UPPSALA UNIVERSITET

Computational Finance: Pricing and Valuation
Assignment 3

Hampus Björklin - *habj8146@student.uu.se*
Oscar Jacobson - *oscar.jacobson.9201@student.uu.se*
Martin Åsell - *martin.asell.0292@student.uu.se*

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1 Introduction

The Black-Scholes model consist of one risky asset S e.g. the share of a company, and one risk free asset B, typically a bond. The price dynamics of the given assets are given as follows.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

$$dB_t = rB_t dt \tag{2}$$

Here W denotes a standard Wiener-process. It is also note that the stock price is modeled by a geometric brownian motion.
A European call with strike price K and expiry T is a derivative which gives the holder the right but not the obligation to buy one unit of the underlying asset S at time T for a predetermined price K, i.e. the payoff at time T is given by

$$C_T = \max(S_T - K, 0) \tag{3}$$

If the option is written with the stock from the Black-Scholes model as underlying asset, the risk neutral valuation of the call is given by the expected payoff under the Q measure, discounted to present time using the risk free rate r. i.e, the analytical price of said call at time t given stock price s at t is given by.

$$V(t,s) = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}{}_{t,s}[C_T] \tag{4}$$

Or equivalently,

$$V(t,s) = e^{-r(T-t)}\,\mathbb{E}_{t,s}[C_T] \tag{5}$$

Where the S dynamics are given by...

$$dS_u = rS_t du + \sigma S_t dW_u \tag{6}$$

$$S_t = s \tag{7}$$

There are multiple ways of solving equation 4 numerically aswell. Lattice methods, which creates a binomial price tree of all assumed possible stock prices at each time step can be used to calculate the option price backwards in time starting from the known payoffs of the assumed stock prices at the final time step T.
The expected payoff at time T can also be approximated using Mote Carlo-methods. By simulating multiple stock paths, their expected payoff can then be discounted to present time to retrieve the numerically calculated price of the option.
It is also possible to make reasonable assumptions about the boundary values of the call price which enables the solution to be found with finite difference methods.

When S has dynamics given by 6 there exists an analytical solution. However, this is not generally the case. If we assume the underlying asset has some arbitrary different dynamics, usually the only available option to price the derivative is by numerical methods such as the Monte Carlo, finite difference or lattice-method above.

2 Numerical methods- Overview

The Black-Scholes model can be extended slightly to the constant elasticity of variance model (CEV-model) to have volatility depend on the price of the underlying. Decreases and increases in volatility are commonly observed on the financial markets when the price of the underlying moves either upwards or downwards. The differential of the underlying asset is in the CEV-model given by.

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t \tag{8}$$

For $\gamma = 1$ we have the standard Black-Scholes model, aswell as an analytical price for a European call. When $\gamma \neq 1$ there are only numerical methods to price a European call.

2.1 Lattice method

The lattice method is one of the most simple numerical methods. It relies on the properties of random walk where the given parameters for step size, variance and drift determines the probability of the stock (in this case) moving up or down. The lattice itself is the discretization of two dimensional time and space. From one point in space the stock can either move up or down once for every step forwards or backwards in time.

$$\begin{aligned} S_{i+1} &= S_i * u \\ &\quad or \\ S_{i+1} &= S_i * d \end{aligned} \tag{9}$$

If u and d are assumed to have the symmetry $u * d = 1$ then the relationship between the two can be written as in equation10

$$\begin{aligned} u &= \beta + \sqrt{\beta^2 - 1} \\ d = 1/u &= \beta - \sqrt{\beta^2 - 1} \\ p_u &= \frac{e^{r\Delta t} - d}{u - d} \\ p_d &= 1 - p_u \end{aligned} \tag{10}$$

Where p is the probability of the stock moving up or down and β is derived from the Black-Scholes equation as equation 11.

$$\beta = \frac{1}{2}(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}) \tag{11}$$

By initiating one initial point S_0 in time and space for a certain underlying price the lattice method can be used to determine the possible price of the stock in the future. The stock either goes up or down which will represent two different prices at time $t_0 + \Delta t$.

Repeating this n times will produce n different possible final prices centered around S_0 at $t = t_n$. The n different values of S produces n different values of any given payoff function from a option. These values can then be brought backwards in time by using the weighted sum of the values and the probabilities p_u and p_d. The value of a point one step backwards in time is therefore given by the equation 12.

$$V_{i-1} = V_i^u * p_d + V_i^d * p_u \tag{12}$$

Where V^u and V^d is the value moving up and down respectively when moving backwards in time. After repeating this for all n steps in time the values all converge on the point where the initial stock price S_0 was set, giving the calculated value of the option at time $t = 0$ for the assumed price S_0.

2.2 Monte-Carlo method

As the dynamics of the underlying are known from equation 8, and the price of underlying can be observed at time $t = 0$. A possible random price path under the Q measure can be simulated by noting that equation 8 is in the discrete case equivalent to

$$\Delta S_t = \mu S_t \Delta t + \sigma S_t^\gamma \Delta W_t \tag{13}$$

Which – as $\Delta W_t = \sqrt{\Delta t} Z$ – is equivalent to

$$\Delta S_t = r S_t \Delta t + \sigma S_t^\gamma \sqrt{\Delta t} Z \tag{14}$$

Where Z $\in \mathcal{N}(0, 1)$.
A random stock path can then be simulated by for $M = \frac{T}{\Delta t}$ time steps update underlying asset price S with

$$S_{t+\Delta t} = S_t + \Delta S_t \tag{15}$$

By simulation large amount of possible stock paths using the dynamics given by 6, it is possible to calculate the mean payoff corresponding to each path.

$$V(t, s) \approx e^{-r(T-t)} \frac{1}{N} \sum_{n=1}^{N} \max(S_{n,T} - K, 0) \tag{16}$$

2.3 Finite difference method

The finite differences method is based on the idea that the partial derivatives are approximated with finite differences. There are a number of ways this can be done, but most common are:

- Explicit scheme.
- Implicit scheme.
- Crank Nicolson scheme.

In this report we will only look at the explicit and the implicit schemes. The only difference between explicit and implicit scheme here is how we approximate the partial time derivative. Let V(s,t)=V_j^n denote the value of the underlying asset, s, at time t. We use central difference approximation for the partial derivative with respect to the asset and standard approximation for the second order derivative i.e.

$$\frac{\partial V}{\partial s} \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\delta s} \tag{17}$$

$$\frac{\partial^2 V}{\partial s^2} \approx \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\delta s^2} \tag{18}$$

For explicit and implicit we have

$$\frac{\partial V}{\partial t} \approx \frac{V_j^n - V_j^{n-1}}{\delta t} \tag{19}$$

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\delta t} \tag{20}$$

Substituting this into the Black-Scholes equation and solving for either V_j^{n-1} or V_j^{n+1} will give us the explicit or implicit scheme needed to solve the PDE.

3 Numerical Results

3.1 Monte Carlo method

The price at time $t = 0$ of a European call expiring at time $T = 0.5$ with strike price $K = 15$ is calculated numerically fore some choices of stock price S_0 at $t = 0$ using the Monte Carlo Method described above. Results together with the analytical solution and the payoff function at $T = 0.5$ are presented below.

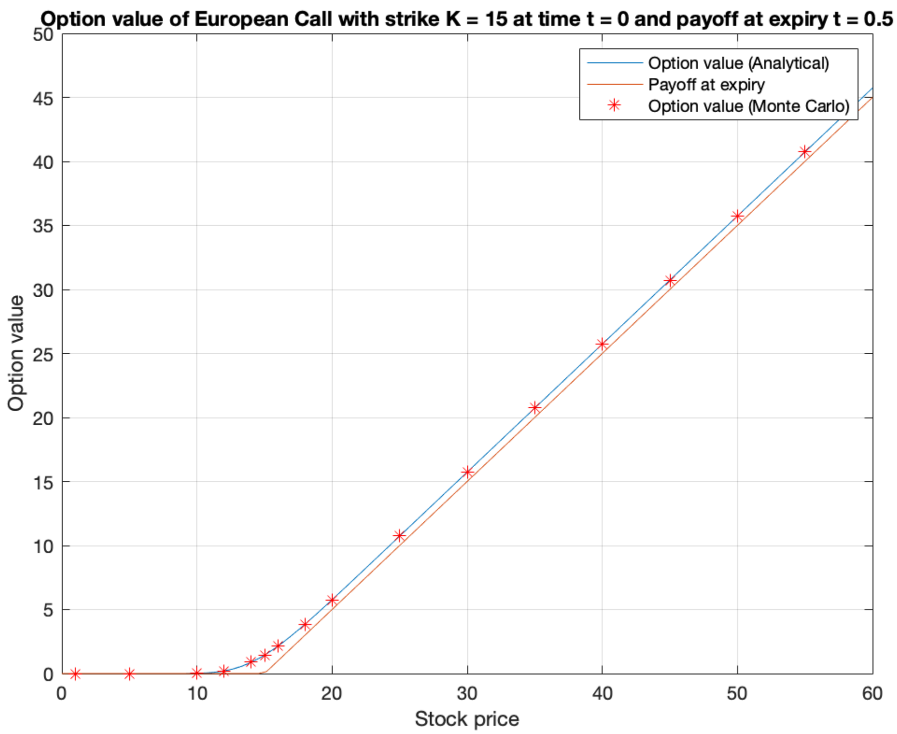


Figure 1: Price of European call with strike $K = 15$ at time $t = 0$ and expiry $T = 0.5$ calculated numerically with Monte Carlo method.

The error consisting of two parts - the sample error due to the fact that an infinite number of paths cannot be used to approximate the expected payoff, and the discretization error due to the fact an infinitely small time step cannot be used - can be calculated with

$$\epsilon = |\text{Numerical price} - \text{Analytical price}| \tag{21}$$

By studying how the error changes when parameters such as the number of samples and time step size are changed the convergence rate can be determined using

$$q = \frac{log(\frac{\epsilon_i}{\epsilon_{i-1}})}{log(\frac{\Delta t_i}{\Delta t_{i-1}})} \tag{22}$$

Results are displayed in figure 2.

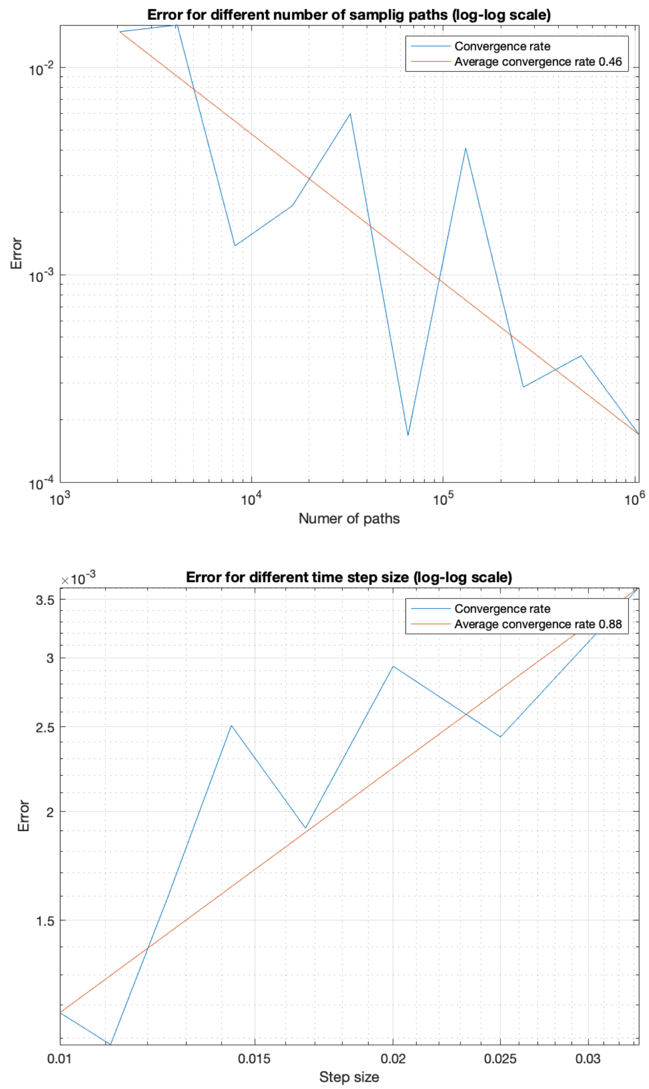


Figure 2: Convergence of sampling error (upper) and discretization error (lower) for Monte Carlo method.

3.2 Finite difference method

The same analysis is repeated for the solution using finite differences.

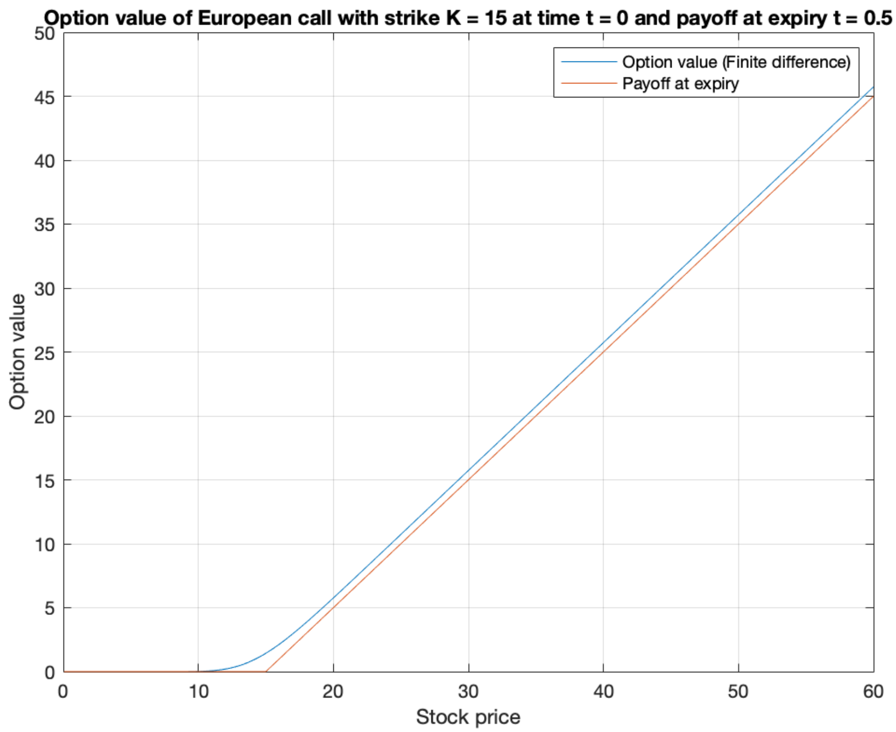
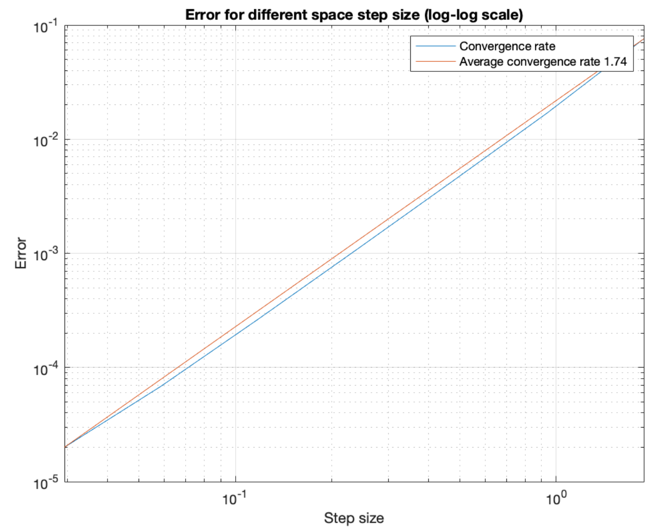


Figure 3: Price of European call with strike $K = 15$ at time $t = 0$ and expiry $T = 0.5$ calculated numerically with finite difference method.



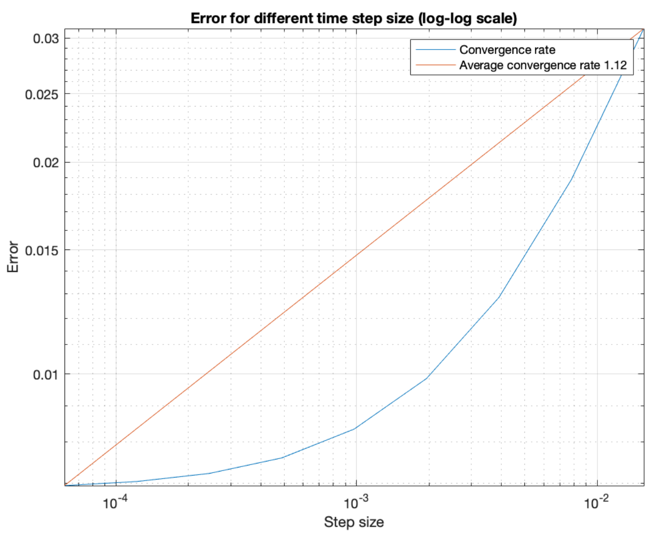


Figure 4: Convergence rate in space (upper) and time (lower) for Euler explicit.

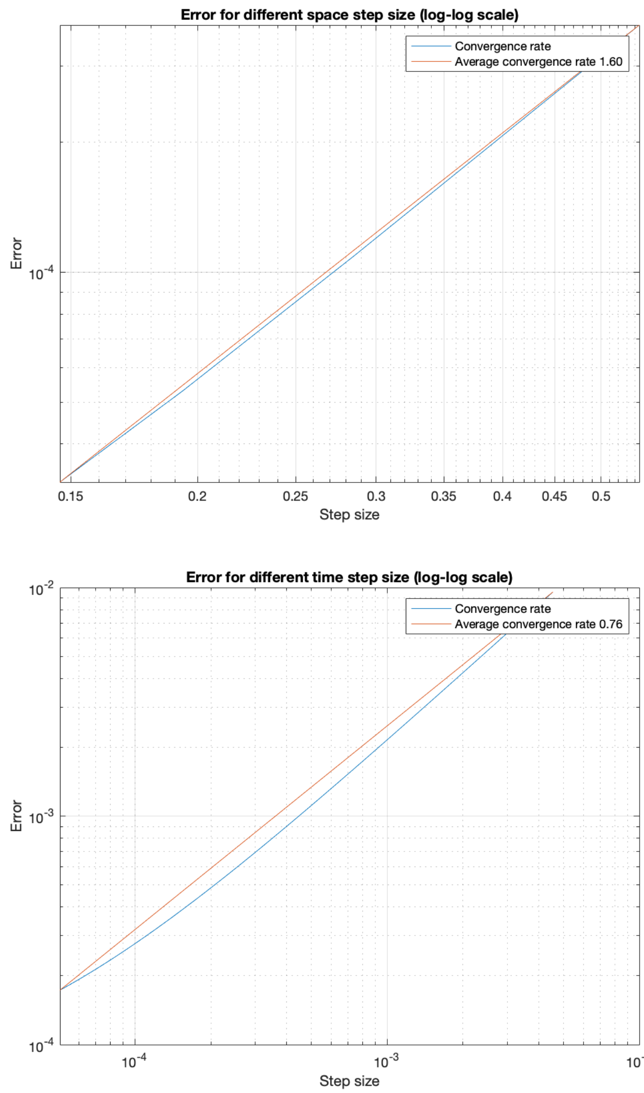


Figure 5: Convergence rate in space (upper) and time (lower) for Euler implicit.

4 Discussion

4.1 Calculating delta

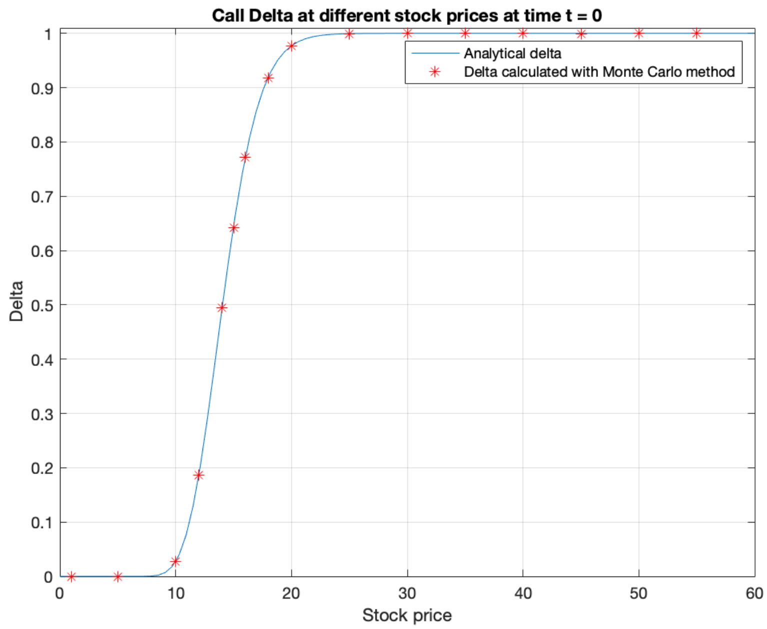


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Figure 6: Delta of European call with strike $K = 15$ at time $t = 0$ and expiry $T = 0.5$ calculated analytically and with Monte Carlo-method.

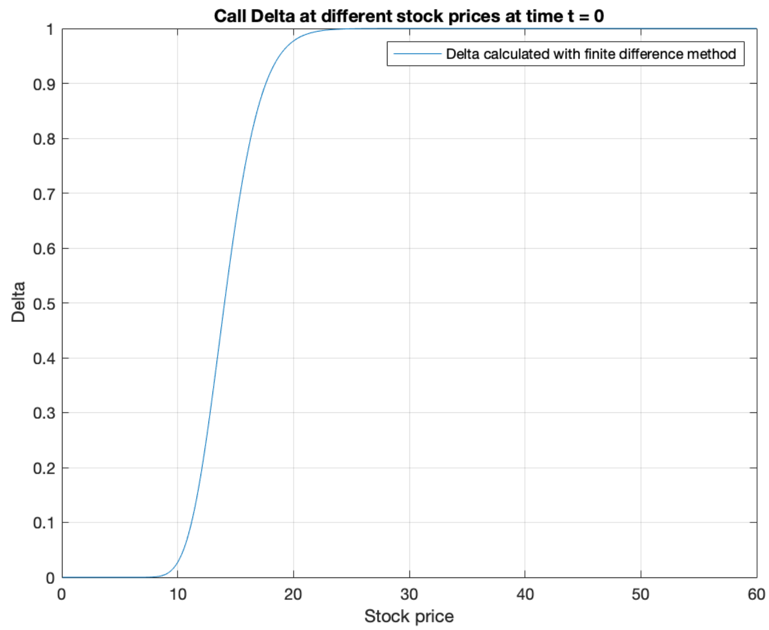

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Figure 7: Delta of European call with strike $K = 15$ at time $t = 0$ and expiry $T = 0.5$ calculated with finite difference method.

4.2 Discussions on dimensionality

This far in the project a Finite difference method and a Monte-Carlo method has been used to solve the expected option price fore some specific underlying constants. This problem is by design modeled in one spatial dimension and one dimension of time. The Monte-Carlo method solves the expected price of an option at **one** specific initial stock price whereas the Finite-difference method solves the price for the entire range of discretized starting prices. The Monte-Carlo method uses M sample paths for N steps in time to calculate one price with the computational complexity of O(N*M) whereas the explicit finite difference method calculates K prices with the computational complexity of O(N) (Assuming square, sparse, diagonal matrix-vector multiplications). The MC method is therefore significantly heavier computationally compared to the Finite difference method in one dimension. This is clearly noted in the execution time.

The explicit Euler finite difference scheme used struggles with stability for some choices of discretization of time and space. The Implicit Euler finite difference method implemented does not but is slightly more computationally complex as the matrix-vector relation has to be solved instead of multiplied. For stable problems, using any of the Finite difference methods assuming sufficiently discretized time and space the solution should converge towards the true solution with a factor of two when dividing the discretization of either time or space by two or with the convergence rate of O($\mathbf{N^2}$). This is not true for the MC method where the central limit theorem allows us to measure the rate of convergence which ends up being O($N^{1/2}$). The MC method is also in this case a worse candidate for solving the pricing of an option.[1]

This far, when pricing an option, the finite difference method has been applied to one-dimensional PDEs. When considering more exotic options though, the one dimensional PDE does not suffice. When using multiple random sources the modeled PDE can in a lot of cases be extended to produce a solution in multiple dimensions. This does, however, require the solution to be computed in the added dimensions multiplicatively. The complexity of such a solution is $O(N^{d+1})$ where the complexity increases with added spatial dimensions and the last dimension being the dimension of time. Comparing this to the original O(N) complexity the usage of finite difference quickly gets infeasible with dimensions. This is called the curse of dimensionality and a lot of numerical solvers suffer from this, including the finite element methods mentioned.[1]

The Monte-Carlo method is by nature, extremely flexible. It can be used to produce a solution for a very large amount of problems as long as there exists a probabilistic model for the underlying variables. This often makes the MC method the only way to price certain kinds of derivatives. As previously mentioned the MC method is very expensive compared to the finite difference methods. This does however come with an upside in this case. The mathematical law of large numbers guarantees convergence for enough samples only using simple averages of simulated sample paths. This coupled with the central limit theorem mentioned previously still provides the same O($N^{1/2}$) convergence independently of dimensions. [1]In practice this means we can price more complex derivatives such as options with a basket of stocks as underlying.

4.3 The greeks

The greeks are often used in close proximity with the pricing of options. The greeks are a way of quantifying risk and the behaviour of price given isolated changes in the underlying asset. By determining how the value of the option will change when parameters such as volatility, interest rate and the price of the underlying asset are altered, the position can be hedged. i.e. the risk related to these changes can be neutralized. The most popular and basic greeks are delta Δ, vega ν, theta Θ and rho ρ which all serve as the quantifier for different variables in the pricing of an option. Δ for example, corresponds to the sensitivity of the option price given a change in the price of the underlying asset. measures the sensitivity to volatility in the underlying, Θ measures the sensitivity to the passage of time and ρ measures sensitivity to the change in the underlying interest rate.[2]

Given a method that can compute the price of an option given all the underlying parameters, the greeks can all be computed by measuring the rate of change in the computed price when only altering the respective parameter. Numerically we can approximate Δ by

$$\Delta \approx = \frac{V(s_{t+\Delta t}, t) - V(s_{t-\Delta t})}{2\Delta t} \tag{23}$$

In figure 6 and 7 the Δ of the European call option can be seen to go from the value 0 to 1 around the strike price K = 15. The value of the option when the underlying is lower than the strike is not increasing ($\Delta = 0$) whereas the value of the option increases linearly after the price of the underlying option reaches above the strike ($\Delta = 1$). The most interesting part of the graph is when the stock price is almost equal to the strike price the Δ becomes a lot harder to predict. For a more exotic option the relationship between underlying and options price could be a lot more unpredictable which would produce a very different graph.

References

[1] Hirsa, Ali. *Computational Methods in Finance, Taylor Francis Group, 2012. ProQuest Ebook Central.* [Online; accessed 22-September-2022].

[2] Wikipedia contributors. *Greeks (finance) — Wikipedia, The Free Encyclopedia.* `https://en.wikipedia.org/w/index.php?title=Greeks_(finance)&oldid=1099292198`. [Online; accessed 23-September-2022]. 2022.