Series 5a

Throughout this series, let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Milstein Scheme for One-Dimensional SDEs

Let $\xi \in \mathbb{R}$, let $\mu : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous, and let $\sigma \in C^1(\mathbb{R};\mathbb{R})$. Consider the SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi.$$

a) Let $M, N \in \mathbb{N}$. Write a MATLAB function `Milstein1D`$(T, \xi, \mu, \sigma, \sigma', W)$ with these inputs:

$$T \in (0,\infty),\ \xi \in \mathbb{R},\ \mu\colon \mathbb{R}^M \to \mathbb{R}^M,\ \sigma\colon \mathbb{R}^M \to \mathbb{R}^M,\ \sigma'\colon \mathbb{R}^M \to \mathbb{R}^M,\ W \in \mathbb{R}^{(N+1)\times M},$$

which returns M realizations $Y_N^N(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Milstein approximation Y_N^N for X_T.

Here: μ, σ, and σ' are function handles, and $W \in \mathbb{R}^{(N+1)\times M}$ is a realisation of M independent 1-dimensional Brownian motions sampled at equally spaced time points $\{n\Delta t \mid n = 0, \ldots, N\}$, i.e.

$$W^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i) \qquad (i = 1, 2, \ldots, M).$$

Hint: You may modify the solution `EulerMaruyama.m` from Series 4a.

b) Investigate the strong error of the Milstein scheme for the one-dimensional SDE

$$\mathrm{d}X_t = X_t\mathrm{d}t + \log\big(1 + X_t^2\big)\,\mathrm{d}W_t, \quad t \in [0,1], \quad X_0 = 1, \tag{1}$$

using $M = 10^5$ realisations and $N = N_\ell = 10 \cdot 2^\ell$ time steps for $\ell \in \{0, 1, \ldots, 4\}$. To this end:

- for each $\ell \in \{0, 1, \ldots, 4\}$, generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$ $(i = 1, \ldots, M)$ of the Milstein approximation $Y_{N_\ell}^{N_\ell}$ for X_T;
- for each $\ell \in \{0, 1, \ldots, 4\}$, compute Monte Carlo approximations for the following expectations:

$$\mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|] \approx \frac{1}{M}\sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T| \quad \text{and} \quad \mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M}\sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T|\right)^{\frac{1}{2}}.$$

Report on the experimental rates of strong convergence in L^1 and L^2. Use a numerical solution of the SDE at level $\ell = 7$ as an approximation of the exact solution.

(You may use the provided template `Milstein_SDE.m`.)

c) Repeat question b) for the following SDE and comment on the results:

$$\mathrm{d}X_t = X_t\mathrm{d}t + \sin\big(1 + X_t^2\big)\,\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = 1. \tag{2}$$

2. Positivity and Simulation of the CIR Process via Drift-Implicit Milstein

Let $a, b, \sigma_v > 0$ and $v_0 \geq 0$. Consider the Cox-Ingersoll-Ross process, given as solution to the SDE

$$\mathrm{d}V_t = a(b - V_t)\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{V_t}\mathrm{d}W_t, \quad V_0 = v_0, \quad t \in [0,T]. \tag{3}$$

It can be shown (e.g., using the Yamada-Watanabe theorem) that the SDE (3) admits a unique solution (up to indistinguishability).

① Milstein Scheme for one-dimensional SDEs

$\xi \in \mathbb{R}$, $\mu : \mathbb{R} \to \mathbb{R}$ glob. Lips. cont, $\sigma \in C^1(\mathbb{R};\mathbb{R})$

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t \in [0,T], \quad X_0 = \xi$$

b) The convergence rate for this case is not 1, as $\sin(1+x^2)$ don't have globally bounded derivative, hence, it doesn't apply this convergence rate.

② Positivity and Simulation of the CIR Process via Drift-implicit Milst.

Let $a, b, \sigma_v > 0$ and $v_0 \geq 0$. Consider the Cox-Ingersoll-Ross process, given as solution to the SDE

$$dV_t = a(b - V_t)\,dt + \sigma_v \sqrt{V_t}\,dW_t, \quad V_0 = v_0, \quad t \in [0,T] \quad (3)$$

It can be shown (using Yamada-Watanabe theorem) that the SDE (3) admits a unique solution up to indistinguishability.

a) Drift-implicit Milstein scheme. Stepsize $\Delta t > 0$, initial value $V_0^N = V_0 > 0$:

$$V_{n+1}^N = V_n^N + a(b - V_{n+1}^N)\Delta t + \sigma_v \sqrt{V_n^N}(W_{t_{n+1}} - W_{t_n}) + \frac{\sigma_v^2}{4}\left((W_{t_{n+1}} - W_{t_n})^2 - \Delta t\right)$$

Now, $\forall n \in \{0, \dots, N-1\}$

$$V_{n+1}^N = V_n^N + a(b - V_{n+1}^N)\Delta t + \sigma_v \sqrt{V_n^N}(W_{t_{n+1}} - W_{t_n}) + \frac{\sigma_v^2}{4}\left((W_{t_{n+1}} - W_{t_n})^2 - \Delta t\right)$$

$$V_{n+1}^N(1 + a\Delta t) = V_n^N + \Delta t\left(ab - \frac{\sigma_v^2}{4}\right) + \sigma_v(W_{t_{n+1}} - W_{t_n})\left(\sqrt{V_n^N} + \frac{\sigma_v}{4}(W_{t_{n+1}} - W_{t_n})\right)$$

$(1 + a\Delta t) > 0$, assume $n = 0$, $V_0^N > 0$

Now, we need to see that the following is ≥ 0:

$$\Delta t\left(ab - \frac{\sigma_v^2}{4}\right) + \sigma_v(W_{t_{n+1}} - W_{t_n})\left(\sqrt{V_n^N} + \frac{\sigma_v}{4}(W_{t_{n+1}} - W_{t_n})\right)$$

Assume if $4ab > \sigma_v^2$, hence, $\left(ab - \frac{\sigma_v^2}{4}\right) > 0$. Then,

$$\left.V_n^N + \sigma_v\sqrt{V_n^N}\,\underbrace{(W_{t_{n+1}} - W_{t_n})}_{x} + \frac{\sigma_v^2}{4}\underbrace{(W_{t_{n+1}} - W_{t_n})}_{x}{}^2\right\} (1)$$

$$\frac{\sigma_v^2}{4}x^2 + \sigma_v\sqrt{V_n^N}\,x + V_n^N = 0$$ We solve the following quadratic equation

$$x = \frac{-\sigma_v\sqrt{V_n^N} \pm \sqrt{\sigma_v^2 V_n^N - \sigma_v^2 V_n^N}}{\frac{\sigma_v^2}{2}} = \frac{-\sigma_v\sqrt{V_n^N}}{\frac{\sigma_v^2}{2}} = -\frac{2\sqrt{V_n^N}}{\sigma_v}$$

So, only exists a $(W_{t_{n+1}} - W_{t_n})$ s.t. $(1) = 0$, but we see that if take $W_{t_{n+1}} - W_{t_n} = 0$, $(1) > 0$, hence we have that $(1) > 0$ $\forall w \in \Omega$ except for $w_0 \in \Omega$ s.t. $(W_{t_{n+1}} - W_{t_n})(w_0) = \frac{2\sqrt{V_n^N}}{\sigma_v}$ which $\{w_0\}$ is a set of measure 0, hence, $P(V_1^N > 0] = 1$

Now, we can use induction and the same exact result to prove that if $V_n^N > 0$, then $V_{n+1}^N > 0$, hence $P(V_n^N > 0)$ $\forall n \in \{0, \dots, N\}$ $\square$

3 Integrability and proof verification of Kolmogorov Backward Equation

Assume setting in S.1.1. with $d = m = 1$ and that $\exists c > 0$ s.t. $\forall t \in [0,T]$, $x \in \mathbb{R}$

$$\sup_{s\in[t,T]} \|\mu(X_s^{t,s})\|_{L^q} + \sup_{s\in[s,T]} \|\sigma(X_s^{t,x})\|_{L^q} < c \quad \text{and} \quad \sup_{s\in[0,T]}\sup_{z\in\mathbb{R}} \frac{|\partial_z u(s,z)|}{(1+|z|)^3} \le c$$

where ∂_z is the partial derivative w.r.t the second argument of u.

$\mathbb{E}_P[(\sigma(X_s^{t,x}) \cdot \partial_z u(t+h, X_s^{t,x}))^2] =$

$$\sup_{s\in[t,T]} \|\sigma(X_s^{t,x})\|_{L^q} < K \underset{\|\cdot\|_{L^2} \le \|\cdot\|_{L^q}}{\Rightarrow} \sup_{s\in[t,T]} E[\sigma(X_s^{t,x})^2] < K \quad (2)$$

Now, $\partial_z u(t+h, X_s^{t,x})$ grows at most polynomially, hence

$$(\partial_z u(t+h, X_s^{t,x}))^2 \le (c \cdot (1+|X_s^{t,x}|))^2 \quad (1)$$

We have to see how $|X_s^{t,x}|$ behaves in a bounded interval $(t, t+h)$

As we know that $X_s^{t,x} = x + \int_t^s \mu(X_{s_1}^{t,x})ds_1 + \int_t^s \sigma(X_{s_1}^{t,x})dW_{s_1}$

$$E[X_s^{t,x}] = x + E\left[\int_t^s \mu(X_{s_1}^{t,x})ds_1\right] \le x + \circledast$$

$\sup_{s_1} \|\mu(X_{s_1}^{t,x})\|_{L^q} \le c' \rightarrow \sup_{s_1} \|\mu(X_{s_1}^{t,x})\|_{L^1} \le c'$. Hence,

$$E\left[\int_t^s |\mu(X_{s_1}^{t,x})| ds_1\right] \le E[(s-t)\cdot c'] < \infty$$

So, we can apply Fubini's theorem to $\circledast$ and get:

$$E[X_s^{t,x}] = x + \int_t^s E[\mu(X_{s_1}^{t,x})]ds_1 \le x + (s-t)\cdot c'$$

This can be ommited, as we know that the solutions of an Ito SDE have finite moments !!

So, $(1) \leq \left(c \cdot \left(1 + (x + (s-t) \cdot c')\right)\right)^2$

Now, we can unify this with (2) and state:

$$E_P[(\sigma(X_s^{t,x}) \cdot \partial_2 u(t+h, X_s^{t,x}))^2] \leq E_P[K \cdot (1 + x + (s-t) \cdot c'))]^2$$

which is finite, $\forall s \in (t, t+h)$, $E_P[(\sigma(X_s^{t,x}) \cdot \partial_2 u(t+h, X_s^{t,x}))^2] \leq K'(s)$

$$\int_t^{t+h} E_P[(\sigma(X_s^{t,x}) \cdot \partial_2 u(t+h, X_s^{t,x}))^2] \leq \int_t^{t+h} \max_{u \in (t, t+h)} K'(u) \, ds$$

$$= h \cdot \max_{u \in (t, t+h)} K'(u) < \infty \quad \square$$

Now we have $u(t,x) = E[u(t+h, X_{t+h}^{t,x})]$ (S.10. of the proof)

Hence, $u(t+h, x) - u(t,x) = u(t+h, x) - E[u(t+h, X_{t+h}^{t,x})]$, now we apply w.t.s.:

$$u(t+h, x) - E[u(t+h, X_{t+h}^{t,x})] = -\int_t^{t+h} E[\frac{\partial u}{\partial x}(t+h, X_s^{t,x}) \cdot \mu(X_s^{t,x})] ds - $$
$$- \int_t^{t+h} \frac{1}{2} E[\frac{\partial^2}{\partial^2 x} u)(t+h, X_s^{t,x}) \cdot \sigma^2(X_s^{t,x})] ds$$

We need to apply Ito's formula:

$$f(X_t) = f(X_0) + \int_0^t \mu(X_s) f'(X_s) ds$$
$$+ \frac{1}{2} \int_0^t f''(X_s) \sigma^2(X_s) ds + \int_0^t f'(X_s) \sigma(X_s) dW_s$$

If we rearrange what we want to show, we see that we want to see this:

$$E[u(t+h, X_{t+h}^{t,x})] = u(t+h, x) + \int_t^{t+h} E[\frac{\partial u}{\partial x}(t+h, X_s^{t,x}) \mu(X_s^{t,x})] ds +$$
$$+ \int_t^{t+h} \frac{1}{2} E[\frac{\partial^2}{\partial^2 x} u(t+h, X_s^{t,x}) \cdot \sigma^2(X_s^{t,x})] ds$$

Now, we can apply Ito to the function $x \mapsto u(t+h, x)$
Which is C^2, as $f \in C^3$ and we have condition (S.1) on the theorem, which ensures integrability, so we can interchange $E[\,]$ with f, and as $f \in C^3$, also is $E[f()]$. (A)

And get:

$$u(t+h, X_{t+h}^{t,x}) = u(t+h, X_t^{t,x}) + \int_t^{t+h} \frac{\partial}{\partial x} u(t+h, X_s^{t,x}) \mu(X_s^{t,x}) ds +$$
$$+ \frac{1}{2} \int_t^{t+h} \frac{\partial^2}{\partial^2 x} u(t+h, X_s^{t,x}) \sigma^2(X_s^{t,x}) ds + \int_t^{t+h} \frac{\partial}{\partial x} u(t+h, X_s^{t,x}) dW_s$$

Now, we apply expectations to the whole expression, and, as we know u has the growth condition, also $\frac{\partial u}{\partial x}$ can be polynomially bounded (seen in last series), hence absolutely bounded in $(t, t+h)$, then, $\mu(X_s)$ is also bounded as $\sup_{s \in [t,T]} \|\mu(X_s)\|_{L^q} < c$

Also, we have seen that $\int_t^{t+h} E_P[(\sigma(X_s^{t,x})\, \partial_2 u(t+h, X_s^{t,x}))^2]\, ds < \infty$
Hence, we apply Fubini's theorem and we can interchange expectations and integrals. This, leads to the following expression:

$$\mathbb{E}[u(t+h, X_{t+h}^{t,x})] = u(t+h, x) + \int_t^{t+h} \mathbb{E}\left[\frac{\partial u}{\partial x} u(t+h, X_s^{t,x})\mu(X_s^{t,x})\right] ds +$$

$$+ \int_t^{t+h} \frac{1}{2} \mathbb{E}\left[\frac{\partial^2 u}{\partial^2 x}(t+h, X_s^{t,x})\sigma^2(X_s^{t,x})\right] ds$$

Which verifies identity (5.11) in the proof

(*) $\frac{d}{d\theta} E[f(X,\theta)] = \mathbb{E}\left[\frac{\partial}{\partial\theta} f(X,\theta)\right]$ if:

1) Continuity on f
2) Differentiability on f
3) Dominated Convergence Theorem: $\exists g(x)$ s.t.
$$\left|\frac{\partial}{\partial\theta} f(x,\theta)\right| \leq g(\theta)$$
where $\mathbb{E}[g(X)] < \infty$.

a) Let $N \in \mathbb{N}$. Assume that $[0, T]$ is discretised using a uniform temporal mesh with $N+1$ nodes, i.e. with time step size $\Delta t = T/N$. The *drift-implicit Milstein scheme* for the stochastic process V with step size Δt and initial value $V_0^N = V_0 > 0$ is given, for $n = 0, \ldots, N-1$, by

$$V_{n+1}^N = V_n^N + a(b - V_{n+1}^N)\Delta t + \sigma_{\mathrm{v}}\sqrt{V_n^N}(W_{t_{n+1}} - W_{t_n}) + \frac{\sigma_{\mathrm{v}}^2}{4}\left((W_{t_{n+1}} - W_{t_n})^2 - \Delta t\right).$$

Show that if $4ab \geq \sigma_{\mathrm{v}}^2$, then $P(V_n^N > 0) = 1$ for all $n \in \{0, \ldots, N\}$.

b) Write a Matlab functio `DriftImplicitMilstein(`$T, N, v_0, a, b, \sigma_{\mathrm{v}}$`)` with inputs $T \in (0, \infty)$, $N \in \mathbb{N}$, $v_0, a, b, \sigma_{\mathrm{v}} > 0$, and output a realization of the drift-implicit Milstein scheme $\{V_0^N, V_1^N, \ldots, V_N^N\}$ for the Cox-Ingersoll-Ross process V. Then, plot a sample path of the stochastic process V using the following parameter choices: $T = 1, N = 10^3, v_0 = 0.5, a = 2, b = 0.5$ and $\sigma_{\mathrm{v}} = 0.25$.

3. Integrability and Proof Verification for the Kolmogorov Backward Equation

We adopt the full setting of Theorem 5.1.1 in the lecture notes with $d = m = 1$. Furthermore, we assume that there exists a constant $c > 0$ such that, for any $t \in [0, T]$ and $x \in \mathbb{R}$,

$$\sup_{s \in [t,T]} \left\|\mu(X_s^{t,x})\right\|_{L^9(P;|\cdot|)} + \sup_{s \in [t,T]} \left\|\sigma(X_s^{t,x})\right\|_{L^9(P;|\cdot|)} < c \quad \text{and} \quad \sup_{s \in [0,T]} \sup_{z \in \mathbb{R}} \frac{|\partial_2 u(s,z)|}{(1+|z|)^3} \leq c,$$

where ∂_2 denotes the partial derivative with respect to the second (spatial) argument of u.

Show that for any $t \in [0, T]$, $x \in \mathbb{R}$, and $h \geq 0$ such that $t + h \leq T$,

$$\int_t^{t+h} \mathbb{E}_P\left[\left(\sigma(X_s^{t,x}) \cdot \partial_2 u(t+h, X_s^{t,x})\right)^2\right] \mathrm{d}s \, < \, \infty, \tag{4}$$

and verify the identity (5.11) in the proof of Theorem 5.1.1.

Submission Deadline: Wednesday, 04 December 2024, by 2:00 PM.