Solutions

NTU math 109

Version 1.0

1. Generating function:

$$\begin{aligned}&(x^a + x^{a+1} + \cdots + x^b)^n \\ =&[x^a(1 + x + x^2 + \cdots + x^{b-a})]^n \\ =&[x^a(\frac{x^{b-a+1}-1}{x-1})]^n\end{aligned} \tag{1}$$

Answer †

$$[x^a(\frac{x^{b-a+1}-1}{x-1})]^n \tag{2}$$

2. **Answer** †

$$(i^j)^{(n^m)} \tag{3}$$

3.

$$\Rightarrow x_1 + x_2 + x_3 + x_4 < 8,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4 \tag{4}$$

Let

$$\begin{aligned}x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0 \\ \Rightarrow y_5 = x_5 - 1,\ y_5 \geq 0\end{aligned} \tag{5}$$

Then,

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + y_5 = 7,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4,\ y_5 \geq 0 \tag{6}$$

Answer †

$$\binom{5+7-1}{7} = \binom{11}{7} = 330 \tag{7}$$

4.

$$\begin{aligned}&\Rightarrow \alpha = 3 \\ &\Rightarrow \begin{cases} a_n^{(h)} &= c \times 3^n \\ a_n^{(p)} &= d \times n + e \end{cases}\end{aligned} \tag{8}$$

We have

$$\begin{aligned} & d \times n + e = 3 \times (d \times (n-1) + e) + n \\ \Rightarrow & \begin{cases} d & = -\frac{1}{2} \\ e & = -\frac{3}{4} \end{cases} \\ \Rightarrow & a_n = c \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \\ \Rightarrow & a_0 = 1 = c - \frac{3}{4} \\ \Rightarrow & c = \frac{7}{4} \\ \Rightarrow & a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \end{aligned} \tag{9}$$

Answer †

$$a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \tag{10}$$

5.

$$\Rightarrow \sum_{i=1}^{n} a_n x^n = 3 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \tag{11}$$

We have

$$\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \tag{12}$$

Then, we have

$$\begin{aligned} & \Rightarrow \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\ & \Rightarrow \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\ & \Rightarrow \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2} \end{aligned} \tag{13}$$

We have the new generating function

$$\begin{aligned} & A(x) - a_0 = 3x \times A(x) + \frac{x}{(1-x)^2} \\ \Rightarrow & A(x) = \frac{x^2 - x + 1}{(1-3x)(1-x)^2} \\ \Rightarrow & A(x) = \frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2} \end{aligned} \tag{14}$$

Answer †

$$\frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2} \tag{15}$$

6. **Answer** † Since

$$\begin{aligned} &\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ \Rightarrow &\binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{\lceil \frac{n}{2} \rceil} \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor} + \cdots + \binom{n}{n} < 1 + n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &\Rightarrow n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^n \\ &\Rightarrow \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \frac{2^n}{n} \end{aligned} \tag{17}$$

7. We have

$$\left[\begin{array}{ccccc|c} 16 & -8 & 4 & -2 & 1 & 150 \\ 1 & -1 & 1 & -1 & 1 & 16 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 18 \\ 16 & 8 & 4 & 2 & 1 & 166 \end{array}\right] \tag{18}$$

Answer † $a,\ b,\ c,\ d,\ e$ are

$$8,\ 1,\ 7,\ 0,\ 2 \tag{19}$$

8. We have

$$\boldsymbol{B}\boldsymbol{B}^\intercal = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$\boldsymbol{B}\boldsymbol{B}^\intercal$ is **symmetric**, so it's **diagonalizable** and $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda)$. We have characteristic polynomial

$$p_{\boldsymbol{B}\boldsymbol{B}^\intercal}(x) = -x^2(x-1)(x-2)(x-4) \tag{21}$$

Answer † The nullities of $\boldsymbol{B}\boldsymbol{B}^\intercal - \lambda \boldsymbol{I}$ for $\lambda = 0,\ 1,\ 2,\ 3,\ 4$ are

$$2,\ 1,\ 1,\ 0,\ 1 \tag{22}$$

since 3 is NOT its eigenvalue.

9. We have

$$\begin{cases} \forall\ x \in U,\ (\boldsymbol{B}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) \\ \forall\ x \in U^{\perp},\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}) \end{cases}$$
$$\Rightarrow \begin{cases} \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) = U \to \mathrm{RS}(\boldsymbol{B}-\boldsymbol{A}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}-\boldsymbol{A}) = 1 \\ \mathrm{N}(\boldsymbol{B}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}) = 3 \end{cases} \tag{23}$$

Let

$$\boldsymbol{B}-\boldsymbol{A} = \begin{bmatrix} \alpha \times (0 & 1 & 0 & -1) \\ \beta \times (0 & 1 & 0 & -1) \\ \gamma \times (0 & 1 & 0 & -1) \\ \delta \times (0 & 1 & 0 & -1) \end{bmatrix} \Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \tag{24}$$

We have

$$U^{\perp} = \left\{ \boldsymbol{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \tag{25}$$

Since $\boldsymbol{n} \in \mathrm{N}(\boldsymbol{B})$, $\boldsymbol{B}\boldsymbol{n} = \boldsymbol{0}$.

$$\begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \boldsymbol{0}$$
$$\Rightarrow \alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1 \tag{26}$$
$$\Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Answer † The numbers of $-2,\ -1,\ 0,\ 1,\ 2$ are

$$0,\ 0,\ 10,\ 4,\ 2 \tag{27}$$

10. $\boldsymbol{B} = \boldsymbol{A}^{+}$. We have characteristic polynomial

$$p_{\boldsymbol{A}^{\intercal}\boldsymbol{A}}(x) = x(x-\frac{1}{4})(x-\frac{1}{2})(x-1) \tag{28}$$

We have SVD of $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\intercal$.

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{V} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{29}$$

We have

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

Then, $\boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^\intercal$.

$$\boldsymbol{A}^+ = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{31}$$

Answer † The numbers of $-2, \ -1, \ 0, \ 1, \ 2$ are

$$1, \ 2, \ 16, \ 1, \ 0 \tag{32}$$

11. We have characteristic polynomial

$$p_{\boldsymbol{A}} = x(x-1)(x-3)^2 \tag{33}$$

Then, we have

$$\operatorname{gm}(3) = \operatorname{nullity}\left(\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2 \end{bmatrix}\right) = 1 \tag{34}$$

Then, we have Jordan form

$$J_{\boldsymbol{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \tag{35}$$

Answer † The numbers of $0,\ 1,\ 2,\ 3,\ 4$ are

$$12,\ 2,\ 0,\ 2,\ 0 \tag{36}$$

Solutions

NTU math 108

VERSION 1.0

1. **Answer** † 1, 3, 7, 9.

2. We have recurrence function

$$\begin{cases} a_n = 2 \times a_{n-2},\ n \geq 3 \\ a_1 = 2,\ a_2 = 2 \end{cases} \tag{1}$$

Then, we have

$$\begin{aligned} &\alpha^2 = 2 \\ &\Rightarrow \alpha = \pm\sqrt{2} \\ &\Rightarrow a_n = c \times (\sqrt{2})^n + d \times (-\sqrt{2})^n \end{aligned} \tag{2}$$

Then, we have

$$\begin{aligned} &\begin{cases} a_1 = 2 = \sqrt{2} \times c - \sqrt{2} \times d \\ a_2 = 2 = 2 \times c + 2 \times d \end{cases} \\ &\Rightarrow c = \frac{\sqrt{2}+2}{2\sqrt{2}},\ d = \frac{\sqrt{2}-2}{2\sqrt{2}} \end{aligned} \tag{3}$$

Answer †

$$a_n = \frac{\sqrt{2}+2}{2\sqrt{2}} \times (\sqrt{2})^n + \frac{\sqrt{2}-2}{2\sqrt{2}} \times (-\sqrt{2})^n \tag{4}$$

3. We have

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = 2 \times \binom{2n+2}{n+1} \tag{5}$$

Answer †

$$A = 2n+2,\ B = n+1 \tag{6}$$

4. We have

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n-(k-1)} = \binom{2n}{n+1} \tag{7}$$

Answer †

$$A = 2n,\ B = n + 1 \tag{8}$$

5. We have

$$\begin{aligned} &\alpha^2 = \alpha + 2 \\ \Rightarrow\ &\alpha = 2 \vee \alpha = -1 \\ \Rightarrow\ &a_n = c \times 2^n + d \times (-1)^n \end{aligned} \tag{9}$$

We have

$$\begin{aligned} &\begin{cases} a_0 = c + d \\ a_1 = 2 \times c - d \end{cases} \\ \Rightarrow\ &c = \frac{a_0 + a_1}{3},\ d = \frac{2 \times a_0 - a_1}{3} \\ \Rightarrow\ &a_n = \frac{2 \times a_0 - a_1}{3} \times (-1)^n + \frac{a_0 + a_1}{3} \times 2^n \end{aligned} \tag{10}$$

Answer †

$$A = \frac{2 \times a_0 - a_1}{3},\ B = \frac{a_0 + a_1}{3},\ X = 1,\ Y = 2 \tag{11}$$

6. We have

$$\begin{aligned} &x_1 + x_2 + \cdots + x_n = r,\ \forall\ x_i \geq n_i + 1,\ 1 \leq i \leq n \\ \Rightarrow\ &y_1 + y_2 + \cdots + y_n = r - ((\sum_{i=1}^{n} n_i) + n),\ \forall\ y_i \geq 0,\ 1 \leq i \leq n \end{aligned} \tag{12}$$

Answer †

$$\binom{n + r - (\sum_{i=1}^{n} n_i) - n - 1}{r - (\sum_{i=1}^{n} n_i) - n} \tag{13}$$

7. **Answer** †

$$\begin{aligned} &[(p \to q) \wedge \neg p] \to \neg q \\ \Longleftrightarrow\ &[(\neg p \vee q) \wedge \neg p] \to \neg q \\ \Longleftrightarrow\ &\neg[(\neg p \vee q) \wedge \neg p] \vee \neg q \\ \Longleftrightarrow\ &[(p \wedge \neg q) \vee p] \vee \neg q \\ \Longleftrightarrow\ &p \wedge (\neg q \vee p) \vee \neg q \\ \Longleftrightarrow\ &(p \vee \neg q) \wedge [(p \vee \neg q) \vee \neg q] \\ \Longleftrightarrow\ &(p \vee \neg q) \wedge [(p \vee \neg q) \vee \neg q] \\ \Longleftrightarrow\ &(p \vee \neg q) \end{aligned} \tag{14}$$

So, it's NOT tautology.

8. We have

- True. Let
$$\begin{aligned} S &= \{a,\ b\} \subset U \\ \Rightarrow \ \operatorname{span}(s) &= c_1 a + c_2 b \subset U \end{aligned} \tag{15}$$

- False. Counterexample:
$$\begin{aligned} R &= \{(1,\ 0),\ (0,\ 1),\ (0,\ 2)\} \\ \Rightarrow (1,\ 0) &\not\ni \operatorname{span}(R \setminus \{(1,\ 0)\}) \end{aligned} \tag{16}$$

- True.
- False. Counterexample: $\operatorname{span}(\mathbf{0}) = \emptyset$, but $\emptyset$ is NOT orthonormal.
- True.

Answer †
$$3 \tag{17}$$

9. We have
   - True.
   - True. $\boldsymbol{A}$ is invertible $\iff$ $\det(\boldsymbol{A}) \neq 0$ $\iff$ $\det(\boldsymbol{A}^{\mathsf{H}}) \neq 0$
   - True.
   - True. $\boldsymbol{A}$ is invertible, so $\operatorname{rank}(\boldsymbol{A}) = m = n = \operatorname{rank}(\boldsymbol{A}^{-1})$.
   - True. $\det(\boldsymbol{A}^{\mathsf{H}}) = \det(\overline{\boldsymbol{A}^{\mathsf{T}}}) = \overline{\det(\boldsymbol{A}^{\mathsf{T}})} = \overline{\det(\boldsymbol{A})}$

Answer †
$$5 \tag{18}$$

10. We have
   - True. $\mathbb{Q}^n$ is the direct sum of eigenspace of $\boldsymbol{A}$ $\iff$ there are n linearly independent eigenvectores of $\boldsymbol{A}$ $\iff$ $\boldsymbol{A}$ is diagonalizable
   - True.
   - False. $\boldsymbol{A}$ may NOT be split.
   - False. If the $\boldsymbol{A}$ is **real** and symmetric, all of its eigenvalues are always real. It can NOT be ensured if $\boldsymbol{A}$ is **complex**.
   - True.

Answer †

$$3 \tag{19}$$

11. We have inverse

$$\begin{bmatrix} \frac{529}{12167} & 0 & \frac{529}{12167} & \frac{529}{12167} \\ 0 & \frac{1587}{12167} & 0 & \frac{1058}{12167} \\ \frac{2116}{12167} & 0 & \frac{1587}{12167} & 0 \\ \frac{1058}{12167} & \frac{1058}{12167} & 0 & \frac{1587}{12167} \end{bmatrix} \tag{20}$$

Answer †

$$6 \tag{21}$$

12. **Answer** †

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

Solutions

NTU math 107

VERSION 1.0

1. We have new question

$$
\begin{aligned}
&(x_1 + x_2 + \cdots + x_n \leq H) - (x_1 + x_2 + \cdots + x_n < L),\ \forall\ x_i \geq 0,\ 1 \leq i \leq n \\
&(\text{Let } x_{n+1} = H - (x_1 + x_2 + \cdots + x_n),\ x_{n+1} \geq 0, \\
&y_{n+1} = L - (x_1 + x_2 + \cdots + x_n),\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\ x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\ - (x_1 + x_2 + \cdots + x_n + y_{n+1} = L,\ \forall\ x_i \geq 0,\ 1 \leq i \leq n,\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\ x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\ - (x_1 + x_2 + \cdots + x_n + z_{n+1} = L - 1,\ \forall\ x_i \geq 0,\ 1 \leq i \leq n,\ z_{n+1} \geq 0)
\end{aligned} \tag{1}
$$

   Answer †

$$
\binom{(n+1) + H - 1}{H} - \binom{(n+1) + (L-1) - 1}{L-1} \tag{2}
$$

2. We have

$$
\begin{aligned}
&\alpha^2 = 2 \times \alpha + 3 \\
\Rightarrow\ &\alpha = 3 \vee \alpha = -1 \\
\Rightarrow\ &a_n = c \times 3^n + d \times (-1)^n
\end{aligned} \tag{3}
$$

   Then, we have

$$
\begin{aligned}
&\begin{cases} a_0 = 1 = c + d \\ a_1 = 1 = 3 \times c - d \end{cases} \\
\Rightarrow\ &c = \frac{1}{2},\ d = \frac{1}{2}
\end{aligned} \tag{4}
$$

   Answer †

$$
a_n = \frac{1}{2} \times 3^n + \frac{1}{2} \times (-1)^n \tag{5}
$$

3. We have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)^2x^n\\ &\overset{\text{integral}}{=}\sum_{n=0}^{\infty}(n+1)x^{n+1}=x\sum_{n=0}^{\infty}(n+1)x\end{aligned}\tag{6}$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)x^n\\ &\overset{\text{integral}}{=}\sum_{n=0}^{\infty}x^{n+1}=\frac{x}{1-x}\end{aligned}\tag{7}$$

Then, we have

$$\begin{aligned}\frac{x}{1-x}&\overset{\text{derivative}}{=}\frac{1}{(1-x)^2}\\ \Rightarrow\sum_{n=0}^{\infty}(n+1)x^{n+1}&=\frac{x}{(1-x)^2}\end{aligned}\tag{8}$$

And, we have

$$\begin{aligned}\frac{x}{(1-x)^2}&\overset{\text{derivative}}{=}\frac{1+x}{(1-x)^3}\\ \Rightarrow\sum_{n=0}^{\infty}(n+1)^2x^n&=\frac{1+x}{(1-x)^3}\end{aligned}\tag{9}$$

Answer †

$$\frac{1+x}{(1-x)^3}\tag{10}$$

4. **Answer** †

$$2^{\binom{m}{2}}\tag{11}$$

5. **Answer** †

$$2^{\frac{n(n+1)}{2}},\ \binom{\binom{n}{2}}{m}\tag{12}$$

6. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x)=x^2-5\times x+4\tag{13}$$

Then, we have

$$\begin{aligned}f(\boldsymbol{A})&=\boldsymbol{A}^4-3\times\boldsymbol{A}^3-6\times\boldsymbol{A}^2+7\times\boldsymbol{A}+2\times\boldsymbol{I}\\ &=(\boldsymbol{A}^2+2\times\boldsymbol{A})(\boldsymbol{A}^2-5\times\boldsymbol{A}+4\times\boldsymbol{I})+(-\boldsymbol{A}+2\times\boldsymbol{I})\\ &=(-\boldsymbol{A}+2\times\boldsymbol{I})\end{aligned}\tag{14}$$

Answer †

$$\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix} \tag{15}$$

7. We have

$$\begin{aligned}
\det(\boldsymbol{A}+t\boldsymbol{I}) &= \begin{bmatrix} t & 0 & 0 & \cdots & a_0 \\ -1 & t & 0 & \cdots & a_1 \\ 0 & -1 & t & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}+t \end{bmatrix}_{n\times n} \\
&= t^{n-1}[(a_{n-1}+t)+\frac{1}{t}a_{n-2}+\frac{1}{t^2}a_{n-3}+\cdots+a_0] \\
&= t^n + t^{n-1}a_{n-1} + t^{n-2}a_{n-2} + \cdots + a_0 \\
&= t^n + \sum_{i=0}^{n-1} a_i t^i
\end{aligned} \tag{16}$$

Answer †

$$t^n + \sum_{i=0}^{n-1} a_i t^i \tag{17}$$

8. By Gram-Schmidt process, we have

$$\begin{aligned}
&u_1 = 1,\ ||u_1|| = \int_0^1 1\times 1dt = 1 \\
&u_2 = t - \frac{\int_0^1 1\times tdt}{1}\times 1 = t-\frac{1}{2},\ ||u_2|| = \int_0^1 (t-\frac{1}{2})^2dt = \frac{1}{12} \\
&u_3 = t^2 - \frac{\int_0^1 1\times t^2dt}{1}\times 1 - \frac{\int_0^1 (t-\frac{1}{2})\times t^2dt}{\frac{1}{12}}\times(t-\frac{1}{2}) = t^2-t+\frac{1}{6}, \\
&\qquad ||u_3|| = \int_0^1 (t^2-t+\frac{1}{6})^2dt = \frac{1}{180}
\end{aligned} \tag{18}$$

We have projection

$$\begin{aligned}
&\frac{\int_0^1 1\times t^3dt}{1}\times 1 + \frac{\int_0^1 (t-\frac{1}{2})\times t^3dt}{\frac{1}{12}}(t-\frac{1}{2}) + \frac{\int_0^1 (t^2-t+\frac{1}{6})\times t^3dt}{\frac{1}{180}}\times(t^2-t+\frac{1}{6}) \\
=&\frac{3}{2}\times t^2 - \frac{3}{5}\times t + \frac{1}{20}
\end{aligned} \tag{19}$$

Answer †

$$\frac{3}{2}\times t^2 - \frac{3}{5}\times t + \frac{1}{20} \tag{20}$$

9. **Answer** † We have

- False.
- False. We have $\det(\boldsymbol{A}^\intercal) = \det(-\boldsymbol{A}) \iff \det(\boldsymbol{A}) = (-1)^n \times \det(\boldsymbol{A})$. ONLY if the dimension of $\boldsymbol{A}$, i.e. n, is odd, then $\boldsymbol{A}$ is singular; otherwise, it's non-singular.
- False. $(\boldsymbol{A}+\boldsymbol{I})^n = (2^n - 1) \times \boldsymbol{A}$.
- True, since symmetric matrix is **orthogonally diagonalizable**, and it's also **diagonalizable**.
- True. Suppose

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} \tag{21}$$

Then, we have

$$\begin{aligned}
&\Rightarrow \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \\
&\Rightarrow \begin{cases} \boldsymbol{BP} + \boldsymbol{CR} = \boldsymbol{I} \\ \boldsymbol{BQ} + \boldsymbol{CS} = \boldsymbol{O} \\ \boldsymbol{DR} = \boldsymbol{O} \to \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{DS} = \boldsymbol{I} \to \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \begin{cases} \boldsymbol{P} = \boldsymbol{B}^{-1} \\ \boldsymbol{Q} = -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{O} & \boldsymbol{D}^{-1} \end{bmatrix}
\end{aligned} \tag{22}$$

10. **Answer** † We have

- True. We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{23}$$

And, we have

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\right) = 2 \tag{24}$$

is non-singular, so $\{u+v,\ v+w,\ w+u\}$ is linearly independent.

- True. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, so $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues.
- False. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, but the eigenvectors may differ.
- False, since if $m \neq n$, $\boldsymbol{B}^{\intercal}\boldsymbol{A}$ may NOT exist.
- False, since
$$(U+W)^{\perp} = U^{\perp} \cap W^{\perp} \tag{25}$$

Solutions

NTU math 106

Version 1.0

1. **Answer** †

$$\begin{bmatrix} 1 & 0 & 1 \\ r & 1 & r \\ 3 & r & 2 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$2 \tag{2}$$

Since $\boldsymbol{A}$ has eigenvalues 0 and 1, it's **idempotent**, $\boldsymbol{A}^2 = \boldsymbol{A}$.

3. Suppose

$$\boldsymbol{A} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

Then, we have projection

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\intercal \boldsymbol{A})^{-1} \boldsymbol{A}^\intercal \tag{4}$$

Answer †

$$\frac{1}{x^2 + y^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \tag{5}$$

4. Suppose

$$\begin{cases} \beta & = \{(1,\ 0),\ (0,\ 1)\} \\ \gamma & = \{\boldsymbol{u} = (1,\ 0),\ \boldsymbol{v} = (a,\ b)\} \end{cases} \tag{6}$$

Then, we have

$$[x]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ [y]_\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ [x]_\gamma = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},\ [y]_\gamma = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \tag{7}$$

Then ,we have transition matrix

$$\begin{aligned} [\boldsymbol{I}]_\gamma^\beta &= \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \\ \Rightarrow [x]_\beta &= \begin{bmatrix} s_1 + s_2 a \\ s_2 b \end{bmatrix}, \ [y]_\beta = \begin{bmatrix} t_1 + t_2 a \\ t_2 b \end{bmatrix} \end{aligned} \tag{8}$$

Then, we have

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &= (s_1 + s_2 a)(t_1 + t_2 a - t_2 b) + s_2 b \times (-t_1 - t_2 a + 4 \times t_2 b) \\ &= s_1 t_1 + s_2 t_2 \\ \Rightarrow a = b = \pm\frac{1}{\sqrt{3}} \end{aligned} \tag{9}$$

Answer †

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \vee \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} \end{bmatrix} \tag{10}$$

5. We have pseudo-inverse

$$\boldsymbol{A}^+ = (\boldsymbol{A}^\intercal \boldsymbol{A})^{-1} \boldsymbol{A}^\intercal \tag{11}$$

Answer †

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \tag{12}$$

6. **Answer** †

$$2^{m^2} - 2^{m^2-m} - 2^{m^2-m} \tag{13}$$

Since both reflexive and irreflexive are 2^{m^2-m}.

7. We have

$$\begin{aligned} &[(p \vee q) \wedge (\neg p \vee r)] \\ \iff &(q \wedge \neg p) \vee (p \wedge r) \vee (q \wedge r) \\ &(\text{Draw the Venn diagram}) \\ \iff &(p \wedge r) \vee (\neg p \wedge q) \end{aligned} \tag{14}$$

Answer †

$$(p \wedge r) \vee (\neg p \wedge q) \tag{15}$$

8. We have

$$\begin{aligned} &\alpha^2 = \alpha + 1 \\ &\Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2} \\ &\Rightarrow a_n = c \times (\frac{1+\sqrt{5}}{2})^n + d \times (\frac{1-\sqrt{5}}{2})^n \end{aligned} \tag{16}$$

And, we have

$$\begin{aligned} &\begin{cases} a_0 = c + d \\ a_1 = c \times (\frac{1+\sqrt{5}}{2}) + d \times (\frac{1-\sqrt{5}}{2}) \end{cases} \\ &\Rightarrow c = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}},\ d = \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \\ &\Rightarrow a_n = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}} \times (\frac{1+\sqrt{5}}{2})^n + \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \times (\frac{1-\sqrt{5}}{2})^n \end{aligned} \tag{17}$$

Answer †

$$A = 2 \times a_1 - (1-\sqrt{5})a_0,\ B = 1+\sqrt{5},\ C = (1+\sqrt{5})a_0 - 2 \times a_1,\ D = 1-\sqrt{5} \tag{18}$$

9. **Answer** † We have

$$\begin{aligned} &\gcd(n, n-1) \\ &= \gcd(n-1,\ 1) \\ &= 1 \end{aligned} \tag{19}$$

10. **Answer** † bipartite

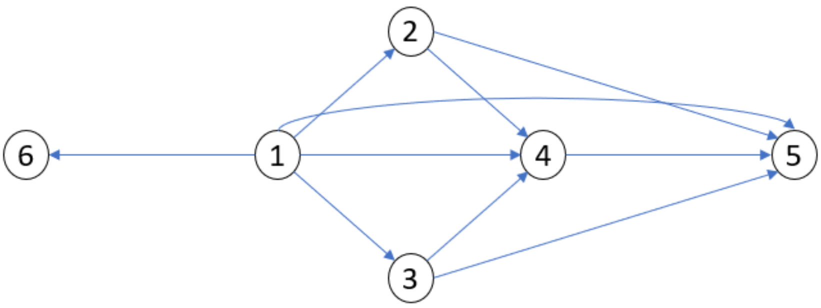
11. We have

$$\begin{aligned} &2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} + \binom{n}{\frac{n+1}{2}} + \cdots + \binom{n}{n} \\ &\Rightarrow \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} = \frac{2^n}{2} = 2^{n-1} \end{aligned} \tag{20}$$

Answer †

$$2^{n-1} \tag{21}$$

12. **Answer** †


2
6
1
4
5
3

Solutions

NTU math 105

Version 1.0

1. We have
   (a) True.
   (b) False.
   (c) True. $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA}) \to \mathrm{tr}(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{B}^{-1}\boldsymbol{BA}) = \mathrm{tr}(\boldsymbol{A})$.
   (d) False.
   (e) True.

   Answer †
$$ace \tag{1}$$

2. We have
   (a) True. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{BA})$
   (b) True.
   (c) True. $\det(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \det(\boldsymbol{B}^{-1}) \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \frac{1}{\det(\boldsymbol{B})} \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{A})$
   (d) True.
   (e) False.

   Answer †
$$abcd \tag{2}$$

3. We have
   (a) False.

(b) False, if $\boldsymbol{R} = \boldsymbol{O}$ is rectangular, but $\boldsymbol{R}^{\intercal}\boldsymbol{R} = \boldsymbol{O}$ is NOT positive definite.

(c) False, the orthogonal set contains $\mathbf{0}$, it's NOT linearly independent.

(d) True.

(e) True.

Answer †

$$de \tag{3}$$

4. We have

(a) False, since it does NOT contain $\mathbf{0}$.

(b) False, since it does NOT contain $\mathbf{0}$.

(c) False, since it does NOT contain $\mathbf{0}$.

(d) True.

(e) False, since it does NOT contain $\mathbf{0}$.

Answer †

$$d \tag{4}$$

5. Suppose

$$f(x) = 1 - x^k = (1 - x)(1 + x + x^2 + \cdots + x^{k-1}) \tag{5}$$

Then, we have

$$f(-\boldsymbol{N}) = \boldsymbol{I} + \boldsymbol{N}^k = (\boldsymbol{I} + \boldsymbol{N})(\boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1}) = \boldsymbol{I} \tag{6}$$

Answer †

$$(\boldsymbol{I} + \boldsymbol{N})^{-1} = \boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1} \tag{7}$$

6. **Answer** †

$$\frac{1}{4} \times x^2 + \frac{1}{9} \times y^2 = 1 \tag{8}$$

7. We have characteristic polynomial

$$p_{\boldsymbol{I}+\boldsymbol{A}}(x) = (x - 2)(x - 7) \tag{9}$$

Then, we have eigenspaces

$$\begin{cases} V(2) &= \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\ V(7) &= \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \end{cases} \tag{10}$$

Answer †

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{11}$$

Since the eigenvectors of $(\boldsymbol{I}+\boldsymbol{A})$ are the same as $(\boldsymbol{I}+\boldsymbol{A})^{100}$.

8. Obviously, 1 is an eigenvalue, which $\mathrm{gm}(1) = n-1$. Then, we have

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n}(1+x_i) = n + \sum_{i=1}^{n} x_i \tag{12}$$

Then, we have the n-th eigenvalue

$$n + (\sum_{i=1}^{n} x_i) - (n-1) \times 1 = 1 + \sum_{i=1}^{n} x_i \tag{13}$$

Answer †

$$\det(\boldsymbol{A}) = 1^{n-1} \times (1 + \sum_{i=1}^{n} x_i) = 1 + \sum_{i=1}^{n} x_i \tag{14}$$

9. We have new problem

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 < 8,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4 \\ & (x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0) \\ \Rightarrow\ & x_1 + x_2 + x_3 + x_4 + y_5 = 8 - 1,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4,\ y_5 \geq 0 \end{aligned} \tag{15}$$

Answer †

$$\binom{5 + (8-1) - 1}{8-1} = 330 \tag{16}$$

10. **Answer** †

$$(1,\ 2,\ 6) \circ (3,\ 5) \circ (4,\ 8) \circ (7) \tag{17}$$

11. We have

$$\Rightarrow \alpha = 2$$
$$\Rightarrow \begin{cases} a_n^{(h)} = c \times 2^n \\ a_n^{(p)} = d \times n + e \end{cases} \tag{18}$$

Then, we have

$$\begin{aligned} d \times n + e &= 2 \times (d \times (n-1) + e) + n \\ \Rightarrow d &= -1,\ e = -2 \\ \Rightarrow a_n &= c \times 2^n - n - 2 \end{aligned} \tag{19}$$

Then, we have

$$\begin{aligned} a_0 &= 4 = c - 0 - 2 \\ \Rightarrow c &= 6 \end{aligned} \tag{20}$$

Answer †

$$a_n = 6 \times 2^n - n - 2 \tag{21}$$

12.

$$\Rightarrow \sum_{i=1}^{n} a_n x^n = 2 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \tag{22}$$

We have

$$\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \tag{23}$$

Then, we have

$$\begin{aligned} &\Rightarrow \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\ &\Rightarrow \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\ &\Rightarrow \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2} \end{aligned} \tag{24}$$

We have the new generating function

$$\begin{aligned} A(x) - a_0 &= 2x \times A(x) + \frac{x}{(1-x)^2} \\ \Rightarrow A(x) &= \frac{4 \times x^2 - 7 \times x + 4}{(1-2x)(1-x)^2} \\ \Rightarrow A(x) &= 6 \times \frac{1}{1-2x} - 2 \times \frac{1}{1-x} - \frac{x}{(1-x)^2} \\ \Rightarrow A(x) &= 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - (\frac{1}{1-x} + \frac{x}{(1-x)^2}) \\ \Rightarrow A(x) &= 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \end{aligned} \tag{25}$$

Answer †

$$6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \tag{26}$$

13. We have

$$(1 + x + x^2 + \cdots)(1 + (x^2) + (x^2)^2 + \cdots)(1 + (x^3) + (x^3)^2 + \cdots)\cdots \tag{27}$$

Since each number can be repeated.

Answer †

$$\prod_{i=1}^{n} \frac{1}{1-x^i} \tag{28}$$

14. **Answer** †

$$2^{2^{m-1}} \tag{29}$$

Since the base means 0 and 1 two values of the codomain, and the index means that if we know the 01-sequence then we know the opposite.

15. We have

$$n \leq 2 \times i + 1 \tag{30}$$

Answer †

$$i \geq \lfloor \frac{n-1}{2} \rfloor \tag{31}$$

Solutions

NTU math 104

Version 1.0

1. We have

$$\begin{aligned} & p \to q \\ \iff & \neg p \vee q \end{aligned} \tag{1}$$

   (a) True.

   (b) False, since

$$\begin{aligned} & \neg p \to \neg q \\ \iff & p \vee \neg q \neq \neg p \vee q \end{aligned} \tag{2}$$

   (c) False, since

$$\begin{aligned} & q \to p \\ \iff & \neg q \vee p \neq \neg p \vee q \end{aligned} \tag{3}$$

   (d) True, since

$$\begin{aligned} & \neg q \to \neg p \\ \iff & q \vee \neg p = \neg p \vee q \end{aligned} \tag{4}$$

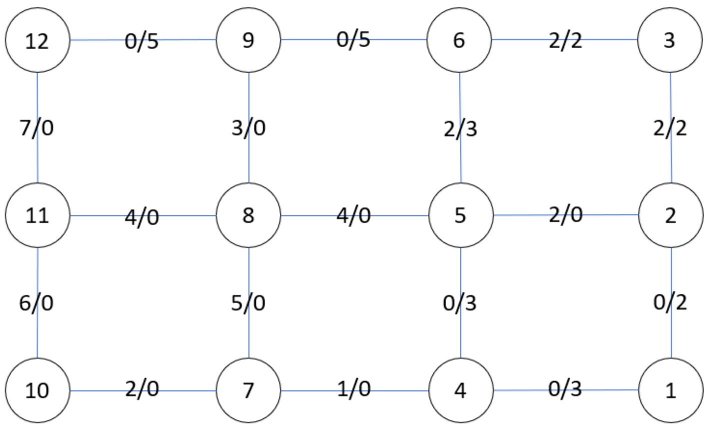
   (e) False, since

$$\begin{aligned} & \neg q \to p \\ \iff & q \vee p \neq \neg p \vee q \end{aligned} \tag{5}$$

   Answer †

$$ad \tag{6}$$

2. We have



Answer †

$$5 \tag{7}$$

3. We have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^n \tag{8}$$

Answer †

$$3^n \tag{9}$$

4. We have

$$120 = 2^3 \times 3^1 \times 5^1 \tag{10}$$

Answer †

$$\Phi(120) = 120 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 32 \tag{11}$$

5. Suppose

$$\begin{aligned} y_1 &= x_1 - 1 \\ y_2 &= x_2 - x_1 \\ y_3 &= x_3 - x_2 \\ &\vdots \\ y_n &= x_n - x_{n-1} \\ y_{n+1} &= r - x_n \end{aligned} \tag{12}$$

We have

$$\begin{aligned} &\begin{cases} y_i \geq 1, \ \forall\ 1 \leq i \leq n \\ x_1 \geq 1, \ x_2 \geq 2, \ \cdots, \ x_{n-1} \geq (n-1) \end{cases} \\ \Rightarrow\ & y_{n+1} = r - x_n = x_1 + x_2 + \cdots + x_{n-1} \geq \frac{n(n-1)}{2} \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned} & y_1 + 2\times y_2 + \cdots + n\times y_n + (n+1)\times y_{n+1} \\ &= (x_1 - 1) + 2\times(x_2 - x_1) + 3\times(x_3 - x_2) + \cdots + n\times(x_n - x_{n-1}) \\ &+ (n+1)\times(r - x_n) \\ &= -1 - x_1 - x_2 - x_3 - \cdots - x_n + (n+1)\times r \\ &= nr - 1 \end{aligned} \tag{14}$$

Then, we have new generating function

$$\begin{aligned} G(x) =& (1 + x + x^2 + \cdots)(x^2 + x^4 + x^6 + \cdots)(x^3 + x^6 + x^9 + \cdots)\cdots \\ & (x^n + x^{2n} + x^{3n} + \cdots)(x^{(n+1)\frac{n(n-1)}{2}} + \cdots) \\ =& \frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}} \end{aligned} \tag{15}$$

Answer † Coefficient of x^{nr-1} of

$$\frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}} \tag{16}$$

6. We have

$$\begin{aligned} &\Rightarrow \alpha = 2 \\ &\Rightarrow \begin{cases} a_n^{(h)} = c\times 2^n \\ a_n^{(p)} = d\times 3^n \end{cases} \end{aligned} \tag{17}$$

Then, we have

$$\begin{aligned} & d\times 3^n = 2\times d\times 3^{n-1} + 3^{n-1} \\ &\Rightarrow d = 1 \\ &\Rightarrow a_n = c\times 2^n + 3^n \end{aligned} \tag{18}$$

Then, we have

$$\begin{aligned} & a_0 = 2 = c + 1 \\ &\Rightarrow c = 1 \end{aligned} \tag{19}$$

Answer †

$$a_n = 2^n + 3^n \tag{20}$$

7. We have

$$\mathrm{tr}(\boldsymbol{XY}) = \mathrm{tr}((\boldsymbol{XY})^{\mathsf{H}}) = \mathrm{tr}(\boldsymbol{Y}^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}}) = \mathrm{tr}(\overline{\boldsymbol{Y}^{\mathsf{T}}\boldsymbol{X}^{\mathsf{T}}}) = \mathrm{tr}(\overline{(\boldsymbol{XY})^{\mathsf{T}}}) = \mathrm{tr}(\overline{\boldsymbol{XY}}) \tag{21}$$

Answer †

$$a \tag{22}$$

8. We have

$$\boldsymbol{A} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \tag{23}$$

We have rank $\boldsymbol{A} = 3$. Then, we have

$$\begin{aligned} &\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) - 5 \le \operatorname{rank}(\boldsymbol{AB}) \\ \Rightarrow\ & 3 + \operatorname{rank}(\boldsymbol{B}) - 5 \le 0 \end{aligned} \tag{24}$$

Answer †

$$\operatorname{rank}(\boldsymbol{B}) \le 2 \tag{25}$$

9. **Answer** † Sum of eigenvalues equals to the trace.

$$2 + 2 + 2 + 2 = 8 \tag{26}$$

10. **Answer** † The problem is **WRONG**, since $\{\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \boldsymbol{A}_3,\ \boldsymbol{B}_1\}$ is linearly independent.

11. We have

$$[f]_\beta = \begin{bmatrix} f(\beta_1,\beta_1) & f(\beta_2,\beta_1) & f(\beta_3,\beta_1) & f(\beta_4,\beta_1) \\ f(\beta_1,\beta_2) & f(\beta_2,\beta_2) & f(\beta_3,\beta_2) & f(\beta_4,\beta_2) \\ f(\beta_1,\beta_3) & f(\beta_2,\beta_3) & f(\beta_3,\beta_3) & f(\beta_4,\beta_3) \\ f(\beta_1,\beta_4) & f(\beta_2,\beta_4) & f(\beta_3,\beta_4) & f(\beta_4,\beta_4) \end{bmatrix} \tag{27}$$

Answer †

$$[f]_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

Solutions

NTU math 103

VERSION 1.0

1. **Answer** † There are 3 As, so we first **permute** other 4 characters, and then **insert** 3 As in the 5 spaces.
$$\frac{4!}{2!} \times \binom{5}{3} \tag{1}$$

2. We have
$$\begin{aligned} x_1 + x_2 + \cdots + x_n = r,\ \forall\ x_i > 0,\ 1 \le i \le n \\ \Rightarrow y_1 + y_2 + \cdots + y_n = r - n,\ \forall\ y_i \ge 0,\ 1 \le i \le n \end{aligned} \tag{2}$$

   Answer †
$$\binom{n + (r-n) - 1}{r-n} \tag{3}$$

3. **Answer** †
$$(2^2)^{(2^m)} = 4^{(2^m)} \tag{4}$$

4. We have
$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{1}{i}x^n &= x + (1+\frac{1}{2})x^2 + (1+\frac{1}{2}+\frac{1}{3})x^3 + \cdots + (1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})x^n + \cdots \\ &= 1 \times (x + x^2 + x^3 + \cdots + x^n + \cdots) + \frac{1}{2} \times (x^2 + x^3 + \cdots + x^n + \cdots) + \cdots \\ &+ \frac{1}{n} \times (x^n + x^{n+1} + \cdots) + \cdots \\ &= \frac{x}{1-x} + \frac{1}{2} \times \frac{x^2}{1-x} + \cdots + \frac{1}{n} \times \frac{x^n}{1-x} + \cdots \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1-x}x^n = \frac{1}{1-x}\sum_{n=1}^{\infty}\frac{1}{n}x^n \end{aligned} \tag{5}$$

And, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}x^n \overset{\text{derivative}}{=} \sum_{n=1}^{\infty}x^{n-1} = \frac{1}{1-x}\\ \Rightarrow\ &\frac{1}{1-x} \overset{\text{integral}}{=} -\ln(1-x)\end{aligned} \tag{6}$$

Then, we have

$$\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{1}{n}x^n = \frac{-\ln(1-x)}{1-x} \tag{7}$$

Answer †

$$\frac{-\ln(1-x)}{1-x} \tag{8}$$

5. We have

$$\begin{aligned}&\Rightarrow \alpha^2 = \alpha + 2\\ &\Rightarrow \alpha = 2 \vee \alpha = -1\\ &\Rightarrow a_n = c \times 2^n + d \times (-1)^n\end{aligned} \tag{9}$$

And, we have

$$\begin{aligned}&\begin{cases} a_0 = 0 = c + d\\ a_1 = 1 = 2 \times c - d\end{cases}\\ \Rightarrow &\begin{cases} c = \frac{1}{3}\\ d = -\frac{1}{3}\end{cases}\end{aligned} \tag{10}$$

Answer †

$$a_n = \frac{1}{3} \times 2^n - \frac{1}{3} \times (-1)^n \tag{11}$$

6. **Answer** †

$$cfjgda \tag{12}$$

7. **Answer** †

(a) If $S = \emptyset$, $\text{span}(S) = \{\mathbf{0}\} \subseteq V$.
Otherwise, if $S \neq \emptyset$, $\mathbf{0} \in \text{span}(S)$, and $\forall\ \boldsymbol{x},\ \boldsymbol{y} \in \text{span}(S)$, let

$$\begin{aligned}&\begin{cases} \text{span}(S) & = \text{span}\{v_1,\ v_2, \cdots,\ v_n\}\\ \boldsymbol{x} & = a_1v_1 + a_2v_2 + \cdots + a_nv_n\\ \boldsymbol{y} & = b_1v_1 + b_2v_2 + \cdots + b_nv_n\end{cases}\\ &\Rightarrow \forall\ \alpha,\ \beta \in \mathbb{R},\ \alpha\boldsymbol{x} + \beta\boldsymbol{y} =\\ &\quad (\alpha a_1 + \beta b_1)v_1 + (\alpha a_2 + \beta b_2)v_2 + \cdots + (\alpha a_n + \beta b_n)v_n \in \text{span}(S)\\ &\Rightarrow\ \text{span}(S) \subseteq V\end{aligned} \tag{13}$$

(b)

$$\begin{aligned} &S \subseteq U,\ \forall\ \boldsymbol{x} = \{x_1,\ x_2, \cdots,\ x_n\} \in S \\ \Rightarrow\ &\text{span}(S) = \{\alpha x_1 + \alpha x_2 + \cdots + \alpha x_n\} \subseteq U \end{aligned} \tag{14}$$

(c) Suppose

$$\exists\ T \subseteq V,\ \text{s.t. } T \subseteq U \tag{15}$$

And, we have

$$\begin{aligned} &S \subseteq \text{span}(S),\ \text{span}(S) \subseteq V \\ \Rightarrow\ &T \subseteq \text{span}(S) \end{aligned} \tag{16}$$

And, we have

$$\begin{aligned} &S \subseteq T,\ \text{span}(S) \subseteq T \\ \Rightarrow\ &T = \text{span}(S) \end{aligned} \tag{17}$$

8.

(a) **Answer** †

$$\text{nullity}(T) + \text{rank}(T) = \dim(V) \tag{18}$$

(b) We have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{\text{rref}}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{19}$$

Answer †

$$4 \times (2,\ 0,\ 1) + 1 \times (2,\ 1,\ -1) - 1 \times (2,\ -1,\ 0) = (8,\ 2,\ 3) \tag{20}$$

(c) **Answer** †

$$7 \times 4 = 28 \tag{21}$$

(d) **Answer** †

$$0,\ 1,\ 2,\ 3,\ 4,\ 5 \tag{22}$$

Since U and V are **distinct**, $U = V = W$ does NOT exist.

9. **Answer** † We have $\boldsymbol{A}^2 = \boldsymbol{I}$, so

$$\begin{aligned} \boldsymbol{A}^{-100} &= (\boldsymbol{A}^2)^{-50} = \boldsymbol{I} \\ \boldsymbol{A}^{101} &= (\boldsymbol{A}^2)^{50} \times \boldsymbol{A} = \boldsymbol{A} \end{aligned} \tag{23}$$

10. Find the minimal solution. We have

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{24}$$

Then, we have

$$\begin{aligned} &(\boldsymbol{A}\boldsymbol{A}^{\mathsf{H}})\boldsymbol{u} = \boldsymbol{b} \\ \Rightarrow &\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix} \\ \Rightarrow &\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \end{aligned} \tag{25}$$

Then, we have

$$\boldsymbol{A}^{\mathsf{H}}\boldsymbol{u} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{26}$$

Answer †

$$\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{27}$$

11. **Answer** †

$$-1, \ 1, \ 2, \ 3 \tag{28}$$

Solutions

NTU math 102

Version 1.0

1. **Answer** † **WRONG**. Counterexample:

$$\begin{aligned} A &= \{1,\ 2,\ 3\} \\ R &= \{(1,\ 2),\ (2,\ 1),\ (1,\ 1),\ (2,\ 2)\} \end{aligned} \tag{1}$$

   R is symmetric and transitive, but R is NOT reflexive.

2. **Answer** † Suppose

$$\begin{aligned} &\Rightarrow \begin{cases} b_n &= n\text{-th character is } 0 \\ c_n &= n\text{-th character is } 1, \text{ and } (n-1)\text{-th character is } 0 \vee 2 \\ d_n &= n\text{-th character is } 2, \text{ and } (n-1)\text{-th character is } 0 \vee 1 \end{cases} \\ &\Rightarrow a_n = b_n + c_n + d_n \\ &\left(\begin{cases} b_n &= a_{n-1} \\ c_n &= a_{n-1} - c_{n-1} \ (\because\ n\text{-th character can NOT be } 1) \\ d_n &= a_{n-1} - d_{n-1} \ (\because\ n\text{-th character can NOT be } 2) \end{cases} \right) \end{aligned} \tag{2}$$

   Then, we have

$$\begin{aligned} a_n &= b_n + c_n + d_n \\ &= a_{n-1} + (a_{n-1} - c_{n-1}) + (a_{n-1} - d_{n-1}) \\ &= 3 \times a_{n-1} - c_{n-1} - d_{n-1} \ (\because\ a_{n-1} = b_{n-1} + c_{n-1} + d_{n-1}) \\ &= 2 \times a_{n-1} + b_{n-1} \\ &= 2 \times a_{n-1} + b_{n-2} \end{aligned} \tag{3}$$

3.

(a) **Answer** †

$$1 \tag{4}$$

Since it needs to contain all edges and all vertices.

(b) **Answer** †

$$\binom{n}{2} \tag{5}$$

Since it needs to be **complete**.

4. **Answer** † Suppose G have n vertices. If $G = (V, E)$ is connected,

$$1 \le \deg(v) \le (n-1),\ \forall\ v \in V \tag{6}$$

Since $|V| = n$, and the possiblities of degree are $(n-1)$,

$$\exists\ u,\ v \in V,\ \text{s.t. } \deg(u) = \deg(v) \tag{7}$$

Otherwise, if $G = (V, E)$ is NOT connected, suppose there exists k vertices $u \in V_1$ such that $\deg(u) = 0$, so other $(n-k)$ vertices $v \in V_2$ are connected, i.e., $V = V_1 + V_2$. Then, we have

$$1 \le v \le (n-k-1),\ \forall\ v \in V_2 \tag{8}$$

Since $|V_2| = n-k$, and the possiblities of degree are $(n-k-1)$,

$$\exists\ v,\ w \in V_2,\ \text{s.t. } \deg(v) = \deg(w) \tag{9}$$

To summary, there are 2 vertices in G having equal degree.

5. **Answer** † We have

$$\begin{aligned} & H \cap K \subseteq H,\ H \cap K \subseteq K\ (|H| = h,\ |K| = k,\ |H \cap K| = m) \\ \Rightarrow\ & m|h,\ m|k \text{ (by Lagrange Theorem)} \\ \Rightarrow\ & m|\gcd(h,\ k) = 1 \\ \Rightarrow\ & m = 1 \end{aligned} \tag{10}$$

6. We have

$$\begin{aligned} & \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \operatorname{tr}_2(\boldsymbol{A}) \\ & = \begin{vmatrix} 1 & 2 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 7 & 6 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 5 & 8 \\ 6 & 7 \end{vmatrix} \end{aligned} \tag{11}$$

Answer †

$$24 \tag{12}$$

7. **Answer** † Eigenvalue matrix:

$$\begin{bmatrix} 3\times\boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & 2\times\boldsymbol{\Lambda} \end{bmatrix} \tag{13}$$

Eigenvector matrix:

$$\begin{bmatrix} \boldsymbol{S} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S} \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned} \mathrm{N}(\boldsymbol{T}) &= \{\boldsymbol{A}\in\mathbb{R}^{n\times n} | \frac{\boldsymbol{A}+\boldsymbol{A}^{\intercal}}{2} = \mathbf{0}\} \\ &= \{\boldsymbol{A}\in\mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^{\intercal}\} \end{aligned} \tag{15}$$

(a) **Answer** †

$$\{\boldsymbol{A}\in\mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^{\intercal}\} \tag{16}$$

(b) **Answer** †

$$(\mathrm{nullity}(T),\ \mathrm{rank}(T)) = (\frac{n(n-1)}{2}, \frac{n(n+1)}{2}) \tag{17}$$

9. We have

$$= \begin{vmatrix} a_0 & a_0 & \cdots & a_0 \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$(\because\ c_{n1}^{-1},\ c_{n2}^{-1},\ \cdots,\ c_{n(n-1)}^{-1})$

$$= \begin{vmatrix} 0 & 0 & \cdots & a_0 \\ p_1(x_1)-p_1(x_n) & p_1(x_2)-p_1(x_n) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1)-p_{n-1}(x_n) & p_{n-1}(x_2)-p_{n-1}(x_n) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$$= (-1)^{n+1}a_0 \begin{vmatrix} a_1(x_1-x_n) & a_1(x_2-x_n) & \cdots & a_1(x_{n-1}-x_n) \\ \sum_{i=1}^{2} a_i(x_1-x_n)^i & \sum_{i=1}^{2} a_i(x_2-x_n)^i & \cdots & \sum_{i=1}^{2} a_i(x_{n-1}-x_n)^i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n-1} a_i(x_1-x_n)^i & \sum_{i=1}^{n-1} a_i(x_2-x_n)^i & \cdots & \sum_{i=1}^{n-1} a_i(x_{n-1}-x_n)^i \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_{12}^{-1},\ r_{13}^{-1},\ \cdots,\ r_{1(n-1)}^{-1},\ r_{23}^{-1},\ r_{24}^{-1},\ \cdots,\ r_{2(n-1)}^{-1},\ r_{34}^{-1},\ \cdots,\ r_{(n-2)(n-1)}^{-1})$$

$$= (-1)^{n+1}a_0 \begin{vmatrix} a_1(x_1-x_n) & a_1(x_2-x_n) & \cdots & a_1(x_{n-1}-x_n) \\ a_2(x_1-x_n)^2 & a_2(x_2-x_n)^2 & \cdots & a_2(x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(x_1-x_n)^{n-1} & a_{n-1}(x_2-x_n)^{n-1} & \cdots & a_{n-1}(x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_1^{\frac{1}{a_1}},\ r_2^{\frac{1}{a_2}},\ \cdots,\ r_{n-1}^{\frac{1}{a_{n-1}}})$$

$$= (-1)^{n+1}\prod_{i=0}^{n-1} a_i \begin{vmatrix} (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ (x_1-x_n)^2 & (x_2-x_n)^2 & \cdots & (x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-1} & (x_2-x_n)^{n-1} & \cdots & (x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)} \tag{18}$$

$$(\because\ c_1^{\frac{1}{x_1-x_n}},\ c_2^{\frac{1}{x_2-x_n}},\ \cdots,\ c_{n-1}^{\frac{1}{x_{n-1}-x_n}})$$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)(\prod_{j=1}^{n-1}(x_j-x_n)) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-2} & (x_2-x_n)^{n-2} & \cdots & (x_{n-1}-x_n)^{n-2} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ \text{Vandermonde matrix})$$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)[(-1)^{n-1}\prod_{j=1}^{n-1}(x_n-x_j)](\prod_{1\le i\le j\le (n-1)}(x_j-x_i))$$

$$= (\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i))$$

Answer †

$$(\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i)) \tag{19}$$

10. We have

$$\begin{aligned} &\Rightarrow \alpha^2 = \frac{1}{2}\times\alpha - \frac{1}{2} \\ &\Rightarrow \alpha = -\frac{1}{2} \vee \alpha = 1 \\ &\Rightarrow B_n = c\times(-\frac{1}{2})^n + d\times(1)^n \end{aligned} \tag{20}$$

And, we have

$$\begin{cases} B_0 = 0 = c + d \\ B_1 = \frac{1}{2} = -\frac{1}{2} \times c + d \end{cases} \tag{21}$$
$$\Rightarrow c = -\frac{1}{3},\ d = \frac{1}{3}$$

Answer †

$$B_k = \frac{1}{3} - \frac{1}{3} \times (-\frac{1}{2})^k,\ \lim_{k\to\infty} B_k = \frac{1}{3} \tag{22}$$

11. We have
    (a) True.
    (b) False.
    (c) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (d) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (e) False.
12. We have
    (a) True.
    (b) True.
    (c) True.
    (d) False.
    (e) False.

Solutions

NTU math 101

VERSION 1.0

1. **Answer** †

$$\begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$3 \tag{2}$$

3. We have

$$\boldsymbol{B} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 \end{bmatrix} \tag{3}$$

   Answer †

$$3 \tag{4}$$

4. We have

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{5}$$

   $\boldsymbol{A}$ is a rotation matrix, which rotates $\frac{\pi}{3}$ **clockwisely**. Then, we have

$$\boldsymbol{A}^{300} = \begin{bmatrix} \cos(100\pi) & \sin(100\pi) \\ -\sin(100\pi) & \cos(100\pi) \end{bmatrix} \tag{6}$$

   Answer †

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

5. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x) = -x^3 + x^2 - 3 \times x + 2 \tag{8}$$

Suppose $\lambda_{\boldsymbol{A}} = a,\ b,\ c$, then we have,

$$\begin{aligned} &\begin{cases} a + b + c = 1 \\ ab + bc + ac = 3 \\ abc = 2 \end{cases} \\ \Rightarrow &\begin{cases} a^2 + b^2 + c^2 = (a+b+c)^2 - 2 \times (ab + bc + ac) = -5 \\ a^2b^2 + b^2c^2 + a^2c^2 = (ab + bc + ac)^2 - 2 \times (abc) \times (a + b + c) = 5 \\ a^2b^2c^2 = (abc)^2 = 4 \end{cases} \end{aligned} \tag{9}$$

Then, we have characteristic polynomial

$$p_{\boldsymbol{A}^2} = \det(\boldsymbol{A}^2 - \boldsymbol{I}\boldsymbol{x}) = -x^3 + (-5) \times x^2 - (5) \times x + 4 \tag{10}$$

Answer †

$$\det(\boldsymbol{x}\boldsymbol{I} - \boldsymbol{A}^2) = -(-x^3 + (-5) \times x^2 - (5) \times x + 4) = x^3 + 5 \times x^2 + 5 \times x - 4 \tag{11}$$

6. **Answer** †

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \tag{12}$$

7. We have

$$\begin{aligned} &-\boldsymbol{H}^3 + \alpha \boldsymbol{H}^2 + \beta \boldsymbol{H} + \gamma \boldsymbol{I} = \boldsymbol{O} \\ \Rightarrow &\begin{cases} \alpha = \operatorname{tr}(\boldsymbol{H}) = 34 \\ \beta = -\operatorname{tr}_2(\boldsymbol{H}) \\ \quad = -(\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 9 & 11 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \end{vmatrix} + \begin{vmatrix} 6 & 8 \\ 14 & 16 \end{vmatrix} + \begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix}) = 80 \\ \gamma = \det(\boldsymbol{H}) = 0 \end{cases} \end{aligned} \tag{13}$$

Answer †

$$\begin{bmatrix} 34 \\ 80 \\ 0 \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned}
&\overset{r_{12}^{1},\, r_{23}^{1},\, \cdots,\, r_{56}^{1}}{=} \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 & 3\times 2^4 \\ 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 \\ 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 \\ 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 \\ 3\times 2^4 & 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 \end{vmatrix} \\
&\overset{r_{2}^{\frac{1}{3}},\, r_{3}^{\frac{1}{3}},\, \cdots,\, r_{6}^{\frac{1}{3}}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 1 & 1 & 2 & 2^2 & 2^3 & 2^4 \\ 2 & 1 & 1 & 2 & 2^2 & 2^3 \\ 2^2 & 2 & 1 & 1 & 2 & 2^2 \\ 2^3 & 2^2 & 2 & 1 & 1 & 2 \\ 2^4 & 2^3 & 2^2 & 2 & 1 & 1 \end{vmatrix} \\
&\overset{r_{12}^{-1},\, r_{23}^{-1},\, \cdots,\, r_{56}^{-1}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 0 & -1 & -2 & -2^2 & -2^3 & -2^4 \\ 1 & 0 & -1 & -2 & -2^2 & -2^3 \\ 2 & 1 & 0 & -1 & -2 & -2^2 \\ 2^2 & 2 & 1 & 0 & -1 & -2 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix} \\
&\overset{r_{65}^{-2},\, r_{54}^{-2},\, \cdots,\, r_{21}^{-2}}{=} 3^5 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 & 0 \\ -6 & -3 & -2 & -1 & 0 & 0 \\ -12 & -6 & -3 & -2 & -1 & 0 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix}
\end{aligned} \tag{15}$$

Answer †

$$-3^5 \tag{16}$$

9. Suppose

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix}, \ \boldsymbol{A}_{100} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}_{100\times 100} \tag{17}$$

$$\Rightarrow q(x_1,\ x_2,\ \cdots,\ x_{100}) = \sum_{k=1}^{99} x_k x_{k+1} = \boldsymbol{x}^\intercal \boldsymbol{A}_{100} \boldsymbol{x}$$

By Rayleigh principle, $\max_{||\boldsymbol{x}||=1} \boldsymbol{x}^\intercal \boldsymbol{A}_n \boldsymbol{x} = \lambda_{\max}(\boldsymbol{A}_n)$. And, we have general tridiagnoal matrix

$$B_n = \begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & a & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & a & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & a \end{bmatrix}_{n\times n} \tag{18}$$

By solving the recurrence function, we have general eigenvalues formula for $\boldsymbol{B}_n$

$$\lambda_k = a + 2 \times b\cos(\frac{k \times \pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{19}$$

Then, we have $a = 0,\ b = \frac{1}{2}$, and get $\boldsymbol{A}_n$'s eigenvalues

$$\lambda_k = \cos(\frac{k \times \pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{20}$$

And, we have

$$\cos(\frac{1 \times \pi}{100+1}) \tag{21}$$

as largest eigenvalue of $\boldsymbol{A}_{100}$.

Answer †

$$\cos(\frac{\pi}{101}) \tag{22}$$

10. (a) has 2 degree-3 vertice and 3 degree-2 vertices, when (b), (c), and (d) have same 4 degree-3 vertice and 1 degree-2 vertices, so (a) can NOT be an isomorphism of others.

    And, we have correspondence

(b)	(c)	(d)
1	4	5
2	1	2
3	5	1
4	3	4
5	2	3

So, $(b)(c)(d)$ are isomorphic.

Answer †

$$(b)(c)(d) \tag{23}$$

11. (a) **Answer** †

$$s_n = s_{n-1} + \frac{n(n-1)}{2} \tag{24}$$

(b) **Answer** †

$$a_0 + a_1 + a_2 + a_3 + \cdots = s_1 = 1 \tag{25}$$

12. Suppose

$$\begin{aligned} &\Rightarrow \begin{cases} x_1 = a - 1 \geq 0 \\ x_2 = b - a \geq 2 \\ x_3 = c - b \geq 2 \\ x_4 = d - c \geq 2 \\ x_5 = 12 - d \geq 0 \end{cases}, \sum_{i=1}^{5} x_i = 11 \\ &\Rightarrow \begin{cases} y_1 = x_1 \geq 0 \\ y_2 = x_2 - 2 \geq 0 \\ y_3 = x_3 - 2 \geq 0 \\ y_4 = x_4 - 2 \geq 0 \\ y_5 = x_5 \geq 0 \end{cases}, \sum_{i=1}^{5} y_i = 11 - 3 \times 2 \end{aligned} \tag{26}$$

Answer †

$$\binom{5 + (11-6) - 1}{(11-6)} = 126 \tag{27}$$

13. (a) We have constraints: m and n must be **even** (≥ 1 Euler circuits), and $m \neq n$ (NO Hamilton cycle).

Answer †

$$2,\ 8 \tag{28}$$

(b) **Answer** †

$$m \text{ and } n \text{ is even}, \text{ and } m \neq n \tag{29}$$

14. **Answer** △

$$\begin{aligned}
&(\Rightarrow)\ \because\ (S,\ +,\ \cdot) \text{ is a ring} \\
&\qquad \therefore\ \forall\ a,\ b \in S,\ a+b \in S,\ a \cdot b \in S \\
&(\Leftarrow)\ \forall\ a \in S,\ -a \in S \\
&\qquad \because (S,\ +,\ \cdot) \text{ is closed},\ \forall\ a \in S,\ a,\ 2a,\ \cdots \in S \\
&\qquad \exists\ i < j,\ ia = ja \to (j-i)a = 0 \to a + (j-i-1)a = 0 \\
&\qquad \therefore\ -a = \begin{cases} 0(=a) \in S &,\ j = i+1 \\ (j-i-1)a \in S &,\ j > i+1 \end{cases}
\end{aligned} \tag{30}$$

Solutions

NTU math 100

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1. **Answer** †

$$1 \tag{1}$$

2. **Answer** †

$$\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \tag{2}$$

3. We have

$$\boldsymbol{B}^3 = \begin{bmatrix} 17 & 6 \\ 18 & -1 \end{bmatrix} = 6 \times \boldsymbol{B} + 5 \times \boldsymbol{I} \tag{3}$$

Answer †

$$(6,\ 5) \tag{4}$$

4. We have

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix})\right) = 2,\ \det\left(\begin{bmatrix} 1 & 5 & 1 \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix})\right) = -31,\ \det\left(\begin{bmatrix} 1 & 4 & 1 \\ 5 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix})\right) = -2 \tag{5}$$

Answer †

$$2 \times \frac{1}{-31} \times (-2) = \frac{4}{31} \tag{6}$$

5. We have

$$\operatorname{tr}(\boldsymbol{A}^2) = (1 + (-2) + 3) + ((-2) + 9 + a) + (3 + a + 0) = 5 \tag{7}$$

Answer †

$$-\frac{7}{2} \tag{8}$$

6. We have

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{w} &= \boldsymbol{w} + \alpha \boldsymbol{w}\boldsymbol{w}^{\intercal}\boldsymbol{w} \\
&= \boldsymbol{w} + 10 \times \alpha \boldsymbol{w} \\
&= (10 \times \alpha + 1)\boldsymbol{w} \\
\Rightarrow 10 \times \alpha + 1 &= 0 (\because \boldsymbol{A} \text{ is singular.})
\end{aligned} \tag{9}$$

And, we have

$$\operatorname{rank}(\boldsymbol{w}\boldsymbol{w}^{\intercal}) = 1 \tag{10}$$

Then, we have eigenvalues of $\boldsymbol{w}\boldsymbol{w}^{\intercal}$

$$0,\ 0,\ 0,\ 0,\ 10 \tag{11}$$

Then, we have eigenvalues of $\boldsymbol{A}$

$$1,\ 1,\ 1,\ 1,\ 0\ (\because\ 1 + 10 \times \alpha) \tag{12}$$

Then, we have $\operatorname{rank}(\boldsymbol{A}) = 4$.

Answer †

$$(-\frac{1}{10},\ 4) \tag{13}$$

7. Suppose

$$\begin{aligned}
\boldsymbol{D} &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \\
\Rightarrow \boldsymbol{D}^{-1} &= \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}
\end{aligned} \tag{14}$$

Then, we have

$$\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} a_{11} & \frac{1}{2} \times a_{12} & \frac{1}{4} \times a_{13} \\ 2 \times a_{21} & a_{22} & \frac{1}{2} \times a_{23} \\ 4 \times a_{31} & 2 \times a_{32} & a_{33} \end{bmatrix} \tag{15}$$

Answer †

$$\begin{bmatrix} 4\times\alpha & 0 & 0 \\ 0 & 2\times\alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix},\ \forall\,\alpha\in\mathbb{R} \tag{16}$$

8. Let

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{I} + \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \\ \Rightarrow \boldsymbol{A} &= \boldsymbol{I} + \boldsymbol{XY} \end{aligned} \tag{17}$$

Since, $\boldsymbol{XY} = \boldsymbol{YX}$ have same eigenvalues, we have

$$\boldsymbol{YX} = \begin{bmatrix} 35 & 55 \\ 55 & 35 \end{bmatrix} \tag{18}$$

Then, $\boldsymbol{YX}$ has eigenvalues $-20,\ 90$, so $\boldsymbol{XY}$ has eigenvalues

$$0,\ 0,\ 0,\ -20,\ 90 \tag{19}$$

Then, $\boldsymbol{A}$ has eigenvalues

$$1+0,\ 1+0,\ 1+0,\ 1+(-20),\ 1+90 \tag{20}$$

Answer †

$$1,\ 1,\ 1,\ -19,\ 91 \tag{21}$$

9. We have

$$
\begin{aligned}
&(\because\ r_{n(n-1)}^{-1},\ r_{n(n-2)}^{-1},\ \cdots,\ r_{n1}^{-1})\\
&=\begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-1}{3\times(n+2)} & \cdots & \frac{n-1}{2n\times(n+1)}\\
\frac{n-2}{3\times(n+1)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{n-2}{2n\times(n+2)}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n\times(n+1)} & \frac{1}{(n+1)\times(n+2)} & \cdots & \frac{1}{2n\times(2n-1)}\\
\frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n}
\end{vmatrix}_{n\times n}\\
&(\because\ r_1^{\frac{1}{n-1}},\ r_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ c_1^{n+1},\ c_2^{n+2},\ \cdots,\ c_n^{2n})\\
&=(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1}\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1}\\
1 & 1 & \cdots & 1
\end{vmatrix}_{n\times n}\\
&(\because\ c_{n(n-1)}^{-1},\ c_{n(n-2)}^{-1},\ \cdots,\ c_{n1}^{-1})\\
&=\frac{n!(n-1)!}{(2n)!}\begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-2}{3\times(n+1)} & \cdots & \frac{1}{n\times(n+1)} & \frac{1}{n+1}\\
\frac{n-1}{3\times(n+2)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{1}{(n+1)\times(n+2)} & \frac{1}{n+2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{n-1}{n\times(2n-1)} & \frac{n-2}{(n+1)\times(2n-1)} & \cdots & \frac{1}{(2n-2)\times(2n-1)} & \frac{1}{2n-1}\\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{n\times n} \qquad (22)\\
&(\because\ c_1^{\frac{1}{n-1}},\ c_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ r_1^{n+1},\ r_2^{n+2},\ \cdots,\ r_{n-1}^{2n-1})\\
&=\frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & 1\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} & 1\\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{(n-1)\times(n-1)}\\
&=\frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n}\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2}
\end{vmatrix}_{(n-1)\times(n-1)}\\
&=\frac{(n!)^2[(n-1)!]^2\times 2n}{[(2n)!]^2}\times P_{n-1}
\end{aligned}
$$

And, we have

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{23}$$

Answer †

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{24}$$

10. We have

$$\begin{cases} \boldsymbol{u}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{v}^{\mathsf{H}}\boldsymbol{v} \\ \boldsymbol{u} \neq \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} = \boldsymbol{I} \\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{v} \end{cases} \tag{25}$$

Let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, and we assume

$$\boldsymbol{A} = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{26}$$

Then, we have

$$\begin{aligned} \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{I}\boldsymbol{u} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{w} = \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})^{\mathsf{H}}(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= \boldsymbol{I} - \frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} + \frac{1}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - [\frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}} + \frac{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}]\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{\boldsymbol{w}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{w}) + \boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{(\boldsymbol{u} - \boldsymbol{v})^{\mathsf{H}}\boldsymbol{v} + \boldsymbol{u}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{v})}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} \end{aligned} \tag{27}$$

Answer †

$$\boldsymbol{A}(\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{28}$$

11. We have

(a) **Answer** †

$$6 \times 6 = 36 \tag{29}$$

Since $K_{6,6}$ has the maximal edges.

(b) **Answer** †

$$e \leq 3 \times 5 - 6 = 9 \tag{30}$$

12. We have

$$\begin{aligned} &\Rightarrow (\alpha - 2)(\alpha - 3) = \alpha^2 - 5 \times \alpha + 6 = 0 \\ &\Rightarrow a_{n+2} - 5 \times a_{n+1} + 6 \times a_n = q_1 \times n + q_2 \end{aligned} \tag{31}$$

And, we have

$$(n + 2 - 7) - 5 \times (n + 1 - 7) + 6 \times (n - 7) = 2 \times n - 17 \tag{32}$$

(a) **Answer** †

$$p_1 = -5,\ p_2 = 6 \tag{33}$$

(b) **Answer** †

$$q_1 = 2,\ q_2 = -17 \tag{34}$$

13. (a) **Answer** †

$$2^{4^2-4} = 4096 \tag{35}$$

(b) **Answer** †

$$2^{\frac{4^2-4}{2}-1} \times 2^4 = 512 \tag{36}$$

14. **Answer** † Suppose

$$x < 50 \wedge y < 50 \rightarrow x + y < 100 \tag{37}$$

contradiction, so

$$x \geq 50 \wedge y \geq 50 \tag{38}$$

15. **Answer** † Let $*$ be operator of G and $\cdot$ be operator of H.

$$\begin{aligned} &\because\ f \text{ is onto} \\ &\therefore\ \forall\ y_1,\ y_2 \in H,\ \exists\ x_1,\ x_2,\ \text{s.t. } f(x_1) = y_1,\ f(x_2) = y_2 \\ &\Rightarrow\ y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 * x_2) = f(x_2 * x_1) = f(x_2) \cdot f(x_1) = y_2 \cdot y_1 \end{aligned} \tag{39}$$

Then, if G is abelian, then H is abelian.