Series 4b

1. Polynomial Growth in Metric Spaces

a) Let (E, d_E) and (F, d_F) be metric spaces with $E \neq \emptyset$, and let $f : E \to F$ be a function. Prove that f grows at most polynomially if and only if there exist $v \in E$ and $w \in F$ such that

$$\limsup_{c \to \infty} \sup_{x \in E} \frac{d_f(w, f(x))}{(1 + d_e(v, x))^c} < \infty. \tag{1}$$

b) Let $k, l, \nu \in \mathbb{N}$ and let $f : \mathbb{R}^k \to \mathbb{R}^l$ be a ν-times continuously differentiable function with at most polynomially growing derivatives. Prove that for all $w = 0, 1, \ldots, \nu$, the w-th derivative

$$f^{(w)} \quad \text{grows at most polynomially.} \tag{2}$$

2. Convergence Types at Time T for Stochastic Processes

For a probability space $(\Omega, \mathcal{F}, P)$ and $T > 0$, consider a sequence of real-valued stochastic processes $Y^N : [0, T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}_0$, defined at time T by

$$Y_T^N = \frac{1}{N} Z, \quad \text{where} \quad Z(P) = \mathcal{N}_{0,1} \quad \text{and} \quad Y_T^0 = 0 \tag{3}$$

(that is, Z is a standard normal random variable). Identify the types of convergence, as defined in Section 4.2, for which $Y_T^N \to Y_T^0$ holds as $N \to \infty$. For each type of convergence that applies, determine the order α of convergence, if applicable.

3. Equivalence of Bounded Derivatives and Quadratic Inequalities

Let $\mu \in C^1(\mathbb{R}, \mathbb{R})$ and $L \in \mathbb{R}$. Prove that:

$$\sup_{x \in \mathbb{R}} \mu'(x) \leq L \quad \text{if and only if} \quad \forall x, y \in \mathbb{R} : \ (x - y)(\mu(x) - \mu(y)) \leq L(x - y)^2. \tag{4}$$

4. Increment-Tamed Euler-Maruyama

(The following does not distinguish between pseudorandom numbers and actual random numbers.) Implement an *Increment-Tamed Euler-Maruyama* approximation $Y \colon \{0, 1, \ldots, N\} \times \Omega \to \mathbb{R}^d$ (see Section 4.5.4) for the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \in [0, T], \quad X_0 = \xi, \tag{5}$$

with time step size T/N, where the SDE (5) is set as in Definition 4.5.7 of the lecture notes.

Do this by writing a MATLAB function `IncrementTamed(`T`,`d`,`m`,`N`,`ξ`,`μ`,`σ`)` which takes as input $T \in (0, \infty)$, $d, m, N \in \mathbb{N}$, $\xi \in \mathbb{R}^d$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d \times m}))$ and returns as output a realisation of an $Y_N(P)$-distributed random variable.

Submission Deadline: Wednesday, 20 November 2024, by 2:00 PM.

① Polynomial growth in Metric Spaces

(E, d_E) and (F, d_F) metric spaces with $E \neq \emptyset$, and let $f: E \to F$ be a function.

$\Rightarrow$) f grows at most polynomially, then, for all $v \in E$, $w \in F$ that there exists a real number $c \in [0, \infty)$ such that for all $x \in E$ it holds

$$d_F(w, f(x)) \leq c\,(1 + d_E(v, x))^{\bar{c}}$$

We take wathever v, w an $\frac{d_f(w, f(x))}{(1 + d_e(v,x))^{\bar{c}}} \leq c$ for a given $c < \infty$ $\forall\, x \in E$.

hence, $\sup_{x \in E} \frac{d_f(w, f(x))}{1 + d_e(v,x))^{\bar{c}}} \leq c$. Now, if we take limit on

$\bar{c}$, as $(1 + d_e(v,x)) \geq 1$, $\limsup_{\bar{c} \to \infty} \sup_{x \in E} \frac{d_f(w, f(x))}{(1 + d_e(v,x))^{\bar{c}}} \leq c$.

$\Leftarrow$) If there exist $v \in E$, $w \in F$ s.t.

$$\limsup_{c \to \infty} \sup_{x \in E} \frac{d_F(w, f(x))}{(1 + d_E(v,x))^{c}} < \infty \text{, then, we have that}$$

Now, for every $c > 0$, $\exists\, C_c > 0$ s.t.

$$\sup_{x \in E} \frac{d_F(w, f(x))}{(1 + d_E(v,x))^{c}} \leq C_c \Rightarrow d_F(w, f(x)) \leq C_c (1 + d_E(v,x))^{c} \quad (1) \quad \forall x \in E$$

Then, for sufficiently large c, C_c has to stabilize, as it cannot be infinite, hence the growth of $d_F(w, f(x))$ is bounded by a polynomial of degree c.

- Now, this has to be used to prove it for all $v' \in E$, $w' \in F$.

$$d_F(w', f(x)) \leq d_F(w', w) + d_F(w, f(x)) \quad \text{(Triangle inequality)}$$

So, using the assumption (1),

$$d_F(w', f(x)) \leq d_F(w', w) + C(1 + d_E(v,x))^{d} \text{ for } C, d > 0 \quad (2)$$

- At the other hand, $d_E(v, x) \leq d_E(v, v') + d_E(v', x)$

thus, $1 + d_E(v,x) \leq (1 + d_E(v, v'))(1 + d_E(v', x))$

So, puting this inequality in (2),

$$d_F(w', f(x)) \leq d_F(w', w) + C\,(1 + d_E(v, v'))(1 + d_E(v', x))$$

And, we see that $d_F(w', w) + C\,(1 + d_E(v, v'))^d = K$ (new constant)

Then,

$$d_F(w', f(x)) \leq K\,(1 + d_E(v', x))^d$$

Hence, the polynomial growth condition is satisfied for all $v' \in E$, $w' \in F_0$.

b) We know that f is v-times continuously differentiable that satisfies that $f^{(v)}$ grows at most polynomially.

Now, we need to see that it holds for all $w \in \{0, \dots, v\}$.

Let's assume $f^{(w)}$ for $w \in \{0, \dots, v-1\}$ don't grow at most polynomially, now, we know that $\forall a \in \mathbb{R}^k$, $\forall x$ around a:

$$f^{(w)}(x) = \sum_{n=0}^{\infty} \frac{f^{(w+n)}(a)}{n!} \cdot (x-a)^n .$$

Note: We consider $f^{(u)}$ as the u-th derivative tensor of f

And $f^{(v)}(x) = \sum_{n=0}^{\infty} \frac{f^{(v+n)}(a)}{n!} (x-a)^n$, so, we can take the terms

$$\underbrace{\sum_{n=v-w}^{\infty} \frac{f^{(w+n)}(a)}{n!} (x-a)^n}_{(1)} \quad \text{and} \quad \underbrace{\sum_{n=0}^{\infty} \frac{f^{(v+n)}(a)}{n!} (x-a)^n}_{(2)}$$

$$\sum_{n=v-w}^{\infty} \frac{f^{(w+n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(v+n)}(a)}{(n+v-w)!} (x-a)^{(n+v-w)} = \sum_{n=0}^{\infty} \frac{f^{(v+n)}(a)}{n!} (x-a)^n \frac{(x-a)^{v-w} \cdot n!}{(n+v-w)!}$$

Now, we see that $\frac{n!}{(n+v-w)!} \to 0$ as $n \to \infty$.

and, $\frac{n!}{(n+v-w)!} \leq \frac{1}{(v-w)!}$

So, $(1) \leq (2) \cdot \frac{1}{(v-w)!}$

But, we know that (1) don't grow at most polynomially, hence (2) don't grow at most polynomally. Hence, we have proved that if $f^{(v)}$ grows at most polynomially, then $f^{(w)}$ too. with $w \in \{0, \dots, v\}$.

② Convergence Types at Time T for Stochastic Processes

$$Y_T^N = \frac{1}{N} Z \quad , \quad Z(P) = N_{0,1} \quad , \quad Y_T^0 = 0$$

a) $Y_T^N \to Y_T^0$

- Convergence in probability, $\forall \varepsilon > 0$

$Y_T^N - Y_T^0 = \frac{1}{N} Z - 0 = \frac{1}{N} \cdot Z \sim N(0, \frac{1}{N^2})$

then, as $P(|\frac{1}{N} Z| \geq \varepsilon) = 2 P(Z > N \cdot \varepsilon)$.

$P(Z > x) \sim \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2}$ as $x \to \infty$ (gaussian decay)

for $x = \varepsilon \cdot N$,

$$P(|Y_T^N - Y_T^0| > \varepsilon) \sim \frac{1}{\sqrt{2\pi}\, N\varepsilon} \cdot e^{-(N\varepsilon)^2/2}$$

Then, we prove that $\limsup_{N \to \infty} P(|Y_T^N - Y_T^0| > \varepsilon) = 0$

- Strong convergence: $\forall p \in (0, \infty)$

$$\mathbb{E}_P[|Y_T^0 - Y_T^N|^p_{\mathbb{R}^1}] = \mathbb{E}_P[|Y_T^N|^p] = \mathbb{E}_P[(Y_T^N)^p \mid Y_T^N \geq 0] +$$
$$\mathbb{E}_P[(-Y_T^N)^p \mid Y_T^N \leq 0] = 2\, \mathbb{E}_P[(Y_T^N)^p \mid Y_T^N \geq 0] =$$
$$= 2\, \mathbb{E}[(\tfrac{1}{N} Z)^p \mid Z \geq 0]$$

Hence, $\limsup_{N \to \infty} \mathbb{E}[(\frac{1}{N} Z)^p \mid Z \geq 0] = \underline{\underline{0}}$ Hence strong convergence holds

To know the rate of convergence,

$\mathbb{E}[\frac{1}{N^p} Z^p \mid Z \geq 0] = \frac{1}{N^p} \mathbb{E}[Z^p \mid Z \geq 0]$. As $\mathbb{E}[Z^p \mid Z \geq 0]$ is bounded and fixed for every given $p \in (0, \infty)$, we have that

$$\| Y_T^0 - Y_T^N \|_{\mathcal{L}^p} \leq C \cdot \frac{1}{N^p} \quad \text{where } C = \mathbb{E}[Z^p \mid Z \geq 0]$$

Then $\alpha = p$ (rate of convergence).

- <u>Almost sure convergence</u>:

First, we see that as N increases
$$\begin{cases} \mathbb{E}[Y_T^N] = \frac{1}{N}\mathbb{E}[Z] = 0 \\ Var(Y_T^N) = \frac{1}{N^2} Var(Z) = \frac{1}{N^2} \to 0 \end{cases}$$

Now, we use Chebyshev's inequality, for any $\varepsilon > 0$
$$P(|Y_T^N| \geq \varepsilon) \leq \frac{Var(Y_T^N)}{\varepsilon^2} = \frac{1/N^2}{\varepsilon^2} = \frac{1}{N^2\varepsilon^2} \xrightarrow{N\to\infty} 0$$

And Borel-Cantelli lemma: if $P(A_n) \to 0$ as $n \to \infty$, and A_n are independent, then $P(\limsup_{n\to\infty} A_n) = 0$.

In this case, it holds for the event $|Y_T^N| \geq \varepsilon$, hence
$$Y_T^N \to Y_T^0 = 0 \quad \text{a.s. as } N \to \infty$$

Rate of convergence. As $\forall N \in \mathbb{N}$, it needs to hold
$$P(|\tfrac{Z}{N}| \leq C(\omega) \cdot N^{-\alpha}) = 1$$

We can find such a C function:
$$C(\omega) \geq \frac{|Z(\omega)|}{N^{1-\alpha}}$$

But this should hold for all $N \in \mathbb{N}$, so we could define $C(\omega)$ as
$$C(\omega) = |Z(\omega)|$$

So, now, $P(|\frac{Z}{N}| \leq |Z(\omega)| \cdot N^{-\alpha}) = P(\frac{1}{N} \leq N^{-\alpha}) = P(\frac{1}{N} \leq \frac{1}{N^\alpha})$

is equal to 1 for $\alpha \leq 1$, so <u>$\alpha = 1$ is the order of convergence.</u>

We have $P(|Y_T^N| \leq C \cdot N^{-1}) = 1 \quad \forall N \in \mathbb{N}$.

- <u>Num. weak convergence</u>: $|\mathbb{E}[\varphi(Y_T^N)]| = |\mathbb{E}[\varphi(\frac{1}{N}Z)]| =$

Let's find the Taylor expansion of $\mathbb{E}[\varphi(\frac{1}{N}Z)]$, given $\mathbb{E}[\frac{Z}{N}] = 0 = \mu_Z$

$$\mathbb{E}[\varphi(\tfrac{Z}{N})] = \varphi(\mu_Z) + \varphi'(\mu_Z)\underbrace{\mathbb{E}[\tfrac{Z}{N} - \mu_Z]}_{0} + \frac{1}{2}\varphi''(0)\,\mathbb{E}[(\tfrac{Z}{N} - \mu_Z)^2] + O(\mathbb{E}[(\tfrac{Z}{N})^3])$$

$$\mathbb{E}[\varphi(\tfrac{Z}{N})] = \varphi(0) + \frac{1}{2}\varphi''(0) \cdot \frac{1}{N^2} + O(\mathbb{E}[(\tfrac{Z}{N})^3])$$

Hence, we all the 3rd order terms are smaller than second order, so, we can find a given $C \in \mathbb{R}$ s.t

$$\left| E\left[\varphi\left(\frac{z}{N}\right)\right]\right| \leq C \cdot \frac{1}{N^2}$$ $\Rightarrow$ There is numerically weak convergence, with order $\alpha = 2$.

3 $\mu \in C^1(\mathbb{R}, \mathbb{R})$, $L \in \mathbb{R}$.

$\Rightarrow$) $\sup_{x \in \mathbb{R}} \mu'(x) \leq L$, now, $\frac{\mu(x) - \mu(y)}{x - y} = \mu'(z)$, $z \in (x, y)$

by MVT, then, $\mu(x) - \mu(y) = \mu'(z)(x-y) \leq L(x-y)$, Hence,
$(x-y)(\mu(x) - \mu(y)) \leq L(x-y)^2$

$\Leftarrow$) If $\forall x, y$ $(x-y)(\mu(x) - \mu(y)) \leq L(x-y)^2$, then,
$\mu(x) - \mu(y) \leq L(x-y)$, so $\frac{\mu(x) - \mu(y)}{x - y} \leq L$. But we know that

$\forall z \in \mathbb{R}$, $\mu'(z) = \lim_{x \to z} \frac{\mu(x) - \mu(z)}{x - z}$ and we have seen that it is $\leq L$.

Hence, $\sup_{z \in \mathbb{R}} \mu'(z) \leq L$.