Solutions

NTU math 100

VERSION 1.0

1. **Answer** †

$$1 \tag{1}$$

2. **Answer** †

$$\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \tag{2}$$

3. We have

$$\boldsymbol{B}^3 = \begin{bmatrix} 17 & 6 \\ 18 & -1 \end{bmatrix} = 6 \times \boldsymbol{B} + 5 \times \boldsymbol{I} \tag{3}$$

Answer †

$$(6,\ 5) \tag{4}$$

4. We have

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix})\right) = 2,\ \det\left(\begin{bmatrix} 1 & 5 & 1 \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix})\right) = -31,\ \det\left(\begin{bmatrix} 1 & 4 & 1 \\ 5 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix})\right) = -2 \tag{5}$$

Answer †

$$2 \times \frac{1}{-31} \times (-2) = \frac{4}{31} \tag{6}$$

5. We have

$$\mathrm{tr}(\boldsymbol{A}^2) = (1 + (-2) + 3) + ((-2) + 9 + a) + (3 + a + 0) = 5 \tag{7}$$

Answer †

$$-\frac{7}{2} \tag{8}$$

6. We have

$$\begin{aligned} \boldsymbol{A}\boldsymbol{w} &= \boldsymbol{w} + \alpha \boldsymbol{w}\boldsymbol{w}^\intercal \boldsymbol{w} \\ &= \boldsymbol{w} + 10 \times \alpha \boldsymbol{w} \\ &= (10 \times \alpha + 1)\boldsymbol{w} \\ \Rightarrow & 10 \times \alpha + 1 = 0 (\because \boldsymbol{A} \text{ is singular.}) \end{aligned} \tag{9}$$

And, we have

$$\text{rank}(\boldsymbol{w}\boldsymbol{w}^\intercal) = 1 \tag{10}$$

Then, we have eigenvalues of $\boldsymbol{w}\boldsymbol{w}^\intercal$

$$0,\ 0,\ 0,\ 0,\ 10 \tag{11}$$

Then, we have eigenvalues of $\boldsymbol{A}$

$$1,\ 1,\ 1,\ 1,\ 0\ (\because\ 1 + 10 \times \alpha) \tag{12}$$

Then, we have rank$(\boldsymbol{A}) = 4$.

Answer †

$$(-\frac{1}{10},\ 4) \tag{13}$$

7. Suppose

$$\begin{aligned} \boldsymbol{D} &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \\ \Rightarrow \boldsymbol{D}^{-1} &= \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \end{aligned} \tag{14}$$

Then, we have

$$\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} a_{11} & \frac{1}{2} \times a_{12} & \frac{1}{4} \times a_{13} \\ 2 \times a_{21} & a_{22} & \frac{1}{2} \times a_{23} \\ 4 \times a_{31} & 2 \times a_{32} & a_{33} \end{bmatrix} \tag{15}$$

Answer †

$$\begin{bmatrix} 4\times\alpha & 0 & 0 \\ 0 & 2\times\alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \forall\, \alpha \in \mathbb{R} \tag{16}$$

8. Let

$$\boldsymbol{A} = \boldsymbol{I} + \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \tag{17}$$
$$\Rightarrow \boldsymbol{A} = \boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y}$$

Since, $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}$ have same eigenvalues, we have

$$\boldsymbol{Y}\boldsymbol{X} = \begin{bmatrix} 35 & 55 \\ 55 & 35 \end{bmatrix} \tag{18}$$

Then, $\boldsymbol{Y}\boldsymbol{X}$ has eigenvalues $-20,\ 90$, so $\boldsymbol{X}\boldsymbol{Y}$ has eigenvalues

$$0,\ 0,\ 0,\ -20,\ 90 \tag{19}$$

Then, $\boldsymbol{A}$ has eigenvalues

$$1+0,\ 1+0,\ 1+0,\ 1+(-20),\ 1+90 \tag{20}$$

Answer †

$$1,\ 1,\ 1,\ -19,\ 91 \tag{21}$$

9. We have

$$
\begin{aligned}
&(\because\ r^{-1}_{n(n-1)},\ r^{-1}_{n(n-2)},\ \cdots,\ r^{-1}_{n1}) \\
&= \begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-1}{3\times(n+2)} & \cdots & \frac{n-1}{2n\times(n+1)} \\
\frac{n-2}{3\times(n+1)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{n-2}{2n\times(n+2)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n\times(n+1)} & \frac{1}{(n+1)\times(n+2)} & \cdots & \frac{1}{2n\times(2n-1)} \\
\frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n}
\end{vmatrix}_{n\times n} \\
&(\because\ r_1^{\frac{1}{n-1}},\ r_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ c_1^{n+1},\ c_2^{n+2},\ \cdots,\ c_n^{2n}) \\
&= (\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n} \frac{1}{n+j})
\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \\
1 & 1 & \cdots & 1
\end{vmatrix}_{n\times n} \\
&(\because\ c^{-1}_{n(n-1)},\ c^{-1}_{n(n-2)},\ \cdots,\ c^{-1}_{n1}) \\
&= \frac{n!(n-1)!}{(2n)!}
\begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-2}{3\times(n+1)} & \cdots & \frac{1}{n\times(n+1)} & \frac{1}{n+1} \\
\frac{n-1}{3\times(n+2)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{1}{(n+1)\times(n+2)} & \frac{1}{n+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{n-1}{n\times(2n-1)} & \frac{n-2}{(n+1)\times(2n-1)} & \cdots & \frac{1}{(2n-2)\times(2n-1)} & \frac{1}{2n-1} \\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{n\times n} \\
&(\because\ c_1^{\frac{1}{n-1}},\ c_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ r_1^{n+1},\ r_2^{n+2},\ \cdots,\ r_{n-1}^{2n-1}) \\
&= \frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n} \frac{1}{n+j})
\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & 1 \\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} & 1 \\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{(n-1)\times(n-1)} \\
&= \frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n} \frac{1}{n+j})
\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2}
\end{vmatrix}_{(n-1)\times(n-1)} \\
&= \frac{(n!)^2[(n-1)!]^2\times 2n}{[(2n)!]^2}\times P_{n-1}
\end{aligned}
\tag{22}
$$

And, we have

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{23}$$

Answer †

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{24}$$

10. We have

$$\begin{cases} \boldsymbol{u}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{v}^{\mathsf{H}}\boldsymbol{v} \\ \boldsymbol{u} \neq \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} = \boldsymbol{I} \\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{v} \end{cases} \tag{25}$$

Let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, and we assume

$$\boldsymbol{A} = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{26}$$

Then, we have

$$\begin{aligned} \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{I}\boldsymbol{u} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{w} = \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})^{\mathsf{H}}(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= \boldsymbol{I} - \frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} + \frac{1}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - [\frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}} + \frac{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}]\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{\boldsymbol{w}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{w}) + \boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{(\boldsymbol{u} - \boldsymbol{v})^{\mathsf{H}}\boldsymbol{v} + \boldsymbol{u}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{v})}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} \end{aligned} \tag{27}$$

Answer †

$$\boldsymbol{A}(\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{28}$$

11. We have

(a) **Answer** †

$$6 \times 6 = 36 \tag{29}$$

Since $K_{6,6}$ has the maximal edges.

(b) **Answer** †

$$e \leq 3 \times 5 - 6 = 9 \tag{30}$$

12. We have

$$\begin{aligned} &\Rightarrow (\alpha - 2)(\alpha - 3) = \alpha^2 - 5 \times \alpha + 6 = 0 \\ &\Rightarrow a_{n+2} - 5 \times a_{n+1} + 6 \times a_n = q_1 \times n + q_2 \end{aligned} \tag{31}$$

And, we have

$$(n + 2 - 7) - 5 \times (n + 1 - 7) + 6 \times (n - 7) = 2 \times n - 17 \tag{32}$$

(a) **Answer** †

$$p_1 = -5,\ p_2 = 6 \tag{33}$$

(b) **Answer** †

$$q_1 = 2,\ q_2 = -17 \tag{34}$$

13. (a) **Answer** †

$$2^{4^2-4} = 4096 \tag{35}$$

(b) **Answer** †

$$2^{\frac{4^2-4}{2}-1} \times 2^4 = 512 \tag{36}$$

14. **Answer** † Suppose

$$x < 50 \wedge y < 50 \rightarrow x + y < 100 \tag{37}$$

contradiction, so

$$x \geq 50 \wedge y \geq 50 \tag{38}$$

15. **Answer** † Let $*$ be operator of G and $\cdot$ be operator of H.

$$\begin{aligned} &\because\ f \text{ is onto} \\ &\therefore\ \forall\ y_1,\ y_2 \in H,\ \exists\ x_1,\ x_2,\ \text{s.t. } f(x_1) = y_1,\ f(x_2) = y_2 \\ &\Rightarrow\ y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 * x_2) = f(x_2 * x_1) = f(x_2) \cdot f(x_1) = y_2 \cdot y_1 \end{aligned} \tag{39}$$

Then, if G is abelian, then H is abelian.