Solutions

NTU math 105

VERSION 1.0

1. We have

   (a) True.

   (b) False.

   (c) True. $\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA}) \to \operatorname{tr}(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{B}^{-1}\boldsymbol{BA}) = \operatorname{tr}(\boldsymbol{A})$.

   (d) False.

   (e) True.

   Answer †

$$ace \tag{1}$$

2. We have

   (a) True. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{BA})$

   (b) True.

   (c) True. $\det(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \det(\boldsymbol{B}^{-1}) \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \frac{1}{\det(\boldsymbol{B})} \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{A})$

   (d) True.

   (e) False.

   Answer †

$$abcd \tag{2}$$

3. We have

   (a) False.

(b) False, if $\boldsymbol{R} = \boldsymbol{O}$ is rectangular, but $\boldsymbol{R}^\intercal \boldsymbol{R} = \boldsymbol{O}$ is NOT positive definite.

(c) False, the orthogonal set contains $\mathbf{0}$, it's NOT linearly independent.

(d) True.

(e) True.

Answer †

$$de \tag{3}$$

4. We have

(a) False, since it does NOT contain $\mathbf{0}$.

(b) False, since it does NOT contain $\mathbf{0}$.

(c) False, since it does NOT contain $\mathbf{0}$.

(d) True.

(e) False, since it does NOT contain $\mathbf{0}$.

Answer †

$$d \tag{4}$$

5. Suppose

$$f(x) = 1 - x^k = (1 - x)(1 + x + x^2 + \cdots + x^{k-1}) \tag{5}$$

Then, we have

$$f(-\boldsymbol{N}) = \boldsymbol{I} + \boldsymbol{N}^k = (\boldsymbol{I} + \boldsymbol{N})(\boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1}) = \boldsymbol{I} \tag{6}$$

Answer †

$$(\boldsymbol{I} + \boldsymbol{N})^{-1} = \boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1} \tag{7}$$

6. **Answer** †

$$\frac{1}{4} \times x^2 + \frac{1}{9} \times y^2 = 1 \tag{8}$$

7. We have characteristic polynomial

$$p_{\boldsymbol{I}+\boldsymbol{A}}(x) = (x - 2)(x - 7) \tag{9}$$

Then, we have eigenspaces

$$\begin{cases} V(2) & = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\ V(7) & = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \end{cases} \tag{10}$$

Answer †

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{11}$$

Since the eigenvectors of $(\boldsymbol{I}+\boldsymbol{A})$ are the same as $(\boldsymbol{I}+\boldsymbol{A})^{100}$.

8. Obviously, 1 is an eigenvalue, which $\mathrm{gm}(1) = n-1$. Then, we have

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n}(1+x_i) = n + \sum_{i=1}^{n} x_i \tag{12}$$

Then, we have the n-th eigenvalue

$$n + (\sum_{i=1}^{n} x_i) - (n-1)\times 1 = 1 + \sum_{i=1}^{n} x_i \tag{13}$$

Answer †

$$\det(\boldsymbol{A}) = 1^{n-1} \times (1 + \sum_{i=1}^{n} x_i) = 1 + \sum_{i=1}^{n} x_i \tag{14}$$

9. We have new problem

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 < 8,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4 \\ & (x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0) \\ \Rightarrow\ & x_1 + x_2 + x_3 + x_4 + y_5 = 8 - 1,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4,\ y_5 \geq 0 \end{aligned} \tag{15}$$

Answer †

$$\binom{5 + (8-1) - 1}{8-1} = 330 \tag{16}$$

10. **Answer** †

$$(1,\ 2,\ 6) \circ (3,\ 5) \circ (4,\ 8) \circ (7) \tag{17}$$

11. We have

$$\begin{aligned} &\Rightarrow \alpha = 2 \\ &\Rightarrow \begin{cases} a_n^{(h)} = c \times 2^n \\ a_n^{(p)} = d \times n + e \end{cases} \end{aligned} \tag{18}$$

Then, we have

$$\begin{aligned} &d \times n + e = 2 \times (d \times (n-1) + e) + n \\ &\Rightarrow d = -1,\ e = -2 \\ &\Rightarrow a_n = c \times 2^n - n - 2 \end{aligned} \tag{19}$$

Then, we have

$$\begin{aligned} &a_0 = 4 = c - 0 - 2 \\ &\Rightarrow c = 6 \end{aligned} \tag{20}$$

Answer †

$$a_n = 6 \times 2^n - n - 2 \tag{21}$$

12.

$$\Rightarrow \sum_{i=1}^{n} a_n x^n = 2 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \tag{22}$$

We have

$$\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \tag{23}$$

Then, we have

$$\begin{aligned} &\Rightarrow \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\ &\Rightarrow \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\ &\Rightarrow \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2} \end{aligned} \tag{24}$$

We have the new generating function

$$\begin{aligned} &A(x) - a_0 = 2x \times A(x) + \frac{x}{(1-x)^2} \\ &\Rightarrow A(x) = \frac{4 \times x^2 - 7 \times x + 4}{(1-2x)(1-x)^2} \\ &\Rightarrow A(x) = 6 \times \frac{1}{1-2x} - 2 \times \frac{1}{1-x} - \frac{x}{(1-x)^2} \\ &\Rightarrow A(x) = 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \left(\frac{1}{1-x} + \frac{x}{(1-x)^2}\right) \\ &\Rightarrow A(x) = 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \end{aligned} \tag{25}$$

Answer †

$$6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \tag{26}$$

13. We have

$$(1 + x + x^2 + \cdots)(1 + (x^2) + (x^2)^2 + \cdots)(1 + (x^3) + (x^3)^2 + \cdots)\cdots \tag{27}$$

Since each number can be repeated.

Answer †

$$\prod_{i=1}^{n} \frac{1}{1-x^i} \tag{28}$$

14. **Answer** †

$$2^{2^{m-1}} \tag{29}$$

Since the base means 0 and 1 two values of the codomain, and the index means that if we know the 01-sequence then we know the opposite.

15. We have

$$n \leq 2 \times i + 1 \tag{30}$$

Answer †

$$i \geq \lfloor \frac{n-1}{2} \rfloor \tag{31}$$