Series 4a

1. An Exponentially Weighted L^p-Space of Stochastic Processes

Given $T \in (0,\infty)$, let $\big(\Omega, \mathcal{F}, P, \mathbb{F} \equiv (\mathbb{F}_t)_{t\in[0,T]}\big)$ be a stochastic basis. For $2 \le p < \infty$ and $d \in \mathbb{N}$, let

$$\mathcal{V} := \left\{ \begin{array}{c} Y\colon [0,T]\times\Omega \to \mathbb{R}^d\colon \\ Y \text{ is } \mathbb{F}\text{-predictable and} \\ \text{a modification of } X \end{array} \,\middle|\, \begin{array}{c} X\colon [0,T]\times\Omega \to \mathbb{R}^d : \\ X \text{ is } \mathbb{F}\text{-predictable and} \\ \sup_{t\in[0,T]} \mathbb{E}_P\big[\|X_t\|_{\mathbb{R}^d}^p\big] < \infty \end{array} \right\}. \tag{1}$$

As usual, we will not distinguish between an $\mathbb{F}$-predictable stochastic process $X\colon [0,T]\times\Omega \to \mathbb{R}^d$ satisfying $\sup_{t\in[0,T]} \mathbb{E}_P\big[\|X_t\|^p\big] < \infty$ and its equivalence class in $\mathcal{V}$. Show that, for each $\lambda \in \mathbb{R}$,

$$\big(\mathcal{V}, \|\cdot\|_{\mathcal{V},\lambda}\big) \quad \text{is a complete normed } \mathbb{R}\text{-vector space} \tag{2}$$

(i.e., an $\mathbb{R}$-Banach space) for the function $\|\cdot\|_{\mathcal{V},\lambda} : \mathcal{V} \to [0,\infty)$ defined by mapping $Y \in \mathcal{V}$ to

$$\|Y\|_{\mathcal{V},\lambda} := \sup_{t\in[0,T]} \Big(e^{\lambda t}\, \|Y_t\|_{L^p(P;\|\cdot\|_{\mathbb{R}^d})} \Big) \equiv \sup_{t\in[0,T]} \Big(e^{\lambda t} \mathbb{E}_P\big[\|Y_t\|_{\mathbb{R}^d}^p\big]^{\frac{1}{p}} \Big). \tag{3}$$

2. Existence, Uniqueness, and Behaviour of Solutions to SDEs

For the following, let $T \in (0,\infty)$ and let $W = (W_t)_{t\in[0,T]}$ be a standard, one-dimensional Brownian motion supported on some stochastic basis $\big(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]}\big)$.

a) Let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some finite $p \ge 2$. Show that the SDE

$$\mathrm{d}X_t = \log(1+X_t^2)\mathrm{d}t + \mathbb{1}_{\{X_t>0\}} X_t \mathrm{d}W_t, \quad t\in[0,T], \quad X_0 = \xi, \tag{4}$$

admits a unique (up to indistinguishability) solution process $X\colon [0,T]\times\Omega \to \mathbb{R}$.

b) Show that there are infinitely many solution processes $X\colon [0,T]\times\Omega \to \mathbb{R}$ to the SDE

$$\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t + 3X_t^{2/3}\mathrm{d}W_t, \quad t\in[0,T], \quad X_0 = 0. \tag{5}$$

Explain which condition of the existence-and-uniqueness theorem (Theorem 3.5.1) is violated.

Hint: Use the function $\theta_a : x \mapsto (x-a)^3 \mathbb{1}_{\{x\ge a\}}$ for some constant $a > 0$.

c) Let $\mu : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and polynomially growing, let $\sigma : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz, and let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for $2 \le p < \infty$. Prove or disprove: μ and σ can be such that

$$\exists\, T \in (0,\infty) \,:\, \mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t\in[0,T), \quad X_0 = \xi, \tag{6}$$

holds on $\big(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]}\big)$ such that, for $S_r := \inf\{t\in[0,T) \,:\, X_t \notin (-r,r)\}$ $(r>0)$, we have

$$P(\forall\, k\in\mathbb{N} \,:\, S_k < T) > 0. \tag{7}$$

3. Euler-Maruyama

In this exercise, we implement the Euler–Maruyama method to approximate the solution of the SDE (4) from Ex. 2a) and empirically investigate the method's convergence rate. (To this end, the following does not distinguish between pseudorandom numbers and actual random numbers.)

a) Let $M, N \in \mathbb{N}$. Write a MATLAB function `EulerMaruyama(`$T, \xi, \hat{W}$`)` with inputs $T > 0$, $\xi \in \mathbb{R}$ and $\hat{W} \in \mathbb{R}^{(N+1)\times M}$, which returns M realizations $Y_N^N(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Euler–Maruyama approximation Y_N^N of X_T. The input $\hat{W} \in \mathbb{R}^{(N+1)\times M}$ shall be a realization of M independent one-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \ldots, N\}$, that is:

$$\hat{W}^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i)$$

for $i = 1, 2, \ldots, M$. You can use the provided template `EulerMaruyama.m`.

b) Investigate the convergence rate of the Euler–Maruyama scheme for the fixed parameters $T = 1$ and $\xi = 1$, using $M = 10^5$ and $N = N_\ell = 10 \cdot 2^\ell$ for $\ell \in \{0, 1, \ldots, 4\}$. To do so, generate M sample paths of the Brownian motion on the finest grid. Then, for every $\ell \in \{0, 1, \ldots, 4\}$ generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Euler-Maruyama approximation $Y_{N_\ell}^{N_\ell}$ of X_T.

Hence, for every $\ell \in \{0, 1, 2, 3\}$, compute a Monte Carlo approximation E_M^ℓ of

$$\mathbb{E}[|Y_{N_{\ell+1}}^{N_{\ell+1}} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M}\sum_{i=1}^{M} |Y_{N_{\ell+1}}^{N_{\ell+1}}(\omega_i) - Y_{N_\ell}^{N_\ell}(\omega_i)|^2\right)^{\frac{1}{2}} =: E_M^\ell$$

based on M samples, and estimate the empirical strong L^2-convergence rate of the Euler–Maruyama scheme by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$ for $\ell \in \{0, 1, 2, 3\}$ (for this, you may use the MATLAB function `polyfit`). Comment on your observed convergence rate.

You may use the provided template `ErrorEM.m`.

Remark: For $N_{\ell+1} > N_\ell$, the triangle inequality yields that, for some positive constant C,

$$\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \leq \mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - X_T|^2]^{\frac{1}{2}} + \mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \leq 2CN_\ell^{-\alpha},$$

where $\alpha > 0$ is the convergence rate of the Euler-Maruyama scheme. Hence, for $N_\ell \approx N_{\ell+1}$, we may assume that

$$\log(2C) - \alpha\log(N_\ell) \approx \frac{1}{2}\log\left(\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2]\right).$$

Submission Deadline: Wednesday, 20 November 2024, by 2:00 PM.