# Solutions

# NTU math 102

Version 1.0

1. **Answer** † **WRONG**. Counterexample:

$$\begin{aligned} A &= \{1,\ 2,\ 3\} \\ R &= \{(1,\ 2),\ (2,\ 1),\ (1,\ 1),\ (2,\ 2)\} \end{aligned} \tag{1}$$

   $R$ is symmetric and transitive, but $R$ is NOT reflexive.

2. **Answer** † Suppose

$$\begin{aligned} &\Rightarrow \begin{cases} b_n & = n\text{-th character is } 0 \\ c_n & = n\text{-th character is } 1,\ \text{and } (n-1)\text{-th character is } 0 \vee 2 \\ d_n & = n\text{-th character is } 2,\ \text{and } (n-1)\text{-th character is } 0 \vee 1 \end{cases} \\ &\Rightarrow a_n = b_n + c_n + d_n \\ &\quad \left( \begin{cases} b_n & = a_{n-1} \\ c_n & = a_{n-1} - c_{n-1}\ (\because\ n\text{-th character can NOT be 1}) \\ d_n & = a_{n-1} - d_{n-1}\ (\because\ n\text{-th character can NOT be 2}) \end{cases} \right) \end{aligned} \tag{2}$$

   Then, we have

$$\begin{aligned} a_n &= b_n + c_n + d_n \\ &= a_{n-1} + (a_{n-1} - c_{n-1}) + (a_{n-1} - d_{n-1}) \\ &= 3 \times a_{n-1} - c_{n-1} - d_{n-1}\ (\because\ a_{n-1} = b_{n-1} + c_{n-1} + d_{n-1}) \\ &= 2 \times a_{n-1} + b_{n-1} \\ &= 2 \times a_{n-1} + b_{n-2} \end{aligned} \tag{3}$$

3.

(a) **Answer** †

$$1 \tag{4}$$

Since it needs to contain all edges and all vertices.

(b) **Answer** †

$$\binom{n}{2} \tag{5}$$

Since it needs to be **complete**.

4. **Answer** † Suppose $G$ have $n$ vertices. If $G = (V, E)$ is connected,

$$1 \le \deg(v) \le (n-1),\ \forall\ v \in V \tag{6}$$

Since $|V| = n$, and the possiblities of degree are $(n-1)$,

$$\exists\ u,\ v \in V,\ \text{s.t. } \deg(u) = \deg(v) \tag{7}$$

Otherwise, if $G = (V, E)$ is NOT connected, suppose there exists $k$ vertices $u \in V_1$ such that $\deg(u) = 0$, so other $(n-k)$ vertices $v \in V_2$ are connected, i.e., $V = V_1 + V_2$. Then, we have

$$1 \le v \le (n-k-1),\ \forall\ v \in V_2 \tag{8}$$

Since $|V_2| = n-k$, and the possiblities of degree are $(n-k-1)$,

$$\exists\ v,\ w \in V_2,\ \text{s.t. } \deg(v) = \deg(w) \tag{9}$$

To summary, there are 2 vertices in $G$ having equal degree.

5. **Answer** † We have

$$\begin{aligned}
&H \cap K \subseteq H,\ H \cap K \subseteq K\ (|H| = h,\ |K| = k,\ |H \cap K| = m)\\
&\Rightarrow m|h,\ m|k \text{ (by Lagrange Theorem)}\\
&\Rightarrow m|\gcd(h,\ k) = 1\\
&\Rightarrow m = 1
\end{aligned} \tag{10}$$

6. We have

$$\begin{aligned}
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \operatorname{tr}_2(\boldsymbol{A})\\
&= \begin{vmatrix} 1 & 2 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 7 & 6 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 5 & 8 \\ 6 & 7 \end{vmatrix}
\end{aligned} \tag{11}$$

**Answer** †

$$24 \tag{12}$$

7. **Answer** † Eigenvalue matrix:

$$\begin{bmatrix} 3\times\boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & 2\times\boldsymbol{\Lambda} \end{bmatrix} \tag{13}$$

Eigenvector matrix:

$$\begin{bmatrix} \boldsymbol{S} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S} \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned} \mathrm{N}(\boldsymbol{T}) &= \{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \frac{\boldsymbol{A}+\boldsymbol{A}^\intercal}{2} = \mathbf{0}\} \\ &= \{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^\intercal\} \end{aligned} \tag{15}$$

(a) **Answer** †

$$\{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^\intercal\} \tag{16}$$

(b) **Answer** †

$$(\mathrm{nullity}(T),\ \mathrm{rank}(T)) = (\frac{n(n-1)}{2},\ \frac{n(n+1)}{2}) \tag{17}$$

9. We have

$$= \begin{vmatrix} a_0 & a_0 & \cdots & a_0 \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$(\because\ c_{n1}^{-1},\ c_{n2}^{-1},\ \cdots,\ c_{n(n-1)}^{-1})$

$$= \begin{vmatrix} 0 & 0 & \cdots & a_0 \\ p_1(x_1)-p_1(x_n) & p_1(x_2)-p_1(x_n) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1)-p_{n-1}(x_n) & p_{n-1}(x_2)-p_{n-1}(x_n) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$$= (-1)^{n+1}a_0 \begin{vmatrix} a_1(x_1-x_n) & a_1(x_2-x_n) & \cdots & a_1(x_{n-1}-x_n) \\ \sum_{i=1}^{2} a_i(x_1-x_n)^i & \sum_{i=1}^{2} a_i(x_2-x_n)^i & \cdots & \sum_{i=1}^{2} a_i(x_{n-1}-x_n)^i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n-1} a_i(x_1-x_n)^i & \sum_{i=1}^{n-1} a_i(x_2-x_n)^i & \cdots & \sum_{i=1}^{n-1} a_i(x_{n-1}-x_n)^i \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_{12}^{-1},\ r_{13}^{-1},\ \cdots,\ r_{1(n-1)}^{-1},\ r_{23}^{-1},\ r_{24}^{-1},\ \cdots,\ r_{2(n-1)}^{-1},\ r_{34}^{-1},\ \cdots,\ r_{(n-2)(n-1)}^{-1})$$

$$= (-1)^{n+1}a_0 \begin{vmatrix} a_1(x_1-x_n) & a_1(x_2-x_n) & \cdots & a_1(x_{n-1}-x_n) \\ a_2(x_1-x_n)^2 & a_2(x_2-x_n)^2 & \cdots & a_2(x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(x_1-x_n)^{n-1} & a_{n-1}(x_2-x_n)^{n-1} & \cdots & a_{n-1}(x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_1^{\frac{1}{a_1}},\ r_2^{\frac{1}{a_2}},\ \cdots,\ r_{n-1}^{\frac{1}{a_{n-1}}})$$

$$= (-1)^{n+1}\prod_{i=0}^{n-1} a_i \begin{vmatrix} (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ (x_1-x_n)^2 & (x_2-x_n)^2 & \cdots & (x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-1} & (x_2-x_n)^{n-1} & \cdots & (x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)} \tag{18}$$

$$(\because\ c_1^{\frac{1}{x_1-x_n}},\ c_2^{\frac{1}{x_2-x_n}},\ \cdots,\ c_{n-1}^{\frac{1}{x_{n-1}-x_n}})$$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)(\prod_{j=1}^{n-1}(x_j-x_n)) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-2} & (x_2-x_n)^{n-2} & \cdots & (x_{n-1}-x_n)^{n-2} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ \text{Vandermonde matrix})$$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)[(-1)^{n-1}\prod_{j=1}^{n-1}(x_n-x_j)](\prod_{1\le i\le j\le (n-1)}(x_j-x_i))$$

$$= (\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i))$$

**Answer** †

$$(\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i)) \tag{19}$$

10. We have

$$\begin{aligned} &\Rightarrow \alpha^2 = \frac{1}{2}\times\alpha - \frac{1}{2} \\ &\Rightarrow \alpha = -\frac{1}{2} \vee \alpha = 1 \\ &\Rightarrow B_n = c\times(-\frac{1}{2})^n + d\times(1)^n \end{aligned} \tag{20}$$

And, we have

$$\begin{cases} B_0 = 0 = c + d \\ B_1 = \frac{1}{2} = -\frac{1}{2} \times c + d \end{cases} \tag{21}$$

$$\Rightarrow c = -\frac{1}{3},\ d = \frac{1}{3}$$

**Answer** †

$$B_k = \frac{1}{3} - \frac{1}{3} \times (-\frac{1}{2})^k,\ \lim_{k\to\infty} B_k = \frac{1}{3} \tag{22}$$

11. We have
    (a) True.
    (b) False.
    (c) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (d) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (e) False.
12. We have
    (a) True.
    (b) True.
    (c) True.
    (d) False.
    (e) False.