\} \quad (1.7)$$

■

1.1.6 Ejercicio (***)

Ejercicio 1.9

Sea

$$V = \mathbb{R}_2[x] = \{a + bx + cx^2, a, b, c \in \mathbb{R}^2\} \quad (1.1)$$

Nótese que $dim(\mathbb{R}_2[x]) = 3$

$$(1.2)$$

$$\text{Una base para } \mathbb{R} \text{ es } 1, x, x^2 \quad (1.3)$$

$$(1.4)$$

■

R No se prueba que genera ya que dim=3

Proposición 1.1.3

(i) Considere $\alpha_1, \alpha_2, \alpha_3$ números reales diferentes y definimos:

$$(1.1)$$

$$L_i : \mathbb{R}_2[x] \mapsto \mathbb{R} \ni \qquad i = 1, 2, 3$$

$$(1.2)$$

$$L_i(p) := p(\alpha_i) \quad (1.3)$$

Los L_i son funcionales lineales. En efecto, sean $p,q \in \mathbb{R}_2[x]$ y $\alpha \in \mathbb{R}$. Entonces:

$$L_i(\alpha p+1) = (\alpha p+1)(\alpha_i) = \alpha p(x_i)+q(\alpha_i) \tag{1.4}$$

$$= \alpha L_i(p)+L_i(q) \tag{1.5}$$

$$\Longrightarrow L_i \text{ es funcional lineal}, i=1,2,3 \tag{1.6}$$

$$\tag{1.7}$$

(ii) $\{L_1,L_2,L_3\}$ es linealmente independiente

$$V=\mathbb{R}_2[x] \qquad \Longrightarrow B=\{,,\} \tag{1.1}$$

$$V^* = \qquad \Longrightarrow B^*=\{L_1,L_2,L_3\} \tag{1.2}$$

Considere:

$$\beta_1L_1+\beta_2L_2+\beta_3L_3=0 \tag{1.3}$$

$$Sip_1(x)=1 \Longrightarrow \quad (\beta_1L_1+\beta_2L_2+\beta_3L_3)(p_1)=0(p_1) \tag{1.4}$$

$$\Longrightarrow \quad \beta_1L_1(p_1)+\beta_2L_2(p_1)+\beta_3L_3(p_1)=0 \tag{1.5}$$

$$\Longrightarrow \quad \beta_1p_1(\alpha_1)+\beta_2p_1(\alpha_2)+\beta_3p_1(\alpha_3)=0 \tag{1.6}$$

$$\Longrightarrow \quad \beta_1(1)+\beta_2(1)+\beta_3(1)=0 \tag{1.7}$$

$$\Longrightarrow \quad \beta_1+\beta_2+\beta_3=0 \tag{1.8}$$

$\{L_1,L_2,L_3\}$ es linealmente independiente $\Longrightarrow$ $\{L_1,L_2,L_3\}$ es una base para V^*

$$\tag{1.9}$$

(iii) Encuentra la base para V de la $\{L_1,L_2,L_3\}$ es dual

$$\text{Si } p_1(x)=1, p_2(x)=x \tag{1.1}$$

$$\Longrightarrow \quad (\beta_1L_1+\beta_2L_2+\beta_3L_3)(p_2)=0(p_2) \tag{1.2}$$

$$\Longrightarrow \quad \beta_1L_1(p_2)+\beta_2L_2(p_2)+\beta_3L_3(p_2)=0 \tag{1.3}$$

$$\Longrightarrow \quad \beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3=0 \tag{1.4}$$

Si $p_1(x)=1, p_2(x)=x, p_3(x)=x^3$

$$\Longrightarrow \quad (\beta_1L_1+\beta_2L_2+\beta_3L_3)p_3=0p_3 \tag{1.5}$$

$$\Longrightarrow \quad \beta_1L_1(p_3)+\beta_2L_2(p_3)+\beta_3L_3(P_3)=0 \tag{1.6}$$

$$\Longrightarrow \quad \beta_1\alpha_1^2+\beta_2\alpha_2^2+\beta_3\alpha_3^2=0 \tag{1.7}$$

Se tiene el sistema:

$$\beta_1+\beta_2+\beta_3=0 \tag{1.1}$$

$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0 \tag{1.2}$$

$$\alpha_1^2\beta_1+\alpha_2^2\beta_2+\alpha_3^2\beta_3=0 \tag{1.3}$$

Es decir, matricialmente:

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.4}$$

Si $\alpha_1,\alpha_2,\alpha_3$ son distintos, entonces:

$$det\begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix} \neq 0 \implies \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.5}$$

$$\implies \quad (\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)=0 \tag{1.6}$$

$$\implies \quad \{L_1,L_2,L_3\} \text{ es linealmente independiente} \tag{1.7}$$

$$\implies \{L_1,L_2,L_3\} \text{ es una base para } V^* \tag{1.8}$$

1.1.7 Ejercicio

Ejercicio 1.10

$\{f_1,f_2,f_3\}$ de $\mathbb{R}_2[x]$, tal que una base dual $B^*=\{L_1,L_2,L_3\}$

Nótese que:

$$\phi_i(x_j)=\delta ij \implies \quad \begin{aligned} L_i(f_j) &= \delta_{ij} && (1.1)\\ f_j(\alpha_i) &= \delta_{ij} && (1.2)\\ f_1(\alpha_{2j}) &= \delta_{ij} && (1.3)\\ f_2(\alpha_i) &= \delta_{ij} && (1.4) \end{aligned}$$

Esto quiere decir:

$$f_1(x) = \frac{(x-\alpha_2)(x-\alpha_3)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} \tag{1.5}$$

$$\implies \quad f_1(\alpha_1) = 1, f_2(\alpha_2) = f_3(x_3) = 0 \tag{1.6}$$

$$\implies \quad f_1(\alpha_i) = \delta_{ij} \tag{1.7}$$

$$f_2(x) = \frac{(x-\alpha_1)(x-\alpha_3)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} \tag{1.8}$$

$$f_3(x) = \frac{(x-\alpha_1)(x-\alpha_2)}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} \tag{1.9}$$

Polinomios de Lagrange (1.10)

■

1.1.8 Ejercicio

Ejercicio 1.11

Dados:

$$L_i = \mathbb{R}_2[x] \mapsto \mathbb{R} \ni \tag{1.1}$$

$$L_i(p) = p(\alpha_i) \tag{1.2}$$

L_i son funcionales lineales.

(1.3)

$\{L_1, L_2, L_3\}$ es linealmente independiente $\implies$ es base de $(\mathbb{R}_2[x])^*$

(1.4)

Se debe cumplir:

$$L_i(p_j) = \delta_{ij} \tag{1.5}$$

$$<=> \quad p_j(\alpha_i) = \delta_{ij} \tag{1.6}$$

$$\implies \quad p_j(\alpha_i) = \begin{cases} 1 & \text{si } j = i \\ 0 & \text{si } j \neq i \end{cases} \tag{1.7}$$

■

Demostración.

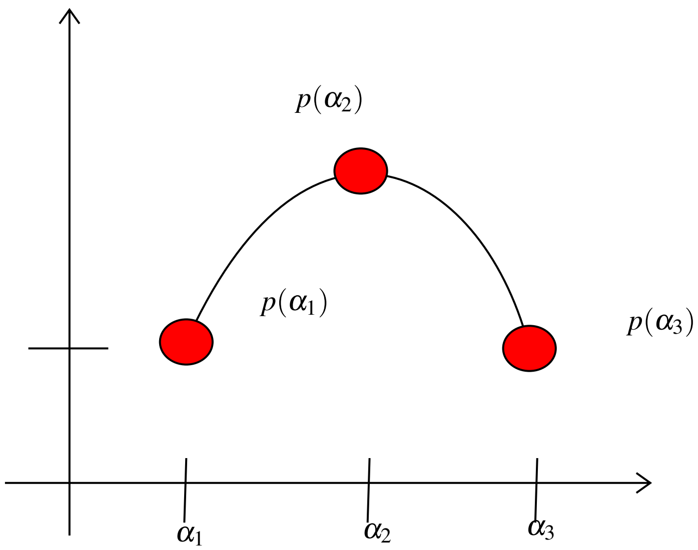
¿Quiénes son $\alpha_1, \alpha_2, \alpha_3$?

$$p_1(\alpha_1) = 1 p_1(\alpha_2) = 0 \qquad p_1(\alpha_3) = 0 \tag{1.1}$$
$$p_2(\alpha_1) = 0 p_2(\alpha_2) = 1 \qquad p_2(\alpha_3) = 0 \tag{1.2}$$
$$p_3(\alpha_1) = 0 p_3(\alpha_2) = 0 \qquad p_3(\alpha_3) = 1 \tag{1.3}$$
$$\Longrightarrow \qquad p_1(x) = \frac{(x-\alpha_2)(x-\alpha_3)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} \tag{1.4}$$
$$p_2(x) = \frac{(x-\alpha_1)(x-\alpha_3)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} \tag{1.5}$$
$$p_3(x) = \frac{(x-\alpha_1)(x-\alpha_2)}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} \tag{1.6}$$

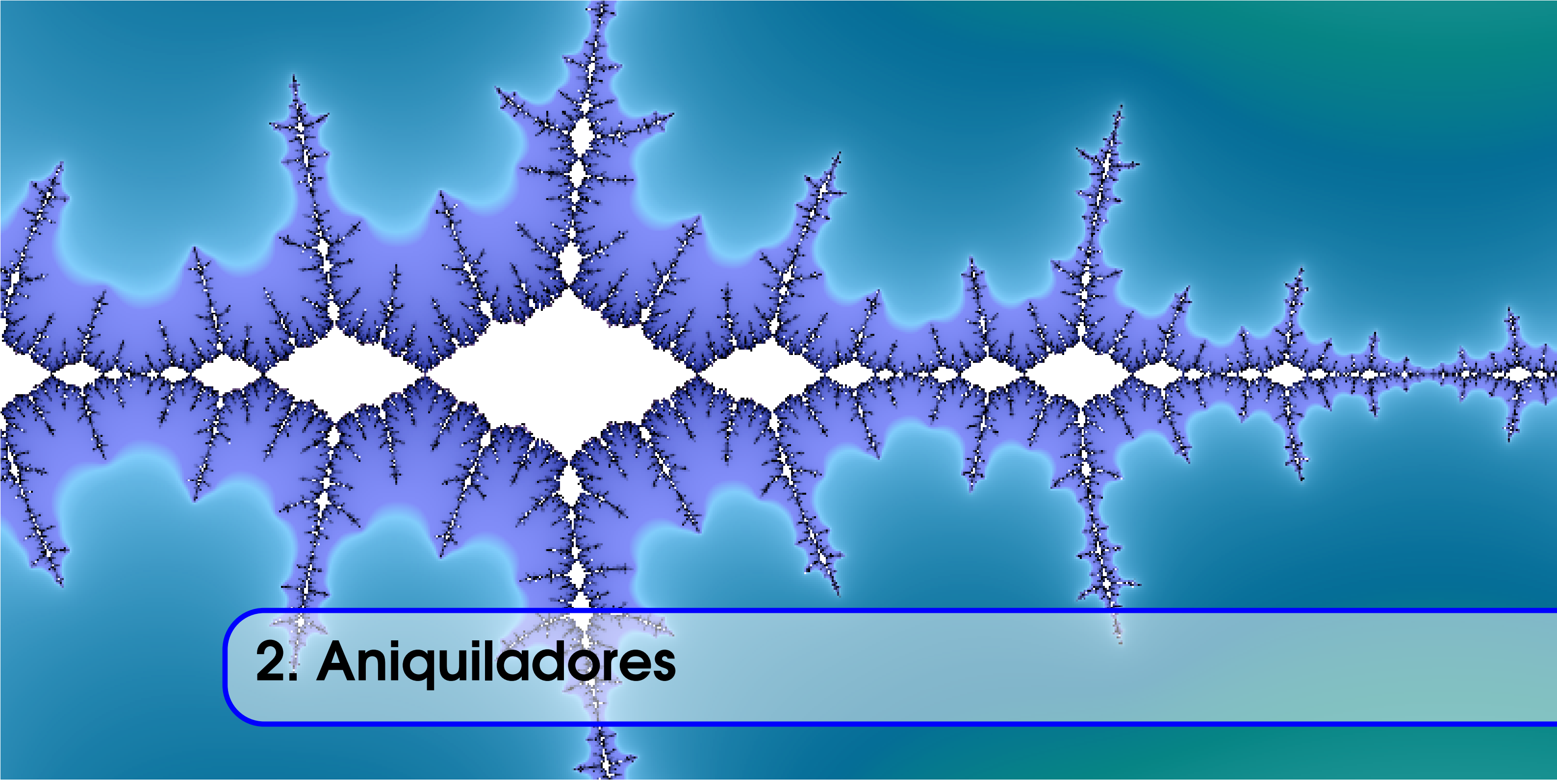
$\{p_1(x), p_2(x), p_3(x)\}$ es la base de $R_2[x] \ni \{L_1, L_2, L_3\}$ es una base dual. Además, si $p \in \mathbb{R}_2[x]$; entonces, $p = p(\alpha_1)p_1 + p(\alpha_2)p_2 + p(\alpha_3)p_3$ (Polinomio interpolante de Lagrange).

$$\tag{1.7}$$

■



II Aniquiladores



2. Aniquiladores