Ecuación de calor con frontera en estado no estacionario sobre un disco

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \phi^2}=\frac{1}{\alpha^2}\frac{\partial \Psi}{\partial t} \tag{1}$$

C.F.:

$$\Psi(r=0,\phi,t)=finito$$

$$\Psi(r=a,\phi,t)=f(\phi)$$

$$\Psi(r,\phi,t)=\Psi(r,\phi+2\pi,t)$$

C.I.:

$$\Psi(r,\phi,t=0)=g(r,\phi)$$

Supongamos que Ψ es una función de la forma:

$$\Psi(r,\phi,t)=\Psi_p(r,\phi,t)+\Psi_c(r,\phi) \tag{2}$$

donde Ψ_p es solución a la ecuación diferencial con condiciones de frontera homogéneas.

Sustituyendo a la ec. 2 en la ec. 1, obtenemos:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi_p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Psi_p}{\partial \phi^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi_c}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Psi_c}{\partial \phi^2}=\frac{1}{\alpha^2}\frac{\partial \Psi}{\partial t}$$

Por lo tanto:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi_c}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Psi_c}{\partial \phi^2}=0 \tag{3}$$

Cuya solución general es:

$$\Psi_c(r,\phi)=b_0 ln(r)+a_0+\sum_{n=1}^{\infty}\left(a_n r^n\cos(n\phi)+b_n r^n\sin(n\phi)\right)+\sum_{n=1}^{\infty}\left(c_n r^{-n}\cos(n\phi)+d_n r^{-n}\sin(n\phi)\right) \tag{4}$$

Dadas las condiciones de frontera:

$$\Psi(r=0,\phi,t)=\Psi_p(r=0,\phi,t)+\Psi_c(r=0,\phi)=finito$$

dado que:

$$\Psi_p(r=0,\phi,t)=finito$$

$$\Rightarrow \Psi_c(r=0,\phi)=finito$$

Esto hace que Ψ_c, sea

$$\Psi_c(r,\phi)=a_0+\sum_{n=1}^{\infty}\left(a_n r^n\cos(n\phi)+b_n r^n\sin(n\phi)\right)$$

Por la otra condición de frontera

$$\Psi(r=a,\phi,t)=\Psi_p(r=a,\phi,t)+\Psi_c(r=a,\phi)=f(\phi)$$

$$0+\Psi_c(r=a,\phi)=f(\phi)$$

$$\Rightarrow a_0 + \sum_{n=1}^{\infty} (a_n a^n \cos(n\phi) + b_n a^n \sin(n\phi)) = f(\phi)$$

$$\Rightarrow A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\phi) + B_n \sin(n\phi)) = f(\phi)$$

$$A_0 = a_0, \quad A_n = a_n a^n, \quad B_n = b_n a^n$$

$$a_n = \frac{A_n}{a^n}, \quad b_n = \frac{B_n}{a^n}$$

$$\Rightarrow A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos(n\phi) + B_n \sin(n\phi)) = f(\phi)$$

de esto, al ser una serie de Fourier, los coeficientes son:

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \cos(n\phi) d\phi$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin(n\phi) d\phi$$

Teniendo esto, llegamos a que la solución completa y general para Ψ es:

$$\boxed{\begin{aligned} \Psi(r,\phi,t) = &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_n(\lambda_{nm} r)[D_{nm}\cos(n\phi) + E_{nm}\sin(n\phi)]e^{-\lambda_{nm}^2 \alpha^2 t} \\ &+ \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos(n\phi) + B_n \sin(n\phi)) + A_0 \end{aligned}} \tag{5}$$

Por último, aplicando la C.I., $\Psi(r,\phi,t=0) = g(r,\phi)$

$$\begin{aligned} \Psi(r,\phi,t=0) = &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\cos(n\phi) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm}\sin(n\phi) J_n(\lambda_{nm} r) \\ &+ \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos(n\phi) + B_n \sin(n\phi)) + A_0 = g(r,\phi) \end{aligned} \tag{6}$$

$$\Rightarrow \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\cos(n\phi) J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm}\sin(n\phi) J_n(\lambda_{nm} r) = g(r,\phi) - \Psi_c(r,\phi)$$

$$\Rightarrow \sum_{n=1}^{\infty} \cos(n\phi) \sum_{m=1}^{\infty} D_{nm} J_n(\lambda_{nm} r) + \sum_{n=1}^{\infty} \sin(n\phi) \sum_{m=1}^{\infty} E_{nm} J_n(\lambda_{nm} r) = g(r,\phi) - \Psi_c(r,\phi)$$

$$\Rightarrow \sum_{n=1}^{\infty} \cos(n\phi) G_1(r) + \sum_{n=1}^{\infty} \sin(n\phi) G_2(r) = g(r,\phi) - \Psi_c(r,\phi)$$

Donde:

$$G_1(r) = \sum_{m=1}^{\infty} D_{nm} J_n(\lambda_{nm} r)$$

$$G_2(r) = \sum_{m=1}^{\infty} E_{nm} J_n(\lambda_{nm} r)$$

$$G_1 = \frac{1}{\pi} \int_0^{2\pi} cos(n\phi)[g - \Psi_c] d\phi$$

$$G_2 = \frac{1}{\pi} \int_0^{2\pi} sin(n\phi)[g - \Psi_c] d\phi$$

Por ende,

$$\begin{aligned} D_{nm} &= \frac{\left\langle \frac{1}{\pi} \int_0^{2\pi} cos(n\phi)[g - \Psi_c] d\phi | J_n(\lambda_{nm} r) \right\rangle}{\langle J_n(\lambda_{nm} r) | J_n(\lambda_{nm} r) \rangle} = \frac{\frac{1}{\pi} \int_0^a \int_0^{2\pi} cos(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}{\int_0^a [J_n(\lambda_{nm} r)]^2 r dr} \\ &= \frac{\frac{1}{\pi} \int_0^a \int_0^{2\pi} cos(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}{\frac{1}{2} a^2 J_{n+1}^2(\lambda_{nm} a)} \end{aligned} \tag{7}$$

$$\boxed{\therefore D_{nm} = \frac{2}{\pi a^2 J_{n+1}^2(\lambda_{nm} a)} \int_0^a \int_0^{2\pi} cos(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}$$

$$\begin{aligned} E_{nm} &= \frac{\left\langle \frac{1}{\pi} \int_0^{2\pi} sin(n\phi)[g - \Psi_c] d\phi | J_n(\lambda_{nm} r) \right\rangle}{\langle J_n(\lambda_{nm} r) | J_n(\lambda_{nm} r) \rangle} = \frac{\frac{1}{\pi} \int_0^a \int_0^{2\pi} sin(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}{\int_0^a [J_n(\lambda_{nm} r)]^2 r dr} \\ &= \frac{\frac{1}{\pi} \int_0^a \int_0^{2\pi} sin(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}{\frac{1}{2} a^2 J_{n+1}^2(\lambda_{nm} a)} \end{aligned} \tag{8}$$

$$\boxed{\therefore E_{nm} = \frac{2}{\pi a^2 J_{n+1}^2(\lambda_{nm} a)} \int_0^a \int_0^{2\pi} sin(n\phi)[g - \Psi_c] J_n(\lambda_{nm} r) dr d\phi}$$