Series 4a

1. An Exponentially Weighted L^p-Space of Stochastic Processes

Given $T \in (0,\infty)$, let $\big(\Omega, \mathcal{F}, P, \mathbb{F} \equiv (\mathbb{F}_t)_{t\in[0,T]}\big)$ be a stochastic basis. For $2 \le p < \infty$ and $d \in \mathbb{N}$, let

$$\mathcal{V} := \left\{ \begin{array}{c} Y\colon [0,T]\times\Omega \to \mathbb{R}^d\colon \\ Y \text{ is } \mathbb{F}\text{-predictable and} \\ \text{a modification of } X \end{array} \,\middle|\, \begin{array}{c} X\colon [0,T]\times\Omega \to \mathbb{R}^d : \\ X \text{ is } \mathbb{F}\text{-predictable and} \\ \sup_{t\in[0,T]} \mathbb{E}_P\big[\|X_t\|^p_{\mathbb{R}^d}\big] < \infty \end{array} \right\}. \tag{1}$$

As usual, we will not distinguish between an $\mathbb{F}$-predictable stochastic process $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ satisfying $\sup_{t\in[0,T]} \mathbb{E}_P\big[\|X_t\|^p\big] < \infty$ and its equivalence class in $\mathcal{V}$. Show that, for each $\lambda \in \mathbb{R}$,

$$\big(\mathcal{V}, \|\cdot\|_{\mathcal{V},\lambda}\big) \quad \text{is a complete normed } \mathbb{R}\text{-vector space} \tag{2}$$

(i.e., an $\mathbb{R}$-Banach space) for the function $\|\cdot\|_{\mathcal{V},\lambda} : \mathcal{V} \to [0,\infty)$ defined by mapping $Y \in \mathcal{V}$ to

$$\|Y\|_{\mathcal{V},\lambda} := \sup_{t\in[0,T]} \Big(e^{\lambda t} \, \|Y_t\|_{L^p(P;\|\cdot\|_{\mathbb{R}^d})} \Big) \equiv \sup_{t\in[0,T]} \Big(e^{\lambda t} \mathbb{E}_P \big[\|Y_t\|^p_{\mathbb{R}^d} \big]^{\frac{1}{p}} \Big). \tag{3}$$

2. Existence, Uniqueness, and Behaviour of Solutions to SDEs

For the following, let $T \in (0,\infty)$ and let $W = (W_t)_{t\in[0,T]}$ be a standard, one-dimensional Brownian motion supported on some stochastic basis $\big(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]}\big)$.

a) Let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some finite $p \ge 2$. Show that the SDE

$$\mathrm{d}X_t = \log(1 + X_t^2)\mathrm{d}t + \mathbb{1}_{\{X_t > 0\}} X_t \mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi, \tag{4}$$

admits a unique (up to indistinguishability) solution process $X\colon [0,T]\times\Omega \to \mathbb{R}$.

b) Show that there are infinitely many solution processes $X\colon [0,T]\times\Omega\to\mathbb{R}$ to the SDE

$$\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t + 3X_t^{2/3}\mathrm{d}W_t, \quad t\in[0,T], \quad X_0 = 0. \tag{5}$$

Explain which condition of the existence-and-uniqueness theorem (Theorem 3.5.1) is violated.

Hint: Use the function $\theta_a : x \mapsto (x-a)^3 \mathbb{1}_{\{x \ge a\}}$ for some constant $a > 0$.

c) Let $\mu : \mathbb{R}\to\mathbb{R}$ be infinitely differentiable and polynomially growing, let $\sigma : \mathbb{R}\to\mathbb{R}$ be globally Lipschitz, and let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for $2 \le p < \infty$. Prove or disprove: μ and σ can be such that

$$\exists\, T \in (0,\infty) \,:\, \mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t\in[0,T), \quad X_0 = \xi, \tag{6}$$

holds on $\big(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]}\big)$ such that, for $S_r := \inf\{t \in [0,T) \,:\, X_t \notin (-r,r)\}$ $(r > 0)$, we have

$$P(\forall\, k \in \mathbb{N} \,:\, S_k < T) > 0. \tag{7}$$

3. Euler-Maruyama

In this exercise, we implement the Euler–Maruyama method to approximate the solution of the SDE (4) from Ex. 2a) and empirically investigate the method's convergence rate. (To this end, the following does not distinguish between pseudorandom numbers and actual random numbers.)

a) Let $M, N \in \mathbb{N}$. Write a MATLAB function `EulerMaruyama(`$T, \xi, \hat{W}$`)` with inputs $T > 0$, $\xi \in \mathbb{R}$ and $\hat{W} \in \mathbb{R}^{(N+1)\times M}$, which returns M realizations $Y_N^N(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Euler–Maruyama approximation Y_N^N of X_T. The input $\hat{W} \in \mathbb{R}^{(N+1)\times M}$ shall be a realization of M independent one-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \ldots, N\}$, that is:

$$\hat{W}^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i)$$

for $i = 1, 2, \ldots, M$. You can use the provided template `EulerMaruyama.m`.

b) Investigate the convergence rate of the Euler–Maruyama scheme for the fixed parameters $T = 1$ and $\xi = 1$, using $M = 10^5$ and $N = N_\ell = 10 \cdot 2^\ell$ for $\ell \in \{0, 1, \ldots, 4\}$. To do so, generate M sample paths of the Brownian motion on the finest grid. Then, for every $\ell \in \{0, 1, \ldots, 4\}$ generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Euler-Maruyama approximation $Y_{N_\ell}^{N_\ell}$ of X_T.

Hence, for every $\ell \in \{0, 1, 2, 3\}$, compute a Monte Carlo approximation E_M^ℓ of

$$\mathbb{E}[|Y_{N_{\ell+1}}^{N_{\ell+1}} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M}\sum_{i=1}^{M} |Y_{N_{\ell+1}}^{N_{\ell+1}}(\omega_i) - Y_{N_\ell}^{N_\ell}(\omega_i)|^2\right)^{\frac{1}{2}} =: E_M^\ell$$

based on M samples, and estimate the empirical strong L^2-convergence rate of the Euler–Maruyama scheme by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$ for $\ell \in \{0, 1, 2, 3\}$ (for this, you may use the MATLAB function `polyfit`). Comment on your observed convergence rate.

You may use the provided template `ErrorEM.m`.

Remark: For $N_{\ell+1} > N_\ell$, the triangle inequality yields that, for some positive constant C,

$$\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \leq \mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - X_T|^2]^{\frac{1}{2}} + \mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \leq 2CN_\ell^{-\alpha},$$

where $\alpha > 0$ is the convergence rate of the Euler-Maruyama scheme. Hence, for $N_\ell \approx N_{\ell+1}$, we may assume that

$$\log(2C) - \alpha\log(N_\ell) \approx \frac{1}{2}\log\left(\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2]\right).$$

Submission Deadline: Wednesday, 20 November 2024, by 2:00 PM.

1 An Exponentially Weigthed $\mathcal{L}^p$-space of Stochastic Processes

$2 \le p < \infty$,

$$\mathcal{V} := \left\{ \begin{array}{l} Y : [0,T] \times \Omega \to \mathbb{R}^d : \\ Y \text{ is } \mathbb{F}\text{-predictable and} \\ \text{a modification of } X \end{array} \middle| \begin{array}{l} X : [0,T] \times \Omega \to \mathbb{R}^d : \\ X \text{ is } \mathbb{F}\text{-predictable and} \\ \sup_{t \in [0,T]} \mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] < \infty \end{array} \right\}$$

And the function $\|\cdot\|_{\mathcal{V},\lambda} : \mathcal{V} \to [0,\infty)$ defined by $Y \in \mathcal{V}$ to

$$\|Y\|_{\mathcal{V},\lambda} := \sup_{t \in [0,T]} \left(e^{\lambda t} \|Y_t\|_{\mathcal{L}^p(P, \|\cdot\|_{\mathbb{R}^d})} \right) = \sup_{t \in [0,T]} \left(e^{\lambda t} \mathbb{E}_P[\|Y_t\|^p_{\mathbb{R}^d}]^{\frac{1}{p}} \right)$$

To prove $(\mathcal{V}, \|\cdot\|_{\mathcal{V},\lambda})$ is a complete normed $\mathbb{R}$-vector space, first we need to see that the norm is well defined in $\mathcal{V}$.

- Given $X, Y \in \mathcal{V}$ s.t. $[X]_\nu = [Y]_\nu$, we want that $\|X\|_{\mathcal{V},\lambda} = \|Y\|_{\mathcal{V},\lambda}$. As they are in the same class in $\mathcal{V}$, Y is $\mathbb{F}$-predictable and a modification of X, hence, $\forall t \in \mathbb{T}$, it holds that there exist an event s.t. $P(A) = 1$ and

  $$A \subseteq \{X_t = Y_t\}$$

  Now, $\|Y\|_{\mathcal{V},\lambda} = \sup_{t \in [0,T]} \left(e^{\lambda t} \mathbb{E}_P[\|Y_t\|^p_{\mathbb{R}^d}]^{\frac{1}{p}} \right)$, but we know that $\forall t$, with prob $= 1$, $X_t = Y_t$, hence $\mathbb{E}_P[\|Y_t\|^p_{\mathbb{R}^d}] =$
  $= \mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] \Rightarrow \|X\|_{\mathcal{V},\lambda} = \|Y\|_{\mathcal{V},\lambda}$

- Now, we need to prove that $\|\cdot\|$ is a norm:
  - $\|\cdot\|_{\mathcal{V},\lambda} \geq 0$ because $\|\cdot\|_{\mathbb{R}^d}$ is a norm, hence $\|\cdot\|_{\mathbb{R}^d} \geq 0 \Rightarrow \|\cdot\|_{\mathcal{V},\lambda} \geq 0$
  - $\forall X \in \mathcal{V}$, $\|a \cdot X\|_{\mathcal{V},\lambda} = \sup_{t \in [0,T]} \left(e^{\lambda t} \mathbb{E}_P[\|a X_t\|^p_{\mathbb{R}^d}]^{\frac{1}{p}} \right)$, but

    $\|\cdot\|_{\mathbb{R}^d}$ is a norm, so $\|a \cdot X\|_{\mathbb{R}^d} = |a| \|X\|_{\mathbb{R}^d}$, and
    $\mathbb{E}_P[|a|^p \|X_t\|^p_{\mathbb{R}^d}]^{\frac{1}{p}} = |a| \mathbb{E}[\|X_t\|^p_{\mathbb{R}^d}]^{1/p} \Rightarrow$

    $$\|a \cdot X\|_{\mathcal{V},\lambda} = |a| \|X\|_{\mathcal{V},\lambda}$$

- $\forall X \in \mathcal{V}$, $\|X\|_{\mathcal{V},\lambda} = 0 \iff \sup_{t\in[0,T]} \mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] = 0$,

And, as $\|\cdot\|_{\mathbb{R}^d}$ is a norm, $\|X\|^p_{\mathbb{R}^d} \geq 0 \Rightarrow$
$\mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] = 0 \iff \|X_t\|_{\mathbb{R}^d} = 0 \iff X = 0$ ✓

- $\forall X, Y \in \mathcal{V}$ $\|X + Y\|_{\mathcal{V},\lambda} = \sup_{t\in[0,T]} \left(e^{\lambda t} \|X_t + Y_t\|_{L^p(P, \|\cdot\|_{\mathbb{R}^d})} \right]$

And we can use Minkowski inequality for $p \geq 2$, and $\|X_t + Y_t\|_{L^p} \leq \|X_t\|_{L^p} + \|Y_t\|_{L^p}$
Hence, $\|X + Y\|_{\mathcal{V},\lambda} \leq \sup_{t\in[0,T]} (e^{\lambda t} \|X_t\|_{L^p}) + \sup_{t\in[0,T]} (e^{\lambda t} \|Y_t\|_{L^p})$

$$= \|X\|_{\mathcal{V},\lambda} + \|Y\|_{\mathcal{V},\lambda}$$

- Next step is to prove that is complete, hence, for any subsequence $\{X^n\}_{n\in\mathbb{N}}$, which $X^n \to X$ in the sense of the norm $\|\cdot\|_{\mathcal{V},\lambda}$, then X is $\mathbb{F}$-predictable and $\sup_{t\in[0,T]} \mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] < \infty$. Let's prove X_t exists considering a Cauchy sequence, and:

$\|X^n - X^m\|_{\mathcal{V},\lambda} = \sup_t \left(e^{\lambda t} \|X_t^n - X_t^m\|_{L^p} \right]$, but, as we fix t, we can narrow our norm to an L^p space, which is complete, hence, when fixed p,

$$X_t^n \to X_t \quad \text{in } L^p \text{ norm}$$

So, we have found X. But we need to prove that X fulfills the two conditions mentioned above:

- $\sup_{t\in[0,T]} \mathbb{E}_P[\|X_t\|^p_{\mathbb{R}^d}] < \infty$ because we know that

  $X_t \in L^p$ and it is a complete norm, hence $\|X_t\|_{L^p} < \infty$
- We now need to prove predictability.
  Since $\{Y^n\}$ is a sequence of $\mathbb{F}$-predictable proc., each Y^n is adapted to the filtration $\mathbb{F}$ and

has càdlàg paths. We know that $\{Y^n\}$ converges to Y in the norm $\|\cdot\|_{\gamma,\lambda}$, we have

$$\sup \left(e^{\lambda t} E[\|Y^n_t - Y_t\|^p]\right) \to 0 \text{ as } n \to \infty$$

then we can extract a subsequence $\{Y^{n_k}\}$ s.t. $Y^{n_k}_t \to Y_t$ almost surely for almost all $t \in [0,T]$.

Since Y^{n_k} is predictable, we can represent it as the limit of simple processes (S^{n_k}) such that $S^{n_k}_m \to Y^{n_k}$ as $m \to \infty$.

Now, by using a diagonal argument, we can construct a sequence of simple predictable processes that converges to Y in probability.

And, since Y is the limit of predictable process in probability, and we can modify Y to be càdlàg (*), therefore Y is $\mathbb{F}$-predictable (because predictability is preserved under convergence in probability).

(*) Can be done, because we're working in the space of equivalence classes of processes.

2) Existence, Uniqueness and Behaviour of Solutions to SDEs.

a) $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for $p \in [2,\infty)$,

$$dX_t = \log(1+X_t^2)\,dt + \mathbb{1}_{\{X_t>0\}} X_t\, dW_t, \quad t \in [0,T], \; X_0 = \xi$$

We consider $\mu(X_t) = \log(1+X_t^2)$ and $\sigma(X_t) = \mathbb{1}_{\{X_t>0\}} X_t$

Now, $\sigma(X_s)$ is linear, so global Lipschitz. And, $\mu'(X_t) = \frac{2X_t}{1+X_t^2}$,

And, as $X_t \to \infty$, $\mu'(X_t) = 0$. Let's calculate extreme values at $\mu'(X_t)$:

$\mu''(X_t) = 0 \Leftrightarrow x = \pm 1$, hence $\sup_{t\in[0,T]} |\mu'(X_t)| = 1 \Rightarrow \mu(X_t)$ is global Lipschitz $(L=1)$

Hence, by Theorem 3.5.1, the SDE admits a unique (up to indistinguish.) solution process $X: [0,T] \times \Omega \to \mathbb{R}$.

b) $dX_t = 3X_t^{1/3}\,dt + 3X_t^{2/3}\,dW_t$, $t \in [0,T]$, $X_0 = 0$

- We prove that the $\mu(X_t) = 3X_t^{1/3}$ and $\sigma(X_t) = 3X_t^{2/3}$ are not local / global Lipschitz.

  - $\mu'(X_t) = \frac{1}{\sqrt[3]{X_t^2}}$ and $\lim_{X_t \to 0} \mu'(X_t) = \infty$

  - $\sigma'(X_t) = \frac{2}{\sqrt[3]{X_t}}$ and $\lim_{X_t \to \infty} \mu'(X_t) = \infty$

So, if we assume that $\mu'(X_t)$ is global Lipschitz, implies that for $L \in \mathbb{R}$,

$|\mu(X_1) - \mu(X_2)| \leq L\,|X_1 - X_2| \quad \forall X_1, X_2 \in \mathbb{R}$

But, we know that $\mu'(X_t) = \lim_{h\to 0} \frac{\mu(X_t) - \mu(X_{t+h})}{X_t - X_{t+h}}$, and as

$\lim_{X_t \to 0} \mu'(X_t) = \infty$, we have that $\lim_{X_t \to 0} \lim_{h \to 0} \frac{\mu(X_t) - \mu(X_{t+h})}{X_t - X_{t+h}} = \infty$

Which contradicts that $\forall X_1, X_2 \in \mathbb{R}$, $|\mu(X_1) - \mu(X_2)| \leq L|X_1 - X_2|$.

Hence, μ is not global Lipschitz, and violates conditions of (Theorem 3.5.1). The same can be proven with $\sigma(X_t)$.

• To prove that there are infinitely many solution process, we consider $f(x) = (x-a)^3 \mathbb{1}_{\{x \geq a\}}, a>0$, W_t is an Itô process with drift $=0$ and difussion $=1$, we can apply Itô's formula: ($f \in C^\infty$)

$$f(W_t) = a^3 + \int_0^t 3(W_s - a)^2 \mathbb{1}_{\{W_t \geq a\}} dW_s + \frac{1}{2}\int_0^t 6(W_s - a)\mathbb{1}_{\{W_t \geq a\}} ds$$

$\uparrow$

$$\left(f'(x) = 3(x-a)^2 \mathbb{1}_{\{x \geq a\}} \quad f''(x) = 6(x-a)\mathbb{1}_{\{x \geq a\}} \right)$$

$\Rightarrow$ if $X_t = (W_t - a)^3 \mathbb{1}_{\{W_t \geq a\}}$

$$dX_t = 3\underbrace{(W_t - a)}\cdot \mathbb{1}_{\{W_t \geq a\}} dt + 3(W_t - a)^2 \mathbb{1}_{\{W_t \geq a\}} dW_t$$

$\hookrightarrow W_t - a = X_t^{1/3}$ (when $W_t \geq a$, 0 otherwise)

$$\boxed{dX_t = 3X_t^{1/3} dt + 3X_t^{2/3} dW_t}$$

$\Rightarrow$ there are infinitelly many solutions for the SDE, i.e. $X_t = (W_t - a)^3$. //

$\hookrightarrow$ In case $X_t = 0$, $dX_t = 0$, hence, fulfills the SDE (S)

c) $S_r := \inf\{t \in [0,T] : X_t \notin (-r, r)\}$ $(r > 0)$

Feller Test for explosions: provides a precise criteria to determine, in terms of μ and σ the conditions under which solutions explode. (We consider $\sigma \equiv 1$)

Now, $\rho(x) = \int_0^x \exp\left(-2\int_0^s b(r)\,dr\right) ds$ and

$$v(x) = 2\int_0^x \frac{\rho(x) - \rho(y)}{\rho'(y)} dy$$

" The explosion time T_e of the solution X of the equation $\begin{cases} dX_t = \mu(X_t)dt + dW_t, \\ X_0 = a \quad t > 0 \end{cases}$

is finite with probability 1 iff one of the following holds:

(i) $v(\infty) < \infty$ and $v(-\infty) < \infty$

(ii) $v(\infty) < \infty$ and $\rho(-\infty) = -\infty$

(iii) $v(-\infty) < \infty$ and $\rho(\infty) = \infty$

So, if we take $dX_t = X_t^2 dt + dW_t$, $X_0 = -1$, $\mu(X_t) = X_t^2$, $\sigma(x) = 1$.

Then, $p(x) = \int_0^x \exp(-2\int_0^s r^2 dr)\, ds = \int_0^x e^{-\frac{2}{3}s^3} ds$

Hence, $v(x) = 2\int_0^x \int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\, dy \Rightarrow \lim_{x\to\infty} v(x) = 2\int_0^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\,dy$

So, we study $\int_0^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\,dy = \left(\int_0^1 + \int_1^\infty\right)\left(\int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\right) dy$

And $\int_1^\infty \int_0^y e^{\frac{2}{3}(z^3-y^3)} dz\,dy = \int_0^1 e^{\frac{2}{3}z^3} dz \int_1^\infty e^{-\frac{2}{3}y^3} dy + \int_1^\infty e^{\frac{2}{3}z^3} \underbrace{\left(\int_z^\infty e^{-\frac{2}{3}y^3} dy\right)} dz$

But $\underbrace{\int_z^\infty e^{-\frac{2}{3}y^3} dy} \leq \int_z^\infty \frac{y^2}{z^2} e^{-\frac{2}{3}y^3} dy = \frac{e^{-\frac{2}{3}z^3}}{2z^3}$. Using this,

$$\int_1^\infty e^{\frac{2}{3}z^3} \underbrace{\left(\int_z^\infty e^{-\frac{2}{3}y^3} dy\right)} dz \leq \int_1^\infty e^{\frac{2}{3}z^3}\left(\frac{e^{-\frac{2}{3}z^3}}{2z^2}\right) dz = \frac{1}{2} < \infty$$

Hence, $\lim_{x\to\infty} v(x) < \infty$ / And $\lim_{x\to-\infty} p(x) = \int_0^{-\infty} e^{-\frac{2}{3}s^3} ds = -\infty$

So, by Feller test, $P(T_e < \infty) = 1$.

So, we have proved that using $\mu(X_t) = X_t^2$ which is infinitely differentiable and polynomially growing, and $\sigma(X_t) = 1$, global Lipschitz, we can find a T_e finite in which $P(\forall k \in \mathbb{N} : S_k < T_e) > 0$ i.e. Solution "explodes".

③ See code attached