1 Stochastic Process

A stochastic process is of the form

$$X_{T+1} - X_T = \theta(T)X_T + f(t) + r_T\sigma(T), \tag{1}$$

where r_T is a random variable drawn from the distribution $P(r, \vec{\lambda})$, where $\vec{\lambda}$ are the parameters that describe the distribution.

$$\begin{aligned} P(X_T) &= \int dX_{T-\Delta T} \int dr_T \; P(X_T, X_{T-\Delta T}, r), & (2) \\ &= \int dX_{T-\Delta T} \int dr_T \; P(X_T | X_{T-\Delta T}, r) P(X_{T-\Delta T}, r_T), & (3) \end{aligned}$$

assuming that $X_{T-\Delta T}$ is independent of the random variable r_T, then this expression becomes

$$P(X_T) = \int dX_{T-\Delta T} \int dr_T \; P(X_T | X_{T-\Delta T}, r_T) P(X_{T-\Delta T}) P(r_T). \tag{4}$$

Next, we apply the constraint where

$$r_T = \frac{X_{T+1} - X_T - \theta(T)X_T - f(T)}{\sigma(T)}, \tag{5}$$

this constraint is enforced through the function

$$P(X_T | X_{T-\Delta T}, r_T) = \delta\left(r_T - \frac{X_T - X_{T-1} - \theta(T-\Delta T)X_{T-\Delta T} - f(T)}{\sigma(T-\Delta T)}\right). \tag{6}$$

Carrying out the integral over r_T results in

$$P(X_T) = \int dX_{T-\Delta T} P(X_{T-\Delta T}) P\left(r_T = \frac{X_T - X_{T-1} - \theta(T-\Delta T)X_{T-\Delta T} - f(T-\Delta T)}{\sigma(T-\Delta T)}\right), \tag{7}$$

this last expression relates the probability distribution of the current value X_T to the value at a time infinitesimally in the past. Carrying out this expression recursively we obtain

$$\begin{aligned} P(X_T) = \int dX_{T-\Delta T} \; P&\left(\frac{X_T - X_{T-\Delta T} - f(T-\Delta T) - \theta(T-\Delta T)X_{T-\Delta T}}{\sigma(T-\Delta T)}\right) \\ \cdots \int dX_0 \; P&\left(\frac{X_1 - X_0 - f(T_0) - \theta(T_0)X_0}{\sigma(T_0)}\right)\delta(X_0 - x_0), \end{aligned} \tag{8}$$

where x_0 is the position at the initial point T_0.

2 Gaussian Noise

This probability can also be expressed as a path integral of the form

$$P(X_T) = \int \mathcal{D}X \ \mathrm{Exp}\left(\int\limits_{T_0}^{T} S(\dot{X}, X, t) dt \right), \tag{9}$$

where the action is

$$S(\dot{X}, X, t) = -\frac{1}{2\sigma^2(t)} \left(\dot{X}(t) - f(t) - \theta(t)X(t) \right)^2 \tag{10}$$

3 Fokker-Plank Equation

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