STA 4102/5106: Homework Assignment #1

(Wednesday, August 27)
Due: Wednesday, September 3

1. Let A be an $n \times n$ real matrix. Prove that if A is symmetric, i.e. $A = A^T$, then all eigenvalues of A are real.

2. Through transformation with orthogonal matrix O, the problem $\hat{b} = \arg\min_b \| y - Xb \|^2$ is equivalent to $\hat{b} = \arg\min_b \| y^* - X^* b \|^2$ where y and y^* are in $\mathbf{R}^m$, X and X^* are in $\mathbf{R}^{m \times n}$ ($m \geq n$), and $y^* = Oy$ and $X^* = OX$. Let $y^* = [y_1^*, y_2^*, \cdots, y_m^*]^T$. Prove that the residual sum of square $\| y - X\hat{b} \|^2 = \sum_{i=n+1}^{m} | y_i^* |^2$.

3. Let O be an $n \times n$ orthogonal real matrix, i.e. $O^T O = I_n$, where I_n is an $n \times n$ identity matrix. Prove that

   i) Any entry in O is between -1 and 1.

   ii) If λ is an eigenvalue of O, then $|\lambda| = 1$.

   iii) $\det(O)$ is either 1 or -1.

4. Let H be an n x n householder matrix given by

$$H = I_n - 2\frac{vv^T}{v^T v},$$ for any non-zero n-length column vector v.

Show that $H = H^T$ and $HH^T = I_n$. In other words, H is a symmetric, orthogonal matrix.