Lectures by Claudio Castelnovo

1 Lecture 1

1.1 Introduction

Aims: Brief introduction to fractionalised quasiparticles that can appear in frustrated magnetism.

Frustrated magnetism: The inability to minimise all terms of Hamiltonian simultanously, cannot be described by Landau-Ginzburg theory.

Has many interesting properties like

- Degeneracy
- Emergent Symmetries
- Topological Order
- Fractional Excitations

Lectures will be structured as series of examples.

1.2 Conventional Magnetism

We have crystal structure, lattice Λ and each site has magnetic moments μ_i. We shall take them to be along 1 axis, and so assume they are of Ising type with only nearest neigbour interactions.

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

where $J > 0$ for ferromagnetic interactions, which favour alignment. This is combatted by entropy, which favours random $\{\sigma_i\}$. The interactions come about from exchange couplings of neighbouring electrons[1]

low sym 1 High symm phase ———*——-¿ T/J

Has $\mathbb{Z}_2$ symmetry in high temperature phase and this symmetry is *broken.*

Remark. Each individual energy term is minimised

Example (Spin correlations)**.** How do we compute correlations? Define partition function

$$Z = \sum_{\{\sigma_i\}} \mathrm{e}^{-\beta H}$$

which then allows us to compute correlation

$$\langle \sigma_l \sigma_m \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} \sigma_l \sigma_k \mathrm{e}^{\beta J \sum_{\langle ij \rangle} \sigma_i \sigma_j}$$

At high temperature $\beta J \ll 1$, so that the lowest terms in the taylor expansion of the probality density dominate, and we obtain

$$\langle \sigma_l \sigma_k \rangle \simeq \frac{2\beta J_{lm}}{Z}$$

[1]Limit of Hubbard model? Must investigate

where $J_{lm} = \begin{cases} J & l, m \text{ are n.neighbours} \\ 0 & \text{otherwise} \end{cases}$ These correlations are *trivial* correlation as the spins only reflect the hamiltonian.

For conventional magnetism, the behaviour is therefore trival everywhere except near critical point. At low temperature the order is trival; If $\sigma = 1$ then most likely all other σ will also be 1. At high temperature the correlations are trivial $\langle \sigma\sigma \rangle \sim \beta H$, where the part of the hamiltonian refered to is the interaction of the two appropriate spins. The kind of interesting behaviour seen near critical point

- Spin correlations decay with power law
- Scale invariance
- Critical scaling of physical properties with distance from C.P. in parameter space. ($C_V \sim |T - T_C|^\alpha$, $\langle \sigma \rangle \sim B^{1/\delta}$)
- Universality

Example (Quantum phase transitions)**.** Now assume quantum spins and add a transverse field Γ (Quantum phase transition arises due to non-commutativity of terms)

$$H = -J \sum_{\langle ij \rangle} \sigma_i^Z \sigma_j^Z + \Gamma \sum_i \sigma_i^X .$$

Now 2D parameter space of T, Γ. phase boundary of ordered and disordered phases is now a line, and at $T = 0$, we have a Quantum critical point. Fluctations near the quantum critical point take much different form from usual, as it will be of $d + 1$ dimensions in contrast to the classical which is at d dimensions. These fluctations spreaed further into the phase diagram into a cone in the quantum critical regime.

Conventional magnetism has trivial correlations apart from critical behaviour. Relies on being minimise all energy terms at the same time. Instead can have competition between lattice geometry and interaction terms.

1.3 Frustrated Magnetism

$$H = \sum_{ij} H_{ij}$$

such that not all H_{ij} can be minimised at the same time. This supresses the critcal temperature significantly $T_c/J \ll 1$, where J is characteristic scale of H_{ij}, and in the region $T_c < T \lesssim J$ non trivial region. One can pseudo quantify the degree of frustration by the ratio $f = \frac{T_c}{|\Theta_{CW}|}$, of the critical temperature and the Curie-Weiss temperature. This region is a new phase called a *spin liquid*.

Remark (Properties of Spin liquids)**.** Spin liquids can both be quantum and classical show

- No long range order
- Strong correlations, as $T < J$

- Have large degeneracy/entanglement
- Emergent Symmetries
- Topological order
- Fractionalised Quantum particles.

and examples of systems exhibiting it include Triangular Ising AFM4, 8-vertex model, and the Quantum Toric Code. Note that the frustration does not always lead to these properties, but it does allow them. Note that it is only bounded by a critical point on the lower bound in classical case, hence it is technically a conventionally defined states. This may not be the case for quantum systems, where the system may be gapped[2]. Note theoretical models can be *fully frustrated* with $f = 0$, though these are not physical, as they have finite 0 temperature entropy.

1.4 Example: Triangular Ising AFM (Both Lecture 1 and 2)

A completely classical anti-ferromagnetic model

$$H = J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

with $J > 0$. On triangular lattice, cannot minimise all terms. Consider a triangle; We can at most have 2 anti-ferromagnetic bonds for every 3, and once the ferromagnetic bond has been chosen, there is two-fold spin degeneracy. These are *2-1 triangles* and can be used to tile the whole lattice.

Remark. For square lattice, can easily minimise all energy terms by making them anti-ferromagnetic.

In particular, can label the ferromagnetic bond of each triangle by a *dimer* going between the centers of the two triangles which the bond touches. This gives a $2:1$ mapping of spins to dimers,called *the dimer model.* The centers of the triangular form a honeycomb *dual lattice* which is where the dimers live. Low energy state has one dimer touching each center. Can also have high energy excitation with 3 ferromagnetic bonds, and so 3 dimers touching a center.

To minimise energy, we need exactly one dimer to touch each center, which is possible at least one way. In fact there are many such ways; take any closed (or ending at the boundary of your lattice) path of alternating dimers and non-dimers, and reverse whether each connection is a dimer or not. This generates an equivalent ground state. Given the number of such possible paths, the total ground state degeneracy is hence exponential in the lattice size; so the ground state entropy is exponential in the number of spins. This is a *fullly frustrated system.* Note there is a way to solve for the degeneracy exactly in the thermodynamic limit.

Definition 1.1 (Full frustration)**.** A fully frustrated system has an exponential (in system size) number of equivalent ground states.

[2] expand on this point?

Now we can reformulate again in a flux language. Divide honeycomb lattice into 2 sublattices A and B. Let all dimers carry 2 flux units; for A they go into B, and for B they emerge from A. The remaining lines carry 1 flux unit so as to make the fluxes divergenceless. We may therefore think of our system as a discretised divergenceless field, like that in magnetostatics, which has an *emergent Gauge Symmetry* (Coulomb phases) [Henley, Ann.Rev.Cond.Mat.2010]. We know how to handle correlators for the magnetostatic case, so if we can back propagate, we may solve our original problem. From Monte-Carlo simulations, this works at long distances!

Note that we have had an emergent Gauge Symmetry because we have effectively remove a large portion of the high-excitation states. This reduced statespace now has a new emergent symmetry. This is completely different from the mechanism of symmetry breaking.

Can we explicitly write Hamiltonian in Dimer Language?

$$\begin{aligned} H &= +J \sum_{\langle ij \rangle} \sigma_i \sigma_j \\ &= H_0 \end{aligned}$$

Is constant in Dimer Language for $2-1$ triangle.

For any biparte lattice, can place dimers to get an emergent Electromagnetic gauge symmetry.

Can we have resonating dimers? Yes Rokhsar-Kivelson model.

2 Lecture 2

2.1 8 Vertex model

Simple model for discussing excitations. Classical model of ising spins with Hamiltonian (note on figures, red is up, blue is down). The lattice sites are S, the spin sites will be i, j, and center of plaquetes will be P. The Hamiltonian is

$$H = -\lambda \sum_S \prod_{i \in S} \sigma_i$$

with $\lambda > 0$. So product of 4 spins sorrounding each site. This makes the model less physical, as it is a four body interaction.

The lowest energy state $\prod_{i \in S} \sigma_i = +1$, which there are 8 possible minimal states per vetex (Hence 8 vertex model). In total there is exponentially many ways of arranging them on lattice, so ground state entropy is finite.

Example. Can you find exact degeneracy?

Note there is complete $\mathbb{Z}_2$ symmetry.

Example (Pauling's estimate of degeneracy)**.** Number of ground states can be estimated as

$$N_{\text{GS}} = \text{Number of states} \times \text{reduction factor}^{\text{Number of terms in hamiltonian}} = 2^N \cdot \left(\frac{8}{16} \right)^{N/2}$$

So entropy is $\sim \frac{N}{2} \ln 2$. This is a usually an informative estimate. What does it neglect? Correlations between constraints, means that we overcount.

It turns out that this estimate is exact as correlations are zero length in the ground state.

Can do better! For any plaquette flip all spins sorrounding it; this flips two spins for all vertices. This generates a new ground state! We can start with the ground state with all spin up. Now we can flip any of the $N/2$ plaquettes to generate all possible ground states. In total $2^{N/2}$ possible operations, but they are not always independent. For example, assuming periodic boundary conditions, flipping all all but one plaquette is equivalent to flipping that plaquette, so there is a total of $2^{N/2-1}$. There are two further independent operations, called *winding loop flips*, which flips horisontal or vertical lines of spins. These give exactly $2^{N/2+1}$ ground states.

The exactness of our estimate, the is due to a coincidence due to zero range correlations.

$$
\begin{aligned}
\langle \sigma_l \sigma_m \rangle_{\mathrm{GS}} \simeq \sum_{\{\sigma_i\}} \sigma_l \sigma_m \\
&= \sum_{\{\sigma_i\}} \frac{1}{2} \left[\sigma_l \sigma_m + \tilde{\sigma}_l \tilde{\sigma}_m \right] \\
&= 0
\end{aligned}
$$

where the tilde are obtained by filpping the plaquette that includes l and not m. This is not a totally trivial model though. In particular, there are *topological correlations* of *Wilson loops*. In particular, let $\gamma_{h,v}$ be horisontal or vertical loops of spins, passing through plauqetes. The product of spins along the lines

$$
\Gamma_{h,v} = \prod_{i \in \gamma_{h,v}} \sigma_i = \pm 1
$$

This is invariant under plaquette flips, but only observable if you know all spins along lines. This is non-local order, which divides the config space into 4 topological sectors. Note that it does not matter which line we choose; only the winding number matters.

2.1.1 Excitations

If we flip a single spin we create two excited vertices. Now can move these around independently by the plaquette flip operations. Now the creates a string, which costs energy on at the end points. However, we can move the string around without energy cost, and is not actually detectible. Hence we have *fractionalised* our spins into two excitations.

Example (Conventional vs frustrated magnetism)**.** Conventional: Excitations are domain walls Frustrated are point like QP on random walks, which have pair creation/annihalation.

2.2 The Quantum Toric Code

Now put spin-1/2 on each bond of a 2D square lattice. Let S be all spins around a vertex and P all spins around a Plaquette. Now define operators

$$A_S = \prod_{i \in S} \sigma_i^z$$
$$B_P = \prod_{i \in P} \sigma_i^x$$

Now B_P effectively implements the plaquette flips from before. Let the Hamiltonian be

$$H = -\Delta_S \sum_S A_S - \Delta_P \sum_P B_P$$

with both constants $\Delta_{S,P} > 0$. All A, B commute, so in fact the ground state is eigenstate of all A, B. Hence the ground state satisfies

$$A_S \left|\phi_0\right\rangle = B_P \left|\phi_0\right\rangle = \left|\phi_0\right\rangle$$

Now the ground state can be found as as a superposition of all ground states of A; and these can be built of $|1 \ldots 1\rangle$ by application of B_p. So if we project out -1 eigenvalues of all B_P, we obtain the ground state

$$\left|\phi_0\right\rangle \sim \prod_P (1 + B_P) \left|1 \ldots 1\right\rangle$$

Is the ground state of unique? No! Think about the winding loop flips from the classical case. Let $\gamma_{h,v}$ be line of spins along edge (note different convention to classical case), and let

$$\Gamma_{h,v} = \prod_{i \in \gamma_{v,h}} \sigma_i^x$$

These comute with A, B so they are simultanously diagonalisable and have eigenvalues ± 1. Hence they define $2^2 = 4$ ground states $|\phi_0^{(m_h, m_v)}\rangle$. These $m_{h,v}$ are *topological quantum numbers*! The Γ are called *Wilson loop operators*. It is not possible to locally distinguish between topological sectors.

Now define line through plaquettes and spins $\tilde{\gamma}_{h,v}$, and corresponding operators

$$\tilde{\Gamma}_{h,v} = \prod_{i \in ga\tilde{m}ma_{h,v}} \sigma_i^z$$

These also commute with A, B, H but not with $\Gamma_{v,h}$. Rather they anti-commute! ($[\tilde{\Gamma}_h, \Gamma_h] = [\tilde{\Gamma}_v, \Gamma_v] = 0$) The information encoded by $(\Gamma_h, \tilde{\Gamma}_v)$ and $(\Gamma_v, \tilde{\Gamma}_h)$ are equivalent to a spin-1/2 algebra; hence the toric code GS encodes 2 toplogical qubits.

2.3 Toric code excitations

Excitations are plaquettes where $A_s = -1$ or $B_p = -1$, which are point-like quasiparticles. Act with σ_i^x operator flips σ_i^z eigenvals, so creates two deconfined $A_s = -1$ stars. Commutes with B_p, so no B_p excitations. Now could also act with σ_i^z flips σ_i^x eigenvals, so creates to deconfined $B_p = -1$ excitations, while leaving all the star excitations unaffected. Note that these excitations live at different places; plaquettes (or visons) or stars (or spinons). Individually, they have the same properties as classical 8-vertex case, hence they are bosons.

Their mutual statistics are very interesting however, as they have *mutual fractional statistics*. In particular if a spinon circles a visons, this induces a -1 phase! When put together, they behave as half a fermion, called a *semion*. Note this is different from the fermion case, since fermions would pick up two -1 phase changes, which in total is no phase at all.

Note this is the same behaviour we would expect if the spionons were electrons, and the visons were particles of π magentic flux.

How do we see this? A state with spinons at the end of strings γ_s and visons at the ends of$\tilde{\gamma}_p$ is $|\psi_1\rangle = \left(\prod_{i\in\gamma_s}\sigma_i^x\right)\left(\prod_{i\in\tilde{\gamma}_p}\sigma_i^z\right)|\psi_0\rangle$. Now define a closed path γ_c intersecting the end of γ_s and circling the vison. Transporting the spinon around the this path gives a new state $|\psi_2\rangle = \left(\prod_{i\in\gamma_c}\sigma_i^x\right)|\psi_1\rangle$. This state has the same excitations in the same place, but the paths connecting them with their partners are different.

$$\begin{aligned}
|\psi_2\rangle &= \left(\prod_{i\in\gamma_c}\sigma_i^x\right)\left(\prod_{i\in\gamma_s}\sigma_i^x\right)\left(\prod_{i\in\tilde{\gamma}_p}\sigma_i^z\right)|\psi_0\rangle \\
&= -\left(\prod_{i\in\gamma_s}\sigma_i^x\right)\left(\prod_{i\in\tilde{\gamma}_p}\sigma_i^z\right)\left(\prod_{i\in\gamma_c}\sigma_i^x\right)|\psi_0\rangle \\
&= -|\psi_1\rangle
\end{aligned}$$

As the the closed loop operator is identity on the ground state, as it is a product of B_p. The anti-commutativity arises because the new string bewteen the spinons intersect the vison string at a single spin, and σ_i^x, σ_i^z anti-commute.

In summary, the visons and spinons behave as semions, analogous to electrons and magentic *pi* fluxes. However, it should be noted that these excitations are *completely static*! We can add dynamics with perturbations, such as transverse fields

$$H = -\Delta_s\sum_s A_s + h_s\sum_i\sigma_i^x - \Delta_p\sum_p B_p + h_p\sum_i\sigma_i^z$$

Note that the ground state is stable for this hamiltoninian as long as $h_s < \Delta_s$ and $h_p < \Delta_p$.

Note semionic behaviour is only possible in $2D$ for point defects. In 3D need eg. line/loop defects.

3 How can we probe Quantum spin liquid?

- Entanglement (not experimentally visible to date)
- Topological degeneracy (possibly in the form of edge modes)
- fractionalised quasiparticles/fractional quantum numbers/statistics. (More promising!)

Consider the Toric code in transverse field at finite temperature

0 $h_p \Delta_p$ h_s $Delta_s$ *—-* ——* —-* — At $T \to \infty$ have paramagnet. At $T \lesssim \Delta_p$ have Quantum spin liquid. At $h_s \lesssim T \lesssim \Delta_s$ have classical 8-vertex model with excitations. If $\Delta_p \ll h_s$ the $\Delta_p \lesssim T \lesssim h_s$ we have interesting finite T regime. In particular the visons are thermally densely populatied $\rho_v \sim e^{-\Delta_p/T} \sim 1$ and they hop incoherently. In contrast the spinons are sparse $\rho_s \sim e^{-\Delta_s/T} \ll 1$ and hop coherently as $T < h_s$. The characteristic time scales are $t_v \sim 1/T$ and $t_s \sim 1/h_s \ll t_v$ Hence we can think of it one of two ways

- Static visions and spinons dynamics
- Born-Oppenheimer of instant spinons and visons relaxing stochastically.

Have a look a papers in lecture 4.

3.0.1 Spinon dynamics

The Spinon dynamics can be modelled say with tight binding with π fluxes in the background, and then average over background. Note that we see emergent disorder, and get weak anderson localisation. If we study

$$\langle r_s^2 \rangle (t) \sim \begin{cases} t^2 & t \ll \xi \\ t^{0.25} & \text{intermediate} \\ \text{Const.} & t \to \infty \end{cases}$$

3.0.2 Visons motion

Use Born-Oppenheimer method to find vison-spinon correlations. Spinons will instantly be in equilibrium. Have Free energy competition

- Spinon kinetic energy favours uniform spread out wavefunction, which is not localised
- Vison entropy favours uniform vison distribution, which favours spinons maximally locallised.

These create a trade-off called a *correlation hole*. Spinons *digs out* a hole in the vison background. Typcically radius ξ controlled by kinetic energy-entropy balance in effective free energy $F \sim \frac{h_s}{\xi^2} + T\xi^2 \ln 2$. Now this gives very interesting properties

- Thermodynamics signatures if you can measure spinon or visons correlation
- Transport and response properties will be affected greatly.