Series 3b

1. Itô vs. Stratonovich

Let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,1]})$ be a stochastic basis and $W\colon [0,1]\times\Omega \to \mathbb{R}^m$ be an m-dimensional standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,1]})$-Brownian motion. Let further $X = (X_t)_{t\in[0,1]}$ be an $(\mathbb{F}_t)_{t\in[0,1]}/\mathcal{B}(\mathbb{R}^{d\times m})$-adapted stochastic process with continuous sample paths (thus, $\int_0^t \|X_s\|^2_{\mathbb{R}^{d\times m}}\,\mathrm{d}s < \infty$ P-a.s.).

Given $t \in [0,1]$, let $(\mathcal{Z}_n^t)_{n\in\mathbb{N}} \subset \mathcal{P}([0,t])$ be any sequence of $[0,t]$-partitions with vanishing mesh, i.e. $\mathcal{Z}_n^t \equiv \{0 = t_0^{(n)} < \cdots < t_{N_{n,t}}^{(n)} = t\}$ with $\|\mathcal{Z}_n^t\| := \max_{j\in\{0,1,\ldots,N_{n,t}-1\}} \left|t_{j+1}^{(n)} - t_j^{(n)}\right| \to 0$ as $n\to\infty$.

a) Show that, for each $t\in[0,1]$, the stochastic integral $\int_0^t X_s\,\mathrm{d}W_s$ (Def. B.4.19) can be written as

$$\int_0^t X_s\,\mathrm{d}W_s = \lim_{n\to\infty} \sum_{j=0}^{N_{n,t}-1} X_{t_j^{(n)}}\big(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\big) \quad \text{in probability.} \tag{1}$$

b) Let $m = 1$ and $\alpha\in[0,1]$. Show that, for each $t\in[0,1]$, the reweighted approximating sum

$$S_\alpha(\mathcal{Z}_n^t) := \sum_{j=0}^{N_{n,t}-1} \big[(1-\alpha)W_{t_j^{(n)}} + \alpha W_{t_{j+1}^{(n)}}\big]\big(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\big) \tag{2}$$

converges in $L^2(P)$ (as $n\to\infty$) and find its limit $L_\alpha(t)$. For which α is $(L_\alpha(t))_{t\in[0,1]}$ a martingale?

Let now $d = m = 1$ (to simplify notation). For an $(\mathbb{F}_t)_{t\in[0,1]}/\mathcal{B}(\mathbb{R})$-adapted Itô process $U = (U_t)_{t\in[0,1]}$ of the form $\mathrm{d}U_t = Y_t\mathrm{d}t + Z_t\mathrm{d}W_t$ (Def. B.5.1), consider the so-called *Stratonovich integral*

$$\int_0^t X_s \circ \mathrm{d}U_s := \int_0^t X_sY_s\,\mathrm{d}s + \int_0^t X_sZ_s\,\mathrm{d}W_s + \frac{1}{2}\langle X, U\rangle_t \qquad (0\le t\le 1), \tag{3}$$

assuming that the covariation $\langle X, U\rangle$ of X and U, as introduced in Remark of Ex. 2/Ser. 3a, exists.

c) Show that, for each $t\in[0,1]$, the stochastic integral (3) has the limiting representation

$$\int_0^t X_s\circ\mathrm{d}W_s = \lim_{n\to\infty}\sum_{j=0}^{N_{n,t}-1} \frac{1}{2}\big[X_{t_j^{(n)}} + X_{t_{j+1}^{(n)}}\big]\big(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\big) \quad \text{in probability.} \tag{4}$$

d) Suppose now that $X = (X^{(1)},\cdots,X^{(d)})$ is in fact an O-valued Itô process, for some open $O\subseteq\mathbb{R}^d$. Then, given any function $f\in C^3(O)$ and $t\in[0,1]$, prove that

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_s)\circ\mathrm{d}X_s^{(i)}. \tag{5}$$

You may accept without proof the fact that the Stratonovich integrals $\int_0^t \frac{\partial f}{\partial x_i}(X_s)\circ\mathrm{d}X_s^{(i)}$ all exist[1].

[1] By the Remark of Ex. 2/Ser. 3a and the fact that every Itô process is also a semimartingale.

① Îto vs. Stratonovich

a) Def B.4.19. (Stochastic Integral). Given the conditions of exercise 1, we denote $\int_a^b X_s dW_s \in L^0(P; \|\cdot\|_{\mathbb{R}^d})$ the set given by

$$\int_a^b X_s\, dW_s = I^W_{a,b}(X) \quad \text{the stochastic integral of } X \text{ from } a \text{ to } b$$

with respect to W.

$$\hookrightarrow I^W_{a,b} : \left\{ X \in \mathbb{M}([0,T]\times\Omega, \mathbb{R}^{d\times m}) : \begin{array}{l} X \text{ is } (\mathbb{F}_t)_t\text{-predictable} \\ P\left(\int_a^b \|X_s\|^2 ds < \infty\right) = 1 \end{array} \right\} \to L^0(P, \|\cdot\|_{\mathbb{R}^d})$$

which satisfies: i) continuity

ii) stochastic integration of simple process.

We want to prove that for each $t \in [0,1]$, the stochastic integral can be rewritten as

$$\int_0^t X_s\, dW_s = \lim_{n\to\infty} \sum_{j=0}^{N_{n,t}-1} X_{t_j^{(n)}} (W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}) \quad \text{in probability}$$

We need to apreciate that, as the partition of $[0,t]$ becomes finer, the right hand side will converge in probability to $\int_0^t X_s dW_s$. So, we define

$$S_n := \sum_{j=0}^{N_{n,t}-1} X_{t_j^{(n)}} (W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}})$$

and we want,

$$\left| \int_0^t X_s\, dW_s - S_n \right| \xrightarrow{P} 0$$

We can first make use of the Itô isommetry: Since W is BM and X is adapted with $\int_0^t \|X_s\|^2_{\mathbb{R}^{d\times m}} ds < \infty$ P-a.s. We have that

$$\mathbb{E}_P\left[\left\| \int_0^t X_s\, dW_s \right\|^2_{\mathbb{R}^d} \right] = \int_0^t \mathbb{E}_P\left[\|X_s\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\, ds \right]$$

Now, we want to show that there is mean-squared convergence of S_n to $\int_0^t X_s dW_s$, as convergence in mean square implies convergence in probability, by Chevyshev's Inequality. Hence, we want:

$$\lim_{n\to\infty} \mathbb{E}\left[\left\| \int_0^t X_s\, dW_s - S_n \right\|^2_{\mathbb{R}^d} \right] = 0$$

So,

$$\lim_{n\to\infty} \mathbb{E}\left[\left\| \int_0^t X_s\, dW_s - \sum_{j=0}^{N_{n,t}-1} X_{t_j^{(n)}} (W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}) \right\|^2_{\mathbb{R}^d} \right] = \lim_{n\to\infty} \mathbb{E}\left[\left\| \int_0^t (X_s - X_{t^{(n)}_{j(s)}})\, dW_s \right\|^2_{\mathbb{R}^d} \right] =$$

$$= \underbrace{\lim_{n\to\infty} \int_0^t \mathbb{E}\left[\|X_s - X_{t^{(n)}_{j(s)}}\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \right] ds}_{\text{Itô's Isommetry}}$$

[where $t^{(n)}_{j(s)}$ is the left endpoint of the interval containing s in the partition $\mathcal{Z}^t_n$.]

Now, since X is continuous and adapted, as the partition becomes finer, $X_s - X_{t_j^{(n)}(s)} \to 0$ almost surely.

So, we only need to put the $\lim_{n\to\infty}$ inside the integral. Concretely, since the continuity of X implies that X is bounded on $[0,t]$, so by Bounded Convergence Theorem,

$$= \int_0^t \lim_{n\to\infty} \mathbb{E}\left[\|X_s - X_{t_j^{(n)}(s)}\|^2_{HS(\mathbb{R}^m;\mathbb{R}^d)}\right] ds \text{ which } \to 0 \text{ as } n\to\infty$$

which means $S_n \to \int_0^t X_s \, dW_s$ in mean square $\Rightarrow$ $\boxed{\lim_{n\to\infty} S_n = \int_0^t X_s \, dW_s \text{ in probability}}$

b) $m=1$ and $\alpha \in [0,1]$. For each $t \in [0,1]$

$$S_\alpha(Z_n^t) := \sum_{j=0}^{N_{n,t}-1} \left[(1-\alpha) W_{t_j^{(n)}} + \alpha W_{t_{j+1}^{(n)}}\right]\left(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\right) =$$

$$= \underbrace{\sum_{j=0}^{N-1} W_{t_j^{(n)}}\left(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\right)}_{(1)} + \alpha \underbrace{\sum_{j=0}^{N-1}\left(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\right)^2}_{(2)}$$

We want to see if it converges in L^2. We can use 3.3) from series 3a and take $X_t = W_t$ and, as $n\to\infty$, $\|Z_n^t\| \to 0$. As $\int_0^t W_s \, dW_s = \frac{W_t^2 - t}{2}$ from theory, we have that

$$(1) = \sum_{j=0}^{N-1} W_{t_j^{(n)}}\left(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\right) \xrightarrow{L^2} \int_0^t W_s \, dW_s = \frac{W_t^2 - t}{2}$$

Now, let's evaluate (2), $\alpha \sum_{j=0}^{N-1}\left(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}}\right)^2 \xrightarrow{L^2} \alpha t$, as per exercise 2 in series 3a.

Hence,
$$\boxed{S_\alpha(Z_n^t) \xrightarrow{L^2} \frac{W_t^2 - t}{2} + \alpha t = \frac{W_t^2}{2} + t\left(\alpha - \frac{1}{2}\right)}$$

$\to$ We have $L_\alpha(t) = \frac{W_t^2}{2} + t\left(\alpha - \frac{1}{2}\right)$

To be a martingale it has to hold:

$$E[L_\alpha(t) \mid \mathcal{F}_s] = L_\alpha(s)$$

$$\mathbb{E}\left[\frac{W_t^2}{2} + t\left(\alpha - \tfrac{1}{2}\right) \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\frac{(W_s + W_t - W_s)^2}{2} + t\left(\alpha - \tfrac{1}{2}\right) \Big| \mathcal{F}_s\right] =$$

$$= \mathbb{E}\left[\frac{W_s^2}{2} + \frac{2W_s(W_t - W_s)}{2} + \frac{(W_t - W_s)^2}{2} + t\left(\alpha - \tfrac{1}{2}\right) \Big| \mathcal{F}_s\right] =$$

$$= \frac{W_s^2}{2} + \frac{t-s}{2} + t\left(\alpha - \tfrac{1}{2}\right)$$

Now, $\cancel{\frac{W_t^2}{2}} + \frac{t-s}{2} + t(\alpha - \frac{1}{2}) = \cancel{\frac{W_s^2}{2}} + s(\alpha - \frac{1}{2})$

$\iff -\frac{s}{2} + t\alpha = s(\alpha - \frac{1}{2}) \iff \boxed{\alpha = 0} \quad \forall s < t$

c) Stratonovich integral: For an $(\mathbb{F}_t)_{t\in[0,1]}/\mathcal{B}(\mathbb{R})$-adapted Itô process $(U_t)_{t\in[0,1]}$ of the form $dU_t = Y_t dt + Z_t dW_t$ (Def B.5.1), consider the Stratonovich integral:

$$\int_0^t X_s \circ dU_s := \int_0^t X_s Y_s ds + \int_0^t X_s Z_s dW_s + \frac{1}{2}\langle X, U\rangle_t$$
$(0 \le t \le 1)$

$\langle X, U\rangle$ the covariation of X and U,

$\langle X, Y\rangle_t := P - \lim_{\|\Delta t\|\to 0} \sum_{n=0}^{N-1} (X_{t_{n+1}} - X_{t_n})(Y_{t_{n+1}} - Y_{t_n})$ "quadratic covariation of X and Y",

(Convergence in probability)

We have (taking $dU_t = dW_t$)

$\int_0^t X_s \circ dW_s = \int_0^t X_s dW_s + \frac{1}{2}\langle X, W\rangle_t =$

↳ by definition of $\langle X, W\rangle_t$

$= \int_0^t X_s dW_s + \frac{1}{2} \lim_{\|\Delta t\|\to 0} \sum_{n=0}^{N-1} (X_{t_{n+1}} - X_{t_n})(W_{t_{n+1}} - W_{t_n})$

And we know from 1.a) that $\int_0^t X_s dW_s \xrightarrow{\mathbb{P}} \lim_{n\to\infty} \sum_{j=0}^{N_{n,t}-1} X_{t_j^{(n)}} (W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}})$

So, if we write both limits in terms of $n \to \infty$ and each of the time values in the mesh with the same representation $t_j^{(n)}$, we get

$$\int_0^t X_s \circ dW_s = \lim_{n\to\infty} \sum_{j=0}^{N_{n,t}-1} \frac{1}{2}(X_{t_j^{(n)}} + X_{t_{j+1}^{(n)}})(W_{t_{j+1}^{(n)}} - W_{t_j^{(n)}})$$

Hint: The following facts about the quadratic covariation $\langle U, V \rangle$ of $\mathbb{R}$-valued Itô processes U and V may be used without proof: (i) if U has sample paths that are continuously differentiable (or, more generally, of bounded variation), then $\langle U, V \rangle = 0$; (ii) if U and V are of the form $\mathrm{d}U_t = Z_t \mathrm{d}W_t$ and $\mathrm{d}V_t = \tilde{Z}_t \mathrm{d}W_t$, then the stochastic process $\langle U, V \rangle$ is of the form $\mathrm{d}\langle U, V \rangle_t = Z_t \tilde{Z}_t \mathrm{d}\langle W, W \rangle_t$. ◇

Based on the results of this exercise: what main differences between the Itô integral (1) and the Stratonovich integral (3) do you observe, and what are the advantages or disadvantages of each?

(For a contextualised comparison of Itô and Stratonovich integrals, see, for example, Section 3.3 (pp. 35ff) in Øksendal, B. (2013): *Stochastic Differential Equations: An Introduction with Applications*, Springer.)

2. Itô's Formula

For $T > 0$, let $(\Omega, \mathcal{F}, P, \mathbb{F})$ be a filtered probability space and let $W\colon [0, T] \times \Omega \to \mathbb{R}$ be an $\mathbb{F}$-adapted one-dimensional standard Brownian motion.

a) Using Itô's formula, find representations of the below processes that do not contain Itô integrals:

$$X_t = \int_0^t \exp(W_s)\,\mathrm{d}W_s \qquad \text{and} \qquad Y_t = \int_0^t W_s \exp(W_s^2)\,\mathrm{d}W_s. \tag{6}$$

b) Let $X = (X_t)_{t \in [0,1]}$ be an Itô process of the form $\mathrm{d}X_t = Y_t \mathrm{d}t + Z_t \mathrm{d}W_t$, that is suppose that

$$X_t = X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s \qquad (0 \leq t \leq T)$$

for $\mathbb{F}/\mathcal{B}(\mathbb{R})$-predictable stochastic processes $Y\colon [0, T] \times \Omega \to \mathbb{R}$ and $Z\colon [0, T] \times \Omega \to \mathbb{R}$ such that $P(\int_0^T |Y_s| + |Z_s|^2 \mathrm{d}s < +\infty) = 1$. Define the *stochastic exponential* of X via

$$\mathcal{E}(X)_t := \exp\left(X_t - X_0 - \frac{1}{2}\int_0^t Z_s^2 \mathrm{d}s\right), \quad t \in [0, T].$$

Use Itô's formula to prove that the stochastic process $\mathcal{E}(X) \equiv (\mathcal{E}(X)_t)_{t \in [0,T]}$ satisfies the equation

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_s Y_s \mathrm{d}s + \int_0^t \mathcal{E}(X)_s Z_s \mathrm{d}W_s \qquad (0 \leq t \leq T).$$

3. Numerical Approximation of an Itô Integral

Consider the stochastic integral $I := \int_0^1 t W_t\,\mathrm{d}W_t$, where $W = (W_t)_{t \in [0,1]}$ is a standard Brownian motion on the interval $[0, 1]$. Write a MATLAB code that approximates $\mathbb{E}[I^\alpha]$ for $\alpha = 1$ and $\alpha = 2$ using different Monte Carlo simulations with $M = 1\,000\,000$ sample paths and $N = 4,\ 8,\ 16,\ \ldots,\ 256$ subdivisions of the interval $[0, 1]$. Then, compare these approximations to the exact values of $\mathbb{E}[I^\alpha]$ for each N. Build your approximations as follows: for each N, approximate $\mathbb{E}[I^\alpha]$ by

$$\frac{1}{M}\sum_{i=1}^{M}\left(\sum_{j=0}^{N-1} t_j^{(N)} W^{(i)}_{t_j^{(N)}}\left(W^{(i)}_{t_{j+1}^{(N)}} - W^{(i)}_{t_j^{(N)}}\right)\right)^{\alpha}, \quad \text{where } t_j^{(N)} := \frac{j}{N}.$$

For simulating the increments of W, note that $W^{(i)}_{t_{j+1}^{(N)}} - W^{(i)}_{t_j^{(N)}} = \eta_j^{(i)}/\sqrt{N}$ with $\eta_j^{(i)} \sim N_{0,1}$.

Submission Deadline: Wednesday, 06 November 2024, by 2:00 PM.

2 a)

- If we take $f(x) = e^x$, we have that f is twice continously differentiable so we can apply Itô's formula to $X_t = W_t$ a 1-dimensional std. B.M:

$$f(X_t) = f(W_t) = f(W_0) + \int_0^t \cancel{f'(W_s)}.0 \, ds + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, ds$$

Hence,

$$e^{W_t} = 1 + \int_0^t \exp(W_s)\, dW_s + \frac{1}{2}\int_0^t \exp(W_s)\, ds$$

$$\Rightarrow \int_0^t \exp(W_s)\, dW_s = e^{W_t} - 1 - \frac{1}{2}\int_0^t \exp(W_s)\, ds \quad //$$

- Now, for $f(x) = \frac{e^{x^2}}{2}$, we have that $f \in C^2$, hence we can apply Itô's formule with same Itô process $X_t = W_t$,

$$\frac{e^{W_s^2}}{2} = 1 + \int_0^t W_s \exp(W_s^2)\, dW_s + \int_0^t W_s^2 \exp(W_s^2)\, ds$$

$$\Rightarrow \int_0^t W_s \exp(W_s)\, dW_s = \frac{e^{W_s^2}}{2} - 1 - \int_0^t W_s^2 \exp(W_s^2)\, ds \quad //$$

b) $X = (X_t)_{t\in[0,T]}$ an Itô process $dX_t = Y_t\, dt + Z_t\, dW_t$, that is suppose that

$$X_t = X_0 + \int_0^t Y_s\, ds + \int_0^t Z_s\, ds \text{ for } \mathbb{F}/\mathcal{B}(\mathbb{R})\text{-predictable st. pr.}$$

Y and Z with $P(\int_0^T |Y_s| + |Z_s|^2 ds < +\infty) = 1$.

We define the stochastic exponential of X via

$$\mathcal{E}(X)_t := \exp\left(X_t - X_0 - \frac{1}{2}\int_0^t Z_s^2\, ds\right),\ t\in[0,T]$$

In order to apply Itô's formula we need to see that $U_t = X_t - X_0 - \frac{1}{2}\int_0^t Z_s^2\, ds$ it's an Itô process. By these means, we know that X_t is an Itô process, so:

$$U_t = \underbrace{X_0 + \int_0^t Y_s \, ds + \int_0^t Z_s \, dW_s}_{X_t} - X_0 - \frac{1}{2}\int_0^t Z_s^2 \, ds$$

$$U_t = 0 + \int_0^t \left(Y_s - \frac{1}{2} Z_s^2\right) ds + \int_0^t Z_s \, dW_s$$

So, U_t is an Itô process with $U_0 = 0$, drift $= Y_s - \frac{1}{2} Z_s^2$ and difussion Z_s.

And, as $Y, Z : [0,T] \times \Omega \to \mathbb{R}$ are $\mathcal{F}/\mathcal{B}(\mathbb{R})$- predictable, our new drift and difussion are also predictable.

Now, we apply Ito for $f(x) = e^x$,

$$f(U_t) = f(U_0) + \int_0^t f'(U_s)\left(Y_s - \frac{1}{2} Z_s^2\right) ds + \int_0^t f'(U_s) Z_s \, dW_s + \frac{1}{2}\int_0^t f''(U_s) Z_s^2 \, ds$$

$$\mathcal{E}(X)_t = e^0 + \int_0^t \left(e^{U_s} Y_s - \frac{1}{2} e^{U_s} Z_s^2\right) ds + \int_0^t e^{U_s} Z_s \, dW_s + \frac{1}{2}\int_0^t e^{U_s} Z_s^2 \, ds$$

So, for $t \in [0,T]$

$$\boxed{\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_s Y_s \, ds + \int_0^t \mathcal{E}(X)_s Z_s \, dW_s}$$

3 Code attached

$$I = \int_0^1 t W_t \, dW_t$$

→ $\alpha = 1$:

We can write I as $I = \int_0^1 f(t, W_t) \, dW_t$ where $f(t, W_t) = t W_t$ which is adapted. And use the fact that I is a Ito integral and has martingale property:

$$E[I] = E\left[\int_0^1 t W_t \, dW_t\right] = \boxed{0}$$

Due to be a bounded and adapted process w.r.t. BM.

→ $\alpha = 2$: We apply Ito's isommetry.

$$E\left[\left(\int_0^1 f(t)\,dW_t\right)^2\right] = E\left[\int_0^1 f(t)^2\,dt\right]$$

In our case $f(t) = tW_t$,

$$E[I^2] = E\left[\int_0^1 (tW_t)^2\,dt\right]$$

And, since W_t is BM, $E[W_t^2] = t$, so

$$E[I^2] = E\left[\int_0^1 t^2 W_t^2\,dt\right] = \int_0^1 t^2 E[W_t^2]\,dt = \int_0^1 t^3\,dt$$

$$\Rightarrow \boxed{E[I^2] = \frac{1}{4}}$$

In our code, we can see how we get really good results that approximate this values.