# STA 5106
# Computational Methods in Statistics I

*Department of Statistics*
Florida State University

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# Review: Importance Sampling

- Instead of sampling from $f(x)$, the **importance sampling** samples from another density $h(x)$, and computes the estimate of $\Theta$ using averages of $g(x)f(x)/h(x)$ instead of $g(x)$ evaluated on those samples.

- Mathematically, we can rearrange the definition of $\Theta$ as follows:

$$\Theta = \int g(x)f(x)dx = \int \frac{g(x)f(x)}{h(x)} h(x)dx.$$

- Generate samples $X_1, X_2, \ldots, X_n$ from the density $h(x)$ and compute the estimate:

$$\hat{\Theta}_n = \frac{1}{n}\sum_{i=1}^{n} \frac{g(X_i)f(X_i)}{h(X_i)}.$$

# 6.5 Tilted Sampling

# Tilted Sampling

- This is a specific case of importance sampling where the sampling distribution is simply a tilted version of the original density function.
- In cases where one is interested in estimating tail probabilities, and where these tails are negligible, it may be useful to tilt the density while raising the tail probability.
- Let $f(x)$ be the original probability density function. Then, the tilted density is given by:

$$f_t(x) = \frac{\exp(tx) f(x)}{M(t)}, \qquad \text{where } M(t) = \int \exp(tx) f(x)\, dx.$$

# Tail Probability

- The amount of tilt is given by the parameter $-\infty < t < \infty$.

- For $t > 0$, the positive values of $x$ increase in probability while the negative values decrease in probability. Opposite happens for $t < 0$.

- Consider the probability where we are interested in estimating the tail probability:

$$\theta = P\{X \geq a\} = \int_a^{\infty} f(x)dx,$$

and let $a$ be much larger than the mean of $X$, i.e. $a >> E[X]$.

# Unbiased Estimator

- Using random sampling from the tilted density $X_1, X_2, \ldots, X_n$ from $f_t(x)$, and the estimator is given by:

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \geq a\}} M(t)\exp(-tX_i).$$

- Compute the expectation of the estimate:

$$E[\hat{\theta}_n] = \frac{1}{n}\sum_{i=1}^{n} E[I_{\{X_i \geq a\}} M(t)\exp(-tX_i)]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \int_a^{\infty} M(t)\exp(-tx_i)\exp(tx_i)\frac{f(x_i)}{M(t)} dx_i$$

$$= \theta$$

# Variance Estimate

- The variance of the estimator is given by:

$$\mathrm{var}[\hat{\theta}_n] = E[(\frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \geq a\}} M(t)\exp(-tX_i) - \theta)^2]$$

$$= \frac{1}{n}[\int I^2_{\{X_i \geq a\}} M(t)^2 \exp(-2tx_i) f_t(x_i) dx_i - \theta^2]$$

- The integration term on the right side simplifies to:

$$\int_a^\infty M(t)^2 \exp(-2tx_i) f_t(x_i)\, dx_i = M(t) \int_a^\infty \exp(-tx_i) f(x_i)\, dx_i$$

- In the interest of reducing estimating variance, we seek a value of *t* that minimizes this term. However, this minimization is not straightforward and leads to an indirect solution.

# Bound

- Assuming $t > 0$, we can bound this term as follows:

$$M(t)\int_a^\infty \exp(-tx_i)f(x_i)dx_i \le M(t)\int_a^\infty \exp(-ta)f(x_i)dx_i$$
$$\le M(t)\exp(-ta)$$

- It is easier to minimize this bounding term as follows:

$$\frac{d}{dt}[M(t)\exp(-ta)] = M'(t)\exp(-ta) - aM(t)\exp(-ta) = 0$$

- This implies that $t$ should be chosen such that

$$a = \frac{M'(t)}{M(t)} = \int x\exp(tx)\frac{f(x)}{M(t)}dx = \int xf_t(x)dx.$$

- i.e. $t$ should be chosen such that $a$ is the mean of the tilted density.

# Example

- Example: For a standard normal random variable $X$, find the probability $P\{X \geq a\}$ for $a >> 0$. The quantity to be estimated is given by:

$$\theta = \int_a^\infty f(x)dx.$$

- The tilted version of $f(x)$ is given by

$$f_t(x) = \frac{\exp(tx)f(x)}{M(t)},$$

where

$$M(t) = \int \exp(tx)f(x)dx = \int \frac{1}{\sqrt{2\pi}}\exp(tx)\exp(-x^2/2)dx$$

$$= \frac{1}{\sqrt{2\pi}}\exp(t^2/2)\int \exp(-\frac{1}{2}(x-t)^2)dx = \exp(t^2/2)$$

# Tilted Density

- In fact, the tilted density can also be written as:

$$f_t(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x-t)^2)$$

- The tilted density is normal with mean $t$ and variance 1.

- Choosing $t$ such that the mean of the tilted density is $a$. That is, set the amount of tilt to be $a$.

- Let $X_i$' s be sampled from the tilted density. Then, $\theta$ can be estimated as follows.

# Estimators

- Since the ratio

$$\frac{f(x)}{f_t(x)} = \exp(-tx)M(t) = \exp(-ax)\exp(a^2/2),$$

- An estimator of $\theta$ is given by:

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \geq a\}} \exp(a^2/2 - aX_i).$$