Many body Quantum dynamics by Vedika Khemani

1 Emergence: More is different

The idea of condensed matter is *emergence*, coined by Anderson in 1972: More is *different*; There are phenoma you can only see when you consider the collective properties of many bodies.

How can we deal with $N \simeq 10^{23}$ particles? There are two key frameworks, Quantum statistical Mechanics and Emergent Quasiparticles.

In Quantum statmech you have system interchanging particles/energy with a bath of much greater size. The system reaches Thermal equilbrium at late times, obtained from max entropy principle. Hence $\rho^S(t) \to \rho^S_{\text{eq}}(T, \mu)$ and the quantitites of interest are $\langle O \rangle = \mathrm{Tr}\left[\rho_{\text{eq}} O\right]$. This is built on the *ergodic hypothesis* of chaotic systems.

Quasi particles are the low energy excitations from the ground state of a system. In particular, these only interact weakly.

1.1 Approach to many-body physics.

Traditionally we study Hamiltonian time evolution with a time independent Hamiltonian $H(t) = H$. Recently we have modified this paradigm by interrupting the unitary time evolution by non-unitary measurements. Generally we study ground or equilibrium states, or small deviations from these at low energies. This has recently been changed to study dynamics of initial states that are not low-energy in any sense. Traditionally we study correlation functions, order parameters for characterizing universal phases and phase transitions, but recently we've moved on to Quantum entanglement, Quantum complexity etc.

The focus of this seminar is the *dynamics* of

- Strongly interacting
- Isolated (Unitary time evolution), so $\rho(t + \delta t) = U\rho(t)U^\dagger$.
- Highly excited (No approaches based on quasiparticles)
- Many body systems (spins, cold atoms, qubits, etc.)

An outline of the lecture is:

- Eigenstate thermalisation hypothesis
- Many body localisation: Breakdown of thermalisation, Useful picture called ℓ-bit picture
- Localisation protected quantum order
- Quantum Chaose and Out of time ordered commutators.
- Q.statmech MBL, Nadkishore Huse, Ann. Rev CMP (2015)
- MBL: Abanin,Altman,Bloch, Serbyn. Rev. Mod. Phys (2019)
- Thermalisation, ETH: D'Alessio,Kafri, Polkovnikov, Rigol, Adv. Phys (2016)

- Localisation protected quantum order, time crystals: Khemani, Moessner,Sondhi, arXiv 1910.10745 (2019)

Definition 1.1 (classes of Models)**.** Will be working with models of the following types

Time ind Hamiltonians Conserved energy, has eigenstates $U = \mathrm{e}^{-iHt}$.

Floquet Periodically driven $H(t+T) = H(t)$, $U(nT) = [U(T)]^n$. Has eigenstates $|\alpha\rangle$ with $U(T)|\alpha\rangle = \mathrm{e}^{i\alpha}|\alpha\rangle$.

Random unitary circuits Layers of random 1 and 2 body gates. Has unitary dynamics with locality.

Why are we changing approach now? Because experimentalists are designing artificial many-body quantum system, including ultracold atoms, superconducting qubits, trapped ions, cavity QED.

2 Eigenstate thermalisation hypothesis

Definition 2.1. Can an isolated strongly interacting many body system act as its own 'bath' and bring its subsystems to thermal equilibirum?

To answer the question we need to answer fundamental questions like What does thermal equilibrium mean in this context? How is thermal equilbrium reached when it is reached, and can it be avoided?

The first example of a system that can be many-body localised and fail to go to equilibrium was Andersons 1958 paper. It turns out there are two distinct possibilities: Either a system can thermalise or it will be many-body localised (Can also be Integrability, scars but these are believed to be *fine tuned*). There is a kind of quantum phase transition between MBL and thermalised.

2.1 Thermalisation in isolation

Actually can have intermidate systems between MBL and thermalising. Under unitary time evolution system *remembers* all details, as $|\psi(t)\rangle = U(t)|\psi_0\rangle$ is reversible. In a thermalising system the memory gets *scrambled* into completely non-local degrees of freedom. So even if it is formally reversible, in practice we can only probe small local subsystems. These subsystems A of $A \cup B$ can therefore lose memory of its initial conditions. A thermal state is a state $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ which in the limit of large system $|A \cup B| \to \infty$ looks like an equilibrium state with the bath traced out $\rho_A \to \mathrm{Tr}_B\, \rho_{\mathrm{eq}}(T, \mu)$.

2.2 Equilibrium states and conservation laws

What is the right equilibrium state for the subsystems to thermalise to? There is no reservoir, so which conserved quantities are interchanged? For systems with only energy conservation it would be Grand Canonical Ensemble $\rho_G = \frac{1}{Z}\mathrm{e}^{-\beta H}$, with energy and particle number it would be the grand canonical ensemble $\rho_{\mathrm{GC}} = \frac{1}{Z}\mathrm{e}^{-\beta(H-\mu N)}$, and in general could have an arbitrary number of conserved quantities like integrable systems, where $\rho_{\mathrm{GGE}} = \frac{1}{Z}\mathrm{e}^{-\beta(H-\mu_1 N_1 - \mu_2 N_2 \dots)}$. Note that not all conserved operators should be accounted for.

Example. For any time independent hamiltonian H the eigenstates $|\alpha\rangle$ produce an exponential number of conserved operators through there projectors $P_\alpha = |\alpha\rangle\langle\alpha|$, as these commute with the hamiltonian $[H, P_\alpha] = 0$. These projectors are not local in any sense.

The only operators we include are the superposition of local operators $N = \sum_i c_i O_i$. The notion of locality is not rigourslу defined, and depend on what kind of observables are. Examples could be finite range, exponentially decaying, power law decaying etc.

What fixed the lagrange multipliers? There is no reservoir to fix β etc. In fact it will be overall density of your conserved quantity. So if start with ρ_0, then $\rho_0(t) = U\rho_0^{AB}U^\dagger$ to reach thermal equilibrium we need that

$$\lim_{t\to infty} \lim_{|AB|\to\infty} \mathrm{Tr}_B\, \rho_0^{AB}(t) = \rho_{eq}^{(A)},$$

To set β need $\langle E\rangle = \mathrm{Tr}(H\rho_0)$, and $\Delta E = \sqrt{\mathrm{Tr}(H^2\rho_0) - (\mathrm{Tr}\, H\rho_0)^2} \propto V^\alpha$ with $\alpha < 1$, so that $\Delta E/\langle E\rangle \to 0$ as volume to ∞. Generally most initial states will have $\alpha = 1/2$, but may be counterexamples.

Example. Let $H|\alpha\rangle = \alpha|\alpha\rangle$ be the eigenstates of a hamiltonian. Now the time depedence of any expected value, given the initial state $|\psi_0\rangle = \sum_\alpha c_\alpha |\alpha\rangle$, will be

$$\langle\psi(t)|O|\psi(t)\rangle = \sum_\alpha |c_\alpha|^2 O_\alpha + \sum_{\alpha\neq\beta} c_\alpha c_\beta^* \langle\beta|O|\alpha\rangle\, \mathrm{e}^{-it(E_\alpha - E_\beta)}$$

The diagonal elements is the *diagonal ensemble* and is responsible of the equilbrium value of the observable. There are two distinct processes at play

Equilibration The expectation value of operators tend toward equilbrium value, as the off-diagonal ensemble loose coherence and average to 0

Thermalisation The equilibrium values are given by the thermal ensemble $\mathrm{Tr}\, O_A\rho_{AB}^{\mathrm{eq}}$.

2.3 Eigenstate thermalisation hypothesis

If all *well defined* initial states reach thermal equilibrium then eigenstates of H must be thermal:

$$H|n\rangle = E_n|n\rangle$$
$$\mathrm{Tr}_B |n\rangle\langle n| = \mathrm{Tr}_B \frac{\mathrm{e}^{-\beta_n H}}{Z}$$

Single-eigenstates are well-defined microcanical ensemble. Strong version: All states are thermal. Weak version: Almost all states are thermal (exceptions are scars).

Example (How to define temperature)**.** Plot entropy or DoS as function of energy density, will get gaussian for local hamiltonian. The slope of this curve is then β. Note that infinite temperature is the energy density is in middle of the spectrum.

2.4 Volume law entanglement

It is very illuminating to think of entanglement as an observable. For subsystem A, with reduced density $\rho_A = \operatorname{Tr}_B \rho$, then the entanglement entropy is $S_A = -\operatorname{Tr}[\rho_A \log \rho_A]$. Now ETH means that this takes on the the thermal value $S_A = V_a s_{\text{th}} + ..$ scaling as volume of A. Note do not need to think of B as reservoir of anything, just as something to get entangled with! The conserved quantities are not essential. For Floquet systems for example, there is no conservation of energy, so $\rho^{\text{eq}}_{\text{Floquet}} \propto 1$.

3 Which systems reach equilibrium?

Do all generic systems reach thermal equilibrium? Most do, but we are aware of one general exception *localisation.* Occur in systems without translational invariance, often they are disordered, but this isn't required.

Definition 3.1. systems retain *local* memory of initial conditions to infinitely late times!

New kind of transition between MBL to thermalising phases. MBL can stabilise new kinds of non-eq order (eg time crystals) An experimental verification is in Choi et al. Science (2016), where atoms retain memory of what region of a trap they occupy.

3.1 What do we know about MBL transitions?

- Anderson: Non-interacting electrons can be localised by disorder (strong in 3D, arbitrarily weak is enough in 1D and 2D)
- MBL: Generalisation of anderson localisation to case of interactions
- Existence of the MBL phase
  - All orders in perturbation theory in small interaction strength,in any dimension (2006)
  - Almost proof including non-perturbative effects in one dimensional lattice models with exponentially decaying interactions. (It is believed that in higher dimensions it may eventually be unstable to non-perturbative effects)
  - Lots of open questions (possible non-perturbative instabilities in higher dimensions, with longer interactions..) (de Roeck, Huveneers).
- Lots of numerical evidence of the thermal phase, but no proof

Example (Single-particle Anderson Localisation)**.** A simple model for the Anderson Localisation is non-interacting fermions hopping in random potential

$$H = \sum_i h_i c_i^\dagger c_i + J(c_i^\dagger c_{i+1} + \text{h.c.}),$$

with $h_i \in [-W, W]$. Locator expansion $J \ll W$ means that wavefunctions will be localised exponentiall around each site $|\phi(r)|^2 \sim e^{-r/\xi}$.

Example (Localisation with interactions)**.** Do a Jordan-Wigner transformation to a spin system (Up is occupied and down is empty), and add interactions to obtain hamiltonian

$$H = \sum_i h_i \sigma_i^z + J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right)$$

There is a lot of numerical exploration of this system, and get a Energy density vs disorder strength W phase diagram. At sufficiently high disorder appear to see localised phase, but a few problems with numerics.

3.2 ℓ-bit Picture of Many body localisation

MBL is sort of an emergent integrability. An integrabel system has an extensive number of conserved quantities. For previous model, if turn off $J = 0$, eigenstates are just spin chains, which are not thermal, and have non-local correlations; they violate ETH. There is no transport, and the only dynamics is Larmor precession. This can be thought of as arising from the fact that we have an extensive number of constants of motion $\{\sigma_i^z\}$, and $[\sigma_i^z, \sigma_i^z] = 0$. What happens when we add interactions? $H = \sum_i h_i \sigma_i^z + J \sum_i \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1}$ for $J << W$ there is a quasi local unitary operator transforming $H = \sum_i \tilde{h}_i \tau_i^z + \sum_i j \tilde{J}_{ij} \tau_i^z \tau_j^z + \ldots$, where $\tau_i^z = U^\dagger \sigma_i^z U$ are an extensive number of conserved quantities. The special aspect is that the unitary transformation is *quasi-local* (exponentially decay) converting from physical bits (p-bits) to local bits (ℓ-bits), so the hamiltonian is still local. The locality follow from the fact that the ground state is gapped, which implies there is a local unitary transforming from the initial ground state to the new ground state. The local unitary is a finite depth circuit V. The finite depth means that we may cut off the tails of our dressed operator at some finite length.

Example (Ising model)**.**

$$H = J \sum_i Z_i Z_{i+1} + h \sum_i X_i$$

In the $J = 0$ limit the ground state is eigenstate of X_i. For $J \neq 0$ can deform operators $\tilde{X}_i$ to become eigenoperators of new ground state. At phase transition this deformation has infinite correlation length, so becomes useless. For MBL it is same pictur, except it is not just the ground state, but rather all the states for which this occurs.

Example (Area law entanglement for MBL eigenstates)**.** In MBL generic eigenstate only area of each subsystem is entangled with rest of system, so $S_A \propto \partial A$.