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Homework #1

1. Prove that the eigenvalues of a symmetric matrix are real.

Assume the the contrary (i.e. that A is symmetric and that A has at least one non real eigenvalue).
By definition, eigenvalues must satisfy

$$Ax = \lambda x$$

By assumption, both sides of the above equation are complex. Therefore, we can now assert that their complex conjugates must also be equal.
Thus, denoting the complex conjugates by $\hat{\lambda}$ and $\hat{x}$, we can see that

$$A\hat{x} = \hat{\lambda}\hat{x}$$

By premultiplying both sides by $\hat{x}'$ and from the multiplication of complex numbers, we gain

$$\hat{x}'Ax - x'A\hat{x} = (\lambda - \hat{\lambda})\hat{x}'x$$

Since A is symmetric, the left side of the equation equals zero. This implies that the right side must also equal zero, but since $\hat{x}'x$ can never be equal to zero it must be that $\lambda = \hat{\lambda}$.
Hence, lambda equals its conjugate.
Thus lambda must be real.

2. Show that $\| y - X\hat{b} \|^2 = \sum_{i=n+1}^{m} | y_i^* |^2$

Let O be an orthogonal matrix such that $X^* = OX$, where X^* is an upper triangular matrix.
Further define $Y^* = OY$ and $\varepsilon^* = O\varepsilon$
Now, partition $X^*, Y^*, and\, \varepsilon^*$such that the regression equation can be rewritten as
$\begin{bmatrix} Y_1^* \\ Y_2^* \end{bmatrix} = \begin{bmatrix} X_1^* \\ 0 \end{bmatrix} b + \begin{bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \end{bmatrix}$, where X_1^* contains the first n rows of X^*
We now have that
$\| Y^* - X^*b \|^2 = \| Y_1^* - X_1^*b \|^2 + \| Y_2^* \|^2$
Now, we can choose b such that $X_1^*b = Y_1^*$
Thus, we now have that
$\| Y^* - X^*b \|^2 = \| Y_2^* \|^2$
Which, after a little algebra, gives
$\|Y_2^* \|^2 = [Y_2^*]' [Y_2^*] = \sum_{i=n+1}^{m} | y_i^* |^2$

3.

i. Show that any entry in O is between -1 and 1

Assume that O is an nxn orthogonal matrix
$\Rightarrow O^T O = I_n$
Since diagonal elements of I_n are all equal to one, the following must be true for all j
$\sum_{j=1}^{n} O_{ij}^2 = 1$
Thus, no single entry in any row i can exceed one in absolute value.
Since this statement is true for all i, then no entry in orthogonal matrix O can exceed 1 in absolute value.

ii. Show that if λ is an eigenvalue of O, then $\mid \lambda \mid = 1$

Since O is orthogonal, it satisfies
$AA^T = I_n$
$\Rightarrow A^{-1} = A^T$
Since the eigenvalues of a matrix are equal to the eigenvalues of that matrix's transpose, we know that the eigenvalues of A are equal to the eigenvalues of A^T
From here, we can see that the eigenvalues of A are equal to the eigenvalues of A^{-1}
We also know that the eigenvalues of an inverse matrix are equal to the reciprocals of that matrix itself.
Thus, the eigenvalues of A are equal to their own reciprocals.
This implies that the eigenvalues of A must be either 1 or -1
Thus, $\mid \lambda \mid = 1$

iii. Show that $det(O) = \pm 1$

Since O is orthogonal, it satisfies
$OO^T = I_n$
$det(OO^T) = det(I_n)$
$det(O)det(O^T) = 1$
$(det(O))^2 = 1$
$det(O) = \pm 1$

4.

a. Show that $H = H^T$

$H = I_n - 2\frac{vv^T}{v^Tv}$
$H^T = \left[I_n - 2\frac{vv^T}{v^Tv}\right]^T$
$H^T = I_n^T - 2\left[\frac{vv^T}{v^Tv}\right]^T$
$H^T = I_n - 2\frac{vv^T}{v^Tv}$
Hence, $H = H^T$

b. Show that$HH^T = I_n$

$HH^T = \left[I_n - 2\frac{vv^T}{v^Tv}\right]\left[I_n - 2\frac{vv^T}{v^Tv}\right]$
$HH^T = I_n - 2\frac{vv^T}{v^Tv} - 2\frac{vv^T}{v^Tv} + 4\left[\frac{vv^Tvv^T}{v^Tvv^Tv}\right]$
Since v^Tv is a scalar, it can be factored out of both the numerator and the denominator
$HH^T = I_n - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv}$
$HH^T = I_n$