STA 5106: Homework Assignment #10

(Tuesday, November 14)
Due: Thursday, November 21

1. Use importance sampling to estimate the quantity:
$$\theta = \int_0^\infty x \frac{e^{-(y-x)^2/2} e^{-3x}}{Z} dx$$
where $Z = \int_0^\infty e^{-(y-x)^2/2} e^{-3x} dx$ and $y = 0.5$. Plot the convergence of your estimator versus the sample size. (Note: you can consider $3e^{-3x}$ as the density for the importance sampling.)

2. Use the technique of importance sampling via tilted densities to estimate the quantity
$$\boldsymbol{\theta = \Pr\{X > a\}}$$
where X is a standard normal random variable. Generate estimates for a = 1, 2, 3, 4, 5. Use sample sizes 1e3, 1e5, and 1e7 in each estimation.

3. Let X be an exponential random variable with mean $1/\lambda$. For constant $a > 0$, our goal is to use the tilted sampling to estimate the probability:
$$\theta = P\{X > a\} = \lambda \int_a^\infty e^{-\lambda x} dx.$$
For a scalar $t > 0$, define the tilted density as:
$$f_t(x) = \frac{e^{tx} f(x)}{M(t)}, \quad \text{where } f(x) = \lambda e^{-\lambda x}.$$
(a) Compute $M(t) = \lambda \int_0^\infty e^{tx} e^{-\lambda x} dx$.
(b) What should be the optimal amount of tilt t to estimate θ for a given a?
(c) Generate estimates for $\lambda = 5$ and a = 1, 2, 3, 4, 5. Use sample sizes 1e3, 1e5, and 1e7 in each estimation.

4, 5 (Optional): Use Python program to finish Problems 1 and 2.