Series 5b

Throughout this series, let $T \in (0,\infty)$ and $d, m, N, K \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W \colon [0,T]\times\Omega \to \mathbb{R}^d$ be a d-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion. Moreover, in this series, we do not distinguish between pseudo-random and actual random numbers.

1. Weak Convergence and Extrapolation

Let $T > 0$ and $d = 1$ and $x_0 \in \mathbb{R}$, and consider the following stochastic differential equation:

$$\mathrm{d}X_t = -\sin(X_t)\cos(X_t)^3\,\mathrm{d}t + \cos(X_t)^2\,\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = x_0. \tag{1}$$

The goal of this exercise is to approximate $\mathbb{E}_P[f(X_T)]$ for the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. To this end, the Talay-Tubaro Expansion (Theorem 5.4.1) ensures the existence of a constant $C_1 \in \mathbb{R}$ with

$$\mathcal{E}^N := \mathbb{E}_P\big[f(X_T)\big] - \mathbb{E}_P\big[f(Y_T^N)\big] = C_1 N^{-1} + \mathcal{O}(N^{-2}), \quad \text{for all } N \in \mathbb{N}\,; \tag{2}$$

here, Y_T^N is the Euler-Maruyama approximation of X with step size $\frac{T}{N}$ at time T.

a) Prove that the stochastic process given by $X_t = \arctan(W_t + \tan(x_0))$, for $t \in [0,T]$, is the (up to indistinguishability) unique solution of (1).

b) How can the numerical value of C_1 be estimated without prior knowledge of $\mathbb{E}_P\big[f(X_T)\big]$?

c) Let from now on $T = 1$ and $x_0 = 1$. Provide a Monte-Carlo approximation $\hat{E}_N$ of $\mathbb{E}_P[f(X_T)]$ such that the absolute error $|\hat{E}_N - \mathbb{E}_P[f(X_T)]|$ is bounded by $5 \cdot 10^{-5}$. Then, let $h_\ell = T/N_\ell$ for $N_\ell := 5 \cdot 2^\ell$ with $\ell \in \mathbb{N}_0$, and repeat the following procedure independently $K = 100$ times:

- For each $\ell = 0, 1, 2, 3$, generate $M = 10^5$ sample paths of the Euler-Maruyama approximation $Y^{N_\ell, i}$ $(i = 1, \ldots, M)$ for the SDE (1), and simulate the error values

$$\mathcal{E}^{h_\ell} := \frac{1}{M}\sum_{i=1}^{M} f(Y_T^{N_\ell,i}) - \mathbb{E}_P[f(X_T)] \quad \text{and} \tag{3}$$

$$\mathcal{R}^{h_\ell} := \frac{1}{M}\sum_{i=1}^{M} \Big[2f(Y_T^{N_{\ell+1},i}) - f(Y_T^{N_\ell,i})\Big] - \mathbb{E}_P[f(X_T)], \tag{4}$$

  where $\mathbb{E}_P[f(X_T)]$ may be approximated independently by the MC-estimate $\hat{E}_N$ above.

Denoting by $\big(\mathcal{E}_j^{h_\ell}\big)_{j=1}^{100}$ and $\big(\mathcal{R}_j^{h_\ell}\big)_{j=1}^{100}$ your $K = 100$ independently drawn samples of (3) and (4), respectively, generate two log-log plots: the first plot should display:

$$E_K(\mathcal{E}^{h_\ell}) := \frac{1}{K}\sum_{j=1}^{K} \mathcal{E}_j^{h_\ell} \ \text{ against } \ N_\ell^{-1}, \qquad \text{for } \ \ell = 0, 1, 2, 3, \tag{5}$$

and the second plot should display:

$$E_K(\mathcal{R}^{h_\ell}) := \frac{1}{K}\sum_{j=1}^{K} \mathcal{R}_j^{h_\ell} \ \text{ against } \ N_\ell^{-1}, \qquad \text{for } \ \ell = 0, 1, 2, 3.$$

Report on the observed experimental rate of convergence. Does the empirical convergence rate observed from (5) coincide with the theoretically established convergence rate in (2)?

(5) Weak convergence and extrapolation

- $dX_t = -\sin(X_t)\cos(X_t)^3 dt + \cos(X_t)^3 dW_t$, $t \in [0,T]$, $X_0 = x_0$
- $f(x) = x^2$

a) We first check $X_t = \arctan(W_t + \tan(x_0))$ is indeed a solution:

We apply Ito's formula using $g(X) = \arctan(X)$

And considering that the ito process $W_t + \tan(x_0)$ is the solution of:

$$\begin{cases} dY_t = dW_t \\ Y_0 = \tan(x_0) \end{cases} \quad \Big| \quad g'(x) = \frac{1}{1+x^2} \quad g''(x) = \frac{-2x}{(1+x^2)^2}$$

Hence, $g(X_t) = g(X_0) + \int_0^t g'(X_s)\cdot 0 \cdot ds + \int_0^t g'(X_s)\cdot dW_s + \frac{1}{2}\int_0^t g''(X_s)\, ds$

$$\Rightarrow X_t = x_0 + \int_0^t \frac{1}{1+(W_s + \tan(x_0))^2} dW_s + \frac{1}{2}\int_0^t \frac{-2(W_s + \tan(x_0))}{1+(W_s+\tan(x_0))^2} ds$$

$$\Rightarrow dX_t = -\frac{(W_t + \tan(x_0))}{1+(W_t + \tan(x_0))^2} dt + \frac{dW_s}{1+(W_t + \tan(x_0))^2}$$

Now, $X_t = \arctan(W_t + \tan(x_0))$, so:

$$W_t + \tan(x_0) = \tan(X_t)$$

Substituting $\tan(X_t)$ into dX_t we get:

$$1 + (W_t + \tan(x_0))^2 = 1 + \tan^2(X_t) = \sec^2(X_t)$$

Thus, $$\frac{1}{1+(W_t + \tan(x_0))^2} = \cos^2(W_t)$$

Similarly, $$\frac{W_t + \tan(x_0)}{(1+(W_t+\tan(x_0))^2)^2} = \frac{\tan(X_t)}{\sec^4(X_t)} = \tan(X_t)\cos^4(X_t)$$

Now, as $\tan(X_t) = \frac{\sin(X_t)}{\cos(X_t)}$, hence we get the SDE:

$$dX_t = -\sin(X_t)\cos^3(X_t)\, dt + \cos^2(X_t)\, dW_t \;//$$

So, we have seen that X_t is solution. Now we need to prove that is unique. We are ok if we prove that

$\mu(X_t) = -\sin(X_t)\cos^3(X_t)$ and $\sigma(X_t) = \cos^2(X_t)$ are globally Lipschitz continuous.

We Know that $\sin(x), \cos(x) \in [-1,1]\ \forall\ x \in \mathbb{R}$. then, $\forall x,y \in \mathbb{R}$

$$|\mu(x)-\mu(y)| + |\sigma(x)-\sigma(y)| =$$

$$= |-\sin(x)\cos^3(x) + \sin(y)\cos^3(y)| + |\cos^2(x) - \cos^2(y)| \leq$$
$$\leq |-\sin(x)\cos^3(x)| + |\sin(y)\cos^3(y)| + |\cos^2(x) - \cos^2(y)|$$
$$\leq |\cos^3(x)|\,|\sin(y) - \sin(x)| + |\cos^2(x) - \cos^2(y)|$$
$$\leq C|x-y| + |\cos^2(x) - \cos^2(y)| \leq (C+K)|x-y|$$

sin is lipschitz due to M.V.T with $C=1$.
And $|\cos^3(x)| \leq 1$

$\cos^2(x)$ is Lipschitz due to MVT

Hence, the coefficients are globally lipschitz continuous. and we can say that $X_t = \arctan(W_t + \tan(X_0))$ is a unique solution under indistinguishability. $\square$

b) We Know that by Talay-Tubero:

$$\varepsilon^N := \mathbb{E}_P[f(X_T)] - \mathbb{E}_P[f(Y_T^N)] = C_1 N^{-1} + O(N^{-2})$$

But we need to find C_1 without Knowing $\mathbb{E}_P[f(X_T)]$

Hence, we can use two levels of approximation N and $2N$, and get:

- $\varepsilon^{N_1} = \dfrac{C_1}{N_1} + O(N_1^{-2})$
- $\varepsilon^{N_2} = \dfrac{C_1}{N_2} + O((N_2)^{-2})$

$$\varepsilon^{N_1} - \varepsilon^{N_2} \approx C_1\left(\frac{1}{N_1} - \frac{1}{N_2}\right) = C_1 \frac{(N_2-N_1)}{N_1 N_2}$$

$$\Rightarrow C_1 \approx \frac{N_1 N_2}{N_2 - N_1}\left(\mathbb{E}_P[f(Y_T^{N_1})] - \mathbb{E}_P[f(Y_T^{N_2})]\right)$$

So, we can compute the two EM approximations and estimate it.

2. Multilevel Monte Carlo

Let $\xi \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, and let $X = (X_t)_{t\in[0,T]}$ be a solution of

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi.$$

Let further $f \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ satisfy $\mathbb{E}_P\big[|f(X_T)|\big] < \infty$. Write a MATLAB function

$$\texttt{MultiLevelMonteCarlo}(T,\xi,\mu,\sigma,\epsilon,\alpha,\beta,\gamma,f) \tag{6}$$

with inputs T, ξ, μ, σ, f as introduced above, and simulation parameters $\epsilon, \alpha, \beta, \gamma > 0$. The function should output a realisation of a multilevel Monte Carlo (MLMC) Euler approximation of $\mathbb{E}_P\big[f(X_T)\big]$, with tolerance ϵ. Recall that the MLMC-Euler estimator is given by

$$\hat{E}^{\mathrm{ML}}(f(Y_{N_L}^{N_L})) = \sum_{\ell=1}^{L} \frac{1}{K_\ell} \sum_{k=1}^{K_\ell} \big(f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k})\big),$$

where $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of X_T with step size $\Delta t = T/N$, $L = \lceil -\log_2(\epsilon) \rceil$, $N_\ell = N_0 2^\ell$ ($\ell = 1, \ldots, L$) with $N_0 = 2T$, and, depending on (α, β, γ),

$$K_\ell = \left\lceil 2^{2\alpha L} \left(\sum_{k=1}^{L} 2^{(\gamma-\beta)k/2} \right) 2^{-(\beta+\gamma)\ell/2} \right\rceil.$$

(You may use the provided template `MultiLevelMonteCarlo.m`.)

Consider then the Black-Scholes model, which models the price process of an underlying S by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad t \in [0,T], \quad S_0 = s_0,$$

for a fixed interest rate $r \in \mathbb{R}$, volatility parameter $\sigma > 0$ and initial price $s_0 > 0$.

a) Test your MATLAB function (6) to evaluate a European call option with:

- strike price $K_{\text{strike}} > 0$,
- payoff at T given by $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$.

To this end, run the multilevel Monte Carlo scheme with tolerance $\epsilon \in \{0.05, 0.02, 0.01, 0.005, 0.002\}$ to estimate the root mean squared error (RMSE):

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT} \sum_{\ell=1}^{L} \frac{1}{K_\ell} \sum_{k=1}^{K_\ell} (f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k})) \right\|_{L^2(P;|\cdot|_{\mathbb{R}})};$$

here, $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Estimate the RMSE by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT} \sum_{\ell=1}^{L} \frac{1}{K_\ell} \sum_{k=1}^{K_\ell} (f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k}))$$

for each ϵ and averaging the squared realisations. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.05, \sigma = 0.1$ and $K_{\text{strike}} = 100$. Estimate the convergence rates of the weak error and of the overall complexity with respect to ϵ. Report on the results.

(You may use the provided template `MultiLevelMonteCarloBSCall.m`.)

Hint: The exact value of the call price $e^{-rT}\mathbb{E}_P[f(S_T)]$ is given by the Black Scholes formula

$$e^{-rT}\mathbb{E}_P[f(S_T)] = S_0\,\Phi\left(\frac{\left(r + \frac{\sigma^2}{2}\right)T + \ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}} \right) - K\,e^{-rT}\,\Phi\left(\frac{\left(r - \frac{\sigma^2}{2}\right)T + \ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}} \right), \tag{7}$$

where $\Phi \colon \mathbb{R} \to \mathbb{R}$ is the cumulative distribution function of the $\mathcal{N}_{0,1}$-distribution function, that is,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\,\mathrm{d}y, \qquad x \in \mathbb{R},$$

which you may use without proof. You may also use the MATLAB function `blsprice()`. Set the parameter `M_RMSE` in the template to `M_RMSE = 1` first to ensure that everything works, and then rerun the experiment with `M_RMSE = 10`.

b) Compare your MATLAB function from (6) with the provided function `MonteCarloEuler()` to evaluate the aforementioned European call option $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$. To this end, run the Monte Carlo-Euler scheme to compute an approximation of $\mathbb{E}_P\big[f(S_T)\big]$ with tolerances $\epsilon \in \{0.05, 0.02, 0.01, 0.005, 0.002\}$. Adjust the number of samples K for each step size $\Delta t = \epsilon$ such that the statistical error and the discretization bias in the root mean squared error (RMSE)

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big) \right\|_{L^2(P;|\cdot|_{\mathbb{R}})} \tag{8}$$

are balanced; here, $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.05, \sigma = 0.1$ and $K_{\text{strike}} = 100$. Estimate the RMSE in display (8) by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big)$$

for each ϵ and averaging the squared realizations. Estimate the convergence rates of the weak error and of the overall complexity with respect to ϵ, and plot the estimated computational times against ϵ in a logarithmic diagram. Report on your results.

(You may use the provided template `MLMCvsMCEBSCall.m`.)

Hint: As for Exercise 2a), initially set the parameter `M_RMSE` in the template to `M_RMSE = 1` to ensure that everything works, and then rerun the experiment with `M_RMSE = 10`.

Submission Deadline: Wednesday, 04 December 2024, by midnight.