

UDACITY 2018: CAPSTONE PROJECT

# Machine learning methods for Forecasting time series

with applications to currency exchange rates

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# 1 Definition

## 1.1 Project Overview

A set of data that is indexed by time is known as a time series. They appear in many different fields, such as statistics, physics, finance, economics, biology, or even business [1]. Because of their wide applicability, it is important to generate accurate forecasts of time series data. These forecasts are generated using specific mathematical models or algorithms which are trained on a subset of the past values of a given time series. For the purpose of simplifying future discussions, we will adopt the following notation for a time series, denoted  $X_t$ , as

$$\{X_t; t = 0, 1, \dots\}. \quad (1)$$

Where  $t$  denotes the time-index of the series. One of the simplest models for a time series is the ARIMA (Auto regressive integrated moving average) model. This model is denoted as ARIMA( $p, d, q$ ) assumes that the time series  $X_t$  has the form

$$X_t = \mu + \epsilon_t + \sum_{i=1}^p \phi_i L^i [(1 - L)^d] X_{t-i} + \sum_{j=1}^q \theta_j \epsilon_{t-j}, \quad (2)$$

where  $\{\phi_i | i = 1, \dots, p\}$ ,  $\{\theta_i | i = 1, \dots, q\}$  are model parameters and  $L$  is the lag operator defined as  $LX_t = X_{t-1}$ . The parameter  $p$ , is known as the autoregressive (AR) order,  $d$  is the differencing order, and  $q$  is the moving average parameter (MA).

The term  $\epsilon_t$  denotes the error terms, assumed to be independent, identically distributed random variables sampled from a zero-mean, normal distribution. The value  $\mu$  denotes the average of this model. ARIMA models can be applied to make forecasts of stationary time series (defined as a time series whose mean, variance and auto correlation does not change over time), or to a time series that can be transformed into a stationary time series. However, there are other state-of-the-art machine learning methods that can be used to model time series methods. Applying some of the methods to time series will be the main goal of this project.

One important type of financial time series is the exchange rate between different currencies (Fig. 1). An exchange rate, is the rate at which one currency will be exchanged for another. There are many factors that can influence this rate, such as balance of payments, interest rate levels, inflation levels and other economical factors which are beyond the scope of this project [3].

## 1.2 Project Statement

The main objective of this project will be to use classical and more recent machine learning techniques to make forecasts of different time series with the goal of applying the best methods to predict currency exchange rates. The simplest model that we will use is the ARIMA model, defined in the previous section as the baseline model. The ARIMA model has been shown to be adequate in estimating the exchange rates of certain currencies Ref. [2]. We will then use different neural networks architectures such as the feed forward and long-short term memory networks, as in Ref. [5, 4, 6], to make predictions of time series

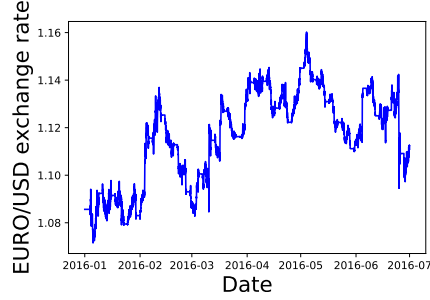


Figure 1: The exchange rate from EUR to USD from Jan 2016 to Jul 2016.

and use our baseline models and root mean square differences to quantify and compare the performance of different forecasts.

### 1.3 Metrics

There are several metrics that can be used to evaluate the predictions of our models Ref. [1], however, for our project we will focus on three commonly used metrics. First we define some terminology, the forecast error,  $e_t$ , is

$$e_t = X_t - F_t, \quad (3)$$

where  $X_t$  is the value of the time series at time step  $t$  and  $F_t$  is the forecasted value at the same time step. The three metrics that we will use for our project are

1. The mean square error (MSE)

- $\text{MSE} = \frac{1}{N} \sum_{t=1}^N e_t^2$

2. The root mean squared error (RMSE)

- $\text{RMSE} = \sqrt{\frac{1}{N} \sum_{t=1}^N e_t^2}$

3. The Akaike information criterion (AIC)

- $\text{AIC} = 2k - \text{Ln}(\hat{L})$

where  $k$  is the number of parameters in the model and  $\hat{L}$  is the maximum value of the likelihood function for the model.

In these three cases, the smaller the value of the RMSE, MSE, and AIC then the better the overall model.

## 2 Analysis

### 2.1 Data Exploration

We focused on three time series data sets. **i.** The international airline passenger data set, containing the total number of airline passengers in thousands from Jan 1949 until Dec 1960; **ii.** The sunspot data set, showing the monthly number of sunspots from 1705-1989; **iii.** The EUR to USD exchange rate from Jan 2016-July 2016. Datasets **i.** and **ii.** both consist of two columns, one for the time steps  $t$  and the values being measured  $X_t$ .

Month ( $t$ )	Int. airline passengers ( $X_t$ )
1949-01	112
1949-02	118
1949-03	132
$\vdots$	$\vdots$

Table 1: The format of datasets **i.** and **ii.**

The third data set, **iii.** contains several columns related to the exchange rate market. The columns are Time, the Opening rate, the maximum value of the rate, the lowest value, the closing exchange rate value, and the total volume of the exchange market.

Time	Open	High	Low	Close	Volume
2010-01-01 00:00	1.43283	1.43293	1.43224	1.43293	608600007.1
2010-01-01 00:15	1.43285	1.43295	1.43229	1.43275	535600003.2
2010-01-01 00:30	1.43280	1.43303	1.43239	1.43281	436299999.2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: A sample of the format of dataset **iii.**

To help us better understand our datasets, in Table 3, we give the computed descriptive statistics of our three data sets in comparison to one-another.

	Data set <b>i.</b>	Data set <b>ii.</b>	Data set <b>iii.</b>
count	144	309	245441
mean	280.298611	49.752104	1.268375
std	119.966317	40.452595	0.112992
min	104.000000	0.000000	1.035580
25%	180.000000	16.000000	1.135360
50%	265.500000	40.000000	1.302350
75%	360.500000	69.800000	1.356560
max	622.000000	190.200000	1.493240

Table 3: Descriptive statistics of the data sets **i.**, **ii.** and **iii.**

## 2.2 Exploratory Visualization

To understand the data in a more visual way, in the figures below, we plot the times series data sets that we will be analyzing.

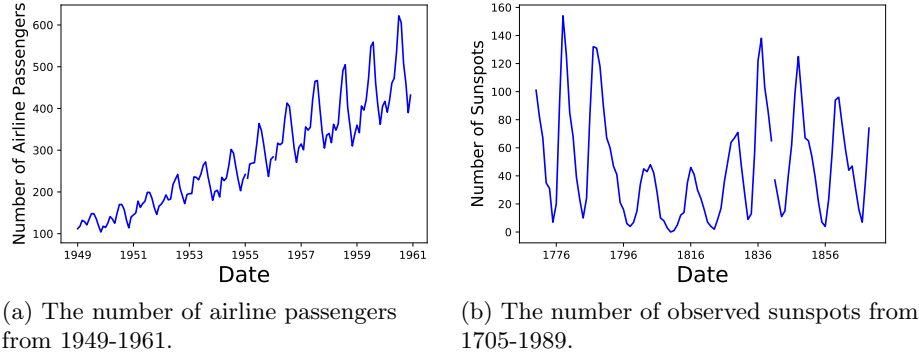


Figure 2: A figure with two subfigures

The airline data set in Fig. 2a shows two interesting patterns, the general increasing number of airline passengers over time, along with a seasonal yearly pattern. The sunspot data set in Fig. 2b shows a cyclical pattern that repeats about every 11-years.

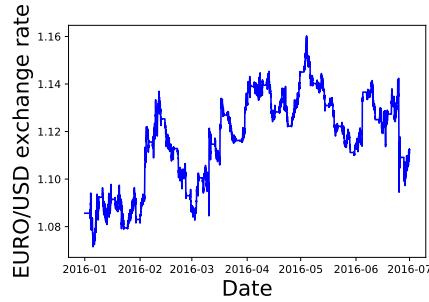


Figure 3: The exchange rate from EUR to USD from Jan 2016 to Jul 2016.

In Fig. 3 we plot the Euro to USD closing price from Jan 2016 to July 2016. There does not seem to be any apparent seasonal or cyclical patterns in this data sample.

## 2.3 Algorithms and Techniques

For this project, we will use the ARIMA model as a benchmark model as described in Sec. 1.1 and implemented as in Sec. 3.2. ARIMA models have been used to predict foreign exchange rates [2] and are a classical method for time series forecasts Ref. [1]. For these reasons, we use the ARIMA model as a benchmark.

Neural networks for time series models have been explored in the literature [1, 4, 5] and have shown good predictive abilities. This motivates us to explore neural networks for our forecasts. The two neural networks that we will use is the feed-forward neural network with 4 nodes along with a long-short-term memory network with 4 LSTM units. These networks are shown schematically in Figs. 4 and 5. The tutorials in Refs. [7, 6, 8] were very instructive in getting started with these models. In this diagram,  $X_{t-1}, X_t$  is the time series pair that

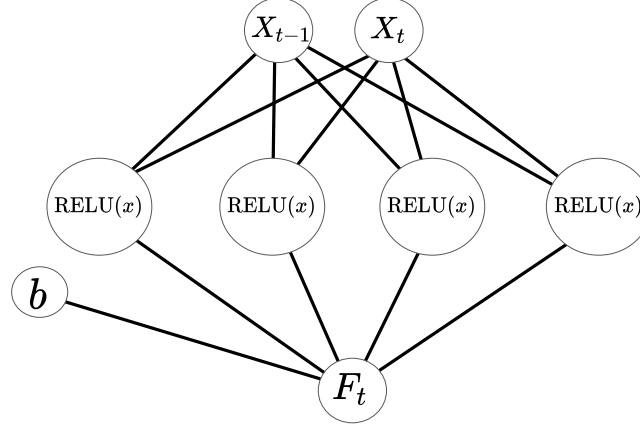


Figure 4: The chosen feed-forward neural network architecture.

is fed into the neural network,  $\text{RELU}(x)$  is the rectified linear unit activation function of the network,  $b$  is the bias parameter, and  $F_t$  is the forecast of the model. As in the previous diagram,  $X_{t-1}, X_t$  is the time series pair. The LSTM

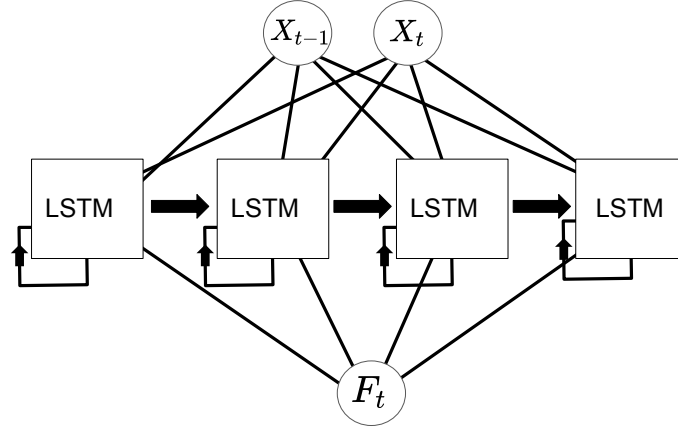


Figure 5: The chosen LSTM neural network architecture.

blocks represent the long-short-term memory units which feed out-put data back into themselves and  $F_t$  is the prediction. In both of the cases, the final model is able to generate a prediction  $F_{t+1}$  for the time series based on the value of the

time series at the current time step  $X_t$ . In Sec. 3.3, we will discuss how these architectures were implemented in python.

## 2.4 Benchmark

The ARIMA model and seasonal ARIMA model were fit with optimized grid-search parameters, as described in Sec. 3.2, with optimized parameters given in Table 5. In Fig. 6a, Fig. 6b and Fig. 7 the curve in blue is the training data used for fitting, the green line in the testing data and the red lines and red shaded area show the average value of the ARIMA forecast with the 95% confidence regions of the predictions. In Fig. 6a, we see that the SARIMA model was able

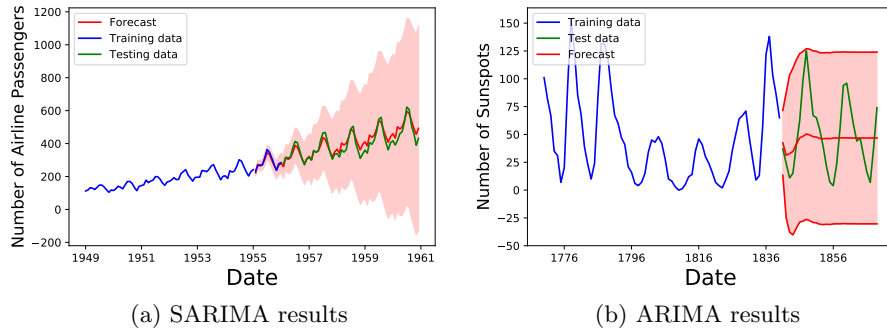


Figure 6: A figure with two subfigures

to correctly find the overall trend and as time increases, the forecast uncertainty also increases as would be expected. Fig. 6b which used a ARIMA model, did not seem to detect the correct trend and produces an uncertainty band that does not change over time. However, the height of the band does overlap with the training data.

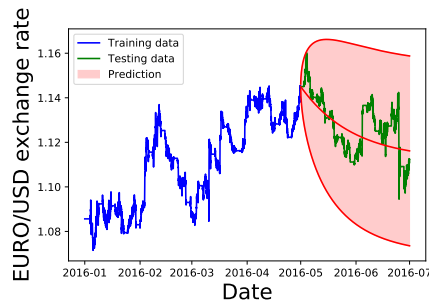


Figure 7: The forecasted exchange rate from EUR to USD from Jan 2016 to Jul 2016.

In Fig. 7 above, we see that the ARIMA model did not do a very good job at reproducing the data and it predicts a decreasing trend over time. The uncertainty band is very large in this case and encompasses a wide scale. In

Table 4, we provide the MSE and RMSE values of the optimized ARIMA and SARIMA models for the above data sets.

Data Set	Model	MSE	RMSE
i.	SARIMA	1878.63101438	43.3431772529
ii.	ARIMA	902.466806466	30.0410853077
iii.	ARIMA	8.67079188384e-05	0.00931170869596

Table 4: The RMS and RMSE values of the benchmark models.

### 3 Methodology

#### 3.1 Data Preprocessing

The forecasting models that we used in this work only require the values  $(t, X_t)$ , since the data sets that we used do not contain any missing values, our data sets did not require any cleanup. However, it was necessary to scale the data, from  $X_t \rightarrow \tilde{X}_t$  for the neural network data so that it was in the range  $[0, 1]$ . This transformation was accomplished using the following scaling function

$$\tilde{X}_t = \frac{X_t - X_{\min}}{X_{\max} - X_{\min}}, \quad (4)$$

where  $X_{\max}, X_{\min}$  are the maximal and minimum values of the data sets, respectively. This is implemented in python as the following code snippet.

---

```
def scale_array(y_vec):

    ymax = np.max(y_vec)
    ymin = np.min(y_vec)

    y_vec_scaled = np.zeros((len(y_vec),1))

    for k in range(0,len(y_vec)):
        y_vec_scaled[k][0] = (y_vec[k][0]-ymin)/(ymax-ymin)

    return y_vec_scaled,ymin,ymax
```

---

To invert the value from the scaling function, we use,

$$X_t = X_{\min} + \tilde{X}_t \cdot (X_{\max} - X_{\min}). \quad (5)$$

The inversion of the transformation was carried out after the model made its predictions so that the predictions would be in the original range of the data set. This is implemented as follows,

---

```
def invert_scaling(y_vec_scaled,ymin,ymax):

    y_vec = np.zeros(y_vec_scaled.shape)

    for k in range(0,len(y_vec_scaled)):
        y_vec[k] = ymin+(ymax-ymin)*y_vec_scaled[k]
    return y_vec
```

---



Another data preprocessing step that we carried out was to separate out the data into training vs testing data sets. The usual amount of training vs testing was about 60-80% for training and 30-40% testing.

### 3.2 ARIMA Model: Implementation

We used the ARIMA and the Seasonal ARIMA models from the python package `statsmodels` (Ref. [10]). The ARIMA model consists of the variables  $(p, d, q)$ , which are the non-seasonal AR order, differencing, and MA order, respectively. In order to fit this model to data, we use grid search from  $(p, d, q) = (0, 0, 0) \rightarrow (p_{\text{Max}}, d_{\text{Max}}, q_{\text{Max}})$ . To fit the ARIMA model we chose to either minimize the AIC, or the root mean square (RMS) value. The following code generates the grid that we will use to search the model space for the best fitting ARIMA model,

---

```
p = range(0,pMax+1)
d = range(0,dMax+1)
q = range(0,qMax+1)

# This creates all combinations of p,q,d
pdq = list(itertools.product(p, d, q))
```

---

For each item in the grid generated with the previous code snippet, we fit ARIMA( $p, d, q$ ) model using the following code.

---

```
for param in pdq:
    arma_mod =
        statsmodels.tsa.arima_model.ARIMA(train_data,order=param).fit()
```

---

Once the model is fit, the AIC or RMS value is computed and the set of parameters  $(p, d, q)$  that generated the best score is taken to be the optimal model.

The seasonal ARIMA model incorporates non-seasonal and seasonal factors as a multiplicative model. Therefore, we can schematically write

$$\text{SARIMA} = \text{ARIMA}(p, d, q) \times S(P, D, Q, T), \quad (6)$$

where the variables  $(p, d, q)$  are the non-seasonal AR order, differencing, and MA order, respectively. The variables  $(P, D, Q)$  denote the same variables, except for the seasonal component  $S$ . The variable  $T$  denotes the number of time steps for a single period. For example, if the time units are in months, then  $T=6$  indicates a half-year seasonal pattern. In order to fit the SARIMA( $p, q, d, P, D, Q, T$ ) model to data, we use a gridsearch. The following code snippet generates a grid of  $(p, d, q) = (0, 0, 0) \rightarrow (p_{\text{Max}}, d_{\text{Max}}, q_{\text{Max}})$  and  $(P, D, Q) = (0, 0, 0) \rightarrow (p_{\text{Max}}, d_{\text{Max}}, q_{\text{Max}}, T)$ , where we take  $T$  as fixed, for simplicity.

---

```
p = range(0,pMax+1)
d = range(0,dMax+1)
q = range(0,qMax+1)
t = [t]

# This creates all combinations of p,q,d
```

---

---

```
pdq = list(itertools.product(p, d, q))

# This creates all combinations of the seasonal variables
seasonal_pdq = [(x[0], x[1], x[2], x[3]) for x in
                 list(itertools.product(p, d, q,t))]
```

---

For each combination,  $(p, q, d, P, D, Q, T)$ , we fit the SARIMA model using the code below.

---

```
for param in pdq:
    for param_seasonal in seasonal_pdq:
        mod = sm.tsa.statespace.SARIMAX(train_data,
                                         order=param,
                                         seasonal_order=param_seasonal,
                                         enforce_stationarity=False,
                                         enforce_invertibility=False)
        results = mod.fit()
```

---

Using the above prescription for fitting the ARIMA and SARIMA models, the following table summarizes the results of the gridsearch.

Data Set	Model	Optimal $(p, d, q)$	Optimal $(P, S, Q, T)$
i. Airline Passengers	SARIMA	(1, 1, 1)	(1, 0, 0, 12)
ii. Sunspots	ARIMA	(5, 0, 4)	-
iii. EUR to USD rate	ARIMA	(2, 0, 2)	-

Table 5: The optimal fitting parameters of the ARIMA, or SARIMA model.

### 3.3 Neural Network Model: Implementation

The feed-forward neural network in Fig. 4 is implemented in Keras with the following code snippet which also fits the model using the `mean_squared_error` metric and the `adam` optimizer.

---

```
model = Sequential()
model.add(Dense(4, input_dim=lags, activation='relu'))
model.add(Dense(1))
model.compile(loss='mean_squared_error', optimizer='adam')
history=model.fit(X_train, y_train, epochs=number_of_epochs,
                 batch_size=2, verbose=1)
```

---

The following is a summary of this implemented architecture

---

Layer (type)	Output Shape	Param #
=====		
dense_1 (Dense)	(None, 4)	8
-----		
dense_2 (Dense)	(None, 1)	5
=====		

Total params: 13  
Trainable params: 13  
Non-trainable params: 0

---

Using the same optimizer and loss function as the feed-forward neural network the LSTM network is implemented as follows

---

```
model = Sequential()
model.add(LSTM(4, input_dim=lags))
model.add(Dense(1))
model.compile(loss='mean_squared_error', optimizer='adam')
history=model.fit(X_train, y_train, epochs=number_of_epochs,
                  batch_size=2, verbose=1)
```

---

Below, we summarize the implemented LSTM network,

---

Layer (type)	Output Shape	Param #
=====		
lstm_1 (LSTM)	(None, 4)	96
-----		
dense_1 (Dense)	(None, 1)	5
=====		
Total params: 101		
Trainable params: 101		
Non-trainable params: 0		

---

## 4 Results

### 4.1 Model Evaluation and Validation

Having implemented the feed forward and LSTM networks in Keras as described in Sec. 3.3 we fit the model and generate the time series predictions  $\{F_t\}$ . One issue that we had using this model was that the predictions of the neural networks do not produce any uncertainties. To try to probe the uncertainties of the model, we carried out the following procedure

1. Train the model using the set  $\{X_{T,\text{Train}}\}$
2. Using the training data, calculate the predictions  $\{F_{T,\text{Train}}\}$
3. Compute the mean squared error (RMSE), and the standard deviation  $\sigma_{\text{RMS}}$  between the training set  $X_{T,\text{Train}}$  and the prediction on the training set  $F_T$ .
4. We generate the perturbation,  $\delta X_T$ , assumed to be a Gaussian random variable sampled from the distribution  $\mathcal{N}(\frac{1}{2}\text{RMS}, \frac{1}{2}\sigma_{\text{RMS}})$
5. We added the perturbation to the test data  $X_T + \delta X_T$  and generate predictions on the testing set,  $\{F_{T,\text{Test}}\}$ .
6. We repeat this process many times, until we generate a distribution of Forecasts. The distribution of the forecasts will the estimated uncertainty of the neural network predictions.

Using this procedure, we generated the uncertainty bands in Figs. 8,9,10. In

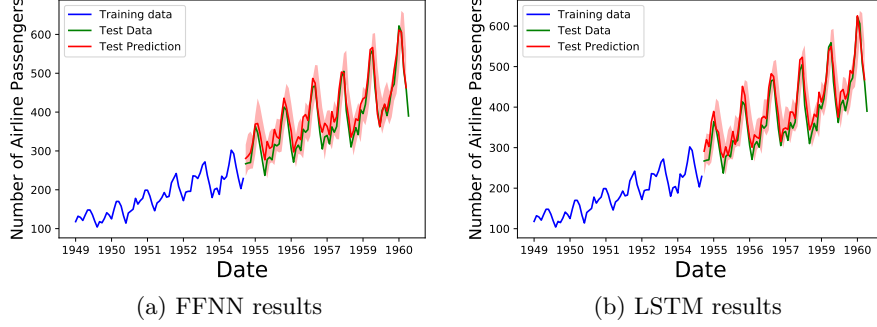


Figure 8: (a) The predictions with uncertainty of the feed forward neural network model. (b) The predictions with uncertainty of the LSTM model.

Fig. 8 we plot the predictions of the feed forward and LSTM neural networks for the Airline passengers dataset. In both cases the red uncertainty band overlap the actual training data, indicating a good fit.

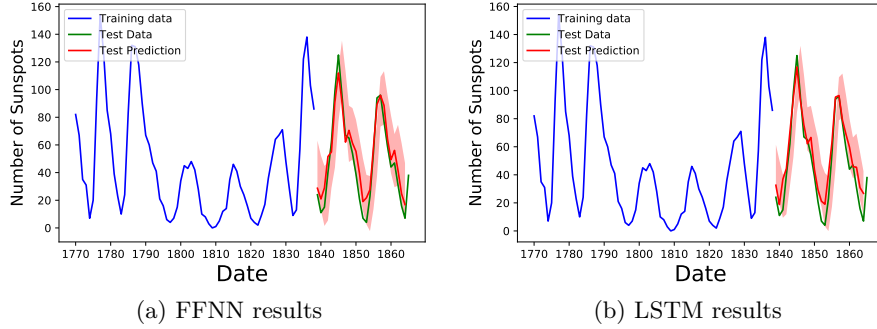


Figure 9: (a) The predictions with uncertainty of the feed forward neural network model. (b) The predictions with uncertainty of the LSTM model.

In Fig. 9 we show the comparison of the FFNN and LSTM results for the sunspot dataset. In this case we also see that the uncertainty bands are in agreement with the training data.

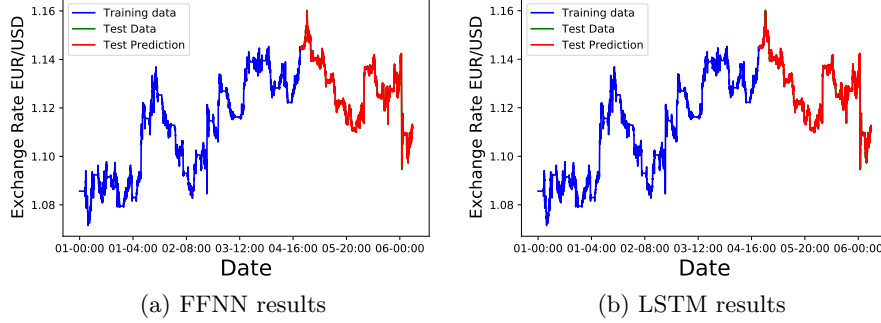


Figure 10: (a) The predictions with uncertainty of the feed forward neural network model. (b) The predictions with uncertainty of the LSTM model.

In Fig. 10 we show the results of the FFNN and LSTM networks on the Euro to USD exchange rate. In this case the estimated uncertainty was very small, therefore the uncertainty bands are not visible in the scale of the figure.

## 4.2 Justification

In Table 6, we give the final MSE and RMSE values of all models considered in this project. The RMSE baseline ratio is the value of the RMSE of the baseline model over the RMSE value of the specific model. The larger this ratio, the better the selected model is compared to the baseline model. In all three cases,

Data Set	Model	MSE	RMSE	RMSE Baseline Ratio
i.	SARIMA	1878.638	43.343	1
i.	FFNN	604.999	24.597	1.76
i.	LSTM	567.813	23.829	1.82
ii.	ARIMA	902.467	30.041	1
ii.	FFNN	101.171	10.058	2.99
ii.	LSTM	119.393	10.927	2.75
iii.	ARIMA	$8.671 \times 10^{-5}$	$9.312 \times 10^{-3}$	1
iii.	FFNN	$7.243 \times 10^{-8}$	$2.691 \times 10^{-4}$	34.6
iii.	LSTM	$8.454 \times 10^{-8}$	$2.908 \times 10^{-4}$	32.0

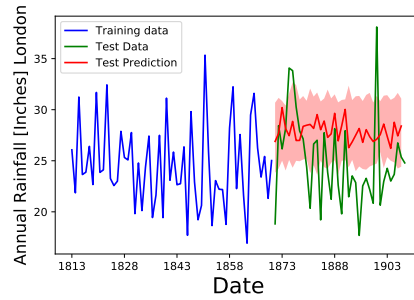
Table 6: The RMS and RMSE values of the Feed Forward (FFNN) and long-short term memory (LSTM) neural network model results.

we have that the FFNN and LSTM models outperformed the baseline SARIMA and ARIMA models. Data sets **i.** outperformed the baseline by a factor of 1.8, data set **ii.** outperformed the baseline roughly by a factor of 3, while dataset **iii.** outperformed the baseline by a factor of about 30. For data set **i.** we have that the LSTM model performed better than the FFNN model. For datasets **ii** and **iii** the FFNN models outperformed the LSTM model. In all cases, the neural networks outperformed the baseline model.

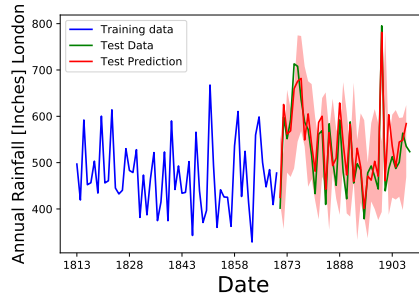
## 5 Conclusion

### 5.1 Free-Form Visualization

One very interesting aspect of this project that I did not have time to explore in more detail was to use different activation functions to test the quality of the generated models. In this project I presented results for the feed forward neural network with a rectified linear unit for the activation function, however, this model seemed to do very poorly for some time series. One such case was the “Annual rainfall in London (in inches)” data set shown in the figures below. When I first looked at this data set, I tried to fit it using the methods that I described in this project, but I found that I only got very poor results, pictured in Fig. 11a. I tried increasing the training set, but it did not fix the model. However, when I tried changing the activation function from RELU to a linear activation, I was able to generate the model in Figure 11b, which reproduces the data very well. This indicates to me that depending on the nature of the time series, different activation functions may be more suitable at generating predictions. I think a thorough study of the using different activation functions for all the data sets would be a very interesting extension of the project.



(a) FFNN with RELU activation



(b) FFNN with linear activation

Figure 11: (a) The predictions with uncertainty of the FFNN model with a RELU activation function; and (b) the predictions with uncertainty of the FFNN model with a linear activation function.

## 5.2 Reflection

In this project we have taken data for time series data  $(t, X_t)$  and we have trained two different neural network models, the feed forward neural network (FFNN) and the Long-short term memory (LSTM) network, denoted as  $G_{\text{FFNN}}$ ,  $G_{\text{LSTM}}$  respectively; to generate a time series forecast given the value of the time series at the previous time step. In other words, we have generated a function  $G$  such that

$$G(X_t) = F_{t+1} \quad (7)$$

where  $F_{t+1}$  should be similar to the  $X_{t+1}$  value of the underlying time series data. This problem was very interesting to me as it introduced me to how complicated and difficult time-series analysis can be, the body of literature that studies this subject is large, so it was great to be able to learn a little about it. The project had some challenges, mainly in getting started and in trying to understand the different approaches to time series modeling. The final model worked better than I expected and it is worth investigating in a more rigorous fashion. The fact that the FFNN and LSTM models outperformed the baseline by such a large margin suggests that perhaps the model was over-fitting. In the future, I would like to carry out more stringent tests to check against over-fitting.

## 5.3 Improvements

There are some further improvements that we could have carried out to the implemented models, such as an in-depth exploration of different neural network architectures. Adding more dense nodes or adding more layers would have been interesting to explore. However I ran out of time before I could carry out such a careful study. In the future I would also like to implement a generative adversarial artificial neural network for time series prediction. There was not enough time to understand and implement this model, but it seemed quite interesting and powerful. I will try to extend my project in the future to do this.

## References

- [1] R. Adhikari and R. K. Agrawal, An Introductory Study on Time Series Modeling and Forecasting, 2013, [arXiv:1302.6613] <https://arxiv.org/abs/1302.6613>.
- [2] T. Mong and U. Ngan, Research Journal of Finance and Accounting [www.iiste.org](http://www.iiste.org) ISSN 2222-1697 (Paper) ISSN 2222-2847 (Online) Vol.7, No.12, 2016 <http://iiste.org/Journals/index.php/RJFA/article/viewFile/31511/32351>.
- [3] P. J. Patel, N. J. Patel and A. R. Patel, IJAIEEM 3, 3, 2014. <http://www.ijaieem.org/volume3issue3/IJAIEEM-2014-03-05-013.pdf>
- [4] B. Oancea, S. Cristian Ciucu, Proceedings of the CKS 2013, [arXiv:1401.1333] <https://arxiv.org/abs/1401.1333>.
- [5] T. D. Chaudhuri and I. Ghosh, Journal of Insurance and Financial Management, Vol. 1, Issue 5, PP. 92-123, 2016, [arXiv:1607.02093] <https://arxiv.org/abs/1607.02093>.
- [6] Pant, N. (2017, September 07). A Guide For Time Series Prediction Using Recurrent Neural Networks (LSTMs). Retrieved from <https://blog.statsbot.co/time-series-prediction-using-recurrent-neural-networks-lstms-807fa6ca7f>.
- [7] D. K. Acatay, (2017, Nov. 21) Part 6: Time Series Prediction with Neural Networks in Python. Retrieved from <http://dakatay.com/data-science/part-6-time-series-prediction-neural-networks-python/>
- [8] Vincent, T. (2018). ARIMA Time Series Data Forecasting and Visualization in Python — DigitalOcean. [online] [digitalocean.com](https://www.digitalocean.com/community/tutorials/a-guide-to-time-series-forecasting-with-arima-in-python-3). Available at: <https://www.digitalocean.com/community/tutorials/a-guide-to-time-series-forecasting-with-arima-in-python-3> [Accessed 29 Jul. 2018].
- [9] C. Esteban, S. L. Hyland and G Rättsch, <https://github.com/ratschlab/RGAN> [arXiv:1706.02633].
- [10] S. Skipper and J. Perktold. “Statsmodels: Econometric and statistical modeling with python.” Proceedings of the 9th Python in Science Conference. 2010. <https://www.statsmodels.org/stable/index.html>