STA 5106
Computational Methods in Statistics I

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Class 3
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Householder Transformation

- **Definition 7** For a vector $v \in \mathbf{R}^m$, an $m \times m$ matrix H of the form

$$H = I_m - 2vv^T/(v^Tv)$$

is called a **Householder reflection matrix**.

- ***v* is called the Householder vector and *H* is an orthogonal matrix.**

- A simple calculation shows that $Hv = -v$.

- It should be noted that the scale (length) of v does not affect H.

How to Find H?

- For a given vector $x \in \mathbf{R}^m$, we want to form an H, a householder matrix, in such a way that Hx has all but the first entry as zeros. In other words, $\boldsymbol{Hx = \lambda e_1}$ **for some constant** $\boldsymbol{\lambda}$**.**

- Since H is an orthogonal matrix, $\lambda = \pm\|x\|$.

- We have

$$\lambda e_1 = Hx = (I - 2vv^T/(v^Tv))x = x - \frac{2v^T x}{v^T v} v$$

- Hence

$$v = \left(\frac{2v^T x}{v^T v}\right)^{-1} (x - \lambda e_1)$$

- To fix the scale of v, we let $2v^Tx/(v^Tv) = 1$. Then $v = x - \lambda e_1$.

How to Find H?

- Let $-\lambda$ have the same sign as x_1, i.e. $\lambda = -\text{sign}(x_1)\|x\|$.
- Then v and Hx have the following forms:

$$v = x + \text{sign}(x_1)\,\|x\|e_1$$
$$Hx = -\text{sign}(x_1)\,\|x\|e_1$$

- We can further normalize v such that $v(1) = 1$ in computation.
- Therefore, all vectors have the following forms:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \qquad Hx = \begin{pmatrix} -sign(x_1)\,\|x\| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Householder Vector Algorithm

- Algorithm: Given an m-vector x compute an m-vector v such that $v(1) = 1$ and $(I - 2vv^T/(v^Tv))x = -\beta e_1$ for a non-zero β.

- **Algorithm 9 (Householder Vector)**

```
function v = house(x);
m = length(x);
mu = norm(x,2);
v = x;
if mu ~= 0
        c = x(1) + sign(x(1))*mu;
        v(2 : m, 1) = v(2 : m, 1)/c;
end
v(1) = 1;
```

Matrix Form

- Let $X \in \mathbf{R}^{m \times n}$ ($n < m$) and H be the householder matrix
$$H = I - 2vv^T/(v^Tv).$$
Then, X is updated to HX.

- **Algorithm:** Given an $m \times n$ matrix ($n < m$) X and an m-vector v overwrite X with HX.

- **Algorithm 10 (Householder Multiplication - Row)**

```
function X = rowhouse(X,v)
X = X - 2*v*v'/(v'*v)*X;
```

Householder Transformation

- We are interested in transforming a given matrix X through multiplication by orthogonal matrices in such a way that it results in an upper triangular matrix, X^*.

- Under the Householder technique, we apply a householder transformation to each of the n columns in an iterative way so that for each column the entries below the diagonal are converted to zero.

- If B is an intermediate result after the first j-transformations then $B(j + 1 : m, j) = 0$, for $1 \leq j \leq n$.

Computational Procedures

- Let H_j ($j = 1, \ldots, m$) be the $m \times m$ matrix generated as:

$$H_j = \begin{pmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{pmatrix}$$

where $\tilde{H}_j = I - 2\frac{\tilde{v}\tilde{v}^T}{\tilde{v}^T\tilde{v}} \in \mathbf{R}^{(m-j+1)\times(m-j+1)}$

- Let X be the original $m \times n$ matrix, $X^{(1)}$ be the result after the first transformation (j =1), $X^{(2)}$ after the second transformation (j = 2) and so on. That is, $X^{(1)} = H_1 X$, $X^{(2)} = H_2 X^{(1)}$.

- Let

$$X = \begin{pmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ & & \ddots & \\ x_{m1} & x_{m2} & & x_{mn} \end{pmatrix}$$

Computational Procedures

- Then

$$X^{(1)} = H_1 X = \begin{pmatrix} x_{11}^1 & x_{12}^1 & & x_{1n}^1 \\ 0 & x_{22}^1 & & x_{2n}^1 \\ & & \ddots & \\ 0 & x_{m2}^1 & & x_{mn}^1 \end{pmatrix}, \quad X^{(2)} = H_2 X^{(1)} = \begin{pmatrix} x_{11}^1 & x_{12}^1 & & x_{1n}^1 \\ 0 & x_{22}^2 & & x_{2n}^2 \\ 0 & 0 & \ddots & \\ 0 & 0 & & x_{mn}^2 \end{pmatrix},$$

- Finally,

$$X^{(n)} = H_n H_{n-1} \cdots H_1 X = \begin{pmatrix} x_{11}^1 & x_{12}^1 & & x_{1n}^1 \\ 0 & x_{22}^2 & & x_{2n}^2 \\ 0 & 0 & \ddots & \\ 0 & 0 & & x_{nn}^n \\ 0 & 0 & & 0 \\ \cdots & & & \\ 0 & 0 & & 0 \end{pmatrix}$$

Householder Transformation Algorithm

- **Algorithm:** Given a $m \times n$ matrix X, convert it into an upper triangular matrix using Householder transformations.

- **Algorithm 12 (Householder Transformation)**

```
function X = householder(X)
[m, n] = size(X);
v = zeros(m, 1);
for j = 1 : n
        v(j : m, 1) = house(X(j : m, j));
        X(j : m, j : n) = rowhouse(X(j : m, j : n), v(j : m, 1));
end
```

Note

- 1. The composite Householder matrix $O = H_n H_{n-1} \ldots H_1$ is never calculated directly.
- 2. Not even the individual transformation matrices H_j' s are calculated explicitly.
- 3. Multiplication of $X^{(j-1)}$ by H_j does not affect the first $j-1$ columns and the first $j-1$ rows of $X^{(j-1)}$.
- 4. Application of j-th transformation (H_j) to the j-th column modifies the diagonal entry in that column and all the other entries below become zero.