Series 3a

1. Lévy-Ciesielski Representation of Brownian Motion (II)

Continuing on from Problem 1 of Series 2b, recall the Schauder functions $(\phi_n \mid n \in \mathbb{N})$ defined by

$$\phi_n : [0,1] \ni t \mapsto \int_0^t \psi_n(s)\,\mathrm{d}s \qquad (n \in \mathbb{N}),$$

with $(\psi_n \mid n \in \mathbb{N})$ the family of Haar functions introduced in Series 2b. Let further $\xi_1, \xi_2, \ldots : \tilde{\Omega} \to \mathbb{R}$ be an iid sequence of $\mathcal{N}_{0,1}$-distributed random variables on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

a) Show that, for any fixed $T > 0$, there exists a $\tilde{P}$-full set $\Omega \in \tilde{\mathcal{F}}$ for which the function

$$B : [0,T] \times \Omega \ni (t,\omega) \mapsto B_t(\omega) := \sqrt{T} \sum_{n=1}^{\infty} \phi_n\big(\tfrac{t}{T}\big)\xi_n(\omega) \tag{1}$$

is an $\mathbb{R}$-valued stochastic process on[1] $(\Omega, \mathcal{F}, P) := (\Omega, \Omega \Cap \tilde{\mathcal{F}}, \tilde{P}\big|_{\mathcal{F}})$ with the property that:

$$(B_t - B_s)(\tilde{P}) = \mathcal{N}_{0,t-s}, \quad \text{for each } \ 0 \le s \le t \le T. \tag{2}$$

Hint: Denoting by Ω_c $(c > 0)$ the nullsets in the hint for Problem 1b) of Series 2b, consider the set $\Omega := \bigcup_{c \in (\sqrt{2},\infty)\cap\mathbb{Q}} \big(\tilde{\Omega} \setminus \Omega_c\big)$. For (2), note $\sum_{n=1}^{\infty} \big(\int_0^1 \psi_n \cdot \varphi \,\mathrm{d}s\big)^2 = \|\varphi\|^2_{L^2([0,1])}$ for all $\varphi \in L^2([0,1])$ (Parseval's identity).

b) Prove that the process (1) is a 1-dimensional standard $(\Omega, \mathcal{F}, P)$-Brownian motion.

Hint: In addition to all the results and hints of Series 2b, you may use without proof the following *fact*:

For any two sequences $(\alpha_j)_{j\in\mathbb{N}}, (\beta_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ with $\max\big\{\sum_{k=1}^{2^j} |\alpha_{2^j+k}|, \sum_{k=1}^{2^j} |\beta_{2^j+k}|\big\} \le 2^{-j/2}$ for all $j \in \mathbb{N}$, the limits $\chi := \sum_{n=1}^{\infty} \alpha_n \xi_n$ and $\eta := \sum_{n=1}^{\infty} \beta_n \xi_n$ (cf. Series 2b, Ex. 1b) are independent if and only if $\mathrm{Cov}(\chi, \eta) = 0$.

2. Quadratic Variation of Standard Brownian Motions

Let $T > 0$, $N \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_N = T$, and let $W : [0,T] \times \Omega \to \mathbb{R}$ be a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Prove that

$$\left\| T - \sum_{n=0}^{N-1} \big(W_{t_{n+1}} - W_{t_n}\big)^2 \right\|_{\mathcal{L}^2(P;|\cdot|_{\mathbb{R}})} \le \sqrt{2T} \left[\max_{n \in \{0,1,\ldots,N-1\}} |t_{n+1} - t_n| \right]^{\frac{1}{2}}. \tag{3}$$

(**Remark (Non-essential information):** *This remark is provided for further context, but can be safely skipped.* More generally, for any adapted *semimartingales* $X = (X_t)_{t\ge0}$ and $Y = (Y_t)_{t\ge0}$ with continuous sample paths on some $(\Omega, \mathcal{F}, P, \mathbb{F})$, there is a unique (up to indistinguishability) continuous and adapted real-valued stochastic process

$$\langle X, Y\rangle \equiv (\langle X,Y\rangle_t)_{t\in[0,T]} \quad \text{given by} \quad \langle X,Y\rangle_t := P - \lim_{\|\Delta_t\|\to 0} \sum_{n=0}^{N-1} \big(X_{t_{n+1}} - X_{t_n}\big)\big(Y_{t_{n+1}} - Y_{t_n}\big), \tag{$*$}$$

where the above convergence is in probability and Δ_t ranges over all possible partitions $(t_n)_{n=0}^N$ $(N \in \mathbb{N})$ of $[0,t]$ normed by their mesh $\|(t_n)\| := \max_{n\in\{0,1,\ldots,N-1\}} |t_{n+1} - t_n|$.

The stochastic process $(*)$ is called the *quadratic covariation* of X and Y. ◆)

3. Product Measurability and Stochastic Integration of L^2-Continuous Processes

a) Let $(I, \mathcal{I})$, $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ be measurable spaces, and let $X : I \times \Omega \to S$ be a $(\mathcal{I} \otimes \mathcal{F})/\mathcal{S}$-measurable function. Show that, for every $\omega \in \Omega$, the map $I \ni i \mapsto X(i,\omega) \in S$ is $\mathcal{I}/\mathcal{S}$-measurable.

b) Let $d \in \mathbb{N}$ and $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,1]})$ be a stochastic basis. Let further $W : [0,1] \times \Omega \to \mathbb{R}$ be a standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,1]})$-Brownian motion, and let $X : [0,1] \times \Omega \to \mathbb{R}^d$ be $(\mathbb{F}_t)_{t\in[0,1]}/\mathcal{B}(\mathbb{R}^d)$-predictable and such that $X \in C([0,1], L^2(P; \|\cdot\|_{\mathbb{R}^d}))$. Using Itô's isometry (see, e.g., Theorem B.4.21 in the lecture notes), prove that

$$\limsup_{n\to\infty} \left\| \int_0^1 X_s \,\mathrm{d}W_s - \left[\sum_{k=0}^{n-1} X_{\frac{k}{n}} \Big(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\Big) \right] \right\|_{L^2(P;\|\cdot\|_{\mathbb{R}^d})} = 0. \tag{4}$$

Submission Deadline: Wednesday, 06 November 2024, by 2:00 PM.

[1] Recall the lecture notes' Definition A.2.9 and Definition B.3.1 for notation and terminology.

1) **Recall:** (Trace set). Let A and $\mathcal{A}$ be sets. Then we denote $A \Cap \mathcal{A}$ the set given by

$$A \Cap \mathcal{A} = \{A \cap B \in \mathcal{P}(A) : B \in \mathcal{A}\} = \{C \in \mathcal{P}(A) : (\exists B \in \mathcal{A} : A \cap B = C)\}$$

(Trace sigma-algebra). Let $(\Omega, \mathcal{A})$ be a measurable space and let $A \subseteq \Omega$ be a subset of Ω. Then, we call $A \Cap \mathcal{A}$ the trace sigma-algebra of A in $\mathcal{A}$.

$$A \Cap \mathcal{A} = \{B \in \mathcal{A} : B \subseteq A\} = \mathcal{P}(A) \cap \mathcal{A}$$

Recall : (P-full set) Given $(\Omega, \mathcal{F}, P)$ a probability space, $A \subseteq \Omega$ is called P-full if:

$$P(A) = 1.$$

a) We consider probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, And use from previous series

$$\Omega_c = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} \{w \in \Omega \mid |\xi_n(w)| > c\sqrt{\log(n)}\}$$

Now, if we take: $\Omega := \bigcup_{c \in (\sqrt{2}, \infty) \cap \mathbb{Q}} (\tilde{\Omega} \setminus \Omega_c)$, we need to proof it has prob $= 1$.

Clearly, $\Omega \in \tilde{\mathcal{F}}$, due to the definition of σ-algebra.

- $\tilde{\Omega} \setminus \Omega_c \in \tilde{\mathcal{F}}$ because $\tilde{\Omega} \in \tilde{\mathcal{F}}$ and the complementary $\in \tilde{\mathcal{F}}$
- $\bigcup_{c \in (\sqrt{2}, \infty) \cap \mathbb{Q}} (\tilde{\Omega} \setminus \Omega_c) \in \tilde{\mathcal{F}}$ as it is a countable intersection.

Now,

$$\Omega = \bigcup_{c \in (\sqrt{2}, \infty) \cap \mathbb{Q}} (\tilde{\Omega} \setminus \Omega_c) = \tilde{\Omega} \setminus \overbrace{\bigcap_{c \in (\sqrt{2}, \infty) \cap \mathbb{Q}} \Omega_c}^{K}, \quad \Omega_c \subseteq \tilde{\Omega} \quad \forall c \in \mathbb{R}.$$

So, $P(\Omega) = P(\tilde{\Omega} \setminus K) = P(\tilde{\Omega}) - P(K) = \boxed{1}$

$P(K) = P\left(\bigcap_{c \in (\sqrt{2}, \infty) \cap \mathbb{Q}} \Omega_c\right) = 0$

$P(\tilde{\Omega}) = 1$ by definition.

To prove B is an $\mathbb{R}$-valued stochastic process on $(\Omega, \underbrace{\Omega \cap \tilde{F}}_{F}, \underbrace{\tilde{P}_{|F}}_{P})$

- $[0,T] \subseteq \mathbb{R}$ ✓
- $\mathcal{U} = (\Omega, F, P)$ is a probability space:
  - P is a measure on (Ω, F), from the definition of $\tilde{P}$ and the fact that $\Omega \cap \tilde{F}$ is a sigma-algebra. We can check the sigma-additivity condition: Given $A_n \cap A_m = \emptyset$, $A_n, A_m \in F$,

$$P\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \tilde{P}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \tilde{P}(A_n) = \sum_{n\in\mathbb{N}} P(A_n)$$

$$\begin{cases} A_n \in F \\ P = \tilde{P}_{|F} \end{cases}$$

  - $P(\Omega) = 1$ (already calculated)
- We take $\mathbb{R}$ and the borel σ-algebra in the image
- Now, we want to see that B is $\mathbb{M}(\mathbb{T} \times \Omega, \mathbb{R})$

Recall $B: [0,T] \times \Omega \to \mathbb{R}$

$$(t,w) \mapsto \sqrt{T} \sum_{n=1}^{\infty} \phi_n\left(\frac{t}{T}\right) \xi_n(w)$$

We know that by "Measurability of pointwise Limit/Sum", an infinite sum of measurable functions is measurable. Moreover, an scalar modification of a measurable function is also measurable. Hence, it only last to prove that $\phi_n: [0,T] \to \mathbb{R}$ and $\xi_n: \Omega \to \mathbb{R}$ are $\mathcal{B}([0,T]) \times F$-measurable.

To see it, for any borel set $A \in \mathcal{B}(\mathbb{R})$,

$$B^{-1}(A) = \{(t,w) \in [0,T] \times \Omega : B(t,w) \in A\}$$

is an element of $\mathcal{B}([0,T]) \times F$.

As ξ_n depends on Ω and ϕ on $[0,T]$, we can study measurability only on its corresponding marginal σ-algebra.

- ξ_n it's a random variable defined over Ω, so, it's clearly F-measurable, and, then, $\mathcal{B}([0,T]) \times F$-measurable.

- $\phi_n : [0,T] \to \mathbb{R}$ it's a continuous function, because ψ_n are piece-wise linear (discontinuous), and its integral its continuous. See how at each discontinuity x_i of ψ_n,

$$\phi_n(x_i) = \int_0^{x_i} \psi_n(t)dt = \lim_{x \to x_i^-} \int_0^x \psi_n(t)\,dt = \lim_{x \to x_i^+} \int_0^x \psi_n(t)\,dt.$$

ϕ_n is continuous.

Hence, any continuous function on $[0,T]$ is measurable w.r.t the Borel σ-algebra on $[0,T]$. Due to the fact that a preimage of an open set in $\mathbb{R}$ under a continuous function is open in the domain. So ϕ_n is $\mathcal{B}([0,T]) \times \mathcal{F}$-measur.

Now, the product of two $\mathbb{M}([0,T]\times\Omega, \mathbb{R})$ functions is $\mathbb{M}([0,T]\times\Omega, \mathbb{R})$.

- It last to prove that for every $t \in [0,T]$,

$$\Omega \ni w \longmapsto B(t,w) \text{ is } (\mathcal{F}, \mathcal{B}(\mathbb{R}))\text{-measurable.}$$

And it is clear, as ξ_n is a random variable: $\Omega \to \mathbb{R}$.

To prove $(B_t - B_s)(\tilde{P}) = N_{0,t-s}$, we have that given $w \in \mathcal{F}$

- $$\mathbb{E}[B_t - B_s] = \mathbb{E}\left[\sqrt{T}\left(\sum_{n=1}^{\infty} \phi_n\left(\frac{t}{T}\right)\xi_n - \sum_{n=1}^{\infty} \phi_n\left(\frac{s}{T}\right)\xi_n\right)\right].$$

$$= \sqrt{T} \sum_{n=1}^{\infty}\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)\mathbb{E}[\xi_n] = 0$$

$\mathbb{E}[\xi_n] = 0$

- $$\mathrm{Var}[B_t - B_s] = \mathrm{Var}\left[\sqrt{T}\sum_{n=1}^{\infty}\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)\xi_n\right] =$$

$$= \sum_{n=1}^{\infty} \mathrm{Var}\left[\sqrt{T}\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)\xi_n\right] =$$

$\mathrm{Var}(\xi_n) = 1$

$\left(\begin{array}{l}\phi_n \text{ deterministic} \\ \{\xi_n\}_n \text{ independent}\end{array}\right)$ $$= \sum_{n=1}^{\infty} T\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)^2 \mathrm{Var}(\xi_n) =$$

$$= \sum_{n=1}^{\infty} T\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)^2 = T\sum_{n=1}^{\infty}\left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)^2$$

(1)

Now, we can use Parseval's identity, which states that, given an orthonormal basis $\{\Psi_n\}_{n\in\mathbb{N}}$ of the Hilbert space $L^2([0,1])$, and $\varphi \in L^2([0,1])$, we have that

$$\sum_{n=1}^{\infty} \left(\int_0^1 \Psi_n \cdot \varphi \, ds\right)^2 = \|\varphi\|^2_{L^2([0,1])}$$

Now, if we take $\varphi = \mathbb{1}_{[\frac{t}{T}, \frac{s}{T}]}$, we have

$$\sum_{n=1}^{\infty} \left(\phi_n\left(\frac{t}{T}\right) - \phi_n\left(\frac{s}{T}\right)\right)^2 = \sum_{n=1}^{\infty} \left(\int_0^t \Psi_n\left(\frac{u}{T}\right) du - \int_0^s \Psi_n\left(\frac{u}{T}\right) du\right)^2 =$$

$$(\text{assuming } t \geq s) = \sum_{n=1}^{\infty} \left(\int_s^t \Psi_n\left(\frac{u}{T}\right) du\right)^2 =$$

$$= \sum_{n=1}^{\infty} \left(\int_0^1 \Psi_n\left(\frac{u}{T}\right) \cdot \mathbb{1}_{[s,t]} \, du\right)^2 \underset{\uparrow}{=} \|\mathbb{1}_{[\frac{s}{T}, \frac{t}{S}]}\|^2_{L^2([0,1])} =$$

(Parseval's Identity)

$$= \int_0^1 \mathbb{1}_{[\frac{s}{T}, \frac{t}{T}]} du = \int_{\frac{s}{T}}^{\frac{t}{T}} \mathbb{1}_{[\frac{s}{T}, \frac{t}{T}]} du = \frac{t-s}{T}$$

Now, we go back to (1):

$$\mathrm{Var}(B_t - B_s) = T \cdot \frac{t-s}{T} = t-s$$

Now, as we know from problemset 2b that $B_t \sim N(0,t)$, and the difference of normally distributed r.v. is indeed normally distributed. We can conclude that $B_t - B_s \sim N(0, t-s)$.

↳ The normality of $B_s - B_t$ can also be proven by seeing

$$\sum_{k=1}^{2^j} \left|\phi_{2^j+k}\left(\frac{s}{T}\right) - \phi\left(\frac{t}{T}\right)_{2^j+k}\right| \leq 2^{-j/2}$$ (from 1b) series 2b)

which is true, because $\sum_{k=1}^{2^j} \left|\phi_{2^j+k}\left(\frac{s}{T}\right) - \phi\left(\frac{t}{T}\right)_{2^j+k}\right| \leq$

$$\leq \sum_{k=1}^{2^j} \left|\phi_{2^j+k}\left(\frac{s}{T}\right)\right| \leq 2^{-\frac{j}{2}} \implies B_s - B_t \sim \text{normally distributed.}$$

b) Prove that B is a 1-dimensional standard $(\Omega, \mathcal{F}, P)$- Brownian motion.

(Hint: For any two sequences $(\alpha_j)_{j\in\mathbb{N}}, (\beta_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ with $\max\{\sum_{k=1}^{2^j} |\alpha_{2^j+k}|, \sum_{k=1}^{2^j} |\beta_{2^j+k}|\} \le 2^{-j/2}$ for all $j \in \mathbb{N}$, the limits

$\mathcal{X} := \sum_{n=1}^{\infty} \alpha_n \xi_n$ and $\eta = \sum_{n=1}^{\infty} \beta_n \xi_n$ (compare series 2b, ex. 1b) are independent $\Leftrightarrow$ $Cov(\mathcal{X}, \eta) = 0$.)

In order to prove that B is a Brownian motion, it last to prove that the increments $B_{t+u} - B_t$, $u \ge 0$ are independent of the past values B_s, $s \le t$.

- Independent increments:

$$B_s(\omega) = \sqrt{T} \sum_{n=1}^{\infty} \phi_n\left(\frac{s}{T}\right) \xi_n(\omega)$$

$$B_{t+u} - B_t = \sqrt{T} \sum_{n=1}^{\infty} \xi_n(\omega) \left(\phi_n\left(\frac{t+u}{T}\right) - \phi_n\left(\frac{t}{T}\right)\right)$$

To prove independence, using the hint, it is enough by proving that $Cov(B_s, B_{t+u} - B_t) = 0$, Because we know that - $\sum_{n=1}^{\infty} |\phi_n(\frac{s}{T})| \le 2^{-j/2}$

- $\sum_{n=1}^{\infty} \left| \phi_n\left(\frac{t+u}{T}\right) - \phi_n\left(\frac{t}{T}\right)\right| \le 2^{-j/2}$ (from section a))

As we know that B_s and $B_{t+u} - B_t$ are normally distributed r.v. with $B_s \sim N(0,s)$, $B_{t+u} - B_t \sim N(0,u)$

See proof of $Cov(B_s, B_{t+u} - B_t) = 0$

Hence, we have to prove $Cov(B_s, B_{t+u} - B_t) = 0$

$$Cov\Big(\underbrace{\sum_{n=1}^{\infty} \phi_n\left(\tfrac{s}{T}\right)\xi_n}_{a}, \underbrace{\sum_{n=1}^{\infty} \xi_n\left(\phi_n\left(\tfrac{t+u}{T}\right) - \phi_n\left(\tfrac{t}{T}\right)\right)}_{b}\Big) =$$

$$= \mathbb{E}[a \cdot b] - \cancel{\underbrace{\mathbb{E}[a]}_{0}\cdot\mathbb{E}[b]} = \mathbb{E}[a \cdot b] =$$

$$= \mathbb{E}\Big[\sum_{n=1}^{\infty} \xi_n^2 \phi_n\left(\tfrac{s}{T}\right)\left(\phi_n\left(\tfrac{t+u}{T}\right) - \phi_n\left(\tfrac{t}{T}\right)\right)\Big] =$$

(as we know that the infinite sum is bounded, we can interchange $\mathbb{E}$ with $\sum_{n=1}^{\infty}$).

$$\sum_{n=1}^{\infty} \mathbb{E}\Big[\xi_n^2 \phi_n\left(\tfrac{s}{T}\right)\left(\phi_n\left(\tfrac{t+u}{T}\right) - \phi_n\left(\tfrac{t}{T}\right)\right)\Big] =$$

$$= \sum_{n=1}^{\infty} \phi_n\left(\tfrac{s}{T}\right)\left(\phi_n\left(\tfrac{t+u}{T}\right) - \phi_n\left(\tfrac{t}{T}\right)\right)$$

$\{\psi_n\}_n$ Haar function

Now, by definition of $\phi_n(t)$ we know that $\phi_n(t) = \langle \mathbb{1}_{\{(0,\frac{t}{T}]\}}, \psi_n \rangle$ because $\phi_n(t) = \int_0^{\frac{t}{T}} \psi_n(s)\,ds$. And $\phi_n\left(\frac{t+u}{T}\right) - \phi_n\left(\frac{t}{T}\right) = \langle \mathbb{1}_{\{(\frac{t}{T}, \frac{t+u}{T}]\}}, \psi_n \rangle$

So, $$\sum_{n=1}^{\infty} \phi_n\left(\tfrac{s}{T}\right)\cdot\left(\phi_n\left(\tfrac{t+u}{T}\right) - \phi_n\left(\tfrac{t}{T}\right)\right) = $$

Perseval Identity: $\langle x, y\rangle = \sum_{b \in B} \langle x, b\rangle\langle y, b\rangle$, B the orthonormal basis.

$$= \sum_{n=1}^{\infty} \langle \mathbb{1}_{\{(0,\frac{s}{T}]\}}, \psi_n \rangle \cdot \langle \mathbb{1}_{\{(\frac{t}{T}, \frac{t+u}{T}]\}}, \psi_n \rangle$$

$$= \langle \mathbb{1}_{\{(0,\frac{s}{T}]\}}, \mathbb{1}_{\{(\frac{t}{T}, \frac{t+u}{T}]\}} \rangle = 0$$

because $\frac{s}{T} \leq \frac{t}{T}$.

Hence,

$$Cov(B_s, B_{t+h} - B_t) = 0 \quad \text{when} \quad s \leq t, \; h \geq 0.$$

② $W_{t_{n+1}} - W_{t_n} \sim N(0, t_{n+1} - t_n) \; / \sim \sqrt{t_{n+1}-t_n}\, Z_n \quad Z_n \sim N(0,1)$

$$\left\| T - \sum_{n=0}^{N-1} (W_{t_{n+1}} - W_{t_n})^2 \right\|_{L^2} \overset{!}{=} \left\| T - \sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 \right\|_{L^2} =$$

$$= \left(E\left[\left(T - \sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 \right)^2 \right] \right)^{1/2}$$

Now, $E\left[\left(T - \sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 \right)^2 \right] =$

$$= E\left[T^2 - 2T \sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 + \left(\sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 \right)^2 \right] =$$

$$= T^2 - 2T \sum_{n=1}^{N-1} (t_{n+1} - t_n) + E\left[\left(\sum_{n=1}^{N-1} (t_{n+1} - t_n) Z_n^2 \right)^2 \right] =$$

(Recall $E[Z_n^4] = 3$)

$$= T^2 - 2T \sum_{n=1}^{N-1} (t_{n+1} - t_n) + E\Big[\sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \underbrace{(t_{n+1} - t_n) Z_n^2 \cdot (t_{m+1} - t_m) Z_m^2}_{\oplus} \Big] =$$

$$= T^2 - 2T \sum_{n=1}^{N-1} (t_{n+1} - t_n) + \sum_{n=1}^{N-1} (t_{n+1} - t_n)^2 E[Z_n^4] + \sum_{n=1}^{N-1} \sum_{\substack{m=1 \\ m \neq n}}^{N-1} E[\oplus]$$

$$= T^2 - 2T \sum_{n=1}^{N} (t_{n+1} - t_n) + 3 \sum_{n=1}^{N-1} (t_{n+1} - t_n)^2 + \sum_{n=1}^{N-1} \sum_{\substack{m=1 \\ m \neq n}}^{N-1} (t_{n+1} - t_n)(t_{m+1} - t_m) \underbrace{E[Z_n^2 \cdot Z_m^2]}_{1}$$

$$= T^2 - 2T \sum_{n=1}^{N} (t_{n+1} - t_n) + 3 \sum_{n=1}^{N-1} (t_{n+1} - t_n)^2 + \sum_{n=1}^{N-1} \sum_{\substack{m=1 \\ m \neq n}}^{N-1} (t_{n+1} - t_n)(t_{m+1} - t_m) =$$

(We have a grid of values t_i from 0 to T, so $\sum_{n=1}^{N} (t_{n+1} - t_n) = T$.)

$$\leq T^2 - 2T^2 + 3 \underbrace{\sum_{n=1}^{N-1} (t_{n+1} - t_n)^2}_{\leq T \max_n |t_{n+1} - t_n|} + \underbrace{\sum_{n=1}^{N-1} \sum_{m=1}^{N-1} (t_{n+1} - t_n)(t_{m+1} - t_m)}_{T^2} - \sum_{n=1}^{N-1} (t_{n+1} - t_n)^2$$

$$\leq T^2 - 2T^2 + 2T \max_n |t_{n+1} - t_n| + T^2$$

$$= \cancel{T^2} - \cancel{2T^2} + 2T \max_n |t_{n+1} - t_n| + \cancel{T^2} = 2T \max_n |t_{n+1} - t_n|$$

Hence, $\left\| T - \sum_{n=0}^{N-1} (W_{t_{n+1}} - W_{t_n})^2 \right\|_{L^2} \leq \sqrt{2T} \left[\max_n |t_{n+1} - t_n| \right]^{1/2}$

(3)

a) $(I, \mathcal{I})$, $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ measurable spaces.

$X : I \times \Omega \to S$ $(\mathcal{I} \otimes \mathcal{F}) / \mathcal{S}$-measurable function.

Now, consider the map $F : I \to S$, $F = X \circ \alpha_w$

$$i \mapsto X(i, w)$$

where $\alpha_w : I \to I \times \Omega$

$$i \mapsto (i, w)$$

To prove the measurability of a composition, both functions need to be measurable. So, for any $C \in \mathcal{I}$, $D \in \mathcal{F}$, when $w \in D$,

$\alpha_w^{-1}(C, D) = I_d^{-1}(C)$, being $I_d : I \to I$ the identity function which is clearly $\mathcal{I}/\mathcal{I}$-measurable.

$\alpha^{-1}_w(C, D) = C \in \mathcal{I}$ if $w \in D$ or $\alpha^{-1}_w(C, D) = \emptyset$ if $w \notin D$.

Both $\emptyset, C \in \mathcal{I}$ by definition. Now, as α_w is $\mathcal{I}/\mathcal{I} \otimes \mathcal{F}$-measurable and X is $(\mathcal{I} \otimes \mathcal{F})/\mathcal{S}$-measurable we have that

$F = X \circ \alpha_w$ is $\mathcal{I}/\mathcal{S}$-measurable. //

b) $d \in \mathbb{N}$ and $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t \in [0,1]})$ a stochastic basis, W a standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t \in [0,1]})$ - Brownian motion.

$X : [0,1] \times \Omega \to \mathbb{R}^d$ be $(\mathbb{F}_t)_{t \in [0,1]} / \mathcal{B}(\mathbb{R}^d)$ - predictable and $X \in C([0,1], L^2)$

[Ito's isometry: $E_P\left[\| \int_a^b Y_s \, dW_s \|^2_{\mathbb{R}^d}\right] = \int_a^b E\left[\| Y_s \|^2_{HS(\mathbb{R}^m, \mathbb{R}^d)}\right] ds$]

We need to construct in a way that we get the convergence in L^2 of the statement in the exercise.

First, we can define the stopped process $X_t^{(n)}$ in such way:

$$X_t^{(n)} := \sum_{j=0}^{n-1} X_{j/n} \mathbb{1}_{[\frac{j}{n}, \frac{j+1}{n}]}$$

this process is $(\mathbb{F}_t)_t / \mathcal{B}(\mathbb{R}^d)$-predictable because a sum of predictable processes is predictable and $X_{\frac{j}{n}}$ is $\mathbb{F}_t$-predictable by being X_t $\mathbb{F}_t$-predictable.

We need to examine the L^2-norm:

$$\left\| \int_0^1 X_s \, dW_s - \sum_{k=0}^{n-1} X_{t_k} (W_{t_{k+1}} - W_{t_k}) \right\|_{L^2(P; \|\cdot\|)}$$

Now, by Ito's isometry, $E\left[\left\| \int_0^1 X_s \, dW_s \right\|^2_{\mathbb{R}^d} \right] =$

$= E\left[\int_0^1 \|X_s\|^2_{\mathbb{R}^d} \, ds \right]$.

Similarly, can be applied to the discrete approximation:

$$\mathbb{E}\left[\left\| \sum^{n-1} X_{t_k} (W_{t_{k+1}} - W_{t_k}) \right\|^2 \right] = \mathbb{E}\left[\sum_{k=0}^{n-1} \|X_{t_k}\|^2_{\mathbb{R}^d} \cdot \frac{1}{n} \right]$$

thus, the discrete sum approximates $\approx \int_0^1 \|X_s\|^2_{\mathbb{R}^d} \, ds$

Now, if we define

$$\mathcal{E}_n := \int_0^1 W_s \, dW_s - \sum_{k=0}^{n-1} X_{t_k} (W_{t_{k+1}} - W_{t_k})$$

We want to see that $\|\mathcal{E}_n\|_{L^2} \to 0$ as $n \to \infty$.

So we use the previous Ito's isometry on the error term:

$\mathbb{E}\left[\|\mathcal{E}_n\|^2_{\mathbb{R}^d} \right] = \mathbb{E}\left[\left\| \int_0^1 (X_s - X_s^{(n)}) \, dW_s \right\|^2_{\mathbb{R}^d} \right] =$

$= \mathbb{E}\left[\int_0^1 \|X_s - X_s^{(n)}\|^2_{\mathbb{R}^d} \, ds \right]$

Now, as $X_s^{(n)}$ converges pointwise to X as $n \to \infty$ (as per X being constant,

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^1 \|X_s - X_s^{(n)}\|^2_{\mathbb{R}^d} \, ds \right] = 0$$

Hence, $\|\mathcal{E}_n\|_{L^2} \to 0$ as $n \to \infty$. And we get the L^2 convergence.