Analyzing internet download speeds with a Gaussian model

First, some housekeeping: loading libraries and setting up colors.

```
options(repr.plot.width=8, repr.plot.height=6, lwd = 4)
library("RColorBrewer") # for pretty colors
library("tidyverse")    # for string interpolation to print variables in plots.
```

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.0 --

## v ggplot2 3.3.2     v purrr   0.3.4
## v tibble  3.0.4     v dplyr   1.0.5
## v tidyr   1.1.3     v stringr 1.4.0
## v readr   1.4.0     v forcats 0.5.1

## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
library("latex2exp")    # the TeX() function makes it possible to print latex math
colors = brewer.pal(12, "Paired")[c(1,2,7,8,3,4,5,6,9,10)];
```

Analyzing internet download speeds using an iid Gaussian model with known variance

Problem and Data

The maximum internet connection speed downstream in my home is 50 Mbit/sec. This maximum will typically never be reached, but my internet service provider (ISP) claims that the average speed is *at least* 20Mbit/sec. To test this, I collect a total of five measurements over the course of five consecutive using an speed testing internet service. I obtained:

```
x = c(15.77, 20.5, 8.26, 14.37, 21.09)
```

Model

The measurements are assumed to be

$$x_1, \ldots, x_n \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2),$$

where θ is the average speed; we ignore for simplicity that the measurements cannot be negative.
The measurements are reported to have a standard deviation of $\sigma = 5$ by speed testing service and we take this as the given σ.

```
sigma2 = 5^2
```

Prior

I will use a prior centered on the average claimed by the ISP, $\mu_0 = 20$, with a prior standard deviation of $\tau_0 = 5$. My prior beliefs are therefore that $\theta \in [10, 30]$ with approximately 95% probability.

```
mu_0 = 20
tau2_0 = 5^2
```

Posterior

A normal prior for a normal model gives us a posterior which is also normal:

$$\theta|\mathbf{x} \sim \mathrm{N}(\mu_n, \tau_n^2),$$

where the **posterior precision** (1/variance) is the sum of the data precision and the prior precision

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2}$$

and the **posterior mean** is a weighted average of the sample mean and the prior mean

$$\mu_n = w\bar{x} + (1-w)\mu_0$$

where the weight is the relative precision of the data and prior information

$$w = \frac{n/\sigma^2}{n/\sigma^2 + 1/\tau_0^2}$$

Let's write a small function that computes the posterior mean and variance, and plots the prior, likelihood and posterior.

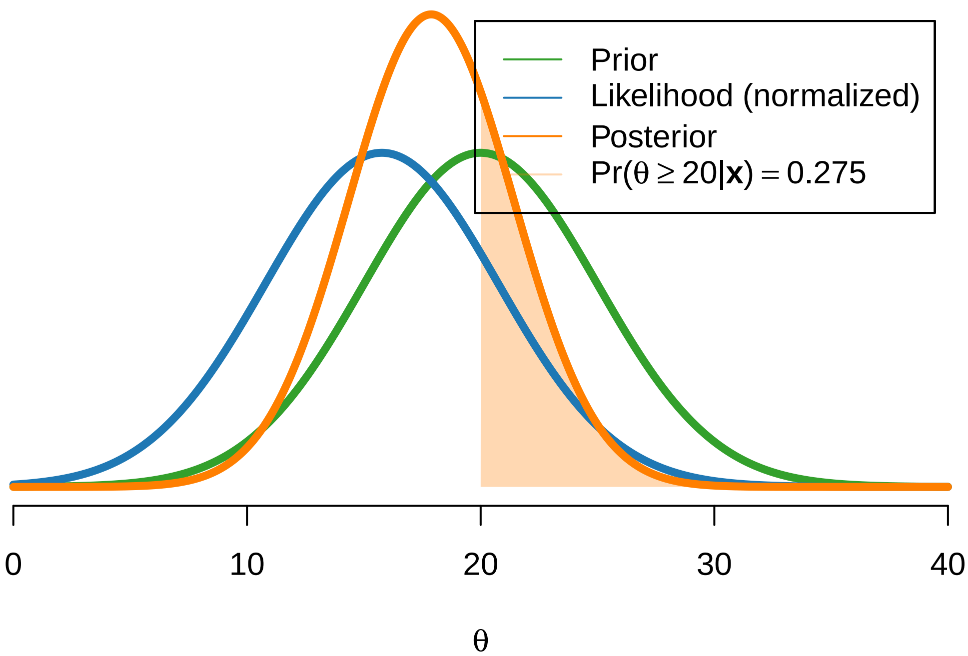
```
postGaussianIID <- function(x, mu_0, tau2_0, sigma2, thetaGrid, areaPoint){

    # compute posterior mean and variance
    n = length(x)
    tau2_n = 1/(n/sigma2 + 1/tau2_0)
    w = (n/sigma2)/(n/sigma2 + 1/tau2_0)
    mu_n = w*mean(x) + (1-w)*mu_0

    # plot PDFs. Likelihood is normalized.
    priorPDF = dnorm(thetaGrid, mean = mu_0, sd = sqrt(tau2_0))
    postPDF = dnorm(thetaGrid, mean = mu_n, sd = sqrt(tau2_n))
    postProbAbovePoint = 1-pnorm(areaPoint, mean = mu_n, sd = sqrt(tau2_n))
    normLikePDF = dnorm(thetaGrid, mean = mean(x[1:n]), sd = sqrt(sigma2/n))
    plot(1, type="n", axes=FALSE, xlab = expression(theta), ylab = "",
         xlim=c(min(thetaGrid),max(thetaGrid)),
         ylim = c(0,max(priorPDF,postPDF,normLikePDF)))
    axis(side = 1)
    polygon(c(thetaGrid[thetaGrid>=areaPoint], max(thetaGrid), areaPoint),
            c(postPDF[thetaGrid>=areaPoint], 0, 0),
            col=adjustcolor(colors[4],alpha.f=0.3), border=NA)
    lines(thetaGrid, priorPDF, type = "l", lwd = 4, col = colors[6])
    lines(thetaGrid, normLikePDF, lwd = 4, col = colors[2])
    lines(thetaGrid, postPDF, lwd = 4, col = colors[4])
    legend(x = "topright", inset=.05, cex = c(1,1,1,1),
           legend = c("Prior", "Likelihood (normalized)", "Posterior",
           TeX(sprintf("$Pr(\\theta \\geq %2.0f | \\mathbf{x}) =  %0.3f$",
           areaPoint, postProbAbovePoint))),
           lty = c(1, 1, 1, 1), pt.lwd = c(3, 3, 3, 3),
           col = c(colors[6], colors[2], colors[4], adjustcolor(colors[4],alpha.f=0.3)))
    cat("Posterior mean is ", round(mu_n,3), "\n")
    cat("Posterior standard deviation is ", round(sqrt(tau2_n),3), "\n")
    cat("The weight on the sample mean is ", round(w,3))
    return(list("mu_n" = mu_n, "tau2_n" = tau2_n, "w" = w))
}
```

Let us start by analyzing just the first observation $x_1 = 15.77$ using this function.

```
thetaGrid = seq(0, 40, length = 1000) # Some suitable grid of values to plot over
areaPoint = 20 # shade the region where theta>= areaPoint (20 in my example)
n = 1
post = postGaussianIID(x[1:n], mu_0, tau2_0, sigma2, thetaGrid, areaPoint)
```

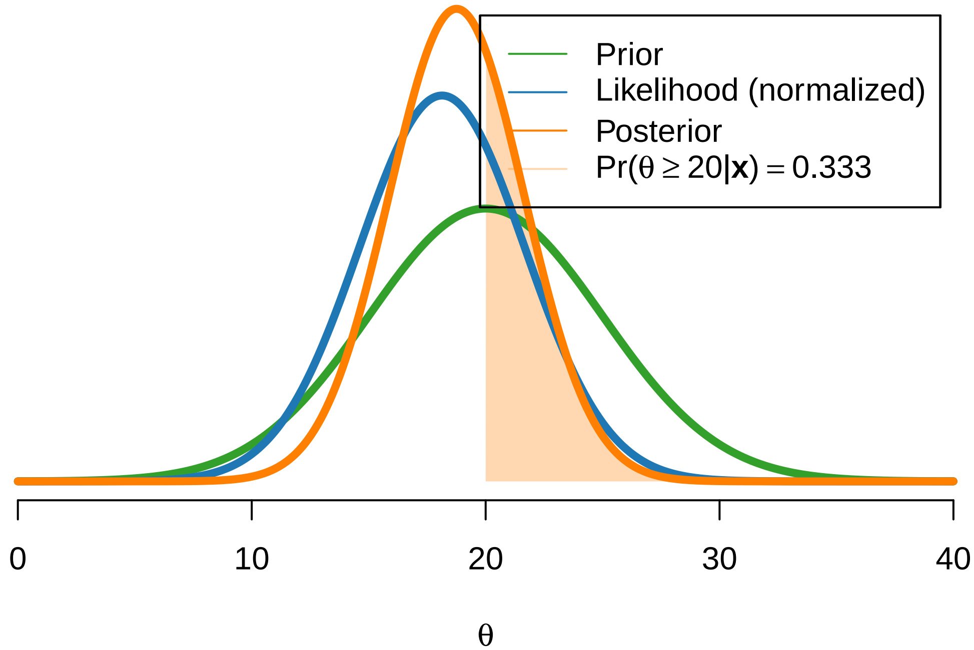



```
## Posterior mean is  17.885
## Posterior standard deviation is  3.536
## The weight on the sample mean is  0.5
```

We see that the prior and data information happen to get the same weight (w) in the posterior. That is a coincidence from the fact that the prior variance τ_0^2 is the same as the data variance σ^2.

Moving on, let's add the next measurement to the analysis:

```
n = 2
post = postGaussianIID(x[1:n], mu_0, tau2_0, sigma2, thetaGrid, areaPoint)
```

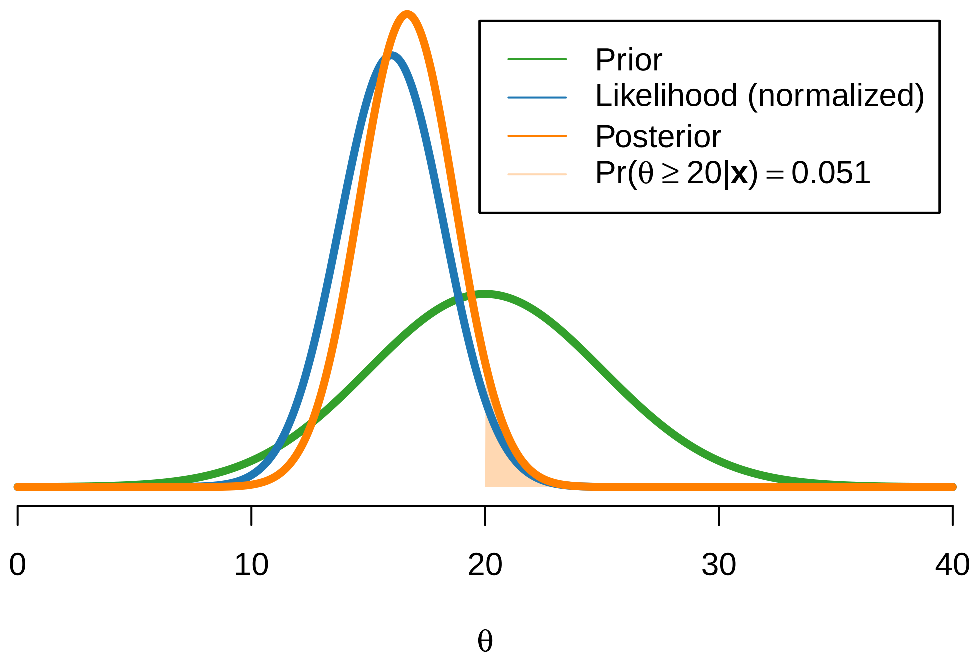



```
## Posterior mean is  18.757
## Posterior standard deviation is  2.887
## The weight on the sample mean is  0.667
```

We now see that the posterior is more affected by the data information than the prior information (w = 0.666).

Finally, adding all $n = 5$ data points gives:

```
n = 5
post = postGaussianIID(x[1:n], mu_0, tau2_0, sigma2, thetaGrid, areaPoint)
```




```
## Posterior mean is  16.665
## Posterior standard deviation is  2.041
## The weight on the sample mean is  0.833
```