Homework 1: PE 2020-2021 [due date: 2020/12/02]

Apellidos: **Nombre:**
Grupo:

1	2	3	4	5	6	7	8	T

1. A Poisson process with rate λ can be defined as a counting process $\{N(t); t \geq 0\}$ with the following properties

   (a) $N(0) = 0$.

   (b) $N(t)$ has independent and stationary increments.

   (c) Let $\Delta N(t) = N(t + \Delta t) - N(t)$ with $\Delta t \to 0^+$. The following relations hold:

$$\mathbb{P}[\Delta N(t) = 0] = 1 - \lambda\Delta t + o(\Delta t) \tag{1}$$

$$\mathbb{P}[\Delta N(t) = 1] = \lambda\Delta t + o(\Delta t) \tag{2}$$

$$\mathbb{P}[\Delta N(t) \geq 2] = o(\Delta t). \tag{3}$$

   From this definition show that

$$\mathbb{P}[N(t) = n] = \frac{1}{n!}\lambda^n t^n e^{-\lambda t}. \tag{4}$$

   To this end, set up a system of differential equations for the quantities $\mathbb{P}[N(t) = 0]$, and $\mathbb{P}[N(t) = n]$ with $n \geq 1$. Then verify that Eq. (4) satisfies the differential equations derived.

   For instance, the differential equation for $\mathbb{P}[N(t) = 0]$ can be derived from the fact that

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0]\,\mathbb{P}[\Delta N(t) = 0] \tag{5}$$

   Using Eq. (1), we obtain

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0] - \mathbb{P}[N(t) = 0]\,\lambda\Delta t + o(\Delta t). \tag{6}$$

   The corresponding differential equation is obtained in the limit $\Delta t \to 0^+$

$$\frac{d}{dt}\mathbb{P}[N(t) = 0] = -\lambda\mathbb{P}[N(t) = 0]. \tag{7}$$

   The solution of this differential equation for the initial condition $\mathbb{P}[N(0) = 0] = 1$ is

$$\mathbb{P}[N(t) = 0] = e^{-\lambda t}. \tag{8}$$

   Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of $\mathbb{P}[N(t) = n]$ given by Eq. (4) for the values $\lambda = 10$, $t = 2$.

2. Simulate a Poisson process with $\lambda = 5.0$. From these simulations show for different values of $n = 1, 2, 5, 10$ that the probability density of the nth arrival is

$$f_{S_n}(t) = \frac{1}{(n-1)!}\lambda^n t^{n-1} e^{-\lambda t}. \tag{9}$$

3. Assume that we have a sample $\{U_i\}_{i=1}^n$ of n iid $U[0, t]$ random variables. The probability density of the order statistics $\left\{U_{(1)} < U_{(2)} < \ldots < U_{(n)}\right\}$ is

$$f_{\left\{U_{(i)}\right\}_{i=1}^n}\left(\left\{u_{(i)}\right\}_{i=1}^n\right) = \frac{n!}{t^n}.$$

   Let $\{N(t); t \geq 0\}$ be a Poisson process with rate λ. Show that conditioned on $N(t) = n$, the distribution of arrival times $\{0 < S_1 < S_2 < \ldots < S_n\}$ coincides with the distribution of order statistics of n iid $U[0, t]$ random variables

$$f_{\left\{S_{(i)}\right\}_{i=1}^n | N(t)}\left(\left\{u_{(i)}\right\}_{i=1}^n | n\right) = \frac{n!}{t^n}.$$

   Hints:

- Use Bayes theorem to calculate the density $f_{\{S_i\}_{i=1}^{n+1}|N(t)}\left(\{s_i\}_{i=1}^{n+1}|n\right)$.
- Use the fact that $N(t) = n$ if and only if $s_n \leq t < s_{n+1}$

$$f_{N(t)|\{S_i\}_{i=1}^{n+1}}\left(n|\,\{s_i\}_{i=1}^{n+1}\right) = \begin{cases} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

- Focus on the the case $s_n \leq t < s_{n+1}$.
- Use the fact that

$$f_{\{S_i\}_{i=1}^{n+1}|N(t)}\left(\{s_i\}_{i=1}^{n+1}|n\right) = f_{S_{n+1}|\{S_i\}_{i=1}^{n},N(t)}\left(s_{n+1}|\,\{s_i\}_{i=1}^{n},n\right) f_{\{S_i\}_{i=1}^{n}|N(t)}\left(\{s_i\}_{i=1}^{n}|n\right).$$

- Use the memoryless property for $s_{n+1} > t$

$$f_{S_{n+1}|\{S_i\}_{i=1}^{n},N(t)}\left(s_{n+1}|\,\{s_i\}_{i=1}^{n},n\right) = f_{S_{n+1}|N(t)}\left(s_{n+1}|n\right)$$

4. Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process $N_1(t)$ with rate $\lambda_1 = 0.02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_2(t)$ with rate $\lambda_2 = 0.03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes.

   (a) Find the probability that no goals are scored.

   (b) Find the probability that at least two goals are scored in the game.

   (c) Find the probability of the final score being Team A:1,Team B:2

   (d) Find the probability that they draw.

   (e) Find the probability that Team B scores the first goal.

   Confirm your results by writing a Python program that simulates the processes and estimate the answers from the simulations.

   Note: In this problem, the series representation of the modified Bessel function of order ν can be useful

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{(2n+\nu)}.$$

5. Consider the process $X(t) = Z\sqrt{t}$ for $t \geq 0$ with the same value of Z for all t

   (a) Show that the distribution of the process at time t is the same as that of a Wiener process: $X(t) \sim N(0, \sqrt{t})$

   (b) What is the mathematical property that allows us to prove that this process not Brownian?

6. Consider the Wiener (standard Brownian) process $W(t)$ in $[0, 1]$,

   (a) From the property of independent increments,

$$\mathbb{E}\left[(W(t_2) - W(t_1))(W(s_2) - W(s_1))\right] = \mathbb{E}\left[(W(t_2) - W(t_1))\right]\mathbb{E}\left[(W(s_2) - W(s_1))\right], \; t_2 \geq t_1 \geq s_2 \geq s1 \geq 0,$$

   show that the autocovariances are given by

$$\gamma(t, s) = \mathbb{E}\left[W(t)W(s)\right] = \min(s, t),$$

   both for $s > t$ and for $t > s$.

   (b) Illustrate this property by simulating a Wiener process in $[0, 1]$ and making a plot of the sample estimate and the theoretical values of $\gamma(t, 0.25)$ as a function of $t \in [0, 1]$.

7. Consider two independent Wiener processes $W(t)$, $W'(t)$. Show that the following processes have the same covariances as the standard Wiener process:

   (a) $\rho W(t) + \sqrt{1-\rho^2}W'(t) \quad t \geq 0$

   (b) $-W(t) \quad t \geq 0$

   (c) $\sqrt{c}W(t/c); \quad t \geq 0, \; c > 0.$

(d) $V(0) = 0; V(t) = tW(1/t); \quad t > 0$

Make a plot of the trajectories of the first three processes to illustrate that they are standard Brownian motion processes. Compare the histogram of the final values of the simulated trajectories with the theoretical density function.

8. Extra point: Make an animation in Python illustrating the evolution of the distribution of a Brownian motion process starting form x_0:

$$\mathbb{P}\left(B(t) = x | B(t_0) = x_0\right).$$

To this end, simulate M trajectories of the process in the interval $[t_0, t0 + T]$ and plot the time evolution of the histogram using as frames a grid of regularly spaced times in that interval. Plot the theoretical form of the density function on the same graph, so that it can be compared with the histogram.