Generalized Additive Model (GAM)

study group

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08 January, 2020

General form

- **Generalized additive model** is a generalized linear model with a linear predictor, involving a sum of smooth functions of covariates.

$g(u_i) = X_i^* \theta + f_1(x_{1i}) + f_2(x_{2i}) + f_3(x_{3i}, x_{4i}) + \ldots$ (eqn. 3.1)

- GAM specifies the model with **smooth functions** rather than detailed parametric relationships.

Univariate smooth functions f

$y_i = f(x_i) + \epsilon_i$ (eqn. 3.2)

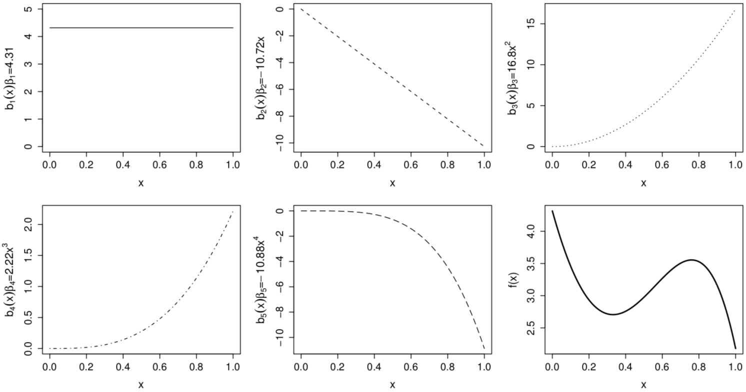
y_i : response variable
f : **smooth function**
x_i : covariate
ϵ_i : i.i.d. $N(0, \sigma^2)$ variable (error)

- Represent the smooth function f in a way that (eqn. 3.2) becomes a linear model.
- The smooth function f can then be expressed as some *basis functions* (b) with parameter β

$f(x_i) = \sum_{j=1}^{q} b_j(x)\beta_j$

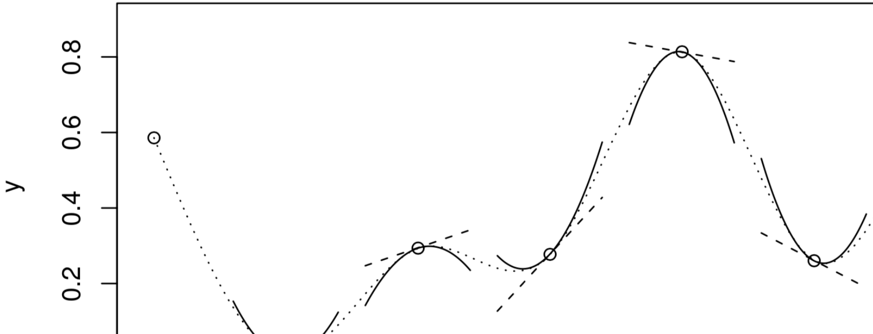
Simple example of smooth function: 4th order polynomial basis

$$f(x) = \beta_1 + x\beta_2 + x^2\beta_3 + x^3\beta_4 + x^4\beta_5$$



Cubic spline basis

- A univariate function represented using a cubic spline.
- A cubic spline is a curve, made up of sections of cubic polynomial, joined together so that they are continuous in value as well as first and second derivatives



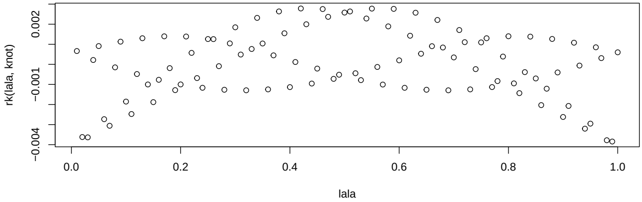
A cubic spline basis can be expressed as...

$$b_1(x) = 1, b_2(x) = x, ..., b_{i+2}(x, x_k^*)$$

- Consist of *one constant*, *one 1st order (linear)* term, and k cubic polynomials
  - k **is the number of knots**
  - knots are evenly spaced or at the quantile of distribution of x $(x_i^*; i = 1 \sim k)$
  - $k + 2$ is the rank of the cubic spline function
- $b_{i+2}(x, x_k^*)$ is a cubic polynomial written as:

$$R(x,z) = \left[(z - 1/2)^2 - 1/12\right]\left[(x - 1/2)^2 - 1/12\right.$$
$$- \left[(|x - z| - 1/2)^4 - 1/2\,(|x - z| - \right.$$

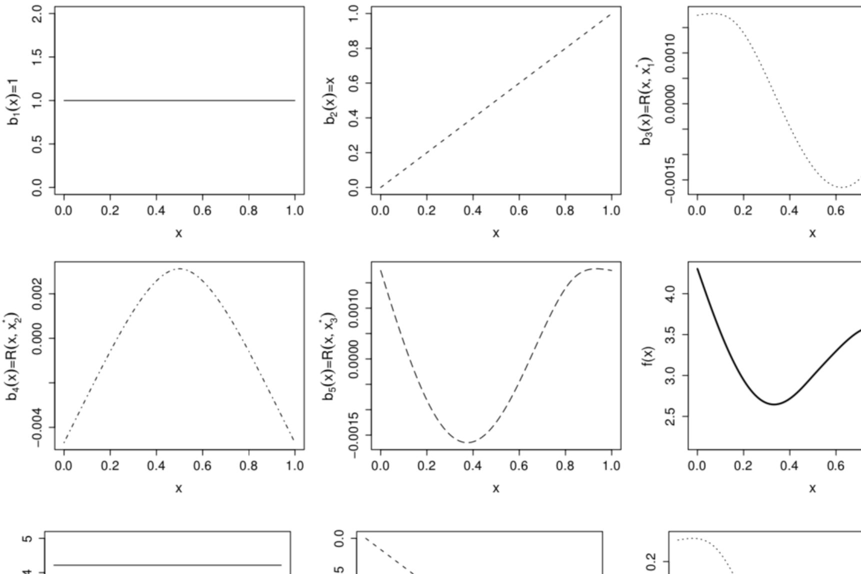
- Each cubic polynomial will peak at the knot
  - e.g. The four cubic polynomials when $k = 4$



The peak of the cubic polynomial is *conceptually* the weight when fitting the y

Example of cubic spline bases

rank 5 cubic regression spline basis



Working example

Data

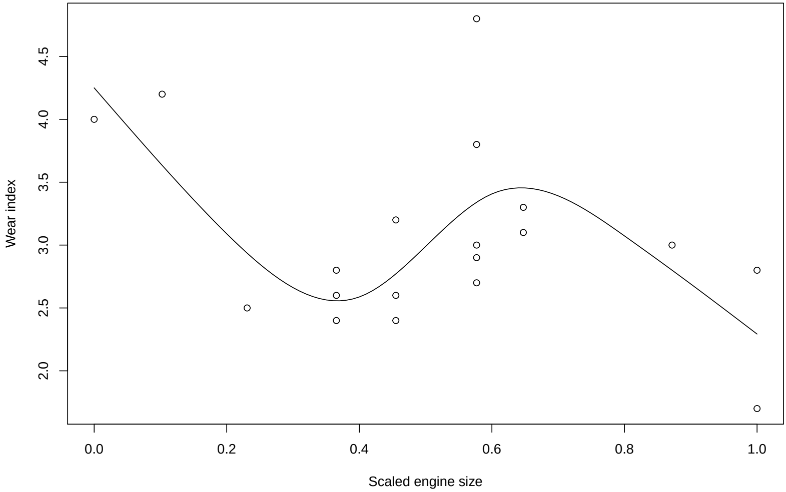
```
size <- c(1.42,1.58,1.78,1.99,1.99,1.99,2.13,2
wear <- c(4.0,4.2,2.5,2.6,2.8,2.4,3.2,2.4,2.6

x <- size - min(size)
x <- x/max(x)
```

Function for generating the model matrix

```
spl.X <- function(x,xk) # set up model matrix
  { q <- length(xk) + 2 # number of parameter
    n <- length(x) # number of data
    X <- matrix(1, n, q) # initialized model
    X[,2] <- x  # set second column to x
```

Results. . .



Knots determine the smoothness

- How many knots?
- **Penalized regression splines**
  Minimize the following... $||y - X\beta|| + \boldsymbol{\lambda \int_0^1 [f''(x)]^2 dx}$
  High λ penalizes high wiggliness ($\int_0^1 [f''(x)]^2 dx$)
  Fig. 3.8

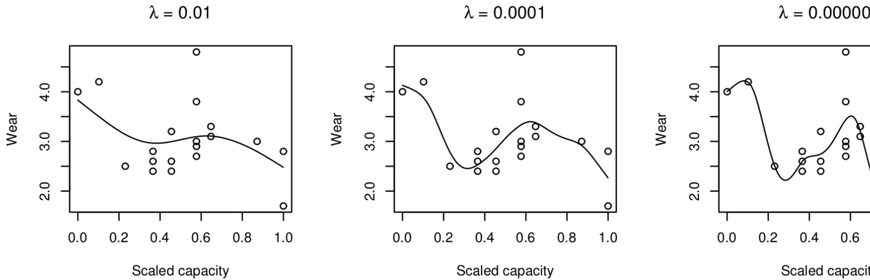
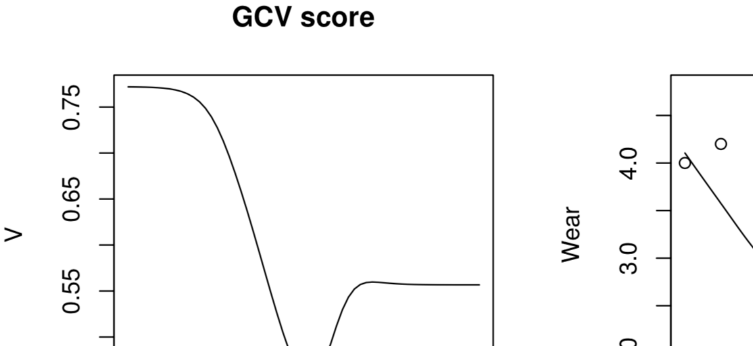


Figure 3.8 *Penalized regression spline fits to the engine wear versus capacity data usin*

To determine λ

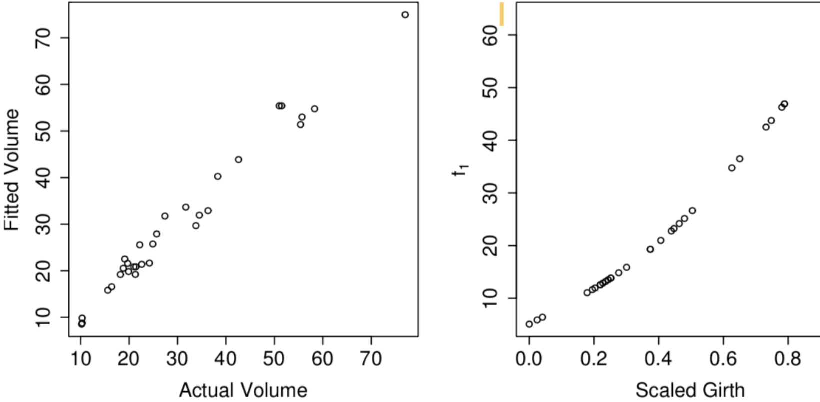
- Ordinary cross validation (OCV)
  Leave one data point at a time when doing cross validation
- Generalized cross validation (GCV)
  Use *influence matrix* to estimate the deviance when leaving one data point out Fig. 3.10



Additive model (more than one x)

$y_i = f_1(x_{1i}) + f_2(x_{2i}) + \epsilon_i$

All calculations are conceptually the same as mentioned above.



Generalized additive model

Due to the link function, penalized likelihood is being maximized.

In practice, **penalized iteratively re-weighted least square (P-IRLS)** is implemented.