# 1.

Let $A$ be an $n \times n$ real matrix. Prove that if $A$ is symmetric, i.e. $A = A^T$, then all eigenvalues of $A$ are real.

*Proof.* Assume, for a contradiction that $A$ is a real, symmetric $n \times n$ matrix and that $\lambda$ is a complex eigenvalue of $A$.
By definition we must have that

$$Av = \lambda v \tag{1}$$

for eigenvector $v$ of $A$. Premultiplying both sides by $\overline{v^T}$:

$$\overline{v^T}Av = \overline{v^T}\lambda v = \lambda \overline{v^T} v$$

Now, taking the conjugate yields:

$$\overline{\overline{v^T}Av} = v^T\overline{A}\overline{v} = \overline{\lambda} v^T \overline{v} = \overline{\lambda \overline{v^T} v}$$

Taking the transpose:

$$(v^T\overline{A}\overline{v})^T = \overline{v^T}\overline{A}^T v = \overline{\lambda}\overline{v^T} v = (\overline{\lambda} v^T \overline{v})^T$$

Now as $A$ is real, symmetric, and through (1):

$$\overline{\lambda}\overline{v^T} v = \overline{v^T}\overline{A}^T v = \overline{v^T}Av = \lambda\overline{v^T} v \Rightarrow \overline{\lambda} = \lambda$$

which is a contradiction unless $\lambda$ is real. Conclusively, all eigenvalues of real, symmetric matrices are themselves real. ■

# 2.

Through transformation with orthogonal matrix $O$ , the problem $\hat{b} = \arg\min \|y - Xb\|^2$ is equivalent to $\hat{b} = \arg\min \|y^* - X^*b\|^2$ where $y$ and $y^*$ are in $\mathbf{R}^{\mathrm{m}}$, $X$ and $X^*$ are in $\mathbf{R}^{\mathrm{m}\times\mathrm{n}}(\mathrm{m} \geq \mathrm{n})$, and $y^* = Oy$ and $X^* = OX$. Let $y^* = [y_1^*, y_2^*, \cdots, y_m^*]^T$ . If $X^*$ is upper-triangular, prove that the residual sum of square is

$$\|y - X\hat{b}\|^2 = \sum_{i=n+1}^{m} |y_i^*|^2$$

*Proof.*

$$\begin{aligned}
\|y - X\hat{b}\|^2 &= \|I_m y - I_m X\hat{b}\|^2 \\
&= \|O^T O y - O^T O X\hat{b}\|^2 \\
&= \|O^T(y^* - X^*\hat{b})\|^2 \\
&= O^T \left( \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_{n-1}^* \\ y_n^* \\ y_{n+1}^* \\ \vdots \\ y_m^* \end{bmatrix} - \begin{bmatrix} x_{1,1}^* & x_{1,2}^* & \dots & x_{1,n}^* \\ 0 & x_{2,2}^* & \dots & x_{2,n}^* \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{n,n}^* \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_{n-1} \\ \hat{b}_n \end{bmatrix} \right)
\end{aligned}$$

Now, through backwards substitution we know that $\hat{b}_n = y_n^*/x_{n,n}^*$ so that

$$y_n^* - x_{n,n}^*\hat{b}_n = y_n^* - y_n^* = 0$$

Similarly, now since $\hat{b}_{n-1} = \left(y_{n-1}^* - x_{n-1,n}^*\hat{b}_n\right)/x_{n-1,n-1}^*$ we have

$$y_{n-1}^* - x_{n-1,n-1}^*\hat{b}_{n-1} - x_{n-1,n}^*\hat{b}_n = y_{n-1}^* - y_{n-1}^* + x_{n-1,n}^*\hat{b}_n - x_{n-1,n}^*\hat{b}_n = 0$$

and following the general formula, $\hat{b}_j = \left(y_j^* - \sum_{i=j+1}^{n} x_{j,i}^*\hat{b}_i\right)/x_{j,j}^*, \quad j = n-1, n-2, \ldots, 1,$ we arrive at values of 0 for all $\tilde{y}_k^*,\ k = 1, 2, \ldots, n$ where $\tilde{y} = y^* - X^*\hat{b}$ and the $k$ subscript denotes the $k^{th}$ entry of $\tilde{y}$. Thus: $\tilde{y} = [0,\ 0,\ \ldots,\ y_{n+1}^*,\ y_{n+2}^*,\ \ldots,\ y_m^*]^T$. Now, observe that

$$\begin{aligned}
\|O^T\tilde{y}\|^2 &= (O^T\tilde{y})^T(O^T\tilde{y}) \\
&= \tilde{y}^T O O^T \tilde{y} \\
&= \tilde{y}^T I_m \tilde{y} \\
&= \tilde{y}^T\tilde{y} \\
&= 0 + 0 + \cdots + (y_{n+1}^*)^2 + (y_{n+2}^*)^2 + \cdots + (y_m^*)^2 \\
&= \sum_{i=n+1}^{m} |y_i^*|^2
\end{aligned}$$

■

# 3.

Let $O$ be an $n \times n$ orthogonal real matrix, i.e. $O^TO = I_n$ where $I_n$ is an $n \times n$ identity matrix. Prove that

i) Any entry in $O$ is between -1 and 1.

ii) If $\lambda$ is an eigenvalue of $O$, then $|\lambda| = 1$.

iii) $\det(O)$ is either 1 or -1.

## i)

*Proof.* Let $O$ be denoted as $O = [v_1 v_2 \ldots v_n]$. Then

$$O^TO = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} v_1^Tv_1 & v_1^Tv_2 & \ldots v_1^Tv_n & \\ v_2^Tv_1 & v_2^Tv_2 & \ldots & v_2^tv_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^Tv_1 & v_n^Tv_2 & \ldots & v_n^Tv_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = I_n$$

Thus, the rows and columns of $O$ are orthonormal. This implies that $v_i^Tv_i = \|v_i\|^2 = 1$, that is each row/column is of unit length. This is only possible if each element $,u_i, j \in v_i,\ j = 1, \ldots, n$, must be between 0 and 1.

■

## ii)

*Proof.* Let $\lambda$ be an eigenvalue of $O$. By definition, we must have that $Ov = \lambda v$.

$$
\begin{aligned}
Ov &= \lambda v \\
(Ov)^T Ov &= (Ov)^T \lambda v \\
v^T O^T Ov &= (\lambda v)^T \lambda v \\
v^T v &= \lambda^2 v^T v \\
\lambda^2 &= 1 \\
&\Rightarrow |\lambda| = 1
\end{aligned}
$$

■

## iii)

*Proof.*

$$
\begin{aligned}
\det(O^T O) &= \det(O^T)\det(O) = 1 = \det(I_n) \\
&= \det(O)\det(O) \ \ = 1 \\
&\Rightarrow \det(O) = \pm 1
\end{aligned}
$$

■

# 4.

Let $H$ be an $n$ x $n$ householder matrix given by

$$H = I_n - 2\frac{vv^T}{v^T v}, \text{ for any non-zero } n \text{ -length column vector } v(\neq 0)\ .$$

Show that $H$ is a symmetric, orthogonal, and reflection matrix. That is, $H$ satisfies

i) $H = H^T$

ii) $HH^T = I_n$

iii) $\det(H) = -1$

## i) Symmetry

*Proof.* Let $a = v^T v$ and note that $a$ is a scalar. Further, observe that

$$
\begin{aligned}
H &= I_n - 2\frac{vv^T}{a} \\
&= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{v_1^2}{a} & \frac{v_1 v_2}{a} & \dots & \frac{v_1 v_n}{a} \\ \frac{v_2 v_1}{a} & \frac{v_2^2}{a} & \dots & \frac{v_2 v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{v_n v_1}{a} & \frac{v_n v_2}{a} & \dots & \frac{v_n^2}{a} \end{bmatrix} \\
H &= \begin{bmatrix} 1 - 2\frac{v_1^2}{a} & -2\frac{v_1 v_2}{a} & \dots & -2\frac{v_1 v_n}{a} \\ -2\frac{v_2 v_1}{a} & 1 - 2\frac{v_2^2}{a} & \dots & -2\frac{v_2 v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ -2\frac{v_n v_1}{a} & -2\frac{v_n v_2}{a} & \dots & 1 - 2\frac{v_n^2}{a} \end{bmatrix}
\end{aligned}
$$

Now as $v_i v_j = v_j v_i$ since $v_i, v_j$ are scalars. Thus $H$ is symmetric, since for any element, $h_{i,j}$ of $H$ in the $i^{th}$ row and $j^{th}$ column will equal its transpose $h_{j,i}$. ■

## ii) Orthogonality

*Proof.* Observe that as $H$ is symmetric, $HH^T = H^T H = HH$. Thus,

$$
HH = \begin{bmatrix} 1 - 2\frac{v_1^2}{a} & -2\frac{v_1 v_2}{a} & \dots & -2\frac{v_1 v_n}{a} \\ -2\frac{v_2 v_1}{a} & 1 - 2\frac{v_2^2}{a} & \dots & -2\frac{v_2 v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ -2\frac{v_n v_1}{a} & -2\frac{v_n v_2}{a} & \dots & 1 - 2\frac{v_n^2}{a} \end{bmatrix} \begin{bmatrix} 1 - 2\frac{v_1^2}{a} & -2\frac{v_1 v_2}{a} & \dots & -2\frac{v_1 v_n}{a} \\ -2\frac{v_2 v_1}{a} & 1 - 2\frac{v_2^2}{a} & \dots & -2\frac{v_2 v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ -2\frac{v_n v_1}{a} & -2\frac{v_n v_2}{a} & \dots & 1 - 2\frac{v_n^2}{a} \end{bmatrix}
$$

The elements on the main diagonal of $HH$, $\tilde{h}_{i,i}$, are:

$$
\begin{aligned}
\tilde{h}_{i,i} &= (1 - 2\frac{v_i^2}{a})(1 - 2\frac{v_i^2}{a}) + \sum_{j \neq i}^{n} (-2\frac{v_i v_j}{a})(-2\frac{v_i v_j}{a}) \\
&= 1 + 4(\frac{v_i^4}{a^2}) + -4\frac{v_i v_j}{a} + \sum_{j \neq i}^{n} 4\left(\frac{v_i v_j}{a}\right)^2 \\
&= 1 + 4\frac{v_i^2}{a}\left(-1 + \sum_{j \neq i} \frac{v_j^2}{a}\right) \\
&= 1 + 4\frac{v_i^2}{a}\left(-1 + \frac{v^T v}{a}\right) \\
&= 1 + 4\frac{v_i^2}{a}\left(-1 + \frac{a}{a}\right) \\
&= 1 + 4\frac{v_i^2}{a}(0) = 1
\end{aligned}
$$

For the elements on the off-diagonal, $\tilde{h}_{i,j},\ i \neq j$, we focus only on the specific case of $\tilde{h}_{1,2}$ since all other off-diagonal elements follow similarly:

$$\begin{aligned}
\tilde{h}_{1,2} &= (1-\frac{v_1^2}{a})(-2\frac{v_1v_2}{a}) + (-2\frac{v_1v_2}{a})(1-\frac{v_2^2}{a}) + \cdots + (-2\frac{v_1v_n}{a})(-2\frac{v_nv_2}{a}) \\
&= -4\frac{v_1v_2}{a} + 4\frac{v_1^3v_2}{a^2} + 4\frac{v_1v_2^3}{a^2} + \cdots + 4\frac{v_1v_n^2v_2}{a^2} \\
&= 4\frac{v_1v_2}{a}\left(-1+\frac{1}{a}(v_1^2+v_2^2+\cdots+v_n^2)\right) \\
&= 4\frac{v_1v_2}{a}\left(-1+\frac{1}{a}(v^Tv)\right) \\
&= 4\frac{v_1v_2}{a}\left(-1+\frac{1}{a}(a)\right) = 4\frac{v_1v_2}{a}(-1+1) = 0
\end{aligned}$$

Consequently, we have that

$$HH = \begin{bmatrix} 1-2\frac{v_1^2}{a} & -2\frac{v_1v_2}{a} & \dots & -2\frac{v_1v_n}{a} \\ -2\frac{v_2v_1}{a} & 1-2\frac{v_2^2}{a} & \dots & -2\frac{v_2v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ -2\frac{v_nv_1}{a} & -2\frac{v_nv_2}{a} & \dots & 1-2\frac{v_n^2}{a} \end{bmatrix} \begin{bmatrix} 1-2\frac{v_1^2}{a} & -2\frac{v_1v_2}{a} & \dots & -2\frac{v_1v_n}{a} \\ -2\frac{v_2v_1}{a} & 1-2\frac{v_2^2}{a} & \dots & -2\frac{v_2v_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ -2\frac{v_nv_1}{a} & -2\frac{v_nv_2}{a} & \dots & 1-2\frac{v_n^2}{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Conclusively, $H$ is orthogonal. ■

## iii) $\det(\mathbf{H}) = -1$

*Proof.* Through our results in *i)* and *ii)* we have established symmetry and orthogonality. Now observe that:

$$\begin{aligned}
H = H^T \Rightarrow HH^T = I_n &\Rightarrow HH = I_n \\
&\Rightarrow \det(HH) = \det(H)\det(H) = \det(I_n) = 1 \\
&\Rightarrow \lambda_H = \pm 1 \\
&\Rightarrow H = v\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} v^T
\end{aligned}$$

Where $\lambda_H$ represents some eigenvalue of $H$. Now, taking the trace of $H$, which is equal to the sum of its eigenvalues:

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i = tr(H) &= tr(I_n - 2\frac{vv^T}{v^Tv}) \\
&= tr(I_n - 2) - 2tr(\frac{vv^T}{v^Tv}) \\
&= n - 2tr(\frac{vv^T}{v^Tv}) \\
&= n - 2tr(\frac{v^Tv}{v^Tv}) \\
&= n - 2tr(1) \\
\sum_{i=1}^{n} \lambda_i &= n - 2
\end{aligned}$$

where the third-to-last line comes from the fact that $tr(AB) = tr(BA)$ for matrices $A$ and $B$. This result implies that there can only be one "$-1$" in the $\lambda_i$'s. Consequently,

$$det(H) = -1 = \prod_{i=1}^{n} \lambda_i$$

■