Series 2b

In this series, we examine hierarchical decompositions of *standard Brownian Motion* $W = (W_t)_{t\in[0,T]}$ for $T > 0$ (recall Definition B.3.1 for $m = 1$), which provide an efficient representation of W that we can leverage for numerical approximations. Using these hierarchical decompositions, we construct Multilevel Monte Carlo (MLMC) estimators to approximate expected values of Brownian functionals with improved computational efficiency.

1. Lévy-Ciesielski Representation of Brownian Motion (I)

Let $H := (\psi_1, \psi_{j,k} \mid j \in \mathbb{N}, k = 1, \ldots, 2^{j-1})$ be the system of Haar functions $\psi_{j,k} : [0,1] \to \mathbb{R}$ defined by

$$\psi_1 \equiv 1 \qquad \text{and} \qquad \psi_{j,k}(x) := \begin{cases} 2^{(j-1)/2} & \text{if } x \in \left[\frac{2k-2}{2^j}, \frac{2k-1}{2^j}\right), \\ -2^{(j-1)/2} & \text{if } x \in \left[\frac{2k-1}{2^j}, \frac{2k}{2^j}\right), \\ 0 & \text{otherwise} \end{cases}$$

(that is, $\psi_{j,k}(x) := 2^{(j-1)/2}\psi(2^{j-1}x-(k-1))$ for $\psi(x) := \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise,} \end{cases}$ the 'mother wavelet').

The family H is an orthonormal basis (i.e., a complete orthonormal system) in $L^2([0,1])$.

The linear enumeration $\psi_n := \psi_{j,k}$ for $n = 2^{j-1} + k$ ($j \in \mathbb{N}$, $1 \le k \le 2^{j-1}$) and with $\psi_1 = 1$ (at times referred to as 'enumeration in lexicographical order') is also used below for convenience.

Consider further the family $S = (\phi_n \mid n \in \mathbb{N})$ of integral functions, also called 'Schauder functions',

$$\phi_n : [0,1] \to \mathbb{R} \quad \text{given by} \quad \phi_n(t) := \int_0^t \psi_n(u)\,\mathrm{d}u \quad \text{for each } t \in [0,1]. \tag{1}$$

a) Show that for each $\phi \in S$ we have $\phi(t) \ge 0$ for all $t \in [0,1]$, and that for any sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ of asymptotic behaviour $(a_n) \in \mathcal{O}(\sqrt{\log(n)})$, the series

$$\sum_{n=1}^{\infty} a_n\phi_n(t) \ \text{ converges absolutely and uniformly in } t \in [0,1].$$

Hint: Throughout, you may use without proof that $\sum_{k=1}^{2^j} \phi_{2^j+k}(t) \le 2^{-\frac{j}{2}-1}$, for each $j \in \mathbb{N}$ and any $t \in [0,1]$. ⋄

b) For an arbitrary sequence $\xi_1, \xi_2, \ldots : \Omega \to \mathbb{R}$ of $\mathcal{N}(0,1)$-distributed random variables over some probability space $(\Omega, \mathcal{F}, P)$, show that $(\xi_n)_{n\in\mathbb{N}} \in \mathcal{O}(\sqrt{\log(n)})$ P-almost surely. Then, assuming in addition that the $\xi_1, \xi_2, \ldots$ are also mutually statistically independent, show the following:

for any sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\sum_{k=1}^{2^j} |\alpha_{2^j+k}| \le 2^{-j/2}$ for all $j \in \mathbb{N}$, the limit

$$\chi := \sum_{n=1}^{\infty} \alpha_n\xi_n \ \text{ exists } P\text{-almost surely}$$

and, in fact, the limit χ is normally distributed with law $\chi(P) = \mathcal{N}_{0,\alpha^2}$ for $\alpha^2 := \sum_{n=1}^{\infty} \alpha_n^2$.

Hint: Use the elementary identity $\frac{1}{x}e^{-x^2/2} = \int_x^\infty \big(1 + \frac{1}{y^2}\big)e^{-y^2/2}\,\mathrm{d}y$ (wich holds for all $x > 0$) together with the Borel-Cantelli lemma to show that, for sufficiently large $c > 0$, the set

$$\bigcap_{i=1}^{\infty}\bigcup_{n=i}^{\infty}\Big\{\omega \in \Omega \;\Big|\; |\xi_n(\omega)| > c\sqrt{\log(n)}\Big\} \quad \text{is a } P\text{-null set.}$$

c) Let $\xi_1, \xi_2, \ldots : \Omega \to \mathbb{R}$ be a sequence of independent and identically distributed random variables over some probability space $(\Omega, \mathcal{F}, P)$ such that $\xi_1(P) = \mathcal{N}_{0,1}$, and let $\phi_1, \phi_2, \ldots : [0,1] \to \mathbb{R}$ be the Schauder functions from (1). Prove that, for any standard Brownian Motion $(W_t)_{t\in[0,T]}$ $(T > 0)$,

$$W_t = \frac{1}{\sqrt{T}}\sum_{n=1}^{\infty}\phi_n\big(\tfrac{t}{T}\big)\xi_n \quad \text{in distribution,} \quad \text{for each } \; t \in [0,T]. \tag{2}$$

Hint: The scaling property of the normal distribution (Proposition A.4.15) implies that if $\tilde{W} = (\tilde{W}_t)_{t\in[0,1]}$ is a standard Brownian Motion over the unit interval then, for any $c > 0$, the scaled process $W^{(c)} := \big(\frac{1}{\sqrt{c}}\tilde{W}_{\frac{t}{c}}\big)_{t\in[0,c]}$ is a standard Brownian motion over the interval $[0, c]$.

2. Multilevel Monte Carlo for Brownian Marginals

Consider the standard Brownian Motion $W = (W_t)_{t\in[0,1]}$ and its associated Schauder (Lévy-Cieselski) decomposition (2) derived in 1c), which states precisely that W_t is equal in distribution to

$$\tilde{W}_t := \phi_1(t)\xi_1 + \sum_{j=1}^{\infty}\sum_{k=1}^{2^{j-1}}\phi_{2^{j-1}+k}(t)\xi_{2^{j-1}+k}, \quad \text{for each } \; t \in [0,T]. \tag{3}$$

We aim to estimate $\mathbb{E}[e^{W_t}]$ for a fixed time $t \in [0,1]$ by using a (3)-informed Monte Carlo scheme. To this end, implement a MATLAB function `SchauderMLMC`(t, n_s, L) that takes $t \in [0,1]$ and $n_s, L \in \mathbb{N}$ as input and returns as output a (2^L-truncated) multilevel Monte Carlo approximation of $\mathbb{E}[e^{W_t}]$ which is computed along the following steps:

- Compute realisations for each of the truncated Schauder approximations

  $$\tilde{W}_t^{(0)} := \phi_1(t)\xi_1 \quad \text{and} \quad \tilde{W}_t^{(\ell)} := \phi_1(t)\xi_1 + \sum_{j=1}^{2^\ell}\sum_{k=1}^{2^{j-1}}\phi_{2^{j-1}+k}(t)\xi_{2^{j-1}+k} \quad (\ell = 1, \ldots, L) \tag{4}$$

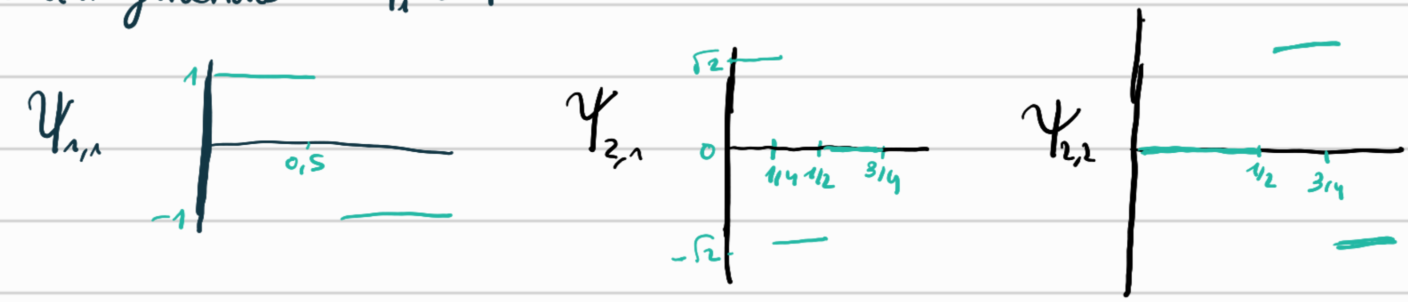
  by using $\lceil\frac{n_s}{2^\ell}\rceil$-many (independently drawn) samples per coefficient ξ_n that appears in $\tilde{W}^{(0)}$ and in the level differences $(\tilde{W}^{(\ell)} - \tilde{W}^{(\ell-1)})$ for $\ell \geq 1$, respectively.

- Use the above-computed sample realisations of (4) to MC approximate the expectations $\mathbb{E}[\Delta_\ell]$ of the differences $\Delta_\ell := e^{\tilde{W}_t^{(\ell)}} - e^{\tilde{W}_t^{(\ell-1)}}$ for $\ell \geq 1$ and $\Delta_0 := e^{\tilde{W}^{(0)}}$, and aggregate these approximations $\hat{\mathbb{E}}[\Delta_\ell]$ of $\mathbb{E}[\Delta_\ell]$ to form the MLMC estimator $E_{\mathrm{MLMC}} = \sum_{\ell=0}^{L}\hat{\mathbb{E}}[\Delta_\ell]$ of $\mathbb{E}[e^{W_t}]$.

Run your algorithm for $(t, n_s, L) = (\frac{1}{\sqrt{2}}, 1000, 10)$, plot the (sample) variances $\mathrm{Var}(\Delta_\ell)$ against the level ℓ, and compare the computational costs and accuracies of your MLMC estimator to a standard Monte Carlo (MC) estimator of $\tilde{W}_t^{(L)}$ using the same number of samples as the MLMC estimator for comparison; discuss your observations.

Submission Deadline: Wednesday, 23 October 2024, by 2:00 PM.

Haar functions: $\psi_1 \equiv 1$



j index squeezes the map and k shifts it.

$\psi_n = \psi_{jk}$ for $n = 2^{j-1}+k$ $j \in \mathbb{N}$, $1 \le k \le 2^{j-1}$

We say that the family of H is an orthonormal basis in $\mathcal{L}^2([0,1])$:

- $\mathcal{L}^2([0,1])$ (the space of all square integrable functions defined on $[0,1]$) is a Hilbert space, a complete inner product space where functions can be compared and decomposed in terms of orthonormal functions.

  ↳ Hilbert space is a vector space H with an inner product $\langle f, g \rangle$ s.t. the norm defined
  $$|f| = \sqrt{\langle f, f \rangle}$$
  turns into complete metric space (which means that every Cauchy sequence converges)

  $\mathcal{L}^2$ is an infinite dimensional Hilbert space s.t.
  $$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x) g(x)\, dx \quad \text{as inner product.}$$

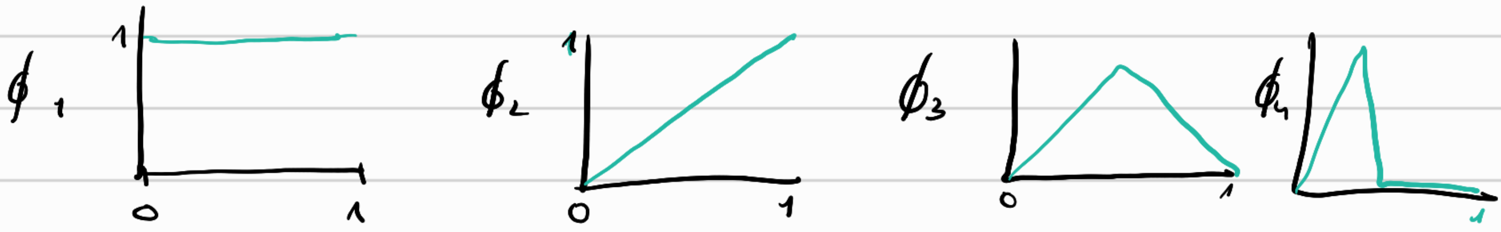
- An orthonormal basis in $\mathcal{L}^2([0,1])$ means 2 things ($\{h_k\}_k$ the basis)

  a) Orthonormality: $\langle h_i, h_j \rangle = \int_0^1 h_i(x) h_j(x)\, dx = \delta_{ij}$
  (mutually orthogonal) and normalized

  b) Completeness: $\{h_k\} \subset \mathcal{L}^2([0,1])$ is complete if any function $f \in \mathcal{L}^2([0,1])$ can be represented as an infinite series of the form: $f(x) = \sum_{k=1}^{\infty} c_k h_k(x)$, where c_k:
  $$c_k = \langle f, h_k \rangle = \int_0^1 f(x) h_k(x)\, dx$$

Now, $S = (\phi_n \mid n \in \mathbb{N})$ are called Schauder functions

$\phi_n [0,1] \to \mathbb{R}$; $\phi_n(t) = \int_0^t \psi_n(u)\,du$ for each $t \in [0,1]$



See Levy decomp. solution

1 a) First, $\phi_n(t) = \int_0^t \psi_n(u)\,du$

So, Consider $n = 2^{j-1} + k$ the lexicographical order of Haar functions

- If $u \in [0, \frac{k}{2^j}) \cup [\frac{k+1}{2^j}, 1]$, $\psi_n(u) = 0$, hence we can write

$$\phi_n(t) = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \psi_n(u)\,du$$

- If $t \in [\frac{k}{2^j}, \frac{k+\frac{1}{2}}{2^j})$, in this interval $\psi_n(u) = 1$

$$\phi_n(t) = \int_{\frac{k}{2^j}}^{x} 1 \, dt = x - \frac{k}{2^j}$$

- If $t \in [\frac{k+\frac{1}{2}}{2^j}, \frac{k+1}{2^j}]$, in this interval $\psi(u) = -1$

hence, $\phi_n(t) = \phi_n\left(\frac{k+1/2}{2^j}\right) + \int_{\frac{k+\frac{1}{2}}{2^j}}^{t} (-1)\,dt$

Since $\phi_n\left(\frac{k+\frac{1}{2}}{2^j}\right) = \frac{1}{2^j}$, $\phi_n(t)$ decreases linearly

but it never becomes negative because $\int_{\frac{k+\frac{1}{2}}{2^j}}^{t} (-1)\,dt \leq \frac{1}{2^j}$ for $t \in [\frac{k+\frac{1}{2}}{2^j}, \frac{k+1}{2^j})$

Hence, $\phi_n(t) \geq 0$ □

b) Consider $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ which has asimptotic behaviour $O(\sqrt{\log(n)})$

Recall: (a_n) is absolutly convergent if:

$$\sum_{n=0}^{\infty} |a_n| = L \text{ for som } L \in \mathbb{R}$$

· (a_n) converges uniformly if for every $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ s.t. for all $n \geq N_\varepsilon$ and all $x \in X$ $|f_n(x) - f(x)| < \varepsilon$.

Let's prove absolute and uniform convergence, to do i will use Weierstrass M-test to prove it.

→ first: $\sum_{n=1}^{\infty} \phi_n(t) = 1 + \sum_{n=2}^{\infty} \phi_n(t) = 1 + \sum_{j=1}^{\infty} \sum_{k=1}^{2j-1} \phi_{j,k}(t) =$

$$= 1 + \phi_{1,1}(t) + \sum_{j=2}^{\infty} \sum_{k=1}^{2^{j-1}} \phi_{2^{j-1}+k}(t) = 1 + \phi_{1,1}(t) + \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \phi_{2^j+k}(t) \leq$$

$$\leq 1 + \phi_{1,1}(t) + \sum_{j=1}^{\infty} 2^{-\frac{j}{2}-1} \leq \underbrace{m}_{\text{a constant}} + \sum_{j=1}^{\infty} \frac{1}{2^{j/2+1}} \leq m + \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2^{j/2}}$$

$$\left(\sum_{k=1}^{2^j} \phi_{2^j m}(t) \leq 2^{-\frac{j}{2}-1} \right) \qquad = m + \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^j$$

We will use this result afterwards in (*)

Now, $\sum_{n=1}^{\infty} a_n \phi_n(t) \leq \sum_{n=1}^{\infty} |a_n \phi_n(t)|$

$$\sum_{n=1}^{\infty} |a_n \phi_n(t)| \underset{\phi_n(t) \geq 0}{=} |a_1| + |a_2| \phi_{1,1}(t) + \sum_{j=2}^{\infty} \sum_{k=1}^{2^{j-1}} |a_{j,k}| \phi_{2^{j-1}+k} \leq$$

(and, we know that $|a_n| \leq c\sqrt{\log(n)}$, so)

$|a_1| + |a_2|\phi_{1,1}(t) \leq b$

$$\leq |a_1| + |a_2| \phi_{1,1}(t) + \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} c\sqrt{\log(2^j + k)}\, \phi_{2^j+k} \leq$$

$$\leq b + \sum_{j=1}^{\infty} c\sqrt{\log(2^j + 2_j)} \sum_{k=1}^{2^j} \phi_{2^j+k} \underset{(*)}{\leq} b + \sum_{j=1}^{\infty} c\sqrt{\log(2^j + 2_j)} \cdot 2^{-\frac{j}{2}-1}$$

$$= b + c \sum_{j=1}^{\infty} \frac{\sqrt{\log(2^j + 2_j)}}{2^{j/2+1}} = b + \frac{c}{2} \sum_{j=1}^{\infty} \frac{\sqrt{\log(2^j + 2_j)}}{2^{j/2}} =$$

$$= b + \frac{c}{2} \sum_{j=1}^{\infty} \left(\frac{\log(2^j \cdot 2_j)}{2^j} \right)^{1/2} \quad (**)$$

And there exists $C' \in \mathbb{R}$ such that $\left(\frac{\log(2^j \cdot 2_j)}{2^j}\right)^{1/2} \leq \left(\frac{C'}{2^j}\right)^{1/2}$

because $\frac{\log(2^j \cdot 2_j)}{2^j}$ decays very rapidly to 0, 2^j dominates $\log(2^n)$

Let's see it formally:

1rst) $\log(2^n \cdot 2n) = \log(2^n) + \log(2n) = n\log(2) + \log(2n)$

so,

$$\frac{\log(2^n \cdot 2n)}{2^n} = \frac{n\log(2)}{2^n} + \frac{\log(2n)}{2^n}$$

2nd) We bound each term: $\begin{cases} \frac{n\log(2)}{2^n} \leq C_1 \frac{1}{2^n} \\ \frac{\log(2n)}{2^n} \leq C_2 \frac{1}{2^n} \end{cases}$

3rd) All together: $\frac{\log(2^n \cdot 2n)}{2^n} \leq C' \frac{1}{2^n}$

Hence, as $\left|\frac{\log(2^j \cdot 2_j)}{2^j}\right|^{1/2} \leq \left(\frac{C'}{2^j}\right)^{1/2}$

if we take $M_j = \left(\frac{C}{2^j}\right)^{1/2}$, we know that

$\sum_{j=1}^{\infty}\left(\frac{C}{2^j}\right)^{1/2} = C^{1/2} \sum_{j=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^j$ which converges, because it's a geometric series with $|r| < 1$

Then, using Weirestrass M-test, we have that

$\sum_{j=1}^{\infty}\left(\frac{\log(2^j \cdot 2_j)}{2^j}\right)^{1/2}$ converges absolutely and uniformly in $t \in [0,1]$.

Hence $\sum_{n=1}^{\infty} a_n \phi_n(t) \leq b + \frac{C}{2}\sum_{j=1}^{\infty}\left(\frac{\log(2^j \cdot 2_j)}{2^j}\right)^{1/2}$ also

from (**)

converges in both senses, and, finally,

$\sum_{n=1}^{\infty} a_n \phi_n(t)$ also converges absolutely and uniformly, due to the inequalities seen before.

b) $\xi_1, \xi_2, \dots : \Omega \to \mathbb{R}$ an arbitrary sequence of $N(0,1)$-distributed random variables over some probability space $(\Omega, \mathcal{F}, P)$,

Now, as $\xi_n \sim N(0,1)$, we know that in this distribution, tail bounds are well defined:

$$P(|\xi_n| \geq t) = 2 \cdot P(\xi_n \geq t) = \frac{2}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{x^2}{2}} dx \leq \frac{2}{t} e^{-\frac{t^2}{2}}$$

So, we want to find a constant C s.t. $|\xi_n| \leq C\sqrt{\log(n)}$ almost surely:

$$P(\limsup_{n \to \infty} |\xi_n| > C\sqrt{\log(n)}) = 0$$

We will prove it using first Borel-Cantelli Lemma:

| If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ occur infinitely often}) = 0$ (being A_n events)

To apply it, we define the events $A_n = \{|\xi_n| > C\sqrt{\log(n)}\}$

And, using the tail bound seen before,

$$P(|\xi_n| > C\sqrt{\log(n)}) \leq \frac{2}{C\sqrt{\log(n)}} e^{-\frac{C^2 \log(n)}{2}}$$ (Similar to Hint)

when n is large, $P(Z \geq t) \approx \frac{1}{t\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

$$\Leftrightarrow \quad P(|\xi_n| > C\sqrt{\log(n)}) \leq \frac{2}{C\sqrt{\log(n)}} \cdot \frac{1}{n^{C^2/2}}$$

So, we have an upper bound for $P(A_n)$ to decide if $\sum_n^{\infty} P(A_n)$ converges.

Now, in the bound, the element that dominates is $\frac{1}{n^{C^2/2}}$,

and it's clear that if $\frac{C^2}{2} > 1$, i.e. $C > \sqrt{2}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{C^2/2}}$ converges, as it's an harmonic series with $p = \frac{C^2}{2}$

Therefore, $\sum_{n=1}^{\infty} P(A_n)$ converges.

$\Longrightarrow$ (By Borel-Cantelli Lemma) $\quad P(\limsup_{n \to \infty} |\xi_n| \geq C\sqrt{\log(n)}) = 0$

$\Rightarrow \xi_n = O(\sqrt{\log(n)})$ P-a.s. for $C > \sqrt{2}$.

Now, we assume independence between ξ_n. If for any sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ s.t. $\sum_{k=1}^{2^j} |\alpha_{2^j+k}| \le 2^{-j/2}$ for all $j \in \mathbb{N}$,

We saw in section a) that $\sum_{n=1}^{\infty} a_n \phi_n(t)$ converges absolutely and unif, $(\xi_n)_n$ is a sequence $\in \mathcal{O}(\sqrt{\log n})$ P-as, $(\alpha_n)_n$ fullfils same conditions as $\phi_n(t)$, we can say that χ exists P-almost surely, i.e.

$$P\left(\sum_{n=1}^{\infty} \alpha_n \xi_n = M\right) = 1 \text{ for some } M \in \mathbb{R}.$$

Because for almost all $w \in \Omega$, we have that ξ_n is $O(\sqrt{\log n})$ so we can assume the result in a).

Now,

$$\left.\begin{aligned} &\cdot\ \mathbb{E}\left(\sum_{n=1}^{\infty} \alpha_n \xi_n\right) = \sum_{n=1}^{\infty} \alpha_n \mathbb{E}(\xi_n) = \boxed{0} \\ &\cdot\ \mathrm{Var}\left(\sum_{n=1}^{\infty} \alpha_n \xi_n\right) = \sum_{n=1}^{\infty} \alpha_n^2 \underbrace{\mathrm{Var}(\xi_n)}_{\mathrm{Var}(\xi_n)=1} = \boxed{\alpha^2} \end{aligned}\right\}$$

$$\alpha^2 := \sum_{n=1}^{\infty} \alpha_n^2$$

where $\sum_{i=1}^{n} \alpha_i^2 < \infty$ because $\sum_{i=1}^{n} \alpha_i$ converges (we saw it in the previous section, using the fact that $\sum_{k=1}^{2^j} |\alpha_{2^j+k}| \le 2^{-j/2}$

$n=1$

and due to the fact that linear combinations of $N(0,1)$ r.v. are r.v, $\sum_{n=1}^{\infty} \alpha_n \xi_n \sim N(0, \alpha^2)$.

c) Let $(\xi_n)_{n\in\mathbb{N}}$ a sequence of $\xi_n \overset{iid}{\sim} N(0,1)$. Let $\phi_1, \phi_2, \ldots : [0,1] \to \mathbb{R}$ the Schauder functions. Consider now $(W_t)_{t\in[0,T]}$ $(T>0)$ any standard Brownian motion, where $W_t \sim N(0,t)$

We have seen that $\sum_{n=1}^{\infty} \alpha_n \xi_n \sim N(0, \alpha^2)$, so, if we take $\alpha_n = \phi_n(\frac{t}{T})$, $\sum_{n=1}^{\infty} \phi_n(\frac{t}{T})\xi_n \sim N(0, a)$ with $\frac{t}{T} \in [0,1]$ and $a \in \mathbb{R}$

$$a = \sum_{n=1}^{\infty} \phi_n^2\left(\frac{t}{T}\right)$$

In order to see what is a, recall $\phi_n(\frac{t}{T}) = \int_0^{\frac{t}{T}} \varphi_n(u)\,du$

so, $a = \sum_{n=1}^{\infty} \left(\int_0^{\frac{t}{T}} \varphi_n(u)\,du\right)^2$, and as $\varphi_n(u)$ form an orthonormal basis in $L^2([0,1])$, We recall Parseval's theorem to find a: Given an orthonormal basis $\{\varphi_n\}$ in $L^2([0,1])$, for any $f \in L^2([0,1])$,

$$\|f\|_{L^2}^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2$$

So, specifically, if we take the function $1_{[0,\frac{t}{T}]}(s) = \begin{cases} 1 & \text{if } s \in [0,\frac{t}{T}] \\ 0 & \text{else} \end{cases}$

we have $\|1_{[0,\frac{t}{T}]}\|_{L^2}^2 = \sum_{n=1}^{\infty} |\langle 1_{[0,\frac{t}{T}]}, \varphi_n \rangle|^2 = \sum_{n=1}^{\infty} \left(\int_0^{\frac{t}{T}} \varphi_n(u)\, du \right)^2$

Hence, $a = \|1_{[0,\frac{t}{T}]}\|_{L^2}^2 = \langle 1_{[0,\frac{t}{T}]}, 1_{[0,\frac{t}{T}]} \rangle = \int_0^{\frac{t}{T}} du = \boxed{\frac{t}{T}}$

So, $\sum_{n=1}^{\infty} \phi_n(\frac{t}{T}) \xi_n \sim N(0, \frac{t}{T})$

And as $\tilde{W}_{\frac{t}{T}} \sim N(0, \frac{t}{T})$ where $\tilde{W}$ is the std. BM over $[0,1]$,

and $W^{(T)} := \left(\frac{1}{\sqrt{T}} \tilde{W}_{\frac{t}{T}} \right)_{t \in [0,T]}$ its a std BM over $[0,T]$, we

can say that $\frac{1}{\sqrt{T}} \sum_{n=1}^{\infty} \phi_n(\frac{t}{T}) \xi_n \sim N(0, \frac{t}{T}) \sim W^{(T)}$

$$\Rightarrow \frac{1}{\sqrt{T}} \sum_{n=1}^{\infty} \phi_n\left(\frac{t}{T}\right) \xi_n \overset{d}{=} W_t$$

2) $W = (W_t)_{t \in [0,1]}$ the standard Brownian Motion and the Schauder decomposition:

$$\tilde{W}_t := \phi_1(t) \xi_1 + \sum_{j=1}^{\infty} \sum_{k=1}^{2^{j-1}} \phi_{2^{j-1}+k}(t) \xi_{2^{j-1}+k}, \text{ for each } t \in [0,1]$$

As we are computing a (2^L-truncated) MLMC approx of $\mathbb{E}[e^{W_1}]$ for each of the ξ_n that appear in the expression of the truncated Schauder approximations, will be computed $\lceil \frac{n_s}{\xi_n} \rceil$ times.
We need have an analytical expression of $\phi_n(t)$ in order to compute it in the code:

Now, $\phi_n(t) = \phi_{j,k}(t) = \int_0^t \psi_{j,k}(s)\,ds =$

We know that $\psi_{j,k}(s) = \begin{cases} 2^{\frac{j-1}{2}} & \text{if } x \in \left[\frac{2k-2}{2^j}, \frac{2k-1}{2^j}\right) \\ -2^{(j-1)/2} & \text{if } x \in \left[\frac{2k-1}{2^j}, \frac{2k}{2^j}\right) \\ 0 & \text{otherwise} \end{cases}$

So, $\phi_{j,k}(t) = \begin{cases} 2^{\frac{j-1}{2}}\left(t - \frac{2k-2}{2^j}\right) & \text{if } t \in \left[\frac{2k-2}{2^j}, \frac{2k-1}{2^j}\right) \\ 2^{\frac{j-1}{2}} \cdot \left(\frac{2k}{2^j} - t\right) & \text{if } t \in \left[\frac{2k-1}{2^j}, \frac{2k}{2^j}\right] \\ 0 & \text{otherwise} \end{cases}$

<u>Obs</u>: The observations regarding the results are in the notebook.