STA 5106
Computational Methods in Statistics I

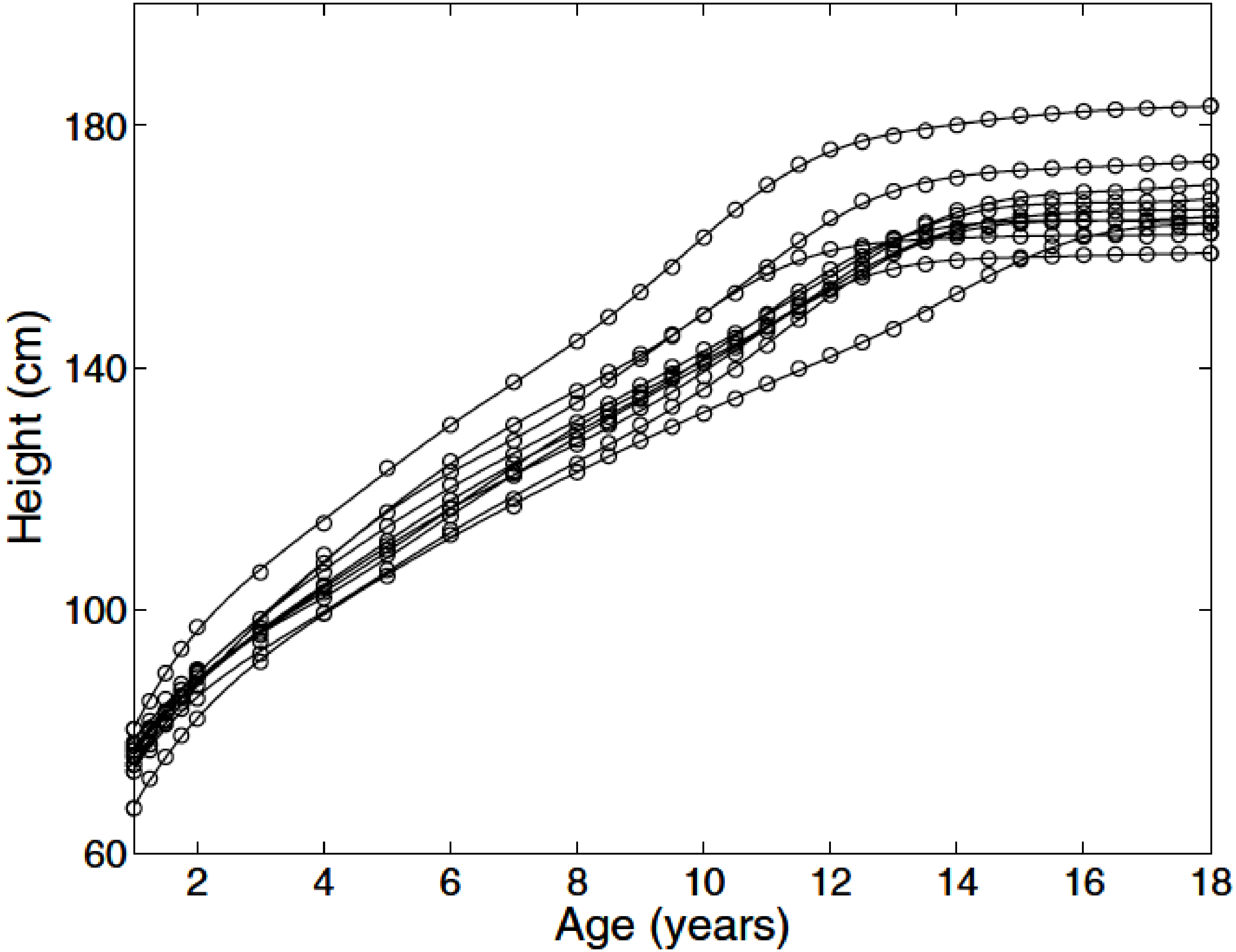
Department of Statistics
Florida State University

Class 24
November 26, 2019

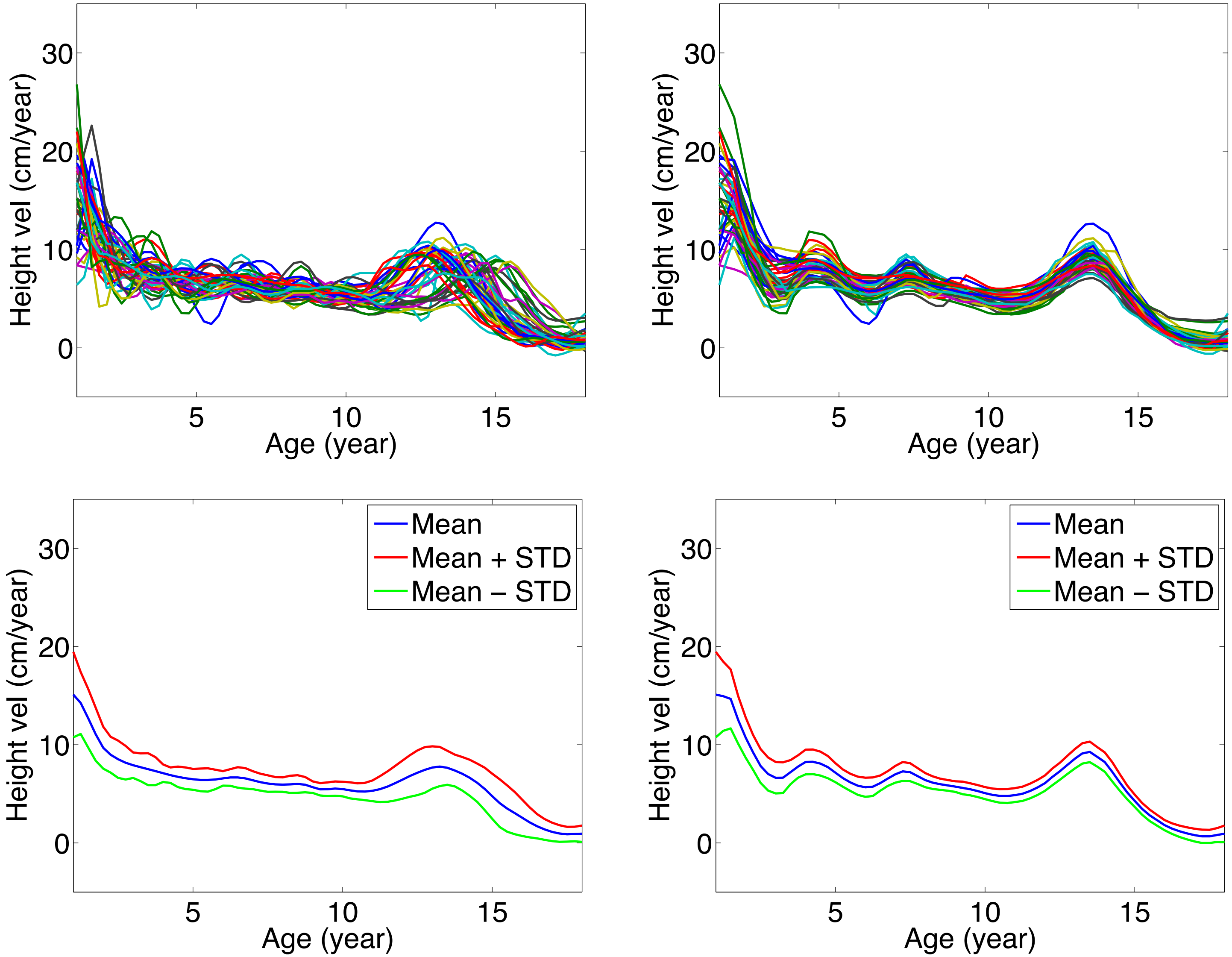
Special Topic 3

Function Registration/Alignment

Berkeley Growth Data

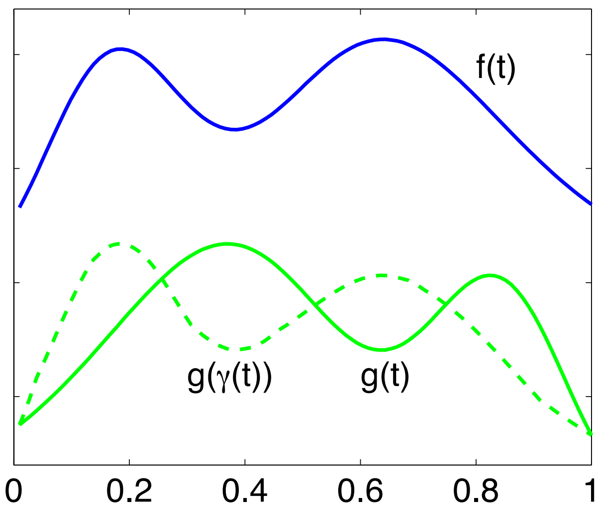


Berkeley Growth Data

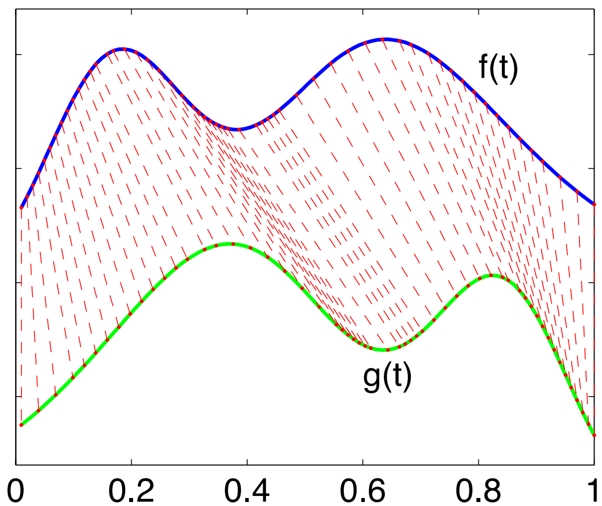


What is Function Registration?

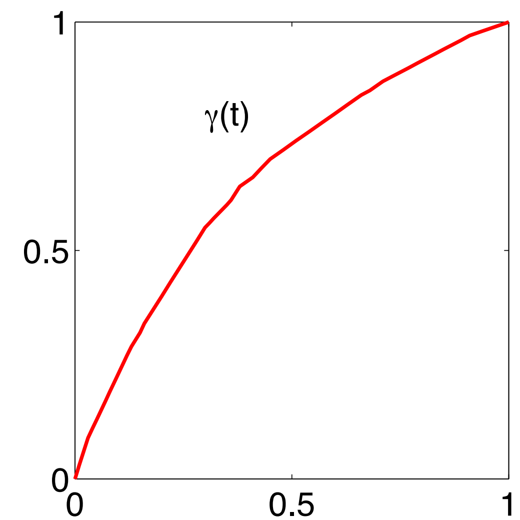
given functions and aligned function



***feature-to-feature* matching (registration)**



time warping function



Time warping: *A continuous*, *strictly increasing*, and *bijective* mapping:

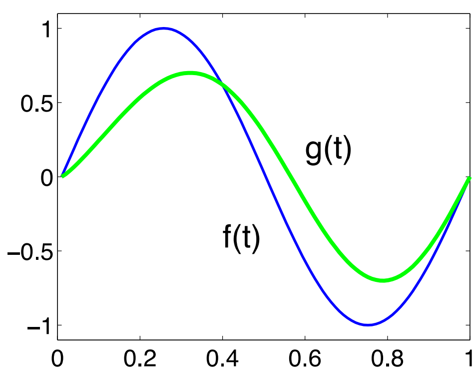
$$\Gamma = \{ \gamma : [0,\ 1] \rightarrow [0,\ 1] \mid \gamma(0) = 0,\ \gamma(1) = 1,\ 0 < \dot{\gamma}(t) < \infty \}.$$

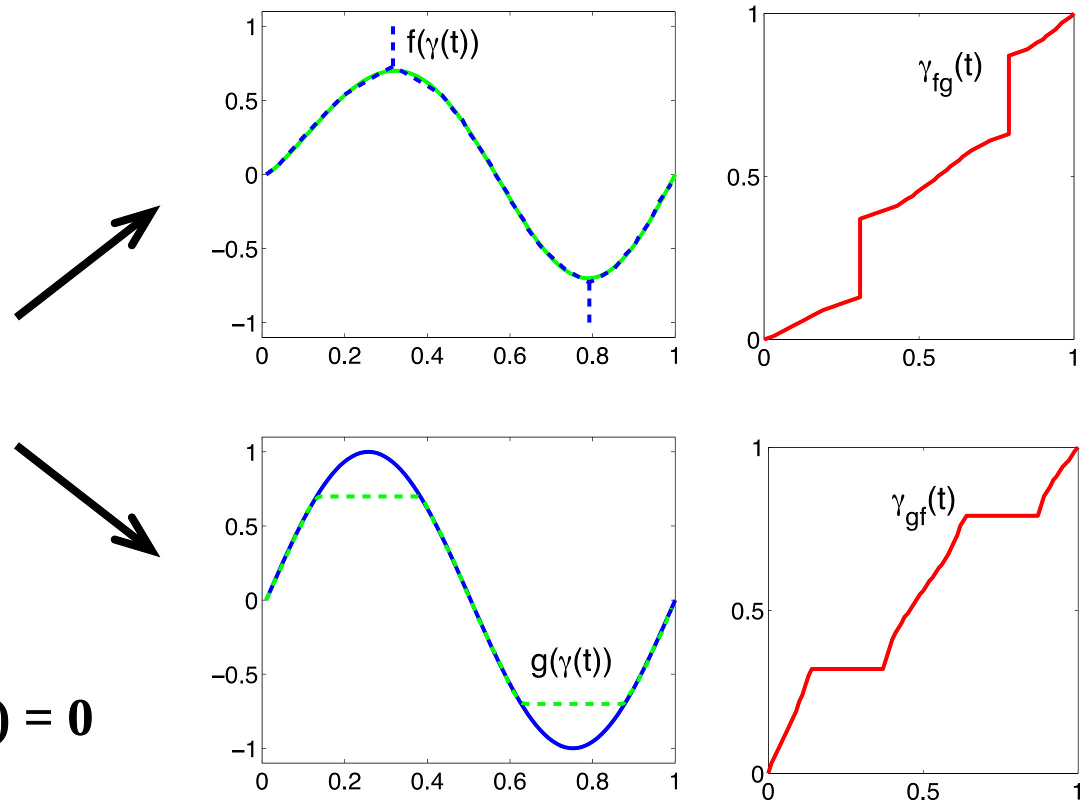
Naïve Matching

- A simple matching between *f* and *g*:

$$d(f,g) = \min_{\gamma} \| g - (f \circ \gamma) \|$$

- Main problems:





Note: 0 < $d(f, g) \neq d(g, f) = 0$

Function Space

- $f:[0,\ 1]\to \mathbf{R}$, an **absolutely continuous** function (denote the set of all such functions as **F**).

- **Square-Root Velocity Function (SRVF)** of f is defined as:

$$q = sign(\dot{f})\sqrt{|\dot{f}|} \in L^2$$

- Note: $q \in \mathbf{L}^2 \Leftrightarrow f \in \mathbf{F}$

- For any $\gamma \in \Gamma$, the SRVF of $f \circ \gamma$:

$$q_{f\circ\gamma} = sign(f \dot{\circ} \gamma)\sqrt{|f \dot{\circ} \gamma|} = sign(\dot{f}\circ\gamma)\sqrt{|\dot{f}\circ\gamma|\dot{\gamma}}$$

$$= (q\circ\gamma)\sqrt{\dot{\gamma}} \equiv (q,\gamma).$$

Equivalent Representations w.r.t. Time Warping

CDF: $f \xrightarrow{\gamma} (f \circ \gamma)$ **Amplitude-preserved**

$\downarrow$ $\downarrow$

SRVF: $q = sign(\dot{f})\,|\,\dot{f}\,| \xrightarrow{\gamma} (q \circ \gamma)\sqrt{\dot{\gamma}}$ **L^2-norm-preserved**

$$\|(q,\gamma)\|^2 = \int_0^1 (q(\gamma(t))\sqrt{\dot{\gamma}(t)})^2\,dt = \int_0^1 (q(\gamma(t))^2\,d(\gamma(t)) = \int_0^1 q(s)^2\,ds = \|q\|^2$$

For any two functions $f_1, f_2 \in \mathbf{F}$ and the corresponding SRVFs $q_1, q_2 \in \mathbf{L}^2$, we define the distance d as follows:

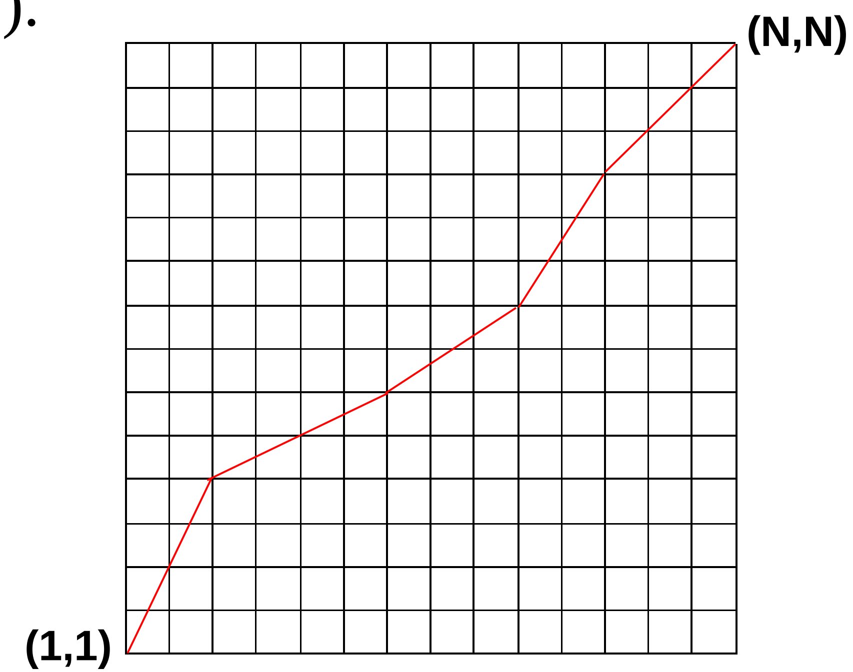
$$d(f_1, f_2) \equiv \min_{\gamma \in \Gamma} \left\| q_1 - (q_2, \gamma) \right\|.$$

Estimation of Optimal Warping Function

- We compute optimal warping $\gamma^* \equiv \underset{\gamma \in \Gamma}{\operatorname{argmin}} \left\| q_1 - (q_2, \gamma) \right\|$ using a **Dynamic Programming.**

- [0, 1] is discretized as N equally-sized time bins, and the warping function $\gamma(t)$ is approximated by a piecewise linear homomorphism with path from (1, 1) to (N, N).

- All segments in the path have positive slope.



Estimation of Optimal Warping Function

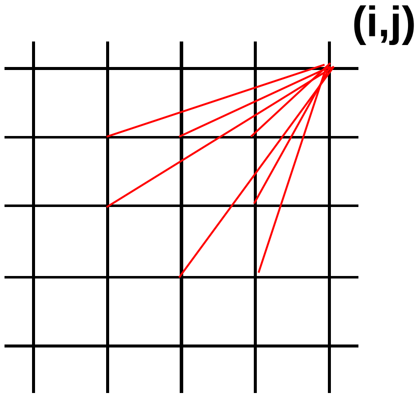
- Now, the problem is reduced to finding the allowable path that minimizes the energy

$$E = \int_0^1 (q_1(t) - q_2(\gamma(t))\sqrt{\dot{\gamma}(t)})^2 dt$$

- For nodes (k, l) and (i, j) where k < i and l < j, the energy in this segment is

$$E(k,l;i,j) = \int_{[k,i]} (q_1(t) - q_2(\gamma(t))\sqrt{\frac{j-l}{i-k}})^2 dt$$

- For efficiency, we restrict (k, l) to a subset $N_{i,j}$. A possible choice has seven nodes (the right plot).



Estimation of Optimal Warping Function

- Let H(i, j) denote the minimum energy from (1, 1) to (i, j). Then H(i, j) can be iteratively computed (computational efficiency: $O(N^2)$)

$$H(1,1) = 0$$

$$H(i, j) = E(\hat{k}, \hat{l}; i, j) + H(\hat{k}, \hat{l})$$

$$\text{where} \quad (\hat{k}, \hat{l}) = \arg\min_{(k,l)\in N_{ij}} (E(k,l;i,j) + N(k,l))$$

- Note: the algorithm works in a discretized space. The warping speed can only take the following values:

  1/3, 1/2, 2/3, 1, 3/2, 2, 3.