Solutions

NTU math 109

Version 1.0

1. Generating function:

$$\begin{aligned}&(x^a + x^{a+1} + \cdots + x^b)^n \\ =&[x^a(1 + x + x^2 + \cdots + x^{b-a})]^n \\ =&[x^a(\frac{x^{b-a+1}-1}{x-1})]^n\end{aligned} \tag{1}$$

Answer †

$$[x^a(\frac{x^{b-a+1}-1}{x-1})]^n \tag{2}$$

2. **Answer** †

$$(i^j)^{(n^m)} \tag{3}$$

3.

$$\Rightarrow x_1 + x_2 + x_3 + x_4 < 8,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4 \tag{4}$$

Let

$$\begin{aligned}&x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0 \\ \Rightarrow\ &y_5 = x_5 - 1,\ y_5 \geq 0\end{aligned} \tag{5}$$

Then,

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + y_5 = 7,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4,\ y_5 \geq 0 \tag{6}$$

Answer †

$$\binom{5+7-1}{7} = \binom{11}{7} = 330 \tag{7}$$

4.

$$\begin{aligned}&\Rightarrow \alpha = 3 \\ &\Rightarrow \begin{cases} a_n^{(h)} &= c \times 3^n \\ a_n^{(p)} &= d \times n + e \end{cases}\end{aligned} \tag{8}$$

We have

$$
\begin{aligned}
& d \times n + e = 3 \times (d \times (n-1) + e) + n \\
\Rightarrow & \begin{cases} d & = -\frac{1}{2} \\ e & = -\frac{3}{4} \end{cases} \\
\Rightarrow & a_n = c \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \qquad (9) \\
\Rightarrow & a_0 = 1 = c - \frac{3}{4} \\
\Rightarrow & c = \frac{7}{4} \\
\Rightarrow & a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4}
\end{aligned}
$$

Answer †

$$
a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \qquad (10)
$$

5.

$$
\Rightarrow \sum_{i=1}^{n} a_n x^n = 3 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \qquad (11)
$$

We have

$$
\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \qquad (12)
$$

Then, we have

$$
\begin{aligned}
\Rightarrow & \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\
\Rightarrow & \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \qquad (13) \\
\Rightarrow & \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2}
\end{aligned}
$$

We have the new generating function

$$
\begin{aligned}
& A(x) - a_0 = 3x \times A(x) + \frac{x}{(1-x)^2} \\
\Rightarrow & A(x) = \frac{x^2 - x + 1}{(1-3x)(1-x)^2} \qquad (14) \\
\Rightarrow & A(x) = \frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2}
\end{aligned}
$$

Answer †

$$
\frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2} \qquad (15)
$$

6. **Answer** † Since

$$\begin{aligned} &\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ \Rightarrow &\binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{\lceil \frac{n}{2} \rceil} \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor} + \cdots + \binom{n}{n} < 1 + n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &\Rightarrow n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^n \\ &\Rightarrow \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \frac{2^n}{n} \end{aligned} \tag{17}$$

7. We have

$$\left[\begin{array}{ccccc|c} 16 & -8 & 4 & -2 & 1 & 150 \\ 1 & -1 & 1 & -1 & 1 & 16 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 18 \\ 16 & 8 & 4 & 2 & 1 & 166 \end{array}\right] \tag{18}$$

Answer † $a,\ b,\ c,\ d,\ e$ are

$$8,\ 1,\ 7,\ 0,\ 2 \tag{19}$$

8. We have

$$\boldsymbol{B}\boldsymbol{B}^\intercal = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$\boldsymbol{B}\boldsymbol{B}^\intercal$ is **symmetric**, so it's **diagonalizable** and $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda)$. We have characteristic polynomial

$$p_{\boldsymbol{B}\boldsymbol{B}^\intercal}(x) = -x^2(x-1)(x-2)(x-4) \tag{21}$$

Answer † The nullities of $\boldsymbol{B}\boldsymbol{B}^\intercal - \lambda \boldsymbol{I}$ for $\lambda = 0,\ 1,\ 2,\ 3,\ 4$ are

$$2,\ 1,\ 1,\ 0,\ 1 \tag{22}$$

since 3 is NOT its eigenvalue.

9. We have

$$
\begin{aligned}
&\begin{cases} \forall\, x \in U,\ (\boldsymbol{B}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) \\ \forall\, x \in U^{\perp},\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}) \end{cases} \\
\Rightarrow &\begin{cases} \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) = U \to \mathrm{RS}(\boldsymbol{B}-\boldsymbol{A}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}-\boldsymbol{A}) = 1 \\ \mathrm{N}(\boldsymbol{B}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}) = 3 \end{cases}
\end{aligned} \tag{23}
$$

Let

$$
\boldsymbol{B}-\boldsymbol{A} = \begin{bmatrix} \alpha \times (0 & 1 & 0 & -1) \\ \beta \times (0 & 1 & 0 & -1) \\ \gamma \times (0 & 1 & 0 & -1) \\ \delta \times (0 & 1 & 0 & -1) \end{bmatrix} \Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \tag{24}
$$

We have

$$
U^{\perp} = \left\{ \boldsymbol{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \tag{25}
$$

Since $\boldsymbol{n} \in \mathrm{N}(\boldsymbol{B})$, $\boldsymbol{B}\boldsymbol{n} = \boldsymbol{0}$.

$$
\begin{aligned}
&\begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \boldsymbol{0} \\
&\Rightarrow \alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1 \\
&\Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{26}
$$

Answer † The numbers of $-2,\ -1,\ 0,\ 1,\ 2$ are

$$
0,\ 0,\ 10,\ 4,\ 2 \tag{27}
$$

10. $\boldsymbol{B} = \boldsymbol{A}^{+}$. We have characteristic polynomial

$$
p_{\boldsymbol{A}^{\intercal}\boldsymbol{A}}(x) = x(x-\frac{1}{4})(x-\frac{1}{2})(x-1) \tag{28}
$$

We have SVD of $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\intercal$.

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{V} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{29}$$

We have

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

Then, $\boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^\intercal$.

$$\boldsymbol{A}^+ = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{31}$$

Answer † The numbers of $-2,\ -1,\ 0,\ 1,\ 2$ are

$$1,\ 2,\ 16,\ 1,\ 0 \tag{32}$$

11. We have characteristic polynomial

$$p_{\boldsymbol{A}} = x(x-1)(x-3)^2 \tag{33}$$

Then, we have

$$\operatorname{gm}(3) = \operatorname{nullity}\left(\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2 \end{bmatrix}\right) = 1 \tag{34}$$

Then, we have Jordan form

$$J_{\boldsymbol{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \tag{35}$$

Answer † The numbers of $0,\ 1,\ 2,\ 3,\ 4$ are

$$12,\ 2,\ 0,\ 2,\ 0 \tag{36}$$