STA 5106: Homework Assignment #1

(Thursday, August 29)
Due: Thursday, September 5

1. Let A be an $n \times n$ real matrix. Prove that if A is symmetric, i.e. $A = A^T$, then all eigenvalues of A are real.

2. Through transformation with orthogonal matrix O, the problem $\hat{b} = \arg\min_b \| y - Xb \|^2$ is equivalent to $\hat{b} = \arg\min_b \| y^* - X^* b \|^2$ where y and y^* are in $\mathbf{R}^m$, X and X^* are in $\mathbf{R}^{m \times n}$ ($m \geq n$), and $y^* = Oy$ and $X^* = OX$. Let $y^* = [y_1^*, y_2^*, \cdots, y_m^*]^T$. If X^* is upper-triangular, prove that the residual sum of square

$$\| y - X\hat{b} \|^2 = \sum_{i=n+1}^{m} | y_i^* |^2 .$$

3. Let O be an $n \times n$ orthogonal real matrix, i.e. $O^T O = I_n$, where I_n is an $n \times n$ identity matrix. Prove that

i) Any entry in O is between -1 and 1.
ii) If λ is an eigenvalue of O, then $|\lambda| = 1$.
iii) $\det(O)$ is either 1 or -1.

4. Let H be an n x n householder matrix given by

$$H = I_n - 2\frac{vv^T}{v^T v}, \text{ for any non-zero } n\text{-length column vector } v \ (\neq 0) .$$

Show that H is a symmetric, orthogonal, and reflection matrix. That is, H satisfies
i) $H = H^T$, ii) $HH^T = I_n$, iii) $\det(H) = -1$.