Interpretation of the classical Lagrangian as an energy trajectory

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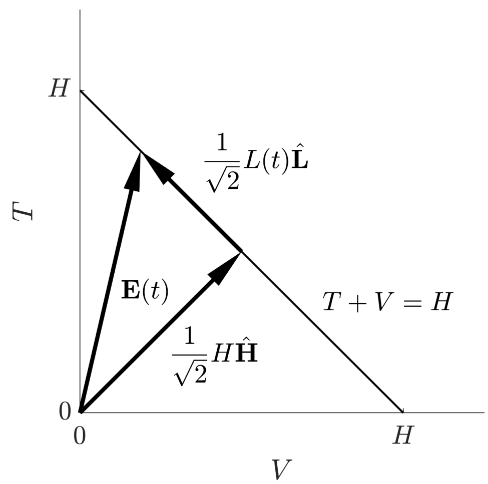

Figure 1: Energy space where the energy trajectory of a given system is represented by the vector $\mathbf{E}(t)$, confined to the constant energy line.

Consider a two-dimensional energy space, as seen in Figure 1, where the axes represent the potential and kinetic energies V and T. A point (V, T) in this energy space then corresponds to some particular energy state of a given system. The energy trajectory of the system can be represented by a vector

$$\mathbf{E}(t) \equiv V(t)\hat{\mathbf{V}} + T(t)\hat{\mathbf{T}} \tag{1}$$

where $\hat{\mathbf{V}}$ and $\hat{\mathbf{T}}$ are unit vectors in the V and T directions. If the energy of the system is conserved, we have that

$$V(t) + T(t) = H \tag{2}$$

with the Hamiltonian H constant. The energy space trajectory is then confined to the corresponding constant energy line and is therefore one-dimensional, such that it can be written in terms of a single energy-variable through a change of basis. It is trivial to prove that an equivalent form of (1) is given by

$$\mathbf{E}(t) = \frac{1}{\sqrt{2}}\left[H\hat{\mathbf{H}} + L(t)\hat{\mathbf{L}}\right] \tag{3}$$

with the new basis

$$\hat{\mathbf{H}} \equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{T}} + \hat{\mathbf{V}}), \tag{4}$$

$$\hat{\mathbf{L}} \equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{T}} - \hat{\mathbf{V}}) \tag{5}$$

and the single energy-variable

$$L(t) \equiv T(t) - V(t) \tag{6}$$

being the Lagrangian of the system, here interpreted as representing its energy space trajectory, as measured along the line of constant energy with respect to the point $\frac{1}{2}(H, H)$.

Furthermore, if one assumes that (given initial conditions) the energy trajectory of the system is uniquely determined, it follows that the average energy-displacement over a time interval

$$\bar{\mathbf{E}} \equiv \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{E}(t)\,\mathrm{d}t \tag{7}$$

is some specific vector, such that any arbitrarily small deviation $\delta\bar{\mathbf{E}}$ must vanish,

$$\delta\bar{\mathbf{E}} = \mathbf{0}. \tag{8}$$

Using (3) and (7), Equation (8) reduces to

$$\delta S = 0 \tag{9}$$

where

$$S \equiv \int_{t_0}^{t_1} L(t)\,\mathrm{d}t \tag{10}$$

is the action. We have thus seen that the principle of stationary action (9) can be derived through the assumptions of energy conservation and classical determinism (uniqueness of trajectories).