Series 5a

Throughout this series, let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Milstein Scheme for One-Dimensional SDEs

Let $\xi \in \mathbb{R}$, let $\mu : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous, and let $\sigma \in C^1(\mathbb{R};\mathbb{R})$. Consider the SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi.$$

a) Let $M, N \in \mathbb{N}$. Write a MATLAB function `Milstein1D`$(T, \xi, \mu, \sigma, \sigma', W)$ with these inputs:

$T \in (0,\infty)$, $\xi \in \mathbb{R}$, $\mu\colon \mathbb{R}^M \to \mathbb{R}^M$, $\sigma\colon \mathbb{R}^M \to \mathbb{R}^M$, $\sigma'\colon \mathbb{R}^M \to \mathbb{R}^M$, $W \in \mathbb{R}^{(N+1)\times M}$,

which returns M realizations $Y_N^N(\omega_i)$ $(i = 1, 2, \ldots, M)$ of the Milstein approximation Y_N^N for X_T.

Here: μ, σ, and σ' are function handles, and $W \in \mathbb{R}^{(N+1)\times M}$ is a realisation of M independent 1-dimensional Brownian motions sampled at equally spaced time points $\{n\Delta t \mid n = 0, \ldots, N\}$, i.e.

$$W^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i) \qquad (i = 1, 2, \ldots, M).$$

Hint: You may modify the solution `EulerMaruyama.m` from Series 4a.

b) Investigate the strong error of the Milstein scheme for the one-dimensional SDE

$$\mathrm{d}X_t = X_t\mathrm{d}t + \log\left(1 + X_t^2\right)\mathrm{d}W_t, \quad t \in [0,1], \quad X_0 = 1, \tag{1}$$

using $M = 10^5$ realisations and $N = N_\ell = 10 \cdot 2^\ell$ time steps for $\ell \in \{0, 1, \ldots, 4\}$. To this end:

- for each $\ell \in \{0, 1, \ldots, 4\}$, generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$ $(i = 1, \ldots, M)$ of the Milstein approximation $Y_{N_\ell}^{N_\ell}$ for X_T;
- for each $\ell \in \{0, 1, \ldots, 4\}$, compute Monte Carlo approximations for the following expectations:

$$\mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|] \approx \frac{1}{M}\sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T| \quad \text{and} \quad \mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M}\sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T|\right)^{\frac{1}{2}}.$$

Report on the experimental rates of strong convergence in L^1 and L^2. Use a numerical solution of the SDE at level $\ell = 7$ as an approximation of the exact solution.

(You may use the provided template `Milstein_SDE.m`.)

c) Repeat question b) for the following SDE and comment on the results:

$$\mathrm{d}X_t = X_t\mathrm{d}t + \sin\left(1 + X_t^2\right)\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = 1. \tag{2}$$

2. Positivity and Simulation of the CIR Process via Drift-Implicit Milstein

Let $a, b, \sigma_v > 0$ and $v_0 \geq 0$. Consider the Cox-Ingersoll-Ross process, given as solution to the SDE

$$\mathrm{d}V_t = a(b - V_t)\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{V_t}\mathrm{d}W_t, \quad V_0 = v_0, \quad t \in [0,T]. \tag{3}$$

It can be shown (e.g., using the Yamada-Watanabe theorem) that the SDE (3) admits a unique solution (up to indistinguishability).

a) Let $N \in \mathbb{N}$. Assume that $[0, T]$ is discretised using a uniform temporal mesh with $N+1$ nodes, i.e. with time step size $\Delta t = T/N$. The *drift-implicit Milstein scheme* for the stochastic process V with step size Δt and initial value $V_0^N = V_0 > 0$ is given, for $n = 0, \dots, N-1$, by

$$V_{n+1}^N = V_n^N + a(b - V_{n+1}^N)\Delta t + \sigma_{\mathrm{v}}\sqrt{V_n^N}(W_{t_{n+1}} - W_{t_n}) + \frac{\sigma_{\mathrm{v}}^2}{4}\left((W_{t_{n+1}} - W_{t_n})^2 - \Delta t\right).$$

Show that if $4ab \geq \sigma_{\mathrm{v}}^2$, then $P(V_n^N > 0) = 1$ for all $n \in \{0, \dots, N\}$.

b) Write a Matlab functio `DriftImplicitMilstein(`$T, N, v_0, a, b, \sigma_{\mathrm{v}}$`)` with inputs $T \in (0, \infty)$, $N \in \mathbb{N}$, $v_0, a, b, \sigma_{\mathrm{v}} > 0$, and output a realization of the drift-implicit Milstein scheme $\{V_0^N, V_1^N, \dots, V_N^N\}$ for the Cox-Ingersoll-Ross process V. Then, plot a sample path of the stochastic process V using the following parameter choices: $T = 1, N = 10^3, v_0 = 0.5, a = 2, b = 0.5$ and $\sigma_{\mathrm{v}} = 0.25$.

3. Integrability and Proof Verification for the Kolmogorov Backward Equation

We adopt the full setting of Theorem 5.1.1 in the lecture notes with $d = m = 1$. Furthermore, we assume that there exists a constant $c > 0$ such that, for any $t \in [0, T]$ and $x \in \mathbb{R}$,

$$\sup_{s\in[t,T]} \left\|\mu(X_s^{t,x})\right\|_{L^9(P;|\cdot|)} + \sup_{s\in[t,T]} \left\|\sigma(X_s^{t,x})\right\|_{L^9(P;|\cdot|)} < c \quad \text{and} \quad \sup_{s\in[0,T]} \sup_{z\in\mathbb{R}} \frac{|\partial_2 u(s,z)|}{(1+|z|)^3} \leq c,$$

where ∂_2 denotes the partial derivative with respect to the second (spatial) argument of u.

Show that for any $t \in [0, T]$, $x \in \mathbb{R}$, and $h \geq 0$ such that $t + h \leq T$,

$$\int_t^{t+h} \mathbb{E}_P\left[\left(\sigma(X_s^{t,x}) \cdot \partial_2 u(t+h, X_s^{t,x})\right)^2\right] \mathrm{d}s \, < \, \infty, \tag{4}$$

and verify the identity (5.11) in the proof of Theorem 5.1.1.

Submission Deadline: Wednesday, 04 December 2024, by 2:00 PM.