Solutions

NTU math 104

VERSION 1.0

1. We have

$$\begin{aligned} & p \to q \\ \iff & \neg p \vee q \end{aligned} \tag{1}$$

   (a) True.

   (b) False, since

$$\begin{aligned} & \neg p \to \neg q \\ \iff & p \vee \neg q \neq \neg p \vee q \end{aligned} \tag{2}$$

   (c) False, since

$$\begin{aligned} & q \to p \\ \iff & \neg q \vee p \neq \neg p \vee q \end{aligned} \tag{3}$$

   (d) True, since

$$\begin{aligned} & \neg q \to \neg p \\ \iff & q \vee \neg p = \neg p \vee q \end{aligned} \tag{4}$$

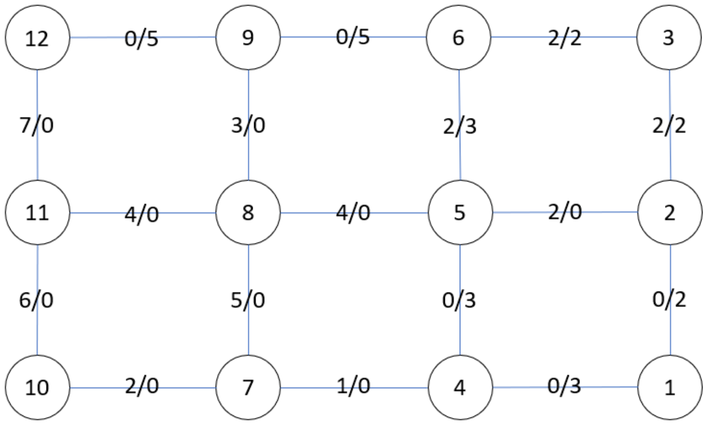
   (e) False, since

$$\begin{aligned} & \neg q \to p \\ \iff & q \vee p \neq \neg p \vee q \end{aligned} \tag{5}$$

   Answer †

$$ad \tag{6}$$

2. We have



Answer †

$$5 \tag{7}$$

3. We have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^n \tag{8}$$

Answer †

$$3^n \tag{9}$$

4. We have

$$120 = 2^3 \times 3^1 \times 5^1 \tag{10}$$

Answer †

$$\Phi(120) = 120 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 32 \tag{11}$$

5. Suppose

$$\begin{aligned} y_1 &= x_1 - 1 \\ y_2 &= x_2 - x_1 \\ y_3 &= x_3 - x_2 \\ &\vdots \\ y_n &= x_n - x_{n-1} \\ y_{n+1} &= r - x_n \end{aligned} \tag{12}$$

We have

$$\begin{aligned} &\begin{cases} y_i \geq 1, \ \forall\ 1 \leq i \leq n \\ x_1 \geq 1, \ x_2 \geq 2, \ \cdots, \ x_{n-1} \geq (n-1) \end{cases} \\ \Rightarrow\ & y_{n+1} = r - x_n = x_1 + x_2 + \cdots + x_{n-1} \geq \frac{n(n-1)}{2} \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned}
& y_1 + 2\times y_2 + \cdots + n \times y_n + (n+1)\times y_{n+1} \\
&= (x_1 - 1) + 2\times(x_2 - x_1) + 3\times(x_3 - x_2) + \cdots + n\times(x_n - x_{n-1}) \\
&+ (n+1)\times(r - x_n) \\
&= \ -1 - x_1 - x_2 - x_3 - \cdots - x_n + (n+1)\times r \\
&= nr - 1
\end{aligned} \tag{14}$$

Then, we have new generating function

$$\begin{aligned}
G(x) =& (1 + x + x^2 + \cdots)(x^2 + x^4 + x^6 + \cdots)(x^3 + x^6 + x^9 + \cdots)\cdots \\
& (x^n + x^{2n} + x^{3n} + \cdots)(x^{(n+1)\frac{n(n-1)}{2}} + \cdots) \\
=& \frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}}
\end{aligned} \tag{15}$$

Answer † Coefficient of x^{nr-1} of

$$\frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}} \tag{16}$$

6. We have

$$\begin{aligned}
& \Rightarrow \alpha = 2 \\
& \Rightarrow \begin{cases} a_n^{(h)} = c\times 2^n \\ a_n^{(p)} = d\times 3^n \end{cases}
\end{aligned} \tag{17}$$

Then, we have

$$\begin{aligned}
& d\times 3^n = 2\times d\times 3^{n-1} + 3^{n-1} \\
& \Rightarrow d = 1 \\
& \Rightarrow a_n = c\times 2^n + 3^n
\end{aligned} \tag{18}$$

Then, we have

$$\begin{aligned}
& a_0 = 2 = c + 1 \\
& \Rightarrow c = 1
\end{aligned} \tag{19}$$

Answer †

$$a_n = 2^n + 3^n \tag{20}$$

7. We have

$$\mathrm{tr}(\boldsymbol{XY}) = \mathrm{tr}((\boldsymbol{XY})^{\mathsf{H}}) = \mathrm{tr}(\boldsymbol{Y}^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}}) = \mathrm{tr}(\overline{\boldsymbol{Y}^{\intercal}\boldsymbol{X}^{\intercal}}) = \mathrm{tr}(\overline{(\boldsymbol{XY})^{\intercal}}) = \mathrm{tr}(\overline{\boldsymbol{XY}}) \tag{21}$$

Answer †

$$a \tag{22}$$

8. We have

$$\boldsymbol{A} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \tag{23}$$

We have rank $\boldsymbol{A} = 3$. Then, we have

$$\begin{aligned} &\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) - 5 \leq \operatorname{rank}(\boldsymbol{AB}) \\ \Rightarrow\ &3 + \operatorname{rank}(\boldsymbol{B}) - 5 \leq 0 \end{aligned} \tag{24}$$

Answer †

$$\operatorname{rank}(\boldsymbol{B}) \leq 2 \tag{25}$$

9. **Answer** † Sum of eigenvalues equals to the trace.

$$2 + 2 + 2 + 2 = 8 \tag{26}$$

10. **Answer** † The problem is **WRONG**, since $\{\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \boldsymbol{A}_3,\ \boldsymbol{B}_1\}$ is linearly independent.

11. We have

$$[f]_\beta = \begin{bmatrix} f(\beta_1, \beta_1) & f(\beta_2, \beta_1) & f(\beta_3, \beta_1) & f(\beta_4, \beta_1) \\ f(\beta_1, \beta_2) & f(\beta_2, \beta_2) & f(\beta_3, \beta_2) & f(\beta_4, \beta_2) \\ f(\beta_1, \beta_3) & f(\beta_2, \beta_3) & f(\beta_3, \beta_3) & f(\beta_4, \beta_3) \\ f(\beta_1, \beta_4) & f(\beta_2, \beta_4) & f(\beta_3, \beta_4) & f(\beta_4, \beta_4) \end{bmatrix} \tag{27}$$

Answer †

$$[f]_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$