Series 4b

1. Polynomial Growth in Metric Spaces

a) Let (E, d_E) and (F, d_F) be metric spaces with $E \neq \emptyset$, and let $f : E \to F$ be a function. Prove that f grows at most polynomially if and only if there exist $v \in E$ and $w \in F$ such that

$$\limsup_{c\to\infty} \sup_{x\in E} \frac{d_f(w, f(x))}{(1 + d_e(v, x))^c} < \infty. \tag{1}$$

b) Let $k, l, \nu \in \mathbb{N}$ and let $f : \mathbb{R}^k \to \mathbb{R}^l$ be a ν-times continuously differentiable function with at most polynomially growing derivatives. Prove that for all $w = 0, 1, \ldots, \nu$, the w-th derivative

$$f^{(w)} \quad \text{grows at most polynomially.} \tag{2}$$

2. Convergence Types at Time T for Stochastic Processes

For a probability space $(\Omega, \mathcal{F}, P)$ and $T > 0$, consider a sequence of real-valued stochastic processes $Y^N : [0, T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}_0$, defined at time T by

$$Y_T^N = \frac{1}{N} Z, \quad \text{where} \quad Z(P) = \mathcal{N}_{0,1} \quad \text{and} \quad Y_T^0 = 0 \tag{3}$$

(that is, Z is a standard normal random variable). Identify the types of convergence, as defined in Section 4.2, for which $Y_T^N \to Y_T^0$ holds as $N \to \infty$. For each type of convergence that applies, determine the order α of convergence, if applicable.

3. Equivalence of Bounded Derivatives and Quadratic Inequalities

Let $\mu \in C^1(\mathbb{R}, \mathbb{R})$ and $L \in \mathbb{R}$. Prove that:

$$\sup_{x\in\mathbb{R}} \mu'(x) \leq L \quad \text{if and only if} \quad \forall x, y \in \mathbb{R} : \ (x - y)(\mu(x) - \mu(y)) \leq L(x - y)^2. \tag{4}$$

4. Increment-Tamed Euler-Maruyama

(The following does not distinguish between pseudorandom numbers and actual random numbers.) Implement an *Increment-Tamed Euler-Maruyama* approximation $Y : \{0, 1, \ldots, N\} \times \Omega \to \mathbb{R}^d$ (see Section 4.5.4) for the stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \in [0, T], \quad X_0 = \xi, \tag{5}$$

with time step size T/N, where the SDE (5) is set as in Definition 4.5.7 of the lecture notes.

Do this by writing a MATLAB function `IncrementTamed(T,d,m,N,ξ,μ,σ)` which takes as input $T \in (0, \infty)$, $d, m, N \in \mathbb{N}$, $\xi \in \mathbb{R}^d$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times m}))$ and returns as output a realisation of an $Y_N(P)$-distributed random variable.

Submission Deadline: Wednesday, 20 November 2024, by 2:00 PM.