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Portfolio construction techniques based on predicted risk, without expected returns, have become popular in the last decade. In terms of individual asset selection, minimum–variance and (more recently) maximum diversification objective functions have been explored, motivated in part by the cross-sectional equity risk anomaly first documented in Ang, Hodrick, Xing, and Zhang [2006]. Application of these objective functions to large (e.g., 1,000 stock) investable sets requires sophisticated estimation techniques for the risk model.

On the other end of the spectrum, the principal of risk parity, traditionally applied to small–set (e.g., 2 to 10) asset allocation decisions, has been proposed for large-set security selection applications. Unfortunately, most of the published research on these low-risk structures is based on standard unconstrained portfolio theory, matched with long-only simulations. The empirical results in such studies are specific to the investable set, time period, maximum weight constraints, and other portfolio limitations, as well as the risk model.

This article compares and contrasts risk-based portfolio construction techniques using long-only analytic solutions. We also provide a simulation of risk-based portfolios for large-cap U.S. stocks, using the CRSP database from 1968 to 2012. We perform this back-test using a single-index model, standard OLS risk estimates, and no maximum position

or other portfolio constraints, which leads to easily replicable results.

The concept of risk parity has evolved over time from the original concept that Bridgewater embedded in research in the 1990s. Initially, an asset allocation portfolio was said to be in parity when weights are proportional to asset-class inverse volatility. For example, if the equity subportfolio has a forecasted volatility of 15 percent and the fixed-income subportfolio has a volatility of just 5 percent, then a combined portfolio of 75 percent fixed income and 25 percent equity (i.e., three times as much fixed–income) is said to be in parity. This early definition of risk parity ignored correlations, even as the concept was applied to more than two asset classes.

Qian [2006] formalized a more complete definition that considers correlations, couching the property in terms of a risk budget where weights are adjusted so that each asset has the same contribution to portfolio risk. Maillard, Roncalli, and Teiletche [2010] call this an "equal risk contribution" portfolio, and analyzed properties of an unconstrained analytic solution. Lee's [2011] equivalent "portfolio beta" interpretation says that risk parity is achieved when weights are proportional to the inverse of their beta, with respect to the final portfolio. Anderson, Bianchi, and Goldberg [2012] analyze the historical track record of risk parity in an asset allocation context, while this article focuses on analytic

solutions in a security selection context. Risk parity is not a traditional mean–variance objective function and has been numerically difficult to implement on large-scale investable sets. Our new analytic solution allows investors to quickly calculate risk parity weights on any size investable set for a general linear risk model.

Chow, Hsu, Kalesnik, and Little [2011] provided a review of minimum-variance portfolios, which have been defined and analyzed from the start of modern portfolio theory in the 1960s. The objective function is minimization of ex ante portfolio risk, irrespective of forecasted returns, so that the minimum-variance portfolio lies on the left-most tip of the ex ante efficient frontier.

Minimum-variance portfolios equalize the marginal contributions of each asset to portfolio risk, in contrast to the risk parity portfolio, which equalizes each asset's total risk contribution. Thus, risk parity portfolios generally lie within the efficient frontier, rather than on it. As a special case of the popular mean–variance objective function, minimum-variance portfolios can be constructed using standard optimization software, given a specified asset covariance matrix. Although substantial analytic work exists on unconstrained portfolios, an analytic solution for long-only constrained minimum variance portfolios was first derived by Clarke, de Silva, and Thorley [2011].

Maximum diversification portfolios use an objective function recently introduced by Choueifaty and Coignard [2008] that maximizes the ratio of weighted-average asset volatilities to portfolio volatility. Like minimum variance, maximum diversification portfolios equalize each asset's marginal contributions, given a small change in the asset's weight.

However, the objective function is motivated by maximizing the portfolio Sharpe ratio, where expected asset returns are assumed to be proportional to asset risk. Thus, the maximum diversification portfolio is the tangent (highest Sharpe ratio) portfolio on the efficient frontier, if average asset returns increase proportionally with risk. On the other hand, if asset returns decrease with risk, the maximum diversification portfolio will be on the lower half of the traditional efficient frontier and clearly suboptimal. In this article, we provide an analytic solution for long-only, constrained maximum diversification portfolios, similar to Clarke, de Silva, and Thorley [2011].

In the first section, we review our analytic solutions for the three risk-based portfolios under the simplifying assumption of a single-factor risk model, with derivations for a more general multi-factor model provided in the technical appendix. We explain how the analytic solutions provide intuition for the comparative weight structure of risk parity, maximum diversification, and minimum-variance portfolios.

The equations show that asset weights decrease with both systematic and idiosyncratic risk for all three portfolios, although the form and impact of these two sources of risk vary. For example, minimum-variance weights are generally proportional to inverse variance (standard deviation squared), while maximum diversification and risk parity weights are generally proportional to inverse volatility (standard deviation). The analytic solutions reveal why long-only maximum diversification and minimum-variance portfolios employ a relatively small portion (e.g., 100 out of 1,000) of the investable set, in contrast to risk parity portfolios, which include all of the assets in the specified set.

In addition to intuition about portfolio composition, the long-only analytic solutions also provide simple numerical recipes for actual portfolio construction using large investable sets. In the second section, we report the performance results for all three risk-based portfolios, using the largest 1,000 U.S. stocks from 1968 to 2012. While numerous back-tests of minimum-variance and maximum-diversification portfolios have been published in recent years, this study includes the first simulation of a large (i.e., 1,000 asset) risk parity portfolio. The third section of this article compares and contrasts the asset weight distribution of the three risk-based portfolios, using the analytic solutions on a specific date: January 2013. The article concludes with a summary of various perspectives on the three risk-based portfolio construction techniques gleaned from the analytic solutions.

## ANALYTIC SOLUTIONS TO RISK-BASED PORTFOLIOS

We first review properties of the single-factor risk model and acquaint the reader with our mathematical notation. Under a single-factor asset covariance matrix, individual assets have only one source of common risk, resulting in the familiar decomposition of the $i$th asset's total risk into systematic and idiosyncratic components

$$\sigma_i^2 = \beta_i^2 \, \sigma_F^2 + \sigma_{\varepsilon,i}^2 \qquad (1)$$

In Equation (1), $\sigma_F$ is the risk of the common factor—the capitalization-weighted market portfolio, for example—and $\sigma_{\varepsilon,i}$ is the $i$th asset's idiosyncratic risk. The asset's exposure to the systematic risk factor, $\beta_i$, is by definition equal to the ratio of asset risk to factor risk, multiplied by the correlation coefficient between the asset and risk factor returns

$$\beta_i = \frac{\sigma_i}{\sigma_F}\rho_i \qquad (2)$$

The single-factor model also yields simple relationships for the pair-wise association between any two assets. Specifically, the covariance between two assets, $i$ and $j$, is $\sigma_{i,j} = \beta_i\beta_j\sigma_F^2$, and the correlation coefficient is the product of their correlations to the common factor, $\rho_{i,j} = \rho_i\rho_j$.

As shown in Clarke, de Silva, and Thorley [2011], and reviewed in the technical appendix, individual asset weights in the long-only minimum-variance portfolio under a single-factor risk model can be written as

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2}\left(1 - \frac{\beta_i}{\beta_L}\right) \quad \text{for} \quad \beta_i < \beta_L \text{ else} = 0 \qquad (3)$$

where $\beta_L$ is a long-only threshold beta and $\sigma_{MV}$ is the risk of the minimum-variance portfolio. According to Equation (3), individual assets are only included in the long-only portfolio if their factor beta is lower than the threshold beta, $\beta_L$, which for large investment sets can exclude a majority of the assets. High idiosyncratic risk in the denominator of the first term in Equation (3) lowers the asset weight, but cannot by itself drive an asset out of solution.

The second term, in Equation (3)'s parentheses, indicates that asset weights increase from zero with the only other asset-specific (i.e., subscripted by $i$) parameter, $\beta_i$, and that the highest weight is given to the lowest beta asset. In fact, in the absence of the cross-sectional variations due to idiosyncratic risk, minimum-variance portfolio weights fall on a kinked line when plotted against factor beta, visually similar to the expiration-date payoff profile to a put option. Specifically, weights are zero above the long-only threshold beta, $\beta_L$, and lie on a negatively sloped line for betas below the long-only threshold.

The technical appendix shows that weights in the maximum-diversification portfolio can be written in a similar form to minimum-variance weights, but are driven by the asset's correlation to the common risk factor:

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2}\frac{\sigma_i}{\sigma_A}\left(1 - \frac{\rho_i}{\rho_L}\right) \quad \text{for} \quad \rho_i < \rho_L \text{ else} = 0 \qquad (4)$$

where $\rho_L$ is a long-only threshold correlation and $\sigma_{MD}$ is the maximum diversification portfolio risk. According to Equation (4), individual assets are only included in the long-only maximum diversification portfolio if their correlation to the common risk factor is lower than the threshold correlation. As in Equation (3) for minimum-variance portfolio weights, high idiosyncratic risk in the denominator of the first term of Equation (4) lowers the asset weight but does not by itself drive an asset out of the maximum diversification portfolio.

Equation (4) also has a middle term: the ratio of the asset risk, $\sigma_i$, to the weighted average risk of the assets in the long-only portfolio, $\sigma_A$. This middle term, like the first term, sizes the weight rather than dictating the asset's inclusion in the maximum diversification portfolio. However, total asset risk in the numerator of the middle term does tend to offset the idiosyncratic risk squared, so that the maximum diversification portfolio weights are approximately proportional to the inverse of asset return standard deviation, as opposed to the inverse of variance in minimum-variance portfolio weights.[1] The result of this structure is that weights in the maximum diversification portfolio tend to be less concentrated than weights in the minimum-variance portfolio, because inverse standard deviation has less extreme values than does inverse variance. Finally, the last term in the parentheses of Equation (4) indicates that asset weights increase from zero with lower correlation, and that the highest weight is given to the lowest correlation asset. In the absence of the variations due to idiosyncratic risk, maximum diversification portfolio weights lie on a kinked line when plotted against correlation rather than beta.

The technical appendix provides a new and conceptually important solution for individual assets weights in a risk parity portfolio. The equation for risk parity weights is somewhat different in algebraic form than the two objective-function-based portfolios

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2} \left[ \left( \frac{\beta_i^2}{\gamma^2} + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2} - \frac{\beta_i}{\gamma} \right] \qquad (5)$$

where $\gamma$ is a constant across all assets, $\sigma_{RP}$ is the risk-parity portfolio risk, and $N$ is the number of assets. High idiosyncratic risk in the denominator of the first term of Equation (5) lowers the asset weight, but the similarity to Equations (3) and (4) ends there. As shown using partial derivatives in the technical appendix, asset weights decline monotonically with asset beta, but all assets have some positive weight, so that the risk-parity portfolio is long-only by definition.

For positive beta assets, the square-brackets term in Equation (5) must be positive, because the square root of squared beta over gamma plus some small (divided by $N$) positive value is greater than the beta over gamma subtracted at the end. On the other hand, if the asset beta is negative, then the weight is clearly positive and large. In the absence of variations due to idiosyncratic risk in Equation (5), risk-parity weights plotted against factor beta lie along a hyperbolic curve, somewhat like the pre-expiration payoff profile for a put option. In other words, risk-parity portfolio weights tend asymptotically towards zero with higher beta, and tend asymptotically towards a line with a negative slope for lower beta.

Equations (3), (4), and (5) for individual asset weights provide insights into risk-based portfolio construction, as well as formulas for determining weights without the need for an optimizer or complex numerical search routines. In Equations (3) and (4), assets need only be sorted by increasing beta and correlation, respectively, and then compared to the threshold parameters to determine which assets are in solution. The sum of raw weights can then be scaled to one to accommodate the portfolio constants outside of the parentheses.

Equation (5) for the risk parity portfolio is inherently different in that portfolio constants, which depend on the final assets' weights, are embedded within the parentheses, making the weights endogenously determined. However, a fast convergence routine can be designed using Equation (5) by initially assuming equal weights, and then iteratively calculating the portfolio parameters and assets weights until they converge. Unlike other numerical routines that have been proposed for risk parity weights, calculating asset weights using Equation (5) is almost instantaneous and can be applied

to large (i.e., 1,000 asset) investment sets in a simple Excel spreadsheet. Although we focus on the intuition of a single-factor risk model in the body of this article, the technical appendix provides a general $K$-factor solution to the risk-parity portfolio, as well as the other two risk-based structures.

## ONE THOUSAND STOCK EMPIRICAL EXAMPLE

In this section, we use Equations (3), (4), and (5) to construct risk-based and benchmark portfolios from the largest 1,000 common stocks in the CRSP database at the end of each month from 1968 to 2012. To keep the risk model estimation process as simple as possible, beta and idiosyncratic risk are based on standard OLS regressions for the prior 60 months, using the value-weighted CRSP index as the common factor. To translate historical betas into predicted betas, the entire set of 1,000 is adjusted ½ towards one each month. Similarly, the set of 1,000 log idiosyncratic risks is adjusted one third toward the average log idiosyncratic risk for that month.[2]

This article's results are generally robust to the specific shrinkage process, although we require some sort of Bayesian adjustment in order to avoid extreme predicted risk parameter values and thus extreme weights. A full 60 months of prior return data are required for a stock to be considered for the portfolios, but no other data scrubbing, stock selection, or maximum-weight constraints are employed.

Although we report full-period performance statistics, we note that the risk structure of the equity market and consequently the composition of the various risk-based portfolios has changed dramatically over the last 45 years. For example, the market portfolio's predicted risk, based on our simple 60-month rolling window, varied between a low of about 10 percent in the early 1960s to highs of about 20 percent in the late 1970s, shortly after the 1987 market crash, and again around the turn of the century. Idiosyncratic risk, on the other hand, has been fairly stable at about 25% to 30% for the average large-cap stock, except for several years around the turn of the century when the average idiosyncratic risk reached a high of about 40 percent.

Although the average predicted market beta naturally remained close to one over time, the spread of predicted betas varied substantially. For example, the lowest systematic-risk stocks of all 1,000 in any given month

typically had betas of 0.3 to 0.5, but the minimum predicted beta went into negative territory for several years in the mid-1990s. For the minimum-variance portfolio, the long-only threshold beta varied between 0.6 and 0.8 over time, meaning that only stocks with lower predicted beta values were admitted into solution each month.

The result is that the long-only minimum-variance portfolio averaged only about 62 stocks in solution, with a low of 30 and a high of 119. Similarly, the long-only maximum-diversification portfolio averaged about 82 stocks in solution, with a low of 50 and a high of 137. Although these counts may indicate higher portfolio concentration than some managers are comfortable with, they are largely a function of the aggressiveness of the shrinkage process used to translate historical security risk values into predicted values.

Exhibit 1 tabulates and Exhibit 2 plots the average performance from 1968 to 2012 of the three risk-based portfolios, as well as the market (value-weighted) portfolio. Based on DeMiguel, Garlappi, and Uppal [2009], we also include an equal-weighted 1,000 stock portfolio as a form of naive diversification. The returns throughout this study are reported in excess of the contemporaneous risk-free rate, measured by one-month Treasury bill returns from Ibbotson Associates.

Although high average returns are not the explicit goal of risk-based portfolio construction, all three risk-based portfolios outperform the excess market return of 5.3 percent. The risk-parity portfolio had an excess return of 7.4 percent, closely matching the return on the market-wide equal-weighted portfolio. The maximum diversification and minimum-variance portfolios had an excess return of 5.7 percent, similar to the capitalization-weighted market portfolio.

As is common practice, the realized returns of each asset class are shown as average annual (multiplication by 12) monthly excess returns, although investors with longer holding periods may be more interested in compound annual returns. Thus, Exhibit 2 also provides an "x" for the position of the compound annual return to each portfolio, calculated over the entire period of 1968 to 2012.[3] The risk-based portfolios' realized average-return performance is interesting, but not part of the portfolio construction process. Conclusions regarding the relative performance of the various portfolio construction techniques depend on the cross-sectional relationship between risk and return among the investable securities during the simulation period.

In order to make this point about historical performance in empirical simulations clear, Exhibit 3 shows the performance of risk-sorted quintile portfolios for the same 1,000 U.S. stocks over the same 1968 to 2012 period. Each quintile portfolio contains two hundred equally weighted stocks, assigned to portfolios each month based on their prior 60-month standard deviation of excess returns. Exhibit 3 is consistent with the low-risk anomaly first documented by Ang, Hodrick, Xing, and Zhang [2006] within the cross-section of U.S. stocks.[4] The lowest and second-to-lowest risk portfolios (i.e., quint 1 and quint 2) have the traditional risk return pattern, but the other three portfolios have an inverse relationship between risk and return.

This potentially perverse relationship between risk and reward is even worse when measured by compound returns, as shown with an "x" for each portfolio in Exhibit 3. The high ex ante risk quintile does in fact have the highest realized risk, at about 27.1 percent, but with a compound annual excess return of only 2.7

## E X H I B I T  1
**Performance of Risk-Based Portfolios from 1968 to 2012**

|  | Market (Value-Weighted) | Equal Weighted | Risk Parity | Maximum Diversification | Minimum Variance |
|---|---|---|---|---|---|
| Average Excess Return | 5.3% | 7.4% | 7.4% | 5.7% | 5.7% |
| Standard Deviation | 15.5% | 17.8% | 16.6% | 19.1% | 12.4% |
| Sharpe Ratio | 0.34 | 0.42 | 0.45 | 0.30 | 0.46 |
| Compound Return | 4.2% | 5.9% | 6.2% | 3.9% | 5.1% |
| Market Beta | 1.00 | 1.09 | 1.01 | 0.94 | 0.51 |
| Average Positions | 1000.0 | 1000.0 | 1000.0 | 81.9 | 61.8 |
| Effective N | 138.5 | 1000.0 | 934.1 | 46.3 | 35.7 |

**Risk-Based Portfolio Performance from 1968 to 2012**
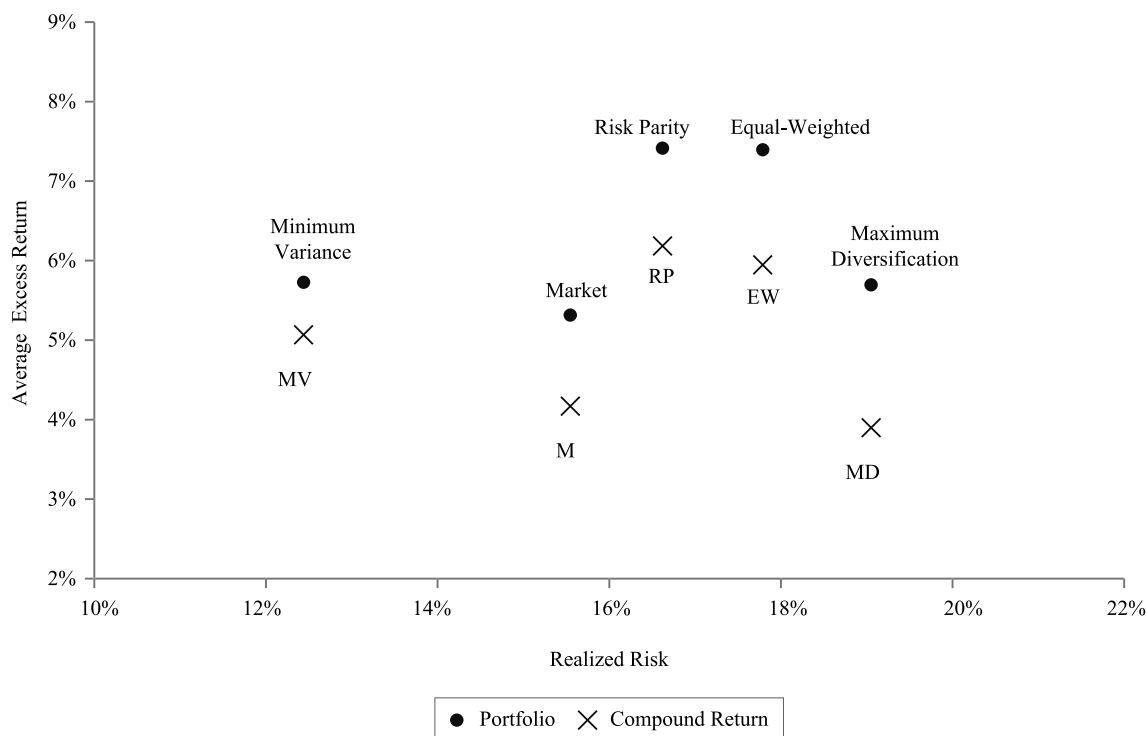


percent over the last 45 years. Although inconsistent with long-standing academic views of risk and return, Baker, Bradley, and Wurgler [2011] provided a reasonable explanation for the low-risk anomaly based on individual investor preferences for high-risk stocks, along with constraints associated with benchmarking that prohibit larger institutions from fully exploiting the anomaly. Frazzini and Pedersen [2013] also provided an explanation based on lack of leverage access for individual investors, who bid up the price (i.e., lower the subsequent return) of high-risk stocks.

For the three risk-based portfolios in Exhibit 1, the explicit objective of low risk is best achieved by the minimum-variance portfolio, with a realized risk of 12.4 percent compared to the market portfolio risk of 15.5 percent. As a result, the Sharpe ratio for 1968 to 2012 is highest for the minimum-variance portfolio, followed by the risk parity and then the maximum diversification portfolios.[5]

Exhibit 1 also reports each portfolio's market beta, the average number of positions over time, and the average effective N, as defined by Strongin, Petsch,

and Sharenow [2000]. The market betas (single time-series regression of realized portfolio returns) for the various portfolios are about 1.00, but with a notably lower beta of 0.51 for the minimum-variance portfolio, a key source of its low realized risk. Though the market portfolio includes all 1,000 securities each month, market-capitalization weighting leads to an average Effective N of only 138.5.

Alternatively, the average effective N of the risk-parity portfolio is 934.1, close to an equally weighted portfolio. The effective N of the maximum diversification and minimum-variance portfolios is even lower than the market portfolio, with average values of 46.3 and 35.7, respectively.

Some investors traditionally view more securities in solution as equivalent to better diversification and lower risk. However, the relatively low number of securities in the maximum diversification and minimum-variance portfolios illustrate that risk reduction is best achieved by selecting fewer, less correlated and less risky securities, rather than just adding more securities.

**Risk-Quintile Portfolio Performance from 1968 to 2012**
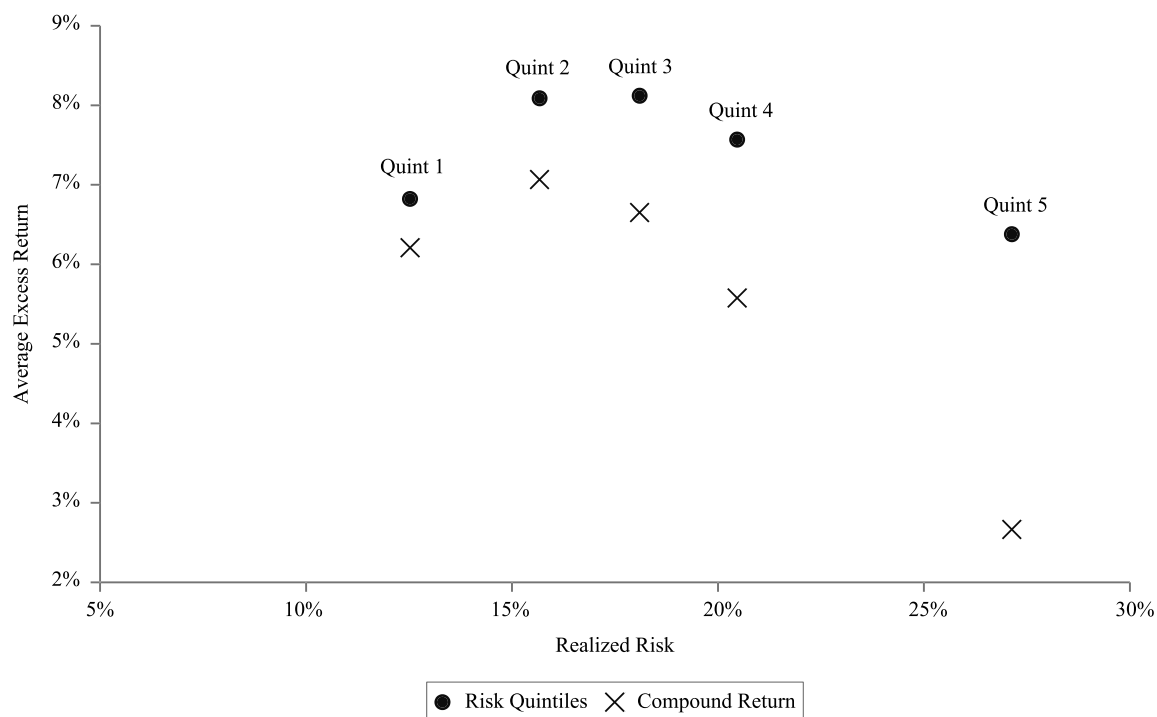


Of course, realized risk minimization depends on the accuracy of security risk forecasts, specifically the spread in systematic and idiosyncratic risk. Alternatively, if a manager does not want to infer much from historical differences in security risks, the predicted security risk parameters would be shrunk towards a common value, and the risk-based portfolios described in Equations (3), (4), and (5) would converge to equal-weighted portfolios.[6]

## WEIGHT DISTRIBUTIONS AS A FUNCTION OF RISK

We now turn to Equations (3), (4), and (5) for a closer examination of the weight structure of the three risk-based portfolios at a specific point in time: January 2013. We chose January 2013 because the weights employ the most recent five years of historical data: January 2008 to December 2012. To support the subsequent weight charts, Exhibit 4 plots the predicted beta and idiosyncratic risk of all 1,000 stocks as of January 2013. Beta ranges from about 0.5 to 2.9, and idiosyncratic risk ranges from about 15 percent to more than 81 percent.

The security-risk ranges for 2013 are high by historical standards, although not as high as the around the turn of the 21st century.

As is typical over time, the scatter-plot in Exhibit 4 shows that the two measures of asset risk are somewhat correlated and that common factor beta, as well as idiosyncratic risk, is positively skewed. The predicted common factor risk (i.e., market portfolio) in January 2013 is 19.5 percent, relatively high by historical standards. As we shall see, the impact of higher systematic risk leads to even more concentration in the minimum-variance and maximum-diversification portfolios than is typical over time.

Exhibit 5 shows asset weights plotted against beta for the three risk-based portfolios; minimum variance (• markers), maximum diversification (◊ markers), and risk parity (+ markers). Only 42 of the 1,000 investable stocks are in solution for the long-only minimum-variance portfolio in January 2013, ranging from a weight of about 8.9 percent to zero.

The stocks with positive minimum-variance weights all have betas below the long-only threshold beta of 0.7 for this period, in accordance with Equation (3). The

## E X H I B I T   4
**Beta and Idiosyncratic Risk for 1,000 U.S. Stocks in 2013**
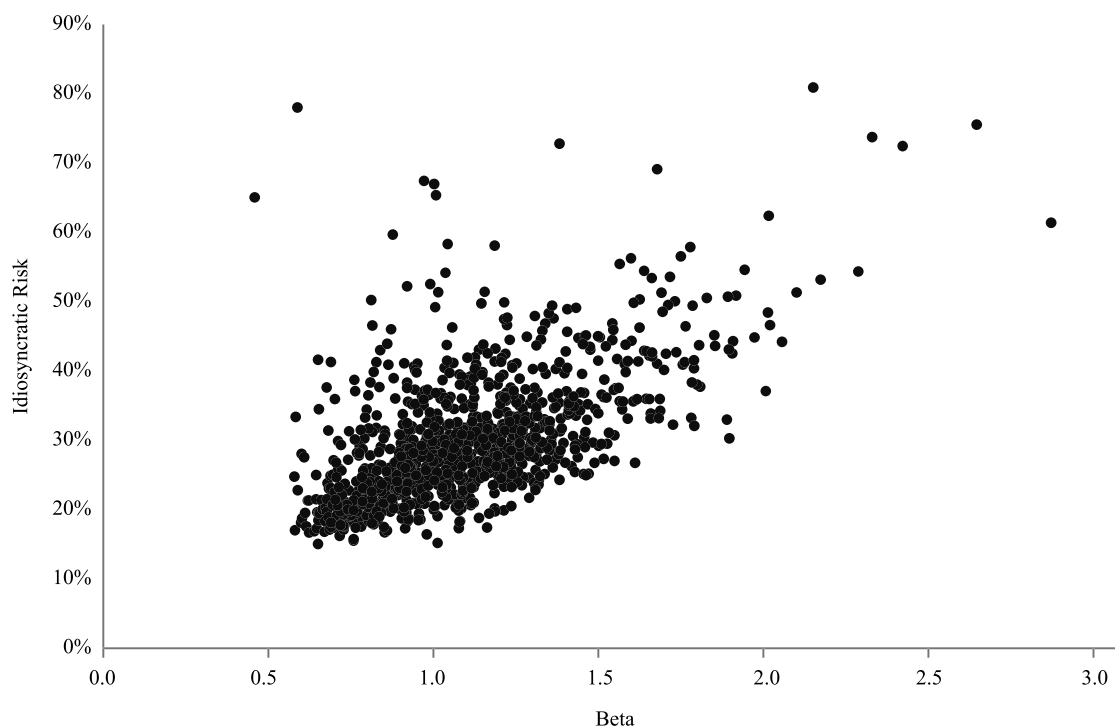


minimum-variance weights tend to fall on a negatively sloped line, also in accordance with Equation (3), although the correspondence is not perfect because of the impact of idiosyncratic risk. For example, the stock with the lowest beta of 0.4 in Exhibit 4 has a minimum-variance weight in Exhibit 5 of about 1.3 percent, much lower than the largest weight of 8.9 percent.

Notice that the impact of idiosyncratic risk on this minimum-variance portfolio weight is greater than for the same 0.4 beta stock in the maximum diversification portfolio, because stock-specific idiosyncratic risk squared in the denominator of the first term in Equation (3) is not offset by total stock risk, as in the numerator of the second term in Equation (4).

The long-only maximum diversification portfolio has 52 of the 1,000 investable stocks in January 2013, with weights ranging from about 6.0 percent to zero. The stocks with positive weights tend to have low betas, but several have high betas, including one maximum diversification weight of about 1.5 percent for a stock with a beta of 1.4. When the maximum diversification weights are plotted against factor correlation (not

shown) instead of factor beta, the positive weights are all associated with stocks that have correlations below the long-only threshold correlation of about 0.4 for this period, in accordance with Equation (4). For example, the stock with a beta of 1.4 mentioned above has the highest idiosyncratic risk in this set leading to a relatively low correlation to the common risk factor and thus a positive maximum diversification weight in Exhibit 5.

As shown using the right-hand scale in Exhibit 5, the risk parity portfolio weights are positive for all 1,000 securities, and the largest weight of about 0.22 percent is given to the stock with the lowest beta. The risk parity weights in Exhibit 5 are remarkably aligned with factor beta along a curve that has the hyperbolic functional form indicated by Equation (6).

Because the risk parity portfolio includes all 1,000 stocks, no single stock has an exceptionally large weight, and the idiosyncratic risk's impact on the weights is negligible. The lower concentration of the risk parity portfolio does not mean, however, that it has less risk ex ante than the other two portfolios. By design, the minimum-variance portfolio has the lowest ex ante risk

**Risk-Based Portfolio Asset Weights and Market Betas in 2013**
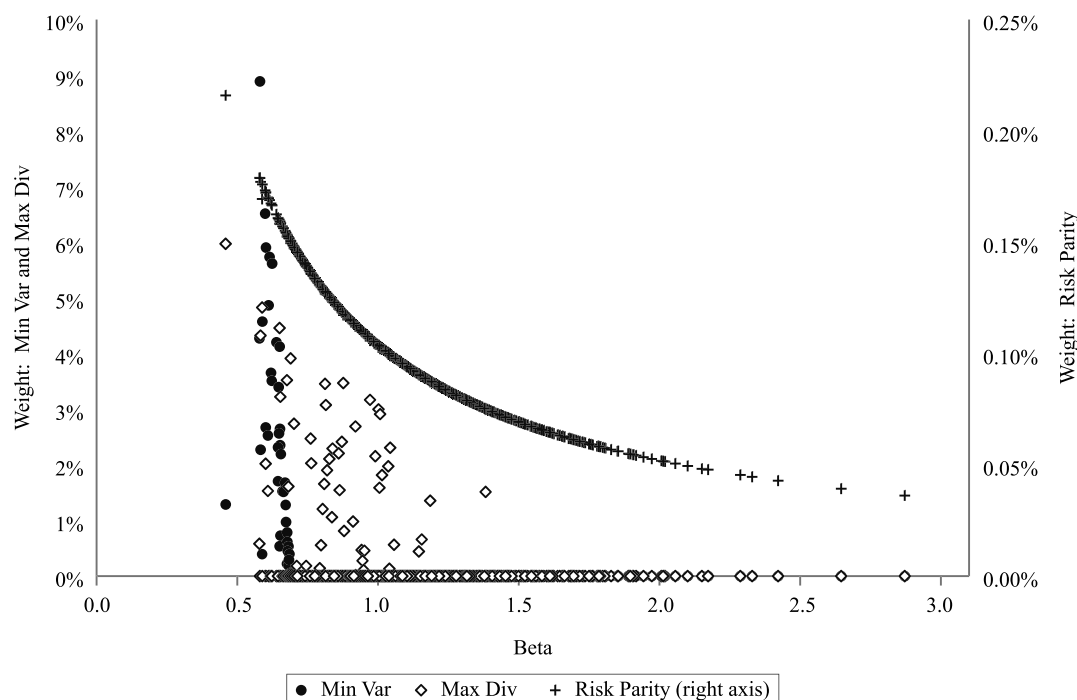


of all three (or any other long-only portfolio) and will generally have the lowest risk ex post depending on the risk model's accuracy.

Finally, as discussed in Lee [2011], the risk parity portfolio weights are perfectly aligned with the inverse of another asset beta: the beta of each stock to the final risk parity portfolio (not shown).

Charts of risk–based portfolio weights similar to Exhibit 5 for dates other than January 2013 illustrate the dynamic nature of market risk, and thus risk–based portfolio construction over time. In the mid–1990s, some of the largest 1,000 U.S. common stocks exhibited negative market factor betas, and are consequently given relatively large weights in each of the three risk-based portfolios, but particularly the risk parity portfolio. For example, the Best Buy Corporation, a potentially countercyclical stock, had a predicted beta of about -0.2 in 1995 and a weight in the risk parity portfolio of more than 1%.

As explained in the technical appendix, risk parity portfolios can only accommodate assets with a limited magnitude of negative common factor beta before the goal of parity across all assets becomes untenable. On the other hand, a key determinant of the concentration in the maximum diversification and minimum variance portfolios is the value of the long-only threshold parameters, which are in turn a function of the level of systematic risk, in addition to the range of betas.

For example, under the simplifying assumption of homogenous idiosyncratic risk (i.e., $\sigma_{\varepsilon,i}$ is the same for all stocks), Equation (A–11) in the technical appendix for the minimum-variance portfolio long-only threshold beta is

$$\beta_L = \frac{\dfrac{\sigma_\varepsilon^2}{\sigma_F^2} + \sum_{\beta_i < \beta_L} \beta_i^2}{\sum_{\beta_i < \beta_L} \beta_i} \qquad (6)$$

Holding the cross-sectional distribution of betas fixed, the long-only threshold beta increases with the ratio of idiosyncratic to systematic risk in Equation (6), increasing the number of securities in solution. Intuitively, higher idiosyncratic risk relative to systematic risk in the marketplace allows for more diversification

through a larger number of positions in the minimum-variance portfolio.

In addition, because the summations in Equation (6) increase with the number of securities, larger investable sets have comparatively lower threshold betas. As a result, the proportions of securities that are included in the long-only maximum diversification and minimum-variance solutions decrease with larger investable sets, for example 100 out of 1,000 investable assets, compared to perhaps 30 out of 100 investable assets.

## SUMMARY AND CONCLUSIONS

High market volatility, increased investor risk aversion, and the provocative findings of risk anomalies within the equity market have prompted a surge of empirical research on risk-based portfolio strategies. Simulations of minimum-variance portfolios over different markets, time periods, and constraint sets have proliferated, with additional interest in maximum diversification portfolios and the application of risk parity to security selection.

Most of these empirical studies confirm the finding that risk-based portfolio structures have historically done quite well, due to one or the other linear versions of the risk anomaly or other dynamic aspects of equity market history. Such studies, however, depend on the specification of the risk model, often proprietary or commercially based, and individual position limits, in addition to the usual vagaries of empirical work. Although some analytic perspectives on the properties of objective-function (i.e., maximum diversification and minimum-variance) portfolios have been published using standard unconstrained portfolio theory, implementation is inevitably in a long-only format, leading to portfolios that are materially different from their long/short counterparts. In addition, little analytic work on large-set, risk parity portfolio construction has been developed, beyond the definitional property that total versus marginal risk contributions define the weight structure.

This article provides analytic solutions to risk-parity and long-only constrained maximum diversification and minimum-variance portfolios. The optimal weight equations are not strictly closed form but provide helpful intuition on the construction of risk-based portfolios. Rather than simply supplying historical data to a constrained optimization routine, the analytic solutions let portfolio managers understand why any given invest-

able asset is included in the risk-based portfolio, as well as the reasons for the magnitude of its weight.

In addition, the single-factor solution allows for simple numerical calculations of long-only weights for large investable sets, yielding an easily replicable, empirical simulation. The application of these equations to the largest 1,000 U.S. stocks over several decades confirms the ex post superiority of minimum-variance portfolios in terms of minimizing risk, as well as the less direct benefit of a higher Sharpe ratio. In addition, the first large-sample empirical results on risk parity equity portfolios reported in this study show promise in terms of a high ex post Sharpe ratio.

The analytic solutions reveal several important aspects of risk-based portfolio construction. Some are intuitive properties elucidated by the algebraic forms of the optimal risk-based weights, while others are subtler.

First, long-only, objective-function-based portfolios exclude a large portion of the investable set, with the proportional exclusion becoming greater with the size of the investable set.

Second, all three risk-based portfolios have asset weights that decrease with both systematic and idiosyncratic risk, but systematic (i.e., non-diversifiable) risk is the dominant factor, especially for risk-parity portfolios.

Third, long-only thresholds on asset risk parameters vary with the ratio of systematic risk to average idiosyncratic risk over time. In particular, risk-based portfolios become more concentrated with higher systematic risk and lower idiosyncratic risk. Higher portfolio concentration does not, however, equate to higher risk ex ante, or even ex post in terms of the empirical track record of the single-factor model.

Fourth, negative beta assets have particularly large weights (intuitively due to hedging properties) in all three risk-based portfolios, although extreme negative beta assets cannot be accommodated under the standard definition of risk parity.

As noted in Scherer [2011], the Ang, Hodrick, Xing, and Zhang [2006] historical risk anomaly in stocks' cross-section can be characterized as asset exposure to systematic risk, idiosyncratic risk, or both. However, the market's intertemporal dynamics over time, as well as the "second moment" nature of risk (i.e., return variance) makes it unlikely that the anomaly can be completely reduced to a simple linear factor such as value or momentum.

In any event, the primary purpose of this study is to provide general analytic solutions to risk-based portfolios, not just another empirical back test. Further examination of the historical data in regards to transaction costs and turnover, for example in Li, Sullivan, and Garcia-Feijoo [2013], may shed more light on the exploitability of the low-risk anomaly. However, the emergence of new objective functions, specifically designed around the anomaly, may constitute a subtle form of data mining. In that regard, minimum-variance portfolios, specified from the beginning of modern portfolio theory in the 1960s, may be more robust. On the other hand, risk parity, which was conceptualized in the 1990s with broad asset allocation in mind, but now applied to large-set security portfolios, may also be less susceptible to an ex post bias.

# A P P E N D I X

The minimum-variance objective function is minimization of ex ante (i.e., estimated) portfolio risk

$$\sigma_P^2 = \mathrm{w'}\,\Omega\,\mathrm{w} \qquad (A\text{-}1)$$

where w is an *N*-by-1 vector of asset weights, and $\Omega$ is an *N*-by-*N* asset covariance matrix. One form of the well-known solution to this optimization problem is

$$\mathrm{w}_{MV} = \sigma_{MV}^2\,\Omega^{-1}\,\iota \qquad (A\text{-}2)$$

where $\iota$ is *N*-by-1 vector of ones. As a practical matter, $\sigma_{MV}^2 = 1/\iota'\,\Omega^{-1}\,\iota$ in Equation (A-2) is simply a scaling parameter that enforces the budget constraint that asset weights sum to one.

The maximum diversification objective is to maximize the diversification ratio

$$D_P = \frac{\mathrm{w'}\,\sigma}{\sqrt{\mathrm{w'}\,\Omega\,\mathrm{w}}} \qquad (A\text{-}3)$$

where $\sigma$ is an *N*-by-1 vector of asset volatilities, the square root of the diagonal terms of $\Omega$. Equation (A-3) has the form of a Sharpe ratio, where the asset volatility vector, $\sigma$, replaces the expected excess returns vector. Using the well-known solution to the tangent (i.e., maximum Sharpe ratio) portfolio with this substitution gives the optimal maximum diversification weight vector as

$$\mathrm{w}_{MD} = \left(\frac{\sigma_{MD}^2}{\sigma_A}\right)\Omega^{-1}\,\sigma \qquad (A\text{-}4)$$

where $\sigma_A$ is the weighted average asset risk. The key conceptual difference between the minimum-variance solution in Equation (A-2) and the maximum diversification solution in Equation (A-4) is not the scaling parameters, but the post multiplication of the inverse covariance matrix by the asset risk vector.

The risk-budgeting interpretation of risk parity is based on the restatement of Equation (A-1) as a double sum

$$\sigma_P^2 = \sum_{i=1}^{N} w_i \sum_{j=1}^{N} w_j\,\Omega_{i,j} \qquad (A\text{-}5)$$

where $\Omega_{i,j}$ are the elements in the asset covariance matrix $\Omega$. A portfolio is said to be in parity if the total (rather than marginal) risk contribution is the same for all assets

$$\frac{w_i \sum_{j=1}^{N} w_j\,\Omega_{i,j}}{\sigma_P^2} = \frac{1}{N} \qquad (A\text{-}6)$$

An equivalent "portfolio beta" interpretation is that risk parity is achieved when weights are equal to the inverse of their beta with respect to the final portfolio

$$w_i = \frac{1}{N\beta_{i,P}} \qquad (A\text{-}7)$$

Note that the asset beta with respect to the risk parity portfolio in Equation (A-7) is not the same as $\beta_i$, the notation we use for the beta of the asset with respect to the common risk factor.

In the body of this study, we focus on a single-factor risk model for the asset covariance matrix, a common simplifying assumption in portfolio theory. Using matrix notation, the *N*-by-*N* asset covariance matrix in a single-factor model is

$$\Omega = \beta\beta'\,\sigma_F^2 + \mathrm{Diag}(\sigma_\varepsilon^2) \qquad (A\text{-}8)$$

where $\beta$ is an *N*-by-1 vector of risk-factor loadings, $\sigma_F^2$ is the risk factor variance, and $\sigma_\varepsilon^2$ is an *N*-by-1 vector of idiosyncratic risks. Using the matrix inversion lemma, the inverse covariance matrix in the single-factor model is analytically solvable

$$\Omega^{-1} = \text{Diag}(1/\sigma^2) - \frac{(\beta/\sigma^2)(\beta/\sigma^2)'}{\frac{1}{\sigma_F^2} + (\beta/\sigma^2)'\beta} \qquad \text{(A-9)}$$

and can be substituted into the optimization solutions in Equations (A-2) and (A-4).

Specifically, the individual optimal weights in the long-only minimum-variance solution are given in Equation (3), where $\beta_L$ is a long-only threshold beta that cannot be exceeded in order for an asset to be in solution. The long-only threshold beta is a function of final portfolio risk estimates

$$\beta_L = \frac{\sigma_{MV}^2}{\beta_{MV}\,\sigma_F^2} \qquad \text{(A-10)}$$

where $\beta_{MV}$ is the risk–factor beta of the long-only minimum-variance portfolio. As a more practical matter, the threshold beta can be calculated from summations of individual asset risk statistics that come into solution,

$$\beta_L = \frac{\dfrac{1}{\sigma_F^2} + \sum_{\beta_i < \beta_L} \dfrac{\beta_i^2}{\sigma_{\varepsilon,i}^2}}{\sum_{\beta_i < \beta_L} \dfrac{\beta_i}{\sigma_{\varepsilon,i}^2}} \qquad \text{(A-11)}$$

Equation (A-11) is more practical than Equation (A-10), in the sense that assets can be sorted in order of ascending factor beta, and then compared to the summations until an individual asset beta exceeds the long-only threshold.

Together with the fact that portfolio risk is simply a scaling factor in Equation (3), numerical search routines are not required to find the set of assets and their optimal weights for long-only minimum-variance portfolios. Scherer [2011] provides analytic work that is similar in form to Equation (3), but for unconstrained long/short portfolios.

For a $K$-factor risk model, the general unconstrained solution to the minimum-variance portfolio is

$$w_{MV,i} = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2}\left(1 - \sum_{k=1}^{K} \frac{\beta_{k,i}}{\beta_k}\right) \qquad \text{(A-12)}$$

where $\beta_k$ is a portfolio (i.e., not asset-specific) parameter, calculated as

$$\beta_k = \frac{\sigma_{MV}^2}{\sum_{l=1}^{K} \beta_{MV,l}\,V_{k,l}} \qquad \text{(A-13)}$$

In Equation (A-13), $\beta_{MV,l}$ is the portfolio beta with respect to the $l^{\text{th}}$ risk factor, and $V_{k,l}$ is an element of the $K$-by-$K$ factor covariance matrix. The $K$-factor solution in Equation (A-12) also has a long-only constrained version, although the criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Substituting the inverse covariance matrix from Equation (A-9) into Equation (A-4) gives a relatively simple solution for individual weights in the maximum diversification portfolio, as shown in Equation (4), which includes the term $\rho_L$ as a long-only threshold correlation that cannot be exceeded for an asset to be in solution. The long-only threshold correlation can be described as a function of final portfolio risk estimates

$$\rho_L = \frac{\sigma_{MD}^2}{\sigma_A\,\beta_{MD}\,\sigma_F} \qquad \text{(A-14)}$$

where $\beta_{MD}$ is the factor beta and $\sigma_A$ is the average asset risk, respectively. As a more practical matter, the long-only threshold correlation can be calculated from summations of individual asset correlations that come into solution,

$$\rho_L = \frac{1 + \sum_{\rho_i < \rho_L} \dfrac{\rho_i^2}{1 - \rho_i^2}}{\sum_{\rho_i < \rho_L} \dfrac{\rho_i}{1 - \rho_i^2}} \qquad \text{(A-15)}$$

Equation (A-15) is more practical than Equation (A-14), in the sense that assets can be sorted in order of ascending factor correlation, and then compared to the summations until the individual asset correlation exceeds the long-only threshold. Note that assets sorted by factor correlation will be in somewhat different order than they would be if sorted by factor beta, due to differences in idiosyncratic risk. Together with the fact that final portfolio risk divided by average asset risk is simply a scaling factor in Equation (4), numerical search routines are not required to find the set of assets and their optimal weights for long-only maximum diversification portfolios. Carvalho, Lu, and Moulin [2012] provided analytic work that produces a form similar to Equation (5), but for unconstrained long/short portfolios.

For a *K*-factor risk model, the general unconstrained solution to the maximum-diversification portfolio is

$$w_{MD,i} = \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2} \frac{\sigma_i}{\sigma_A} \left(1 - \sum_{k=1}^{K} \frac{\rho_i}{\rho_k}\right) \qquad (A\text{-}16)$$

where $\rho_k$ is a portfolio (i.e., not asset-specific) parameter

$$\rho_k = \frac{\sigma_{MD}^2}{\sigma_A \sum_{l=1}^{K} \beta_{MD,l} V_{k,l}} \qquad (A\text{-}17)$$

In Equation (A-17), $\beta_{MD,l}$ is the portfolio beta with respect to the $l^{\text{th}}$ risk factor, and $V_{k,l}$ is an element of the *K*-by-*K* factor covariance matrix. The *K*-factor solution in Equation (A-16) has a long-only constrained version, although criterion for inclusion in the portfolio involves multiple sorts and is therefore more complicated to calculate.

Risk-parity portfolio weights under a single-factor risk model can be derived by substituting the individual covariance matrix terms of Equation (A-8) into Equation (A-6), and then applying the quadratic formula. Using the positive root in that formula gives Equation (5) in the body of the article, where

$$\gamma = \frac{2\,\sigma_{RP}^2}{\beta_{RP}\,\sigma_F^2} \qquad (A\text{-}18)$$

is a constant term across asset weights that includes $\beta_{RP}$, the risk parity portfolio's beta with respect to the risk factor.

Unlike the equations for individual weights in the minimum-variance and maximum diversification portfolios, Equation (5) cannot be used to perform a simple asset sort to calculate optimal weights, because final portfolio terms are embedded in the equation, not simply part of the scaling factor.

However, Equation (5) does suggest a quick numerical routine for even large investable sets. The numerical process starts with an equally weighted portfolio, and then iteratively calculates the portfolio parameters ($\beta_{RP}$ and $\sigma_{RP}$) and asset weights until the weights converge to one over *N* times their beta with respect to the final portfolio, $\beta_{i,RP}$, in accordance with Equation (A-7). We also note that the quadratic formula allows for a negative root that is real. However, the positive root specified in Equation (5) is the only root that maintains the budget constraint that the asset weights sum to one.

For a general K-factor risk model, the solution to the risk parity portfolio is

$$w_{RP,i} = \frac{\sigma_{RP}^2}{\sigma_{\varepsilon,i}^2} \left[ \left( \left(\sum_{k=1}^{K} \frac{\beta_{k,i}}{\gamma_k}\right)^2 + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2}\right)^{1/2} - \sum_{k=1}^{K} \frac{\beta_{k,i}}{\gamma_k} \right] \qquad (A\text{-}19)$$

where

$$\gamma_k = \frac{2\sigma_{RP}^2}{\sum_{l=1}^{K} \beta_{RP,l} V_{k,l}} \qquad (A\text{-}20)$$

The general *K*-factor risk parity portfolio is long-only by definition and can be used to inform a quick calculation routine similar to the one described for the single-factor model.

Under the single-factor model, the implications of differences in beta and idiosyncratic risk on the relative magnitude of the weights in the minimum-variance and maximum diversification portfolios is fairly evident from the algebraic form of Equations (3) and (4). The more complex form for risk parity portfolio weights in Equation (5) makes the impact of the different asset risk parameters less apparent, motivating formal calculus. The partial derivative of the risk parity asset weight in Equation (5) with respect to asset beta is

$$\frac{\partial w_{RP,i}}{\partial \beta_i} = \frac{-w_{RP,i}}{\gamma \left( \frac{\beta_i^2}{\gamma^2} + \frac{1}{N} \frac{\sigma_{\varepsilon,i}^2}{\sigma_{RP}^2} \right)^{1/2}} \qquad (A\text{-}21)$$

and thus always negative, because weights in the long-only solution are positive, as is the square-root term in the numerator of Equation (A-21).

The derivative is partial in the sense that Equation (A-21) applies to the magnitude of asset weights compared to one another in a risk-balanced set. A full derivative would be more complex, because a change in any single asset's risk parameter will change the risk parity weights for the entire set.

The functional form of Equation (5) (i.e., squared terms and square-roots) for risk parity portfolios indicates that weights asymptotically approach zero with high beta and increase with low beta, including negative betas (if any). Indeed, under the assumption of homogenous idiosyncratic risk, risk parity weights form the positive side of a non-rectangular hyperbola, centered at the origin, with linear asymptotes of the X-axis for larger beta stocks, and a line given by

$$w\left(asymptote\right) = -\frac{\beta_{RP}\,\sigma_F^2}{\sigma_\varepsilon^2} \qquad (A\text{-}22)$$

for lower beta stocks. As an indication of the eccentricity of the hyperbolic curve for weights, an asset with a factor beta of exactly zero would have a weight of $\sigma_{RP}/\sigma_\varepsilon \sqrt{N}$.

Assets with a large negative beta with respect the single risk factor, $\beta_i$, may have a negative beta with respect to the final portfolio, $\beta_{i,RP}$. However, negative weight assets were likely not envisioned by those who formalized the risk-parity condition. Specifically, an asset with a small positive $\beta_{i,RP}$ would have a relatively large positive weight, according to the definition in Equation (A-7). But a slightly different asset with a small negative $\beta_{i,RP}$ would have a large negative weight, a nonsensical result for an asset with the potential to hedge the single risk factor.

The lower limit for $\beta_i$ on any single asset that still allows for a risk-parity portfolio involving all investable assets is complex, although it can be shown that under the assumption of homogenous idiosyncratic risk, the lower limit is the arithmetic inverse of the hyperbolic asymptote in Equation (A-22). The number of iterations for convergence in the numerical process specified above for risk-parity portfolios increases for investable sets that have betas that approach that limit, although betas of that magnitude were not encountered anywhere in the 540 months (1968 to 2012) considered for the 1,000 largest U.S. stocks in the empirical part of this study.

## ENDNOTES

[1]The inverse volatility property of maximum diversification portfolio weights was also demonstrated by Maillard, Roncalli, and Teiletche [2010], given constant correlation in the asset covariance matrix, a simplifying assumption for risk-model analysis first introduced by Elton and Gruber [1973]. In fact, under the simplifying assumption of constant correlation, the maximum diversification portfolio is equivalent to the risk parity portfolio, as discussed in Choueifaty and Coignard [2008].

[2]The shrinkage of historical betas for purposes of risk prediction is similar to the Bloomberg rule of adjusting historical beta values ⅓ towards one. We use ½ instead of ⅓, based on observed values of the coefficient in cross-sectional regressions of 60-month realized betas on 60-month historical betas for 1,000 stocks. The choice of ⅓ shrinkage towards the mean for log idiosyncratic risk is also based on historical regression values. We shrink using logs because idiosyncratic risk is by definition a positively valued variable, and thus highly skewed.

[3]Excess compound returns are calculated as the compound total return, minus the compound Treasury bill return from 1968 to 2012. A common rule of thumb, depending on the distribution of returns, is that compound returns are equal to the arithmetic average return, minus half the return

variance. The compound returns reported in this study are fairly consistent with that rule.

[4]Note that the initial identification of the low-risk anomaly by Ang et al. [2006] used a short-horizon risk estimate (daily returns for the prior month), but quantitative portfolio managers generally attempt to exploit the anomaly using longer-horizon risk estimates (monthly returns for the past three to five years.)

[5]Consistent with Choueifaty and Coignard [2008], we find in sensitivity analysis that the maximum-diversification portfolio fares better using the top five hundred (i.e., S&P 500) rather than the top 1,000 (e.g., Russell 1000), although still not as well as the other two portfolios, similar to the findings of Linzmeier [2011]. We also note that correlations, which are critical to maximum diversification portfolio structure, may be better estimated with a multi-factor risk model.

[6]Quantitative portfolio management has a long tradition of shrinking forecasted or expected returns, using the ex ante information coefficient in the Grinold–Kahn framework or equilibrium-expected returns in the Black–Litterman approach, for example. As portfolio construction techniques that rely solely on risk parameters become more common, managers are learning to similarly shrink the cross-sectional spread of historical risk. A formal statistical approach using the Bayesian theory of Ledoit and Wolf [2004] varies around ½, but we choose to leave the shrinkage parameter at exactly ½ over time for ease of study replication.

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