國立臺灣大學 109 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1. (5%) Consider

$$x_1+x_2+\cdots+x_n=r,$$

where $a\le x_i\le b$ for $1\le i\le n$. What is the generating function for the number of integer solutions to the above equation (where the desired count appears as the coefficient of x^r, where $r=0,1,\ldots$)?

2. (5%) What is the number of functions from $\{1,2,\ldots,n\}^m$ to $\{1,2,\ldots,i\}^j$?

3. (10%) Consider

$$x_1+x_2+\cdots+x_n<r,$$

where $x_i\ge 0$ for $1\le i\le n$. What is its number of nonnegative integer solutions when $n=4$ and $r=8$?

4. (10%) Derive the solution for a_n that satisfies the recurrence equation $a_n=3a_{n-1}+n$ with $a_0=1$.

5. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _________. (Note that expressions like

$$\frac{x-8}{(x-3)^2}-\frac{9}{x-1}$$

are *not* partial fraction decompositions.)

6. (10%) Prove the following inequality:

$$\binom{n}{\lfloor n/2\rfloor}\ge\frac{2^n}{n}$$

where $2\le n$.

7. (10%) If the polynomial function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ satisfies

$$\begin{aligned} f(-2) &= 150 \\ f(-1) &= 16 \\ f(0) &= 2 \\ f(1) &= 18 \\ f(2) &= 166, \end{aligned}$$

then a, b, c, d, e are ____, ____, ____, ____, ____, respectively.

8. (10%) The nullities of the matrices $BB^T - \lambda I$ for $\lambda = 0, 1, 2, 3, 4$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

are ____, ____, ____, ____, ____, respectively.

9. (10%) Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Let

$$U = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The numbers of elements -2, -1, 0, 1, 2 in a matrix B with

$$Bx = \begin{cases} Ax & \text{if } x \in U \\ 0 & \text{if } x \in U^{\perp}. \end{cases}$$

are ____, ____, ____, ____, ____, respectively.

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10. (10%) If

$$A = \begin{bmatrix} 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

then the numbers of elements -2, -1, 0, 1, 2 in a matrix B with

$$\begin{aligned} ABA &= A \\ BAB &= B \\ (AB)^T &= AB \\ (BA)^T &= BA. \end{aligned}$$

are ____, ____, ____, ____, ____, respectively.

11. (10%) The numbers of elements 0, 1, 2, 3, 4 in a Jordan normal form of the matrix

$$A = \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

are ____, ____, ____, ____, ____, respectively.

Solutions

NTU math 109

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1. Generating function:

$$
\begin{aligned}
&(x^a + x^{a+1} + \cdots + x^b)^n \\
=&[x^a(1 + x + x^2 + \cdots + x^{b-a})]^n \\
=&[x^a(\frac{x^{b-a+1}-1}{x-1})]^n
\end{aligned} \tag{1}
$$

Answer †

$$[x^a(\frac{x^{b-a+1}-1}{x-1})]^n \tag{2}$$

2. **Answer** †

$$(i^j)^{(n^m)} \tag{3}$$

3.

$$\Rightarrow x_1 + x_2 + x_3 + x_4 < 8,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4 \tag{4}$$

Let

$$
\begin{aligned}
x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0 \\
\Rightarrow y_5 = x_5 - 1,\ y_5 \geq 0
\end{aligned} \tag{5}
$$

Then,

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + y_5 = 7,\ x_i \geq 0,\ \forall\ 1 \leq i \leq 4,\ y_5 \geq 0 \tag{6}$$

Answer †

$$\binom{5+7-1}{7} = \binom{11}{7} = 330 \tag{7}$$

4.

$$
\begin{aligned}
&\Rightarrow \alpha = 3 \\
&\Rightarrow \begin{cases} a_n^{(h)} &= c \times 3^n \\ a_n^{(p)} &= d \times n + e \end{cases}
\end{aligned} \tag{8}
$$

We have

$$
\begin{aligned}
& d \times n + e = 3 \times (d \times (n-1) + e) + n \\
\Rightarrow & \begin{cases} d & = -\frac{1}{2} \\ e & = -\frac{3}{4} \end{cases} \\
\Rightarrow & a_n = c \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \\
\Rightarrow & a_0 = 1 = c - \frac{3}{4} \\
\Rightarrow & c = \frac{7}{4} \\
\Rightarrow & a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4}
\end{aligned} \tag{9}
$$

Answer †

$$
a_n = \frac{7}{4} \times 3^n - \frac{1}{2} \times n - \frac{3}{4} \tag{10}
$$

5.

$$
\Rightarrow \sum_{i=1}^{n} a_n x^n = 3 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \tag{11}
$$

We have

$$
\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \tag{12}
$$

Then, we have

$$
\begin{aligned}
\Rightarrow & \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\
\Rightarrow & \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\
\Rightarrow & \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2}
\end{aligned} \tag{13}
$$

We have the new generating function

$$
\begin{aligned}
& A(x) - a_0 = 3x \times A(x) + \frac{x}{(1-x)^2} \\
\Rightarrow & A(x) = \frac{x^2 - x + 1}{(1-3x)(1-x)^2} \\
\Rightarrow & A(x) = \frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2}
\end{aligned} \tag{14}
$$

Answer †

$$
\frac{7}{4} \times \frac{1}{1-3x} - \frac{1}{4} \times \frac{1}{1-x} - \frac{1}{2} \times \frac{1}{(1-x)^2} \tag{15}
$$

6. **Answer** † Since

$$\begin{aligned} &\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ \Rightarrow &\binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{\lceil \frac{n}{2} \rceil} \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor} + \cdots + \binom{n}{n} < 1 + n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &\Rightarrow n \times \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^n \\ &\Rightarrow \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \frac{2^n}{n} \end{aligned} \tag{17}$$

7. We have

$$\left[\begin{array}{ccccc|c} 16 & -8 & 4 & -2 & 1 & 150 \\ 1 & -1 & 1 & -1 & 1 & 16 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 18 \\ 16 & 8 & 4 & 2 & 1 & 166 \end{array}\right] \tag{18}$$

Answer † $a,\ b,\ c,\ d,\ e$ are

$$8,\ 1,\ 7,\ 0,\ 2 \tag{19}$$

8. We have

$$\boldsymbol{B}\boldsymbol{B}^\intercal = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$\boldsymbol{B}\boldsymbol{B}^\intercal$ is **symmetric**, so it's **diagonalizable** and $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda)$. We have characteristic polynomial

$$p_{\boldsymbol{B}\boldsymbol{B}^\intercal}(x) = -x^2(x-1)(x-2)(x-4) \tag{21}$$

Answer † The nullities of $\boldsymbol{B}\boldsymbol{B}^\intercal - \lambda \boldsymbol{I}$ for $\lambda = 0,\ 1,\ 2,\ 3,\ 4$ are

$$2,\ 1,\ 1,\ 0,\ 1 \tag{22}$$

since 3 is NOT its eigenvalue.

9. We have

$$\begin{aligned}
&\begin{cases} \forall\ x \in U,\ (\boldsymbol{B}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) \\ \forall\ x \in U^{\perp},\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0} \to \boldsymbol{x} \in \mathrm{N}(\boldsymbol{B}) \end{cases} \\
\Rightarrow &\begin{cases} \mathrm{N}(\boldsymbol{B}-\boldsymbol{A}) = U \to \mathrm{RS}(\boldsymbol{B}-\boldsymbol{A}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}-\boldsymbol{A}) = 1 \\ \mathrm{N}(\boldsymbol{B}) = U^{\perp} \to \mathrm{rank}(\boldsymbol{B}) = 3 \end{cases}
\end{aligned} \tag{23}$$

Let

$$\boldsymbol{B}-\boldsymbol{A} = \begin{bmatrix} \alpha \times (0 & 1 & 0 & -1) \\ \beta \times (0 & 1 & 0 & -1) \\ \gamma \times (0 & 1 & 0 & -1) \\ \delta \times (0 & 1 & 0 & -1) \end{bmatrix} \Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \tag{24}$$

We have

$$U^{\perp} = \left\{ \boldsymbol{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \tag{25}$$

Since $\boldsymbol{n} \in \mathrm{N}(\boldsymbol{B})$, $\boldsymbol{B}\boldsymbol{n} = \boldsymbol{0}$.

$$\begin{aligned}
&\begin{bmatrix} 2 & \alpha & 0 & 2\times\alpha \\ 0 & \beta & 0 & -\beta \\ 0 & \gamma & 0 & -\gamma \\ 2 & \delta & 0 & (2-\delta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \boldsymbol{0} \\
&\Rightarrow \alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1 \\
&\Rightarrow \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{26}$$

Answer † The numbers of $-2,\ -1,\ 0,\ 1,\ 2$ are

$$0,\ 0,\ 10,\ 4,\ 2 \tag{27}$$

10. $\boldsymbol{B} = \boldsymbol{A}^{+}$. We have characteristic polynomial

$$p_{\boldsymbol{A}^{\intercal}\boldsymbol{A}}(x) = x(x-\frac{1}{4})(x-\frac{1}{2})(x-1) \tag{28}$$

We have SVD of $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\intercal$.

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{V} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{29}$$

We have

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

Then, $\boldsymbol{A}^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^\intercal$.

$$\boldsymbol{A}^+ = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{31}$$

Answer † The numbers of $-2, -1, 0, 1, 2$ are

$$1, \ 2, \ 16, \ 1, \ 0 \tag{32}$$

11. We have characteristic polynomial

$$p_{\boldsymbol{A}} = x(x-1)(x-3)^2 \tag{33}$$

Then, we have

$$\mathrm{gm}(3) = \mathrm{nullity}\left(\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2 \end{bmatrix} \right) = 1 \tag{34}$$

Then, we have Jordan form

$$J_{\boldsymbol{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \tag{35}$$

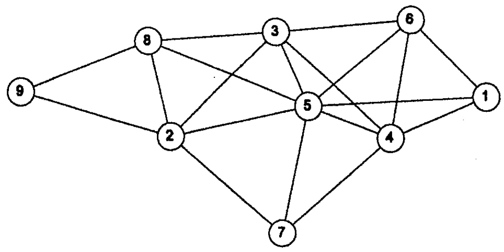
Answer † The numbers of $0,\ 1,\ 2,\ 3,\ 4$ are

$$12,\ 2,\ 0,\ 2,\ 0 \tag{36}$$

請在答案卷可作答部分的第一頁繪製十二列的表格如下並填入各題的答案

1	
2	
⋮	
11	
12	

1. (5%) Consider the following undirected graph. Write down 4 nodes which form an independent set.



2. (5%) A palindrome is a sequence of symbols that reads the same left to right as right to left. What is the number of palindromic binary numbers of length n?

3. (5%) Derive

$$A = \_\_\_\_$$
$$B = \_\_\_\_$$

in

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{\binom{A}{B}}{2}.$$

4. (10%) Derive

$$A = \_\_\_\_$$
$$B = \_\_\_\_$$

in

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{A}{B}.$$

見背面

5. (10%) The solution to the recurrence equation

$$a_{n+2} = a_{n+1} + 2 \times a_n$$

is of the form:

$$a_n = A(-X)^n + BY^n.$$

Derive

$$\begin{aligned} A &= \underline{\quad\quad}, \\ B &= \underline{\quad\quad}, \\ X &= \underline{\quad\quad}, \\ Y &= \underline{\quad\quad}, \end{aligned}$$

in terms of (the arbitrary initial conditions) a_0 and a_1.

6. (10%) Consider

$$x_1 + x_2 + \cdots + x_n = r,$$

where $x_i > n_i$ for $1 \le i \le n$. The number of positive integer solutions is _____.

7. (5%) Is the following a tautology:

$$[(p \to q) \land \neg p] \to \neg q$$

Why?

8. (10%) How many of the following five statements are correct? __________

- Let V be a vector space. Let S be a subset of V. Let U be a subspace of V. If $S \subseteq U$, then $\operatorname{span}(S) \subseteq U$.
- If R is a linearly dependent subset of a vector space, then $x \in \operatorname{span}(R \setminus \{x\})$ holds for each vector $x \in R$.
- Based on any consistent axiom set for set theory, any vector space admits a basis.
- Based on the standard ZFC axioms for set theory, each inner-product space admits an orthonormal basis.
- For any complex vector spaces V and W with $\dim(W) < \infty$, if T is a linear surjection from V to W, then $\dim(V) = \operatorname{nullity}(T) + \operatorname{rank}(T)$.

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9. (10%) How many of the following five statements are correct? __________

- The solution set for a system of homogeneous linear equations having an $m \times n$ rational coefficient matrix is a rational vector space.
- An $m \times n$ complex matrix A is invertible if and only if A^* is invertible.
- If A is an $m \times n$ rational matrix, then $\mathrm{rank}(A) = \mathrm{rank}(A^t)$.
- If A is an $m \times n$ invertible real matrix, then $\mathrm{nullity}(A) = \mathrm{nullity}(A^{-1})$.
- If A is an $n \times n$ complex matrix, then $\det(A^*) = \overline{\det(A)}$.

10. (10%) How many of the following five statements are correct? __________

- Let A be an $n \times n$ rational matrix. If the rational n-tuple vector space $\mathbb{Q}^n$ is the direct sum of the eigenspaces of A, then A can be diagonalized.
- If A is an $n \times n$ complex matrix with $A^*A = AA^*$, then the eigenspaces of A^* equal the eigenspaces of A.
- If A is an $n \times n$ real matrix with $A^t = A$, then the characteristic polynomial of A can be written as a product of degree-one polynomials with real coefficients.
- If A is an $n \times n$ complex matrix with $A^t = A$, then all eigenvalues of A are real.
- If A and B are unitarily equivalent $n \times n$ complex matrices, then the trace of A^*A equals the trace of B^*B.

11. (10%) How many zero entries are there in the inverse of the following matrix? _____

$$\begin{pmatrix} -15 & -6 & 5 & 9 \\ -12 & 9 & 4 & -2 \\ 20 & 8 & 1 & -12 \\ 18 & -2 & -6 & 3 \end{pmatrix}$$

12. (10%) Give a basis for the vector space of the linear transformations from the vector space $\mathbb{R}^3$ of real triples to the vector space $\mathbb{R}^2$ of real pairs: ____________________.

試題隨卷繳回

Solutions

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1. **Answer** † 1, 3, 7, 9.

2. We have recurrence function

$$\begin{cases} a_n = 2 \times a_{n-2},\ n \geq 3 \\ a_1 = 2,\ a_2 = 2 \end{cases} \tag{1}$$

Then, we have

$$\begin{aligned} &\alpha^2 = 2 \\ &\Rightarrow \alpha = \pm\sqrt{2} \\ &\Rightarrow a_n = c \times (\sqrt{2})^n + d \times (-\sqrt{2})^n \end{aligned} \tag{2}$$

Then, we have

$$\begin{aligned} &\begin{cases} a_1 = 2 = \sqrt{2} \times c - \sqrt{2} \times d \\ a_2 = 2 = 2 \times c + 2 \times d \end{cases} \\ &\Rightarrow c = \frac{\sqrt{2}+2}{2\sqrt{2}},\ d = \frac{\sqrt{2}-2}{2\sqrt{2}} \end{aligned} \tag{3}$$

Answer †

$$a_n = \frac{\sqrt{2}+2}{2\sqrt{2}} \times (\sqrt{2})^n + \frac{\sqrt{2}-2}{2\sqrt{2}} \times (-\sqrt{2})^n \tag{4}$$

3. We have

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = 2 \times \binom{2n+2}{n+1} \tag{5}$$

Answer †

$$A = 2n+2,\ B = n+1 \tag{6}$$

4. We have

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n-(k-1)} = \binom{2n}{n+1} \tag{7}$$

Answer †

$$A = 2n,\ B = n + 1 \tag{8}$$

5. We have

$$\begin{aligned} &\alpha^2 = \alpha + 2 \\ &\Rightarrow \alpha = 2 \vee \alpha = -1 \\ &\Rightarrow a_n = c \times 2^n + d \times (-1)^n \end{aligned} \tag{9}$$

We have

$$\begin{aligned} &\begin{cases} a_0 = c + d \\ a_1 = 2 \times c - d \end{cases} \\ &\Rightarrow c = \frac{a_0 + a_1}{3},\ d = \frac{2 \times a_0 - a_1}{3} \\ &\Rightarrow a_n = \frac{2 \times a_0 - a_1}{3} \times (-1)^n + \frac{a_0 + a_1}{3} \times 2^n \end{aligned} \tag{10}$$

Answer †

$$A = \frac{2 \times a_0 - a_1}{3},\ B = \frac{a_0 + a_1}{3},\ X = 1,\ Y = 2 \tag{11}$$

6. We have

$$\begin{aligned} &x_1 + x_2 + \cdots + x_n = r,\ \forall\, x_i \geq n_i + 1,\ 1 \leq i \leq n \\ &\Rightarrow y_1 + y_2 + \cdots + y_n = r - ((\sum_{i=1}^{n} n_i) + n),\ \forall\, y_i \geq 0,\ 1 \leq i \leq n \end{aligned} \tag{12}$$

Answer †

$$\binom{n + r - (\sum_{i=1}^{n} n_i) - n - 1}{r - (\sum_{i=1}^{n} n_i) - n} \tag{13}$$

7. **Answer** †

$$\begin{aligned} &[(p \to q) \wedge \neg p] \to \neg q \\ \iff\ &[(\neg p \vee q) \wedge \neg p] \to \neg q \\ \iff\ &\neg[(\neg p \vee q) \wedge \neg p] \vee \neg q \\ \iff\ &[(p \wedge \neg q) \vee p] \vee \neg q \\ \iff\ &p \wedge (\neg q \vee p) \vee \neg q \\ \iff\ &(p \vee \neg q) \wedge [(p \vee \neg q) \vee \neg q] \\ \iff\ &(p \vee \neg q) \wedge [(p \vee \neg q) \vee \neg q] \\ \iff\ &(p \vee \neg q) \end{aligned} \tag{14}$$

So, it's NOT tautology.

8. We have

- True. Let
$$\begin{aligned} S &= \{a,\ b\} \subset U \\ \Rightarrow\ \text{span}(s) &= c_1 a + c_2 b \subset U \end{aligned} \tag{15}$$
- False. Counterexample:
$$\begin{aligned} R &= \{(1,\ 0),\ (0,\ 1),\ (0,\ 2)\} \\ \Rightarrow (1,\ 0) &\notni \text{span}(R \setminus \{(1,\ 0)\}) \end{aligned} \tag{16}$$
- True.
- False. Counterexample: $\text{span}(\mathbf{0}) = \emptyset$, but $\emptyset$ is NOT orthonormal.
- True.

Answer †
$$3 \tag{17}$$

9. We have
   - True.
   - True. $\boldsymbol{A}$ is invertible $\iff$ $\det(\boldsymbol{A}) \neq 0$ $\iff$ $\det(\boldsymbol{A}^{\mathsf{H}}) \neq 0$
   - True.
   - True. $\boldsymbol{A}$ is invertible, so $\text{rank}(\boldsymbol{A}) = m = n = \text{rank}(\boldsymbol{A}^{-1})$.
   - True. $\det(\boldsymbol{A}^{\mathsf{H}}) = \det(\overline{\boldsymbol{A}^{\mathsf{T}}}) = \overline{\det(\boldsymbol{A}^{\mathsf{T}})} = \overline{\det(\boldsymbol{A})}$

Answer †
$$5 \tag{18}$$

10. We have
   - True. $\mathbb{Q}^n$ is the direct sum of eigenspace of $\boldsymbol{A}$ $\iff$ there are n linearly independent eigenvectores of $\boldsymbol{A}$ $\iff$ $\boldsymbol{A}$ is diagonalizable
   - True.
   - False. $\boldsymbol{A}$ may NOT be split.
   - False. If the $\boldsymbol{A}$ is **real** and symmetric, all of its eigenvalues are always real. It can NOT be ensured if $\boldsymbol{A}$ is **complex**.
   - True.

Answer †

$$3 \tag{19}$$

11. We have inverse

$$\begin{bmatrix} \frac{529}{12167} & 0 & \frac{529}{12167} & \frac{529}{12167} \\ 0 & \frac{1587}{12167} & 0 & \frac{1058}{12167} \\ \frac{2116}{12167} & 0 & \frac{1587}{12167} & 0 \\ \frac{1058}{12167} & \frac{1058}{12167} & 0 & \frac{1587}{12167} \end{bmatrix} \tag{20}$$

Answer †

$$6 \tag{21}$$

12. **Answer** †

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

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1. (10%) The number of nonnegative integer solutions to

$$L \leq x_1 + x_2 + \cdots + x_n \leq H$$

is _________.

2. (10%) The solution to the recurrence equation

$$a_n = 2a_{n-1} + 3a_{n-2}$$

with $a_0 = 1$ and $a_1 = 1$ is $a_n =$ _________.

3. (10%) The generating function for the square numbers $1^2, 2^2, 3^2, \ldots$ is _________.

4. (10%) The number of reflexive symmetric relations on A where $|A| = m$ is _________.

5. (10%) The number of simple, labeled graphs (with self-loops allowed) with n nodes is _________.

   The number of simple, labeled graphs with n nodes and m edges is _________.

6. (10%) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Please find $f(A)$, where $f(t) = t^4 - 3t^3 - 6t^2 + 7t + 2$.

7. (10%) Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & 0 & a_1 \\ 0 & -1 & 0 & 0 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & a_{n-1} \end{bmatrix}_{n\times n}$ and I_n be the $n\times n$ identity matrix. Find $det(A + tI_n)$.

8. (10%) Consider a subspace $V = \mathrm{P}_2(\mathrm{t})$ of $\mathrm{P(t)}$ with inner product defined as

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

   Please find an orthogonal set of $\{1, t, t^2\}$ with integer coefficients and the projection of t^3 onto V.

9. (10%) Your answer will be considered correct only if all the true statements are selected.
   Let A be an $n\times n$ matrix.
   (a) If x_1 and x_2 are the eigenvectors of A, then $x_1 + x_2$ is also an eigenvector of A.
   (b) If $A^T = -A$, then A is singular.
   (c) If $A^2 = A$. $(A + I)^n = I + (2^n + 1)A$.
   (d) If $A = A^T$, then A is diagonalizable.
   (e) If $A = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$ and B and D are invertible, then $A^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ O & D^{-1} \end{bmatrix}$.

10. (10%) Your answer will be considered correct only if all the true statements are selected.
   (a) Suppose $\{u, v, w\}$ is linearly independent, then $\{u + v, v + w, w + u\}$ is also linearly independent.
   (b) If two matrices A and B are similar, then they have the same eigenvalues.
   (c) If two matrices A and B are similar, then they have the same eigenvectors.
   (d) Let V be a vector space of $m\times n$ matrices over $\mathcal{R}$. $\langle A, B\rangle = tr(B^T A)$ defines an inner product in V.
   (e) If U and W are subspaces of a finite-dimensional inner product space V, then $(U + W)^\perp = U^\perp \cup W^\perp$.

試題隨卷繳回

Solutions

NTU math 107

VERSION 1.0

1. We have new question

$$
\begin{aligned}
&(x_1 + x_2 + \cdots + x_n \leq H) - (x_1 + x_2 + \cdots + x_n < L),\ \forall\, x_i \geq 0,\ 1 \leq i \leq n \\
&(\text{Let } x_{n+1} = H - (x_1 + x_2 + \cdots + x_n),\ x_{n+1} \geq 0, \\
&y_{n+1} = L - (x_1 + x_2 + \cdots + x_n),\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\, x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\quad - (x_1 + x_2 + \cdots + x_n + y_{n+1} = L,\ \forall\, x_i \geq 0,\ 1 \leq i \leq n,\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\, x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\quad - (x_1 + x_2 + \cdots + x_n + z_{n+1} = L - 1,\ \forall\, x_i \geq 0,\ 1 \leq i \leq n,\ z_{n+1} \geq 0)
\end{aligned} \tag{1}
$$

Answer †

$$
\binom{(n+1) + H - 1}{H} - \binom{(n+1) + (L-1) - 1}{L-1} \tag{2}
$$

2. We have

$$
\begin{aligned}
&\alpha^2 = 2 \times \alpha + 3 \\
\Rightarrow\ &\alpha = 3 \lor \alpha = -1 \\
\Rightarrow\ &a_n = c \times 3^n + d \times (-1)^n
\end{aligned} \tag{3}
$$

Then, we have

$$
\begin{aligned}
&\begin{cases} a_0 = 1 = c + d \\ a_1 = 1 = 3 \times c - d \end{cases} \\
\Rightarrow\ &c = \frac{1}{2},\ d = \frac{1}{2}
\end{aligned} \tag{4}
$$

Answer †

$$
a_n = \frac{1}{2} \times 3^n + \frac{1}{2} \times (-1)^n \tag{5}
$$

3. We have

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+1)^2x^n \\ &\overset{\text{integral}}{=} \sum_{n=0}^{\infty}(n+1)x^{n+1} = x\sum_{n=0}^{\infty}(n+1)x \end{aligned} \tag{6}$$

Then, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+1)x^n \\ &\overset{\text{integral}}{=} \sum_{n=0}^{\infty}x^{n+1} = \frac{x}{1-x} \end{aligned} \tag{7}$$

Then, we have

$$\begin{aligned} &\frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\ &\Rightarrow \sum_{n=0}^{\infty}(n+1)x^{n+1} = \frac{x}{(1-x)^2} \end{aligned} \tag{8}$$

And, we have

$$\begin{aligned} &\frac{x}{(1-x)^2} \overset{\text{derivative}}{=} \frac{1+x}{(1-x)^3} \\ &\Rightarrow \sum_{n=0}^{\infty}(n+1)^2x^n = \frac{1+x}{(1-x)^3} \end{aligned} \tag{9}$$

Answer †

$$\frac{1+x}{(1-x)^3} \tag{10}$$

4. **Answer** †

$$2^{\binom{m}{2}} \tag{11}$$

5. **Answer** †

$$2^{\frac{n(n+1)}{2}}, \quad \binom{\binom{n}{2}}{m} \tag{12}$$

6. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x) = x^2 - 5\times x + 4 \tag{13}$$

Then, we have

$$\begin{aligned} f(\boldsymbol{A}) &= \boldsymbol{A}^4 - 3\times\boldsymbol{A}^3 - 6\times\boldsymbol{A}^2 + 7\times\boldsymbol{A} + 2\times\boldsymbol{I} \\ &= (\boldsymbol{A}^2 + 2\times\boldsymbol{A})(\boldsymbol{A}^2 - 5\times\boldsymbol{A} + 4\times\boldsymbol{I}) + (-\boldsymbol{A} + 2\times\boldsymbol{I}) \\ &= (-\boldsymbol{A} + 2\times\boldsymbol{I}) \end{aligned} \tag{14}$$

Answer †

$$\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix} \tag{15}$$

7. We have

$$\begin{aligned} \det(\boldsymbol{A}+t\boldsymbol{I}) &= \begin{bmatrix} t & 0 & 0 & \cdots & a_0 \\ -1 & t & 0 & \cdots & a_1 \\ 0 & -1 & t & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}+t \end{bmatrix}_{n\times n} \\ &= t^{n-1}[(a_{n-1}+t)+\frac{1}{t}a_{n-2}+\frac{1}{t^2}a_{n-3}+\cdots+a_0] \\ &= t^n+t^{n-1}a_{n-1}+t^{n-2}a_{n-2}+\cdots+a_0 \\ &= t^n+\sum_{i=0}^{n-1}a_it^i \end{aligned} \tag{16}$$

Answer †

$$t^n+\sum_{i=0}^{n-1}a_it^i \tag{17}$$

8. By Gram-Schmidt process, we have

$$\begin{aligned} u_1 &= 1,\ ||u_1|| = \int_0^1 1\times 1dt = 1 \\ u_2 &= t - \frac{\int_0^1 1\times tdt}{1}\times 1 = t-\frac{1}{2},\ ||u_2|| = \int_0^1 (t-\frac{1}{2})^2dt = \frac{1}{12} \\ u_3 &= t^2 - \frac{\int_0^1 1\times t^2dt}{1}\times 1 - \frac{\int_0^1 (t-\frac{1}{2})\times t^2dt}{\frac{1}{12}}\times(t-\frac{1}{2}) = t^2-t+\frac{1}{6}, \\ ||u_3|| &= \int_0^1 (t^2-t+\frac{1}{6})^2dt = \frac{1}{180} \end{aligned} \tag{18}$$

We have projection

$$\begin{aligned} &\frac{\int_0^1 1\times t^3dt}{1}\times 1 + \frac{\int_0^1 (t-\frac{1}{2})\times t^3dt}{\frac{1}{12}}(t-\frac{1}{2}) + \frac{\int_0^1 (t^2-t+\frac{1}{6})\times t^3dt}{\frac{1}{180}}\times(t^2-t+\frac{1}{6}) \\ =&\frac{3}{2}\times t^2-\frac{3}{5}\times t+\frac{1}{20} \end{aligned} \tag{19}$$

Answer †

$$\frac{3}{2}\times t^2-\frac{3}{5}\times t+\frac{1}{20} \tag{20}$$

9. **Answer** † We have

   - False.
   - False. We have $\det(\boldsymbol{A}^\intercal) = \det(-\boldsymbol{A}) \iff \det(\boldsymbol{A}) = (-1)^n \times \det(\boldsymbol{A})$. ONLY if the dimension of $\boldsymbol{A}$, i.e. n, is odd, then $\boldsymbol{A}$ is singular; otherwise, it's non-singular.
   - False. $(\boldsymbol{A}+\boldsymbol{I})^n = (2^n - 1) \times \boldsymbol{A}$.
   - True, since symmetric matrix is **orthogonally diagonalizable**, and it's also **diagonalizable**.
   - True. Suppose

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} \tag{21}$$

Then, we have

$$\begin{aligned}
&\Rightarrow \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \\
&\Rightarrow \begin{cases} \boldsymbol{BP} + \boldsymbol{CR} = \boldsymbol{I} \\ \boldsymbol{BQ} + \boldsymbol{CS} = \boldsymbol{O} \\ \boldsymbol{DR} = \boldsymbol{O} \to \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{DS} = \boldsymbol{I} \to \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \begin{cases} \boldsymbol{P} = \boldsymbol{B}^{-1} \\ \boldsymbol{Q} = -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{O} & \boldsymbol{D}^{-1} \end{bmatrix}
\end{aligned} \tag{22}$$

10. **Answer** † We have

   - True. We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{23}$$

And, we have

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\right) = 2 \tag{24}$$

is non-singular, so $\{u+v,\ v+w,\ w+u\}$ is linearly independent.

- True. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, so $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues.
- False. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, but the eigenvectors may differ.
- False, since if $m \neq n$, $\boldsymbol{B}^{\intercal}\boldsymbol{A}$ may NOT exist.
- False, since
$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp} \tag{25}$$

國立臺灣大學 106 學年度碩士班招生考試試題

1. (10%) What is the inverse of the following real matrix?

$$\begin{pmatrix} r^2-2 & -r & 1 \\ -r & 1 & 0 \\ 3-r^2 & r & -1 \end{pmatrix}$$

2. (10%) What is the smallest positive integer k such that

$$A^k = A$$

holds for any n-by-n diagonalizable complex matrix A whose eigenvalues are 0 and 1.

3. (10%) Let (x, y) be a point in $\mathbb{R}^2$ that is not the origin $(0,0)$. Let L be the line of $\mathbb{R}^2$ passing (x, y) and $(0,0)$. What is the matrix representation of the orthogonal projection of $\mathbb{R}^2$ on L with respect to the standard basis of $\mathbb{R}^2$?

4. (10%) Define function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x, y) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2,$$

for any vector x (respectively, y) with standard coordinate (x_1, x_2) (respectively, (y_1, y_2)). Let

$$u = (1, 0).$$

Find a vector v such that

$$f(x, y) = s_1t_1 + s_2t_2,$$

where (s_1, s_2) (respectively, (t_1, t_2)) is the coordinate of x (respectively, y) with respect to the ordered basis of $\mathbb{R}^2$ consisting of u and v.

5. (10%) Find the pseudo-inverse of

$$\frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

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國立臺灣大學 106 學年度碩士班招生考試試題

6. (10%) For $|A| = m$, how many relations on A are neither reflexive nor irreflexive.

7. (10%) Please fill in the blanks:

$$[(p \vee q) \wedge (\neg p \vee r)] \Rightarrow (\_\_ \vee \_\_).$$

8. (10%) The solution to the recurrence equation

$$a_{n+2} = a_{n+1} + a_n$$

is of the form:

$$a_n = \left(\frac{A}{2\sqrt{5}}\right)\left(\frac{B}{2}\right)^n + \left(\frac{C}{2\sqrt{5}}\right)\left(\frac{D}{2}\right)^n.$$

Derive

$$\begin{aligned} A &= \_\_\_\_ \\ B &= \_\_\_\_ \\ C &= \_\_\_\_ \\ D &= \_\_\_\_ \end{aligned}$$

in terms of (arbitrary initial conditions) a_0 and a_1 .

9. (5%) Prove that $n-1$ and n are relatively prime for $n \geq 2$.

10. (5%) A graph is ________ if and only if all its cycles have an even length.

11. (5%) For odd n, simplify

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}}.$$

12. (5%) Consider the following directed graph (first plot). Draw its transitive closure in the second plot.

接次頁

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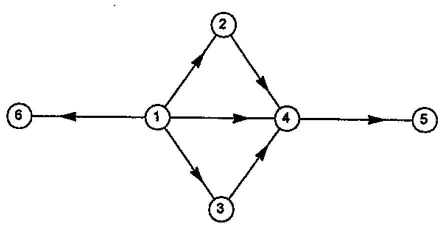
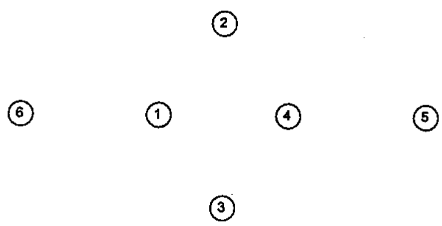




試題隨卷繳回

Solutions

NTU math 106

Version 1.0

1. **Answer** †

$$\begin{bmatrix} 1 & 0 & 1 \\ r & 1 & r \\ 3 & r & 2 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$2 \tag{2}$$

Since $\boldsymbol{A}$ has eigenvalues 0 and 1, it's **idempotent**, $\boldsymbol{A}^2 = \boldsymbol{A}$.

3. Suppose

$$\boldsymbol{A} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

Then, we have projection

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\intercal \boldsymbol{A})^{-1} \boldsymbol{A}^\intercal \tag{4}$$

Answer †

$$\frac{1}{x^2 + y^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \tag{5}$$

4. Suppose

$$\begin{cases} \beta & = \{(1,\ 0),\ (0,\ 1)\} \\ \gamma & = \{\boldsymbol{u} = (1,\ 0),\ \boldsymbol{v} = (a,\ b)\} \end{cases} \tag{6}$$

Then, we have

$$[x]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ [y]_\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ [x]_\gamma = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},\ [y]_\gamma = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \tag{7}$$

Then ,we have transition matrix

$$
\begin{aligned}
[\boldsymbol{I}]_\gamma^\beta &= \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \\
\Rightarrow [x]_\beta &= \begin{bmatrix} s_1 + s_2 a \\ s_2 b \end{bmatrix}, \ [y]_\beta = \begin{bmatrix} t_1 + t_2 a \\ t_2 b \end{bmatrix}
\end{aligned} \tag{8}
$$

Then, we have

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{y}) &= (s_1 + s_2 a)(t_1 + t_2 a - t_2 b) + s_2 b \times (-t_1 - t_2 a + 4 \times t_2 b) \\
&= s_1 t_1 + s_2 t_2 \\
\Rightarrow a = b = \pm \frac{1}{\sqrt{3}}
\end{aligned} \tag{9}
$$

Answer †

$$
\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \vee \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} \end{bmatrix} \tag{10}
$$

5. We have pseudo-inverse

$$
\boldsymbol{A}^+ = (\boldsymbol{A}^\intercal \boldsymbol{A})^{-1} \boldsymbol{A}^\intercal \tag{11}
$$

Answer †

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \tag{12}
$$

6. **Answer** †

$$
2^{m^2} - 2^{m^2-m} - 2^{m^2-m} \tag{13}
$$

Since both reflexive and irreflexive are 2^{m^2-m}.

7. We have

$$
\begin{aligned}
& [(p \vee q) \wedge (\neg p \vee r)] \\
\iff & (q \wedge \neg p) \vee (p \wedge r) \vee (q \wedge r) \\
& \text{(Draw the Venn diagram)} \\
\iff & (p \wedge r) \vee (\neg p \wedge q)
\end{aligned} \tag{14}
$$

Answer †

$$
(p \wedge r) \vee (\neg p \wedge q) \tag{15}
$$

8. We have

$$\begin{aligned} &\alpha^2 = \alpha + 1 \\ &\Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2} \\ &\Rightarrow a_n = c \times (\frac{1+\sqrt{5}}{2})^n + d \times (\frac{1-\sqrt{5}}{2})^n \end{aligned} \tag{16}$$

And, we have

$$\begin{aligned} &\begin{cases} a_0 = c + d \\ a_1 = c \times (\frac{1+\sqrt{5}}{2}) + d \times (\frac{1-\sqrt{5}}{2}) \end{cases} \\ &\Rightarrow c = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}},\ d = \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \\ &\Rightarrow a_n = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}} \times (\frac{1+\sqrt{5}}{2})^n + \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \times (\frac{1-\sqrt{5}}{2})^n \end{aligned} \tag{17}$$

Answer †

$$A = 2 \times a_1 - (1-\sqrt{5})a_0,\ B = 1+\sqrt{5},\ C = (1+\sqrt{5})a_0 - 2 \times a_1,\ D = 1-\sqrt{5} \tag{18}$$

9. **Answer** † We have

$$\begin{aligned} &\gcd(n, n-1) \\ &= \gcd(n-1,\ 1) \\ &= 1 \end{aligned} \tag{19}$$

10. **Answer** † bipartite

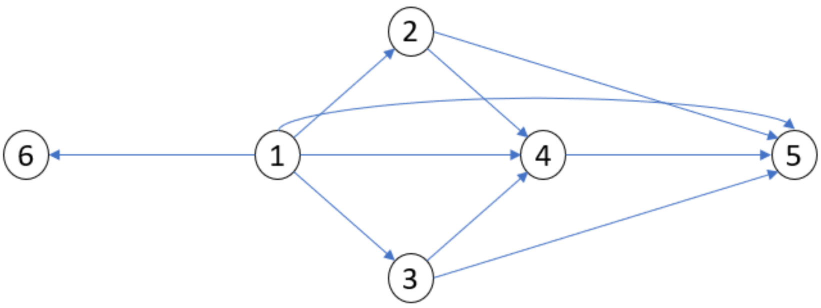
11. We have

$$\begin{aligned} &2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} + \binom{n}{\frac{n+1}{2}} + \cdots + \binom{n}{n} \\ &\Rightarrow \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} = \frac{2^n}{2} = 2^{n-1} \end{aligned} \tag{20}$$

Answer †

$$2^{n-1} \tag{21}$$

12. **Answer** †


2
6
1
4
5
3

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

From Problem 1 to Problem 4, your answer will be considered correct only if all the true statements are selected.

1. (5%) Let $A, B \in M_{n\times n}(F)$, where $F = \boldsymbol{C}$ or $\boldsymbol{R}$. Which of the following statements are true?
   (a) $tr(AB) = tr(BA)$
   (b) $tr(AB) = tr(A)tr(B)$
   (c) $tr(B^{-1}AB) = tr(A)$
   (d) $tr(A^k) = (tr(A))^k$
   (e) $tr(A \pm B) = tr(A) \pm tr(B)$

2. (5%) Let $A, B \in M_{n\times n}(F)$, where $F = \boldsymbol{C}$ or $\boldsymbol{R}$. Which of the following statements are true?
   (a) $det\,(AB) = det(BA)$
   (b) $det(AB) = det(A)det(B)$
   (c) $det(B^{-1}AB) = det(A)$
   (d) $det(A^k) = (det\,(A))^k$
   (e) $det(A \pm B) = det(A) \pm det(B)$

3. (5%) Which of the following statements are true?
   (a) If A and B are invertible matrices, then the matrix $C = A^{-1} + B^{-1}$ is also invertible.
   (b) If R is a rectangular matrix, then $A = R^T R$ is positive definite.
   (c) Every orthogonal set is linearly independent.
   (d) $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$, where T_{11} is a $p\times p$ matrix, T_{22} is a $q\times q$ matrix, and T_{12} is a $p\times q$ matrix. The set of eigenvalues of T is the union of the sets of eigenvalues of T_{11} and T_{22}.
   (e) If $T: V \to W$ is a linear transformation with V and W are vector spaces over a field F. Then, *ker(T)* is a vector space.

4. (10%) Let vectors $\boldsymbol{x} \in \boldsymbol{R}^n$ be represented as $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$. Which of the following sets is a subspace of $\boldsymbol{R}^n$?
   (a) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid \sum_{i=1}^{n} x_i^2 = n^2\}$
   (b) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_1 = 1\}$
   (c) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid \sum_{i=1}^{n} \alpha_i x_i = min_i(\alpha_i)\}$ for some known constants $\alpha_i \in \boldsymbol{R}$.
   (d) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_{2i} = x_{2i-1}, \forall i = 1, \ldots, n/2\}$, where n is assumed to be even in this case.
   (e) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_{2i} - x_{2i-1} = 1, \forall i = 1, \ldots, n/2\}$, where n is assumed to be even in this case.

5. (5%) Let N be an $n\times n$ matrix over $\boldsymbol{R}$ or $\boldsymbol{C}$ such that $N^k = 0$ for some integer k. Is $I+N$ invertible? If yes, find its inverse. If not, provide the reason why.

6. (5%) Define an inner product in $\boldsymbol{R}^2$ as $\langle u, v\rangle = \frac{1}{4}u_1v_1 + \frac{1}{9}u_2v_2$. What is the equation of the unit circle?

7. (5%) Please find the eigenvectors of $(I + A)^{100}$ given $A = \begin{bmatrix} -4 & -5 \\ 10 & 11 \end{bmatrix}$.

見背面

8. (10%) Please find the determinant of A.

$$A = \begin{pmatrix} 1+x_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & 1+x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & 1+x_3 & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & 1+x_n \end{pmatrix}$$

9. (10%) Consider

$$x_1 + x_2 + \mathrm{L} \ + x_n < r,$$

where $x_i \geq 0$ for $1 \leq i \leq n$. When $n = 4$ and $r = 8$, the number of nonnegative integer solutions is _________.

10. (5%) The cycle decomposition of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 5 & 8 & 3 & 1 & 7 & 4 \end{pmatrix}$$

is __________________.

11. (10%) The solution to the recurrence equation

$$a_n = 2a_{n-1} + n$$

with $a_0 = 4$ is $a_n =$ _________.

12. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _________. (Note that expressions like

$$\frac{x-8}{(x-3)^2} - \frac{9}{x-1}$$

are *not* partial fraction decompositions.)

13. (5%) Let $p(m)$ denote the number of partitions of $m \in \mathbb{Z}^+$. For example, the number of partitions of $m = 5$ is $p(5) = 7$:

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1

The generating function for $\{p(n)\}_{n=0,1,\mathrm{K}}$ is _________.

14. (5%) A Boolean function *f* is **self-dual** if

$$f(x_1, x_2, \ldots, x_m) = f(\neg x_1, \neg x_2, \ldots, \neg x_m).$$

There are _________ self-dual Boolean functions of *m* variables.

15. (5%) For any binary tree with *n* nodes and *i* internal nodes, the relation between *n* and *i* is _________ $\leq i$.

試題隨卷繳回

Solutions

NTU math 105

Version 1.0

1. We have

   (a) True.

   (b) False.

   (c) True. $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA}) \to \mathrm{tr}(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{B}^{-1}\boldsymbol{BA}) = \mathrm{tr}(\boldsymbol{A})$.

   (d) False.

   (e) True.

   Answer †

$$ace \tag{1}$$

2. We have

   (a) True. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{BA})$

   (b) True.

   (c) True. $\det(\boldsymbol{B}^{-1}\boldsymbol{AB}) = \det(\boldsymbol{B}^{-1}) \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \frac{1}{\det(\boldsymbol{B})} \times \det(\boldsymbol{A}) \times \det(\boldsymbol{B}) = \det(\boldsymbol{A})$

   (d) True.

   (e) False.

   Answer †

$$abcd \tag{2}$$

3. We have

   (a) False.

(b) False, if $\boldsymbol{R} = \boldsymbol{O}$ is rectangular, but $\boldsymbol{R}^{\intercal}\boldsymbol{R} = \boldsymbol{O}$ is NOT positive definite.

(c) False, the orthogonal set contains $\mathbf{0}$, it's NOT linearly independent.

(d) True.

(e) True.

Answer †

$$de \tag{3}$$

4. We have

(a) False, since it does NOT contain $\mathbf{0}$.

(b) False, since it does NOT contain $\mathbf{0}$.

(c) False, since it does NOT contain $\mathbf{0}$.

(d) True.

(e) False, since it does NOT contain $\mathbf{0}$.

Answer †

$$d \tag{4}$$

5. Suppose

$$f(x) = 1 - x^k = (1 - x)(1 + x + x^2 + \cdots + x^{k-1}) \tag{5}$$

Then, we have

$$f(-\boldsymbol{N}) = \boldsymbol{I} + \boldsymbol{N}^k = (\boldsymbol{I} + \boldsymbol{N})(\boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1}) = \boldsymbol{I} \tag{6}$$

Answer †

$$(\boldsymbol{I} + \boldsymbol{N})^{-1} = \boldsymbol{I} + (-\boldsymbol{N}) + (-\boldsymbol{N})^2 - \cdots + (-\boldsymbol{N})^{k-1} \tag{7}$$

6. **Answer** †

$$\frac{1}{4} \times x^2 + \frac{1}{9} \times y^2 = 1 \tag{8}$$

7. We have characteristic polynomial

$$p_{\boldsymbol{I}+\boldsymbol{A}}(x) = (x - 2)(x - 7) \tag{9}$$

Then, we have eigenspaces

$$\begin{cases} V(2) & = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \\ V(7) & = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \end{cases} \tag{10}$$

Answer †

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{11}$$

Since the eigenvectors of $(\boldsymbol{I} + \boldsymbol{A})$ are the same as $(\boldsymbol{I} + \boldsymbol{A})^{100}$.

8. Obviously, 1 is an eigenvalue, which $\mathrm{gm}(1) = n - 1$. Then, we have

$$\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n}(1 + x_i) = n + \sum_{i=1}^{n} x_i \tag{12}$$

Then, we have the n-th eigenvalue

$$n + (\sum_{i=1}^{n} x_i) - (n-1) \times 1 = 1 + \sum_{i=1}^{n} x_i \tag{13}$$

Answer †

$$\det(\boldsymbol{A}) = 1^{n-1} \times (1 + \sum_{i=1}^{n} x_i) = 1 + \sum_{i=1}^{n} x_i \tag{14}$$

9. We have new problem

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 < 8,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4 \\ & (x_5 = 8 - (x_1 + x_2 + x_3 + x_4),\ x_5 > 0) \\ \Rightarrow\ & x_1 + x_2 + x_3 + x_4 + y_5 = 8 - 1,\ \forall\ x_i \geq 0,\ 1 \leq i \leq 4,\ y_5 \geq 0 \end{aligned} \tag{15}$$

Answer †

$$\binom{5 + (8-1) - 1}{8 - 1} = 330 \tag{16}$$

10. **Answer** †

$$(1,\ 2,\ 6) \circ (3,\ 5) \circ (4,\ 8) \circ (7) \tag{17}$$

11. We have

$$
\begin{aligned}
&\Rightarrow \alpha = 2 \\
&\Rightarrow \begin{cases} a_n^{(h)} = c \times 2^n \\ a_n^{(p)} = d \times n + e \end{cases}
\end{aligned} \tag{18}
$$

Then, we have

$$
\begin{aligned}
& d \times n + e = 2 \times (d \times (n-1) + e) + n \\
\Rightarrow\ & d = -1,\ \ e = -2 \\
\Rightarrow\ & a_n = c \times 2^n - n - 2
\end{aligned} \tag{19}
$$

Then, we have

$$
\begin{aligned}
& a_0 = 4 = c - 0 - 2 \\
\Rightarrow\ & c = 6
\end{aligned} \tag{20}
$$

Answer †

$$
a_n = 6 \times 2^n - n - 2 \tag{21}
$$

12.

$$
\Rightarrow \sum_{i=1}^{n} a_n x^n = 2 \times \sum_{i=1}^{n} a_{n-1} x^n + \sum_{i=1}^{n} n x^n \tag{22}
$$

We have

$$
\sum_{i=1}^{n} n x^n = x \sum_{i=1}^{n} n x^{n-1} \tag{23}
$$

Then, we have

$$
\begin{aligned}
&\Rightarrow \sum_{i=1}^{n} n x^{n-1} \overset{\text{integral}}{=} \sum_{i=1}^{n} x^n = \frac{x}{1-x} \\
&\Rightarrow \frac{x}{1-x} \overset{\text{derivative}}{=} \frac{1}{(1-x)^2} \\
&\Rightarrow \sum_{i=1}^{n} n x^n = \frac{x}{(1-x)^2}
\end{aligned} \tag{24}
$$

We have the new generating function

$$
\begin{aligned}
& A(x) - a_0 = 2x \times A(x) + \frac{x}{(1-x)^2} \\
\Rightarrow\ & A(x) = \frac{4 \times x^2 - 7 \times x + 4}{(1-2x)(1-x)^2} \\
\Rightarrow\ & A(x) = 6 \times \frac{1}{1-2x} - 2 \times \frac{1}{1-x} - \frac{x}{(1-x)^2} \\
\Rightarrow\ & A(x) = 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - (\frac{1}{1-x} + \frac{x}{(1-x)^2}) \\
\Rightarrow\ & A(x) = 6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2}
\end{aligned} \tag{25}
$$

Answer †

$$6 \times \frac{1}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2} \tag{26}$$

13. We have

$$(1+x+x^2+\cdots)(1+(x^2)+(x^2)^2+\cdots)(1+(x^3)+(x^3)^2+\cdots)\cdots \tag{27}$$

Since each number can be repeated.

Answer †

$$\prod_{i=1}^{n} \frac{1}{1-x^i} \tag{28}$$

14. **Answer** †

$$2^{2^{m-1}} \tag{29}$$

Since the base means 0 and 1 two values of the codomain, and the index means that if we know the 01-sequence then we know the opposite.

15. We have

$$n \leq 2 \times i + 1 \tag{30}$$

Answer †

$$i \geq \lfloor \frac{n-1}{2} \rfloor \tag{31}$$

國立臺灣大學104學年度碩士班招生考試試題

請在答案卷上可作答部份中的第一頁繪製表格如下，並將各題答案填入表格。

1		2		3	
4		5		6	
7		8		9	
10		11			

1. (5%) Which of the following are equivalent to $p \to q$? ________

   (a) $\neg p \vee q$

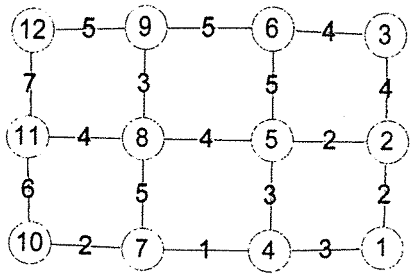
   (b) $\neg p \to \neg q$

   (c) $q \to p$

   (d) $\neg q \to \neg p$

   (e) $\neg q \to p$

2. (5%) The maximum flow from node 12 to node 1 of the following network is ________.



3. (10%) Simplify $\sum_{i=0}^{n} \binom{n}{i} 2^i$: ________.

4. (10%) The Euler function $\phi(120) =$ ________.

5. (10%) The number of nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_n = r,$$

   where $1 \le x_1 < x_2 < \cdots < x_n \le r$ is ________.

6. (10%) Solve the recurrence equation $a_n = 2 \times a_{n-1} + 3^{n-1}$ with $a_0 = 2$ for a_n.

見背面

7. (10%) Alice writes down the following function f in her email to Bob and claims that f is an inner-product function for the vector space consisting of all 2×2 complex matrices:

$$f(X, Y) = \mathrm{tr}(XY).$$

Specifically, for any 2×2 matrices X and Y whose elements are complex numbers, the value of $f(X, Y)$ is the trace of the product of X and Y. Bob immediately notices that there must be something wrong with Alice's email. What can be the possible reason or reasons for Bob's observation?

(a) Bob notices that f has this nice property that $f(X, Y) = f(Y, X)$ holds for all X and Y. This nice property prevents f from being a legal inner-product function for a vector space over complex numbers.

(b) Bob notices that there are X and Y such that $f(X, Y)$ is a complex number. Since inner-product functions are defined to measure distances, lengths, and areas, the values of $f(X, Y)$ are not allowed to be complex numbers.

(c) In order for f to be a legal inner-product function, for any matrix Y, the function $g_Y(X) = f(X, Y)$ has to be a linear function. Bob notices that there is a matrix Y such that g_Y is not linear.

(d) In order for f to be a legal inner-product function, for any matrix X, the function $h_X(Y) = f(X, Y)$ has to be a linear function. Bob notices that there is a matrix X such that h_X is not linear.

8. (10%) Let

$$A = \begin{pmatrix} -1 & 1 & 3 & -1 & 0 \\ 3 & -1 & -5 & 1 & -6 \\ 1 & 0 & -1 & 2 & 1 \\ -2 & 1 & 4 & -1 & 3 \end{pmatrix}.$$

What is the maximum rank of all 5×5 real matrices B such that AB is the 4×5 zero matrix?

9. (10%) What is the sum of the eigenvalues of the following matrix?

$$\begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}$$

接次頁

10. (10%) Consider the vector space V over real numbers that is spanned by three 2×2 matrices A_1, A_2, and A_3, where

$$A_1 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix},\ A_2 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix},\ \text{and } A_3 = \begin{pmatrix} -1 & -9 \\ 5 & -1 \end{pmatrix}.$$

Let

$$B_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & -1 \\ 2 & -2 \end{pmatrix} \text{ and } B_2 = \frac{1}{3\sqrt{2}} \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}.$$

Find a matrix B_3 such that B_1, B_2, and B_3 form an orthonormal basis of V with respect to the standard inner product for V defined by

$$\left\langle \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \right\rangle = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4.$$

11. (10%) Let V be the vector space consisting of all 2×2 real matrices. Let β be the standard ordered basis consisting of $\beta_1, \ldots, \beta_4$ in that order, where

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \beta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \beta_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \beta_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The function f defined by

$$f(X, Y) = \operatorname{tr}(X) \cdot \operatorname{tr}(Y)$$

for any matrices X and Y in V is known to be in bilinear form. What is the matrix representation of the function f in bilinear form with respect to the ordered basis β? (The answer is a 4×4 real matrix. That is why you need a larger cell for this problem in the table for your answers on the first page of your answer sheet.)

試題隨卷繳回

Solutions

NTU math 104

Version 1.0

1. We have

$$\begin{aligned} & p \to q \\ \iff & \neg p \vee q \end{aligned} \tag{1}$$

   (a) True.

   (b) False, since

$$\begin{aligned} & \neg p \to \neg q \\ \iff & p \vee \neg q \neq \neg p \vee q \end{aligned} \tag{2}$$

   (c) False, since

$$\begin{aligned} & q \to p \\ \iff & \neg q \vee p \neq \neg p \vee q \end{aligned} \tag{3}$$

   (d) True, since

$$\begin{aligned} & \neg q \to \neg p \\ \iff & q \vee \neg p = \neg p \vee q \end{aligned} \tag{4}$$

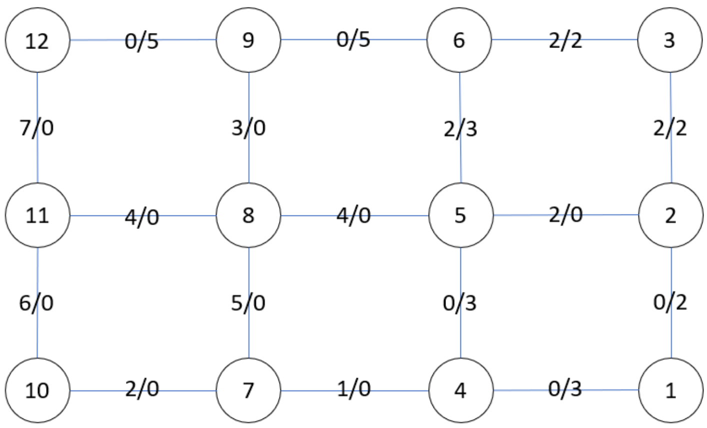
   (e) False, since

$$\begin{aligned} & \neg q \to p \\ \iff & q \vee p \neq \neg p \vee q \end{aligned} \tag{5}$$

   Answer †

$$ad \tag{6}$$

2. We have



Answer †

$$5 \tag{7}$$

3. We have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^n \tag{8}$$

Answer †

$$3^n \tag{9}$$

4. We have

$$120 = 2^3 \times 3^1 \times 5^1 \tag{10}$$

Answer †

$$\Phi(120) = 120 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 32 \tag{11}$$

5. Suppose

$$\begin{aligned} y_1 &= x_1 - 1 \\ y_2 &= x_2 - x_1 \\ y_3 &= x_3 - x_2 \\ &\vdots \\ y_n &= x_n - x_{n-1} \\ y_{n+1} &= r - x_n \end{aligned} \tag{12}$$

We have

$$\begin{cases} y_i \geq 1, \ \forall\ 1 \leq i \leq n \\ x_1 \geq 1, \ x_2 \geq 2, \ \cdots, \ x_{n-1} \geq (n-1) \end{cases} \tag{13}$$
$$\Rightarrow y_{n+1} = r - x_n = x_1 + x_2 + \cdots + x_{n-1} \geq \frac{n(n-1)}{2}$$

Then, we have

$$\begin{aligned}
& y_1 + 2 \times y_2 + \cdots + n \times y_n + (n+1) \times y_{n+1} \\
&= (x_1 - 1) + 2 \times (x_2 - x_1) + 3 \times (x_3 - x_2) + \cdots + n \times (x_n - x_{n-1}) \\
&+ (n+1) \times (r - x_n) \\
&= \ -1 - x_1 - x_2 - x_3 - \cdots - x_n + (n+1) \times r \\
&= nr - 1
\end{aligned} \tag{14}$$

Then, we have new generating function

$$\begin{aligned}
G(x) =& (1 + x + x^2 + \cdots)(x^2 + x^4 + x^6 + \cdots)(x^3 + x^6 + x^9 + \cdots)\cdots \\
& (x^n + x^{2n} + x^{3n} + \cdots)(x^{(n+1)\frac{n(n-1)}{2}} + \cdots) \\
=& \frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}}
\end{aligned} \tag{15}$$

Answer † Coefficient of x^{nr-1} of

$$\frac{1}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}\cdots\frac{x^n}{1-x^n}\frac{x^{\frac{(n^2-1)}{2}}}{1-x^{n+1}} \tag{16}$$

6. We have

$$\begin{aligned}
&\Rightarrow \alpha = 2 \\
&\Rightarrow \begin{cases} a_n^{(h)} = c \times 2^n \\ a_n^{(p)} = d \times 3^n \end{cases}
\end{aligned} \tag{17}$$

Then, we have

$$\begin{aligned}
& d \times 3^n = 2 \times d \times 3^{n-1} + 3^{n-1} \\
\Rightarrow\ & d = 1 \\
\Rightarrow\ & a_n = c \times 2^n + 3^n
\end{aligned} \tag{18}$$

Then, we have

$$\begin{aligned}
& a_0 = 2 = c + 1 \\
\Rightarrow\ & c = 1
\end{aligned} \tag{19}$$

Answer †

$$a_n = 2^n + 3^n \tag{20}$$

7. We have

$$\mathrm{tr}(\boldsymbol{XY}) = \mathrm{tr}((\boldsymbol{XY})^{\mathsf{H}}) = \mathrm{tr}(\boldsymbol{Y}^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}}) = \mathrm{tr}(\overline{\boldsymbol{Y}^{\intercal}\boldsymbol{X}^{\intercal}}) = \mathrm{tr}(\overline{(\boldsymbol{XY})^{\intercal}}) = \mathrm{tr}(\overline{\boldsymbol{XY}}) \tag{21}$$

Answer †

$$a \tag{22}$$

8. We have

$$\boldsymbol{A} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \tag{23}$$

We have rank $\boldsymbol{A} = 3$. Then, we have

$$\begin{aligned} &\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) - 5 \leq \operatorname{rank}(\boldsymbol{AB}) \\ \Rightarrow\ & 3 + \operatorname{rank}(\boldsymbol{B}) - 5 \leq 0 \end{aligned} \tag{24}$$

Answer †

$$\operatorname{rank}(\boldsymbol{B}) \leq 2 \tag{25}$$

9. **Answer** † Sum of eigenvalues equals to the trace.

$$2 + 2 + 2 + 2 = 8 \tag{26}$$

10. **Answer** † The problem is **WRONG**, since $\{\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \boldsymbol{A}_3,\ \boldsymbol{B}_1\}$ is linearly independent.

11. We have

$$[f]_\beta = \begin{bmatrix} f(\beta_1,\beta_1) & f(\beta_2,\beta_1) & f(\beta_3,\beta_1) & f(\beta_4,\beta_1) \\ f(\beta_1,\beta_2) & f(\beta_2,\beta_2) & f(\beta_3,\beta_2) & f(\beta_4,\beta_2) \\ f(\beta_1,\beta_3) & f(\beta_2,\beta_3) & f(\beta_3,\beta_3) & f(\beta_4,\beta_3) \\ f(\beta_1,\beta_4) & f(\beta_2,\beta_4) & f(\beta_3,\beta_4) & f(\beta_4,\beta_4) \end{bmatrix} \tag{27}$$

Answer †

$$[f]_\beta = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

國立臺灣大學 103 學年度碩士班招生考試試題

1. (10%) How many ways are there to arrange TALLAHA with no adjacent As?

2. (10%) The number of positive-integer solutions to $x_1 + x_2 + \cdots + x_n = r$, where $r > 0$, is ______________. (That is, all x_i must be positive integers to qualify as one solution.)

3. (5%) How many functions from $\{0,1\}^m$ (an m-dimensional boolean vector) to $\{0,1\}^2$ (a 2-dimensional boolean vector) are there?

4. (10%) The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

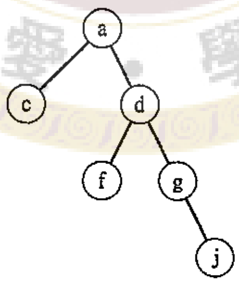
is the generating function for the sequence $\{a_i\}_{i=0,1,\ldots}$. The harmonic numbers $\{H_i\}_{i=0,1,2,\ldots}$ are defined by

$$\begin{aligned} H_0 &= 0, \\ H_i &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i} \quad (i \geq 1). \end{aligned}$$

Derive the closed-form generating function for the harmonic numbers.

5. (10%) Solve the recurrence equation $a_{n+2} = a_{n+1} + 2a_n$ with $a_0 = 0$ and $a_1 = 1$.

6. (5%) The postorder traversal of the following rooted binary tree is ______________.



7. (10%) Let V be a vector space over a scalar field F. For any subset S of V, let span(S) consist of the vectors of V that can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ with $a_1, a_2, \ldots, a_n \in F$ and $x_1, x_2, \ldots, x_n \in S$ for some positive integer n. Prove that span(S) is the unique subspace of V such that any subspace of V containing S has to contain span(S). Specifically, you have to show that (a) span(S) is a subspace of V, (b) if U is a subspace of V with $S \subseteq U$, then span(S) $\subseteq U$, and (c) there is no other subspace of V satisfying property (b).

見背面

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8. (10%)

   (a) Let T be a linear transformation from V to W, where V and W are finite-dimensional vector spaces over a comon scalar field F. We have nullity(T) + rank(T) = dim(__________).

   (b) Let R consist of the real numbers. Let function $f : R^3 \to R^3$ be defined as

   $$f(x,y,z) = (x+y+z, x-y, y-z).$$

   If $g : R^3 \to R^3$ is a linear function with $g(1,1,0) = (2,0,1)$, $g(1,0,1) = (2,1,-1)$, and $g(0,1,1) = (2,-1,0)$, then $g(5,3,0) =$ __________.

   (c) If the dimension of the vector space $M_{7\times4}(C)$ of matrices with seven rows and four columns over the field C of complex numbers equals the dimension of the vector space R^n of n-tuples over the field R of real numbers, then $n =$ __________.

   (d) Let A be an $m \times n$ matrix over the field R of real numbers. If the m rows of A are linearly independent, then the dimension of the vector space spanned by the n rows of A is __________.

   (e) If U and V are two distinct subspaces of a vector space W with $\dim(W) = 6$, then $\dim(U \cap V)$ is either __________ or __________.

9. (10%) Let

   $$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

   Find A^{-100} and A^{101}.

10. (10%) Consider the following system of linear equations:

    $$\begin{aligned} 2x+y+z &= 4 \\ 4x+2y+2z &= 8 \\ 5x+y &= 19. \end{aligned}$$

    Find the solution (x,y,z) to the above system of linear equations that minimizes $x^2+y^2+z^2$.

11. (10%) Find the eigenvalues of the following matrix:

    $$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Solutions

NTU math 103

Version 1.0

1. **Answer** † There are 3 As, so we first **permute** other 4 characters, and then **insert** 3 As in the 5 spaces.
$$\frac{4!}{2!} \times \binom{5}{3} \tag{1}$$

2. We have
$$\begin{aligned} x_1 + x_2 + \cdots + x_n = r,\ \forall\ x_i > 0,\ 1 \le i \le n \\ \Rightarrow y_1 + y_2 + \cdots + y_n = r - n,\ \forall\ y_i \ge 0,\ 1 \le i \le n \end{aligned} \tag{2}$$
Answer †
$$\binom{n + (r-n) - 1}{r-n} \tag{3}$$

3. **Answer** †
$$(2^2)^{(2^m)} = 4^{(2^m)} \tag{4}$$

4. We have
$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} x^n &= x + (1 + \frac{1}{2})x^2 + (1 + \frac{1}{2} + \frac{1}{3})x^3 + \cdots + (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})x^n + \cdots \\ &= 1 \times (x + x^2 + x^3 + \cdots + x^n + \cdots) + \frac{1}{2} \times (x^2 + x^3 + \cdots + x^n + \cdots) + \cdots \\ &+ \frac{1}{n} \times (x^n + x^{n+1} + \cdots) + \cdots \\ &= \frac{x}{1-x} + \frac{1}{2} \times \frac{x^2}{1-x} + \cdots + \frac{1}{n} \times \frac{x^n}{1-x} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-x} x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{1}{n} x^n \end{aligned} \tag{5}$$

And, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}x^n &\overset{\text{derivative}}{=} \sum_{n=1}^{\infty}x^{n-1} = \frac{1}{1-x}\\ \Rightarrow \frac{1}{1-x} &\overset{\text{integral}}{=} -\ln(1-x)\end{aligned} \tag{6}$$

Then, we have

$$\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{1}{n}x^n = \frac{-\ln(1-x)}{1-x} \tag{7}$$

Answer †

$$\frac{-\ln(1-x)}{1-x} \tag{8}$$

5. We have

$$\begin{aligned}&\Rightarrow \alpha^2 = \alpha + 2\\ &\Rightarrow \alpha = 2 \vee \alpha = -1\\ &\Rightarrow a_n = c\times 2^n + d\times(-1)^n\end{aligned} \tag{9}$$

And, we have

$$\begin{aligned}&\begin{cases} a_0 = 0 = c + d\\ a_1 = 1 = 2\times c - d\end{cases}\\ \Rightarrow &\begin{cases} c = \frac{1}{3}\\ d = -\frac{1}{3}\end{cases}\end{aligned} \tag{10}$$

Answer †

$$a_n = \frac{1}{3}\times 2^n - \frac{1}{3}\times(-1)^n \tag{11}$$

6. **Answer** †

$$cfjgda \tag{12}$$

7. **Answer** †

(a) If $S = \emptyset$, $\mathrm{span}(S) = \{\mathbf{0}\} \subseteq V$.
Otherwise, if $S \neq \emptyset$, $\mathbf{0} \in \mathrm{span}(S)$, and $\forall\ \boldsymbol{x},\ \boldsymbol{y} \in \mathrm{span}(S)$, let

$$\begin{aligned}&\begin{cases}\mathrm{span}(S) & = \mathrm{span}\{v_1,\ v_2, \cdots,\ v_n\}\\ \boldsymbol{x} & = a_1v_1 + a_2v_2 + \cdots + a_nv_n\\ \boldsymbol{y} & = b_1v_1 + b_2v_2 + \cdots + b_nv_n\end{cases}\\ &\Rightarrow \forall\ \alpha,\ \beta \in \mathbb{R},\ \alpha\boldsymbol{x} + \beta\boldsymbol{y} =\\ &\quad (\alpha a_1 + \beta b_1)v_1 + (\alpha a_2 + \beta b_2)v_2 + \cdots + (\alpha a_n + \beta b_n)v_n \in \mathrm{span}(S)\\ &\Rightarrow\ \mathrm{span}(S) \subseteq V\end{aligned} \tag{13}$$

(b)

$$\begin{aligned} &S \subseteq U,\ \forall\ \boldsymbol{x} = \{x_1,\ x_2, \cdots,\ x_n\} \in S \\ \Rightarrow\ &\mathrm{span}(S) = \{\alpha x_1 + \alpha x_2 + \cdots + \alpha x_n\} \subseteq U \end{aligned} \tag{14}$$

(c) Suppose

$$\exists\ T \subseteq V,\ \text{s.t. } T \subseteq U \tag{15}$$

And, we have

$$\begin{aligned} &S \subseteq \mathrm{span}(S),\ \mathrm{span}(S) \subseteq V \\ \Rightarrow\ &T \subseteq \mathrm{span}(S) \end{aligned} \tag{16}$$

And, we have

$$\begin{aligned} &S \subseteq T,\ \mathrm{span}(S) \subseteq T \\ \Rightarrow\ &T = \mathrm{span}(S) \end{aligned} \tag{17}$$

8.

(a) **Answer** †

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V) \tag{18}$$

(b) We have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{\text{rref}}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{19}$$

Answer †

$$4 \times (2,\ 0,\ 1) + 1 \times (2,\ 1,\ -1) - 1 \times (2,\ -1,\ 0) = (8,\ 2,\ 3) \tag{20}$$

(c) **Answer** †

$$7 \times 4 = 28 \tag{21}$$

(d) **Answer** †

$$0,\ 1,\ 2,\ 3,\ 4,\ 5 \tag{22}$$

Since U and V are **distinct**, $U = V = W$ does NOT exist.

9. **Answer** † We have $\boldsymbol{A}^2 = \boldsymbol{I}$, so

$$\begin{aligned} \boldsymbol{A}^{-100} &= (\boldsymbol{A}^2)^{-50} = \boldsymbol{I} \\ \boldsymbol{A}^{101} &= (\boldsymbol{A}^2)^{50} \times \boldsymbol{A} = \boldsymbol{A} \end{aligned} \tag{23}$$

10. Find the minimal solution. We have

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{24}$$

Then, we have

$$\begin{aligned} &(\boldsymbol{A}\boldsymbol{A}^{\mathsf{H}})\boldsymbol{u} = \boldsymbol{b} \\ \Rightarrow &\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix} \\ \Rightarrow &\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \end{aligned} \tag{25}$$

Then, we have

$$\boldsymbol{A}^{\mathsf{H}}\boldsymbol{u} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{26}$$

Answer †

$$\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{27}$$

11. **Answer** †

$$-1, \ 1, \ 2, \ 3 \tag{28}$$

國立臺灣大學 102 學年度碩士班招生考試試題

1 Is the following argument correct or wrong? Why? (10%)

Suppose that $\mathfrak{R}$ is a binary relation on a non-empty set A. If $\mathfrak{R}$ is symmetric and transitive, then $\mathfrak{R}$ is reflexive.

Proof. Let $(x, y) \in \mathfrak{R}$. By the symmetric property, we have $(y, x) \in \mathfrak{R}$. Then, with $(x, y), (y, x) \in \mathfrak{R}$, it follows by the transitive property that we have $(x, x) \in \mathfrak{R}$. As a consequence, $\mathfrak{R}$ is reflexive.

2 Consider ternary strings with symbols 0, 1, 2 used. For $n \geq 1$, let a_n count the number of ternary strings of length n, where there are no consecutive 1's and no consecutive 2's. Show that a_n can be expressed recursively as $2a_{n-1} + a_{n-2}$. (10%)

3 Suppose that G is an undirected simple graph of n vertices. (10%)
(a) Find the number of spanning subgraphs of G that are also induced subgraphs of G.
(b) If every induced subgraph of G is connected, then find the number of edges in G.

4 If G is an undirected simple graph, then there are two vertices in G having equal degree. Why? (10%)

5 Suppose that G is a group, and H, K are two subgroups of G. Prove that if $\gcd(|H|, |K|) = 1$, then $H \cap K = \{e\}$, where e is the identity of G. (10%)

6 If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are eigenvalues of matrix A:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 1 & 4 & 5 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix}$$

Then $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 =$________ (5%).

7 If $A = S\Lambda S^{-1}$, then the eigenvalue matrix and eigenvector matrix of $B = \begin{bmatrix} 3A & 0 \\ 0 & 2A \end{bmatrix}$ are

_______ and ________, respectively (5%).

8 Define $T(A) = \frac{A+A^T}{2}$ where A is a $n \times n$ matrix. Then

(a) ker(T)=________ (5%).
(b) (nullity(T), rank(T))=(_____, _____) (5%).

見背面

國立臺灣大學 102 學年度碩士班招生考試試題

9 Suppose that $p_k(x)$ is a polynomial of order k with leading coefficients, a_k, $k = 0, \cdots, n-1$. That is, $p_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, $k = 0, \cdots, n-1$. Then

$$\begin{vmatrix} p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) \\ p_1(x_1) & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{vmatrix} = \_\_\_\_\_\_\_\_ \text{(10\%)}.$$

10 Let a sequence B_k with $B_o = 0, B_1 = \frac{1}{2}$ and $B_{k+2} = \frac{B_{k+1}+B_k}{2}$, $k = 0, 1, 2 \cdots$. Please find the general expression for B_k=______ (7%) and $\lim_{k\to\infty} B_k$=_______ (3%).

[True or false] Credits will be given only if all the answers are correct.

11 (5%) Let W_1 and W_2 be subspaces of a vector space V over $\mathbb{R}$.

(a) $W_1 \cap W_2$ is a subspace of V.
(b) $W_1 \cup W_2$ is a subspace of V.
(c) $(V - W_1) \cap W_2$ is a subspace of V.
(d) $V - W_1$ is a subspace of V.
(e) If $W_1 \perp W_2$ then $W_1 = (W_2)^\perp$

12 (5%) Suppose that $A, B \in M_{n\times n}$.

(a) A and A^T have the same eigenvalues.
(b) If A is diagonalizable, so is its transpose A^T.
(c) AB and BA have the same eigenvalues.
(d) If α is an eigenvalues of A and β is an eigenvalue of B, then $\alpha\beta$ must be the eigenvalue of AB.
(e) If A and B are both diagonalizable, so is A-B.

試題隨卷繳回

Solutions

NTU math 102

Version 1.0

1. **Answer** † **WRONG**. Counterexample:

$$\begin{aligned} A &= \{1,\ 2,\ 3\} \\ R &= \{(1,\ 2),\ (2,\ 1),\ (1,\ 1),\ (2,\ 2)\} \end{aligned} \tag{1}$$

R is symmetric and transitive, but R is NOT reflexive.

2. **Answer** † Suppose

$$\begin{aligned} &\Rightarrow \begin{cases} b_n &= n\text{-th character is } 0 \\ c_n &= n\text{-th character is } 1, \text{ and } (n-1)\text{-th character is } 0 \vee 2 \\ d_n &= n\text{-th character is } 2, \text{ and } (n-1)\text{-th character is } 0 \vee 1 \end{cases} \\ &\Rightarrow a_n = b_n + c_n + d_n \\ &\left(\begin{cases} b_n &= a_{n-1} \\ c_n &= a_{n-1} - c_{n-1}\ (\because\ n\text{-th character can NOT be } 1) \\ d_n &= a_{n-1} - d_{n-1}\ (\because\ n\text{-th character can NOT be } 2) \end{cases} \right) \end{aligned} \tag{2}$$

Then, we have

$$\begin{aligned} a_n &= b_n + c_n + d_n \\ &= a_{n-1} + (a_{n-1} - c_{n-1}) + (a_{n-1} - d_{n-1}) \\ &= 3 \times a_{n-1} - c_{n-1} - d_{n-1}\ (\because\ a_{n-1} = b_{n-1} + c_{n-1} + d_{n-1}) \\ &= 2 \times a_{n-1} + b_{n-1} \\ &= 2 \times a_{n-1} + b_{n-2} \end{aligned} \tag{3}$$

3.

(a) **Answer** †

$$1 \tag{4}$$

Since it needs to contain all edges and all vertices.

(b) **Answer** †

$$\binom{n}{2} \tag{5}$$

Since it needs to be **complete**.

4. **Answer** † Suppose G have n vertices. If $G = (V, E)$ is connected,

$$1 \leq \deg(v) \leq (n-1), \ \forall\, v \in V \tag{6}$$

Since $|V| = n$, and the possiblities of degree are $(n-1)$,

$$\exists\, u,\ v \in V,\ \text{s.t. } \deg(u) = \deg(v) \tag{7}$$

Otherwise, if $G = (V, E)$ is NOT connected, suppose there exists k vertices $u \in V_1$ such that $\deg(u) = 0$, so other $(n-k)$ vertices $v \in V_2$ are connected, i.e., $V = V_1 + V_2$. Then, we have

$$1 \leq v \leq (n-k-1), \ \forall\, v \in V_2 \tag{8}$$

Since $|V_2| = n-k$, and the possiblities of degree are $(n-k-1)$,

$$\exists\, v,\ w \in V_2,\ \text{s.t. } \deg(v) = \deg(w) \tag{9}$$

To summary, there are 2 vertices in G having equal degree.

5. **Answer** † We have

$$\begin{aligned} &H \cap K \subseteq H,\ H \cap K \subseteq K\ (|H| = h,\ |K| = k,\ |H \cap K| = m) \\ &\Rightarrow m|h,\ m|k \text{ (by Lagrange Theorem)} \\ &\Rightarrow m|\gcd(h,\ k) = 1 \\ &\Rightarrow m = 1 \end{aligned} \tag{10}$$

6. We have

$$\begin{aligned} &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \operatorname{tr}_2(\boldsymbol{A}) \\ &= \begin{vmatrix} 1 & 2 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 7 & 6 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 5 & 8 \\ 6 & 7 \end{vmatrix} \end{aligned} \tag{11}$$

Answer †

$$24 \tag{12}$$

7. **Answer** † Eigenvalue matrix:

$$\begin{bmatrix} 3 \times \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & 2 \times \mathbf{\Lambda} \end{bmatrix} \tag{13}$$

Eigenvector matrix:

$$\begin{bmatrix} \boldsymbol{S} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S} \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned} \mathrm{N}(\boldsymbol{T}) &= \{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \frac{\boldsymbol{A} + \boldsymbol{A}^\intercal}{2} = \mathbf{0}\} \\ &= \{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^\intercal\} \end{aligned} \tag{15}$$

(a) **Answer** †

$$\{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \boldsymbol{A} = -\boldsymbol{A}^\intercal\} \tag{16}$$

(b) **Answer** †

$$(\mathrm{nullity}(T),\ \mathrm{rank}(T)) = (\frac{n(n-1)}{2},\ \frac{n(n+1)}{2}) \tag{17}$$

9. We have

$$= \begin{vmatrix} a_0 & a_0 & \cdots & a_0 \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$(\because\ c_{n1}^{-1},\ c_{n2}^{-1},\ \cdots,\ c_{n(n-1)}^{-1})$

$$= \begin{vmatrix} 0 & 0 & \cdots & a_0 \\ p_1(x_1) - p_1(x_n) & p_1(x_2) - p_1(x_n) & \cdots & p_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) - p_{n-1}(x_n) & p_{n-1}(x_2) - p_{n-1}(x_n) & \cdots & p_{n-1}(x_n) \end{vmatrix}_{n\times n}$$

$$= (-1)^{n+1} a_0 \begin{vmatrix} a_1(x_1 - x_n) & a_1(x_2 - x_n) & \cdots & a_1(x_{n-1} - x_n) \\ \sum_{i=1}^{2} a_i(x_1 - x_n)^i & \sum_{i=1}^{2} a_i(x_2 - x_n)^i & \cdots & \sum_{i=1}^{2} a_i(x_{n-1} - x_n)^i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n-1} a_i(x_1 - x_n)^i & \sum_{i=1}^{n-1} a_i(x_2 - x_n)^i & \cdots & \sum_{i=1}^{n-1} a_i(x_{n-1} - x_n)^i \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_{12}^{-1},\ r_{13}^{-1},\ \cdots,\ r_{1(n-1)}^{-1},\ r_{23}^{-1},\ r_{24}^{-1},\ \cdots,\ r_{2(n-1)}^{-1},\ r_{34}^{-1},\ \cdots,\ r_{(n-2)(n-1)}^{-1})$$

$$= (-1)^{n+1}a_0 \begin{vmatrix} a_1(x_1-x_n) & a_1(x_2-x_n) & \cdots & a_1(x_{n-1}-x_n) \\ a_2(x_1-x_n)^2 & a_2(x_2-x_n)^2 & \cdots & a_2(x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(x_1-x_n)^{n-1} & a_{n-1}(x_2-x_n)^{n-1} & \cdots & a_{n-1}(x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$(\because\ r_1^{\frac{1}{a_1}},\ r_2^{\frac{1}{a_2}},\ \cdots,\ r_{n-1}^{\frac{1}{a_{n-1}}})$$

$$= (-1)^{n+1}\prod_{i=0}^{n-1} a_i \begin{vmatrix} (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ (x_1-x_n)^2 & (x_2-x_n)^2 & \cdots & (x_{n-1}-x_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-1} & (x_2-x_n)^{n-1} & \cdots & (x_{n-1}-x_n)^{n-1} \end{vmatrix}_{(n-1)\times(n-1)} \tag{18}$$

$$(\because\ c_1^{\frac{1}{x_1-x_n}},\ c_2^{\frac{1}{x_2-x_n}},\ \cdots,\ c_{n-1}^{\frac{1}{x_{n-1}-x_n}})$$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)(\prod_{j=1}^{n-1}(x_j-x_n)) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (x_1-x_n) & (x_2-x_n) & \cdots & (x_{n-1}-x_n) \\ \vdots & \vdots & \ddots & \vdots \\ (x_1-x_n)^{n-2} & (x_2-x_n)^{n-2} & \cdots & (x_{n-1}-x_n)^{n-2} \end{vmatrix}_{(n-1)\times(n-1)}$$

$(\because$ Vandermonde matrix$)$

$$= (-1)^{n+1}(\prod_{i=0}^{n-1} a_i)[(-1)^{n-1}\prod_{j=1}^{n-1}(x_n-x_j)](\prod_{1\le i\le j\le (n-1)}(x_j-x_i))$$

$$= (\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i))$$

Answer †

$$(\prod_{i=0}^{n-1} a_i)(\prod_{1\le i\le j\le n}(x_j-x_i)) \tag{19}$$

10. We have

$$\begin{aligned} &\Rightarrow \alpha^2 = \frac{1}{2}\times\alpha - \frac{1}{2} \\ &\Rightarrow \alpha = -\frac{1}{2} \lor \alpha = 1 \\ &\Rightarrow B_n = c\times(-\frac{1}{2})^n + d\times(1)^n \end{aligned} \tag{20}$$

And, we have

$$\begin{cases} B_0 = 0 = c + d \\ B_1 = \frac{1}{2} = -\frac{1}{2} \times c + d \end{cases} \tag{21}$$

$$\Rightarrow c = -\frac{1}{3},\ d = \frac{1}{3}$$

Answer †

$$B_k = \frac{1}{3} - \frac{1}{3} \times (-\frac{1}{2})^k,\ \lim_{k\to\infty} B_k = \frac{1}{3} \tag{22}$$

11. We have
    (a) True.
    (b) False.
    (c) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (d) False, since $\mathbf{0} \in W_1 \to \mathbf{0} \neq (V - W_1)$.
    (e) False.
12. We have
    (a) True.
    (b) True.
    (c) True.
    (d) False.
    (e) False.

國立臺灣大學101學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1-10 題爲填充題

1. If $\begin{bmatrix} 10 & 1 \\ 19 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ and $a, b, c \in \mathbf{R}$, then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} =$__________.(5%)

2. If $A = \begin{bmatrix} 2 & 1 & d \\ 1 & 0 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\det(A) = 0$, then $d =$__________.(5%)

3. If $B = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 4 \\ 1 & 2 & 3 & -1 & 1 & 5 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 \end{bmatrix}$ then $\operatorname{rank}(B) =$__________.(5%)

4. $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^{300} =$__________.(5%)

5. Suppose $A \in \mathbf{R}^{3\times3}$ and $\det(xI_3 - A) = x^3 - x^2 + 3x - 2$, then $\det(xI_3 - A^2) =$__________.(5%)

6. Let $A \in \mathbf{R}^{3\times3}$ and $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. If $AP = P\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $A^T\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

   then $\begin{bmatrix} x \\ y \end{bmatrix} =$__________.(5%)

7. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 13 \\ -1 & 6 \end{bmatrix}$. If $AC + CA = B$ then the matrix $C =$__________.(5%)

8. Let $H = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Suppose $H^3 = \alpha H^2 + \beta H + \gamma I_4$, $\alpha, \beta, \gamma \in \mathbf{R}$, then

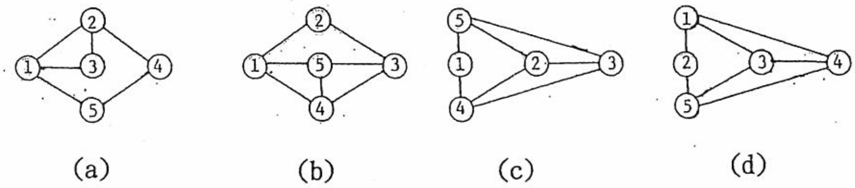
   $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} =$__________.(5%)

9. If $T = \begin{bmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 2 & 1 & 2 & 2^2 & 2^3 & 2^4 \\ 2^2 & 2 & 1 & 2 & 2^2 & 2^3 \\ 2^3 & 2^2 & 2 & 1 & 2 & 2^2 \\ 2^4 & 2^3 & 2^2 & 2 & 1 & 2 \\ 2^5 & 2^4 & 2^3 & 2^2 & 2 & 1 \end{bmatrix}$ then $\det(T) =$__________.(5%)

10 Let $S = \{\sum_{k=1}^{99} x_k x_{k+1} \mid x_1, x_2, \cdots, x_{100} \in \mathbf{R} \text{ and } x_1^2 + x_2^2 + \cdots + x_{100}^2 = 1\}$ then the largest number in S is__________.(5%)

見背面

11 Determine which pairs of the following graphs are isomorphic. Also give an isomorphism for each isomorphic pair. (10%)



12 For $n \geq 1$, the nth *triangular number* t_n is defined by $t_n = 1+2+\cdots+n$.

(a) Find a recurrence relation for s_n, where $n \geq 1$ and $s_n = t_1 + t_2 + \cdots + t_n$. (5%)

(b) Compute $a_0 + a_1 + a_2 + a_3 + \cdots$, where $s_n = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \cdots$. (5%)

13 Suppose $1 \leq a < b < c < d \leq 12$. How many sets $\{a, b, c, d\}$ are there, where no consecutive integers (e.g., 1 and 2, 2 and 3, 3 and 4, ...) appear in $\{a, b, c, d\}$? (10%)

14 A graph $G=(V, E)$ is bipartite, if the vertex set V can be partitioned into two subsets V_1, V_2 such that each edge in E connects a vertex in V_1 with a vertex in V_2. Further, a bipartite graph G is complete, if it has a maximal number, i.e., $|V_1| \times |V_2|$, of edges. Usually, a complete bipartite graph G is denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$.

(a) Suppose $m \times n = 16$ and $m \leq n$. Find the values of m, n such that $K_{m,n}$ has one or more Euler circuits, but has no Hamilton cycle? (5%)

(b) Generalize the result of (a), i.e., give conditions of m, n under which $K_{m,n}$ has one or more Euler circuits, but has no Hamilton cycle? (5%)

15 Suppose that $(R, +, \cdot)$ is a ring and S is a nonempty subset of R. Then, $(S, +, \cdot)$ is a ring if and only if

- for all $a, b \in S$, $a+b \in S$ and $a \cdot b \in S$;
- for all $a \in S$, $-a \in S$.

Please show that when S is finite, $(S, +, \cdot)$ is a ring if and only if for all $a, b \in S$, $a+b \in S$ and $a \cdot b \in S$. (10%)

試題隨卷繳回

Solutions

NTU math 101

VERSION 1.0

1. **Answer** †

$$\begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$3 \tag{2}$$

3. We have

$$\boldsymbol{B} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 \end{bmatrix} \tag{3}$$

   Answer †

$$3 \tag{4}$$

4. We have

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{5}$$

   $\boldsymbol{A}$ is a rotation matrix, which rotates $\frac{\pi}{3}$ **clockwisely**. Then, we have

$$\boldsymbol{A}^{300} = \begin{bmatrix} \cos(100\pi) & \sin(100\pi) \\ -\sin(100\pi) & \cos(100\pi) \end{bmatrix} \tag{6}$$

   Answer †

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

5. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x) = -x^3 + x^2 - 3 \times x + 2 \tag{8}$$

Suppose $\lambda_{\boldsymbol{A}} = a,\ b,\ c$, then we have,

$$\begin{aligned}
&\begin{cases} a+b+c=1 \\ ab+bc+ac=3 \\ abc=2 \end{cases} \\
\Rightarrow &\begin{cases} a^2+b^2+c^2=(a+b+c)^2-2\times(ab+bc+ac)=-5 \\ a^2b^2+b^2c^2+a^2c^2=(ab+bc+ac)^2-2\times(abc)\times(a+b+c)=5 \\ a^2b^2c^2=(abc)^2=4 \end{cases}
\end{aligned} \tag{9}$$

Then, we have characteristic polynomial

$$p_{\boldsymbol{A}^2} = \det(\boldsymbol{A}^2 - \boldsymbol{I}\boldsymbol{x}) = -x^3 + (-5) \times x^2 - (5) \times x + 4 \tag{10}$$

Answer †

$$\det(\boldsymbol{x}\boldsymbol{I} - \boldsymbol{A}^2) = -(-x^3 + (-5) \times x^2 - (5) \times x + 4) = x^3 + 5 \times x^2 + 5 \times x - 4 \tag{11}$$

6. **Answer** †

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \tag{12}$$

7. We have

$$\begin{aligned}
&-\boldsymbol{H}^3 + \alpha\boldsymbol{H}^2 + \beta\boldsymbol{H} + \gamma\boldsymbol{I} = \boldsymbol{O} \\
\Rightarrow &\begin{cases} \alpha = \operatorname{tr}(\boldsymbol{H}) = 34 \\ \beta = -\operatorname{tr}_2(\boldsymbol{H}) \\ \quad = -(\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 9 & 11 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \end{vmatrix} + \begin{vmatrix} 6 & 8 \\ 14 & 16 \end{vmatrix} + \begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix}) = 80 \\ \gamma = \det(\boldsymbol{H}) = 0 \end{cases}
\end{aligned} \tag{13}$$

Answer †

$$\begin{bmatrix} 34 \\ 80 \\ 0 \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned}
&\overset{r_{12}^{1},\, r_{23}^{1},\, \cdots,\, r_{56}^{1}}{=} \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 & 3\times 2^4 \\ 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 \\ 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 \\ 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 \\ 3\times 2^4 & 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 \end{vmatrix} \\
&\overset{r_2^{\frac{1}{3}},\, r_3^{\frac{1}{3}},\, \cdots,\, r_6^{\frac{1}{3}}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 1 & 1 & 2 & 2^2 & 2^3 & 2^4 \\ 2 & 1 & 1 & 2 & 2^2 & 2^3 \\ 2^2 & 2 & 1 & 1 & 2 & 2^2 \\ 2^3 & 2^2 & 2 & 1 & 1 & 2 \\ 2^4 & 2^3 & 2^2 & 2 & 1 & 1 \end{vmatrix} \\
&\overset{r_{12}^{-1},\, r_{23}^{-1},\, \cdots,\, r_{56}^{-1}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 0 & -1 & -2 & -2^2 & -2^3 & -2^4 \\ 1 & 0 & -1 & -2 & -2^2 & -2^3 \\ 2 & 1 & 0 & -1 & -2 & -2^2 \\ 2^2 & 2 & 1 & 0 & -1 & -2 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix} \\
&\overset{r_{65}^{-2},\, r_{54}^{-2},\, \cdots,\, r_{21}^{-2}}{=} 3^5 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 & 0 \\ -6 & -3 & -2 & -1 & 0 & 0 \\ -12 & -6 & -3 & -2 & -1 & 0 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix}
\end{aligned} \tag{15}$$

Answer †

$$-\,3^5 \tag{16}$$

9. Suppose

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix}, \ \boldsymbol{A}_{100} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}_{100\times 100} \tag{17}$$

$$\Rightarrow q(x_1,\ x_2,\ \cdots,\ x_{100}) = \sum_{k=1}^{99} x_k x_{k+1} = \boldsymbol{x}^\intercal \boldsymbol{A}_{100} \boldsymbol{x}$$

By Rayleigh principle, $\max_{||\boldsymbol{x}||=1} \boldsymbol{x}^\intercal \boldsymbol{A}_n \boldsymbol{x} = \lambda_{\max}(\boldsymbol{A}_n)$. And, we have general tridiagnoal matrix

$$B_n = \begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & a & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & a & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & a \end{bmatrix}_{n\times n} \tag{18}$$

By solving the recurrence function, we have general eigenvalues formula for $\boldsymbol{B}_n$

$$\lambda_k = a + 2 \times b \cos(\frac{k \times \pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{19}$$

Then, we have $a = 0,\ b = \frac{1}{2}$, and get $\boldsymbol{A}_n$'s eigenvalues

$$\lambda_k = \cos(\frac{k \times \pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{20}$$

And, we have

$$\cos(\frac{1 \times \pi}{100+1}) \tag{21}$$

as largest eigenvalue of $\boldsymbol{A}_{100}$.

Answer †

$$\cos(\frac{\pi}{101}) \tag{22}$$

10. (a) has 2 degree-3 vertice and 3 degree-2 vertices, when (b), (c), and (d) have same 4 degree-3 vertice and 1 degree-2 vertices, so (a) can NOT be an isomorphism of others.

And, we have correspondence

(b)	(c)	(d)
1	4	5
2	1	2
3	5	1
4	3	4
5	2	3

So, $(b)(c)(d)$ are isomorphic.

Answer †

$$(b)(c)(d) \tag{23}$$

11. (a) **Answer** †

$$s_n = s_{n-1} + \frac{n(n-1)}{2} \tag{24}$$

(b) **Answer** †

$$a_0 + a_1 + a_2 + a_3 + \cdots = s_1 = 1 \tag{25}$$

12. Suppose

$$\begin{aligned} &\Rightarrow \begin{cases} x_1 = a - 1 \geq 0 \\ x_2 = b - a \geq 2 \\ x_3 = c - b \geq 2 \\ x_4 = d - c \geq 2 \\ x_5 = 12 - d \geq 0 \end{cases} , \sum_{i=1}^{5} x_i = 11 \\ &\Rightarrow \begin{cases} y_1 = x_1 \geq 0 \\ y_2 = x_2 - 2 \geq 0 \\ y_3 = x_3 - 2 \geq 0 \\ y_4 = x_4 - 2 \geq 0 \\ y_5 = x_5 \geq 0 \end{cases} , \sum_{i=1}^{5} y_i = 11 - 3 \times 2 \end{aligned} \tag{26}$$

Answer †

$$\binom{5 + (11-6) - 1}{(11-6)} = 126 \tag{27}$$

13. (a) We have constraints: m and n must be **even** (≥ 1 Euler circuits), and $m \neq n$ (NO Hamilton cycle).

Answer †

$$2,\ 8 \tag{28}$$

(b) **Answer** †

$$m \text{ and } n \text{ is even, and } m \neq n \tag{29}$$

14. **Answer** △

$$\begin{aligned}
&(\Rightarrow)\ \because\ (S,\ +,\ \cdot) \text{ is a ring}\\
&\qquad \therefore\ \forall\ a,\ b \in S,\ a+b \in S,\ a\cdot b \in S\\
&(\Leftarrow)\ \forall\ a \in S,\ -a \in S\\
&\qquad \because (S,\ +,\ \cdot) \text{ is closed, } \forall\ a \in S,\ a,\ 2a,\ \cdots \in S\\
&\qquad \exists\ i<j,\ ia = ja \to (j-i)a = 0 \to a + (j-i-1)a = 0\\
&\qquad \therefore\ -a = \begin{cases} 0(=a) \in S & ,\ j = i+1 \\ (j-i-1)a \in S & ,\ j > i+1 \end{cases}
\end{aligned} \tag{30}$$

1-10 題爲填充題,請依題號,將答案填寫於答案卷上。

1. If $A=\begin{bmatrix}5 & a\\ 1 & 4\end{bmatrix}$ and $A^T=A$, then $a=$_________. (5%)

2. If $\begin{bmatrix}1 & -1 & -1\\ 2 & 1 & -3\\ -1 & 1 & 2\end{bmatrix}x=\begin{bmatrix}1\\ 2\\ 2\end{bmatrix}$, then $x=$_________. (5%)

3. If $B=\begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}$ and $B^3=\alpha B+\beta I_2$, then $(\alpha,\beta)=$________. (5%)

4. If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 2 & 3 & 2\end{bmatrix}\begin{bmatrix}1 & 5 & 1\\ 1 & 1 & 2\\ -2 & 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}1 & 4 & 1\\ 5 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}$, then $\det(A)=$________. (5%)

5. Let $A=\begin{bmatrix}1 & 2 & 1\\ -1 & 3 & a\\ 3 & 1 & 0\end{bmatrix}$. If $\operatorname{trace}(A^2)=5$, then $a=$_____________. (5%)

6. Suppose $w^T=[2 \quad -1 \quad 0 \quad 2 \quad 1]$ and the matrix $A=I_5+\alpha ww^T$ is singular, then $(\alpha,\operatorname{rank}(A))=$___________. (5%)

7. Let the diagonal matrix $D=$_____________. Then for any

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\in R^{3\times 3}, \text{ we have } D^{-1}AD=\begin{bmatrix}a_{11} & \frac{1}{2}a_{12} & \frac{1}{4}a_{13}\\ 2a_{21} & a_{22} & \frac{1}{2}a_{23}\\ 4a_{31} & 2a_{32} & a_{33}\end{bmatrix}. \text{ (5\%)}$$

8. If $u=\begin{bmatrix}1\\ 2\\ 3\\ 4\\ 5\end{bmatrix}$, $v=\begin{bmatrix}5\\ 4\\ 3\\ 2\\ 1\end{bmatrix}$, and $A=I_5+[u \quad v]\begin{bmatrix}v^T\\ u^T\end{bmatrix}$,

then all the eigenvalues of A are___________. (5%)

9. If $P_n=\begin{bmatrix}\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1}\\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n}\end{bmatrix}$, then $\frac{\det(P_{n+1})}{\det(P_n)}=$___________. (5%)

10. Given two complex vectors $u,v\in C^n$ with $u^Hu=v^Hv\neq 0$ and $u\neq v$. If the complex matrix $A(u,v)=$_________, then $A(u,v)^HA(u,v)=I_n$, and $A(u,v)u=v$. (5%)

見背面

11 Find the possibly maximal number of edges contained in

(a) a bipartite graph with 12 vertices and (5%)

(b) a planar graph with 5 vertices. (5%)

12 Suppose that $c_1 2^n + c_2 3^n + n - 7$ is the general solution to $a_{n+2} + p_1 a_{n+1} + p_2 a_n = q_1 n + q_2$, where $n \geq 0$ and p_1, p_2, q_1, q_2 are constants. Find

(a) p_1 and p_2; (5%)

(b) q_1 and q_2. (5%)

13 Suppose $A = \{w, x, y, z\}$. Find the number of relations on A that are

(a) reflexive; (5%)

(b) symmetric and contain (x, y). (5%)

14 Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$. (10%)

15 Suppose that $f: G \to H$ is a group homomorphism and f is onto. Prove that if G is abelian, then H is abelian. (10%)

Solutions

NTU math 100

Version 1.0

1. **Answer** †

$$1 \tag{1}$$

2. **Answer** †

$$\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \tag{2}$$

3. We have

$$\boldsymbol{B}^3 = \begin{bmatrix} 17 & 6 \\ 18 & -1 \end{bmatrix} = 6 \times \boldsymbol{B} + 5 \times \boldsymbol{I} \tag{3}$$

Answer †

$$(6,\ 5) \tag{4}$$

4. We have

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix})\right) = 2,\ \det\left(\begin{bmatrix} 1 & 5 & 1 \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix})\right) = -31,\ \det\left(\begin{bmatrix} 1 & 4 & 1 \\ 5 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix})\right) = -2 \tag{5}$$

Answer †

$$2 \times \frac{1}{-31} \times (-2) = \frac{4}{31} \tag{6}$$

5. We have

$$\mathrm{tr}(\boldsymbol{A}^2) = (1 + (-2) + 3) + ((-2) + 9 + a) + (3 + a + 0) = 5 \tag{7}$$

Answer †

$$-\frac{7}{2} \tag{8}$$

6. We have

$$\begin{aligned} \boldsymbol{A}\boldsymbol{w} &= \boldsymbol{w} + \alpha \boldsymbol{w}\boldsymbol{w}^{\intercal}\boldsymbol{w} \\ &= \boldsymbol{w} + 10 \times \alpha \boldsymbol{w} \\ &= (10 \times \alpha + 1)\boldsymbol{w} \\ \Rightarrow 10 \times \alpha + 1 &= 0 (\because \boldsymbol{A} \text{ is singular.}) \end{aligned} \tag{9}$$

And, we have

$$\text{rank}(\boldsymbol{w}\boldsymbol{w}^{\intercal}) = 1 \tag{10}$$

Then, we have eigenvalues of $\boldsymbol{w}\boldsymbol{w}^{\intercal}$

$$0,\ 0,\ 0,\ 0,\ 10 \tag{11}$$

Then, we have eigenvalues of $\boldsymbol{A}$

$$1,\ 1,\ 1,\ 1,\ 0\ (\because\ 1 + 10 \times \alpha) \tag{12}$$

Then, we have $\text{rank}(\boldsymbol{A}) = 4$.

Answer †

$$(-\frac{1}{10},\ 4) \tag{13}$$

7. Suppose

$$\begin{aligned} \boldsymbol{D} &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \\ \Rightarrow \boldsymbol{D}^{-1} &= \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \end{aligned} \tag{14}$$

Then, we have

$$\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} a_{11} & \frac{1}{2} \times a_{12} & \frac{1}{4} \times a_{13} \\ 2 \times a_{21} & a_{22} & \frac{1}{2} \times a_{23} \\ 4 \times a_{31} & 2 \times a_{32} & a_{33} \end{bmatrix} \tag{15}$$

Answer †

$$\begin{bmatrix} 4\times\alpha & 0 & 0 \\ 0 & 2\times\alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix},\ \forall\,\alpha\in\mathbb{R} \tag{16}$$

8. Let

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{I} + \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \\ \Rightarrow \boldsymbol{A} &= \boldsymbol{I} + \boldsymbol{X}\boldsymbol{Y} \end{aligned} \tag{17}$$

Since, $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}$ have same eigenvalues, we have

$$\boldsymbol{Y}\boldsymbol{X} = \begin{bmatrix} 35 & 55 \\ 55 & 35 \end{bmatrix} \tag{18}$$

Then, $\boldsymbol{Y}\boldsymbol{X}$ has eigenvalues $-20,\ 90$, so $\boldsymbol{X}\boldsymbol{Y}$ has eigenvalues

$$0,\ 0,\ 0,\ -20,\ 90 \tag{19}$$

Then, $\boldsymbol{A}$ has eigenvalues

$$1+0,\ 1+0,\ 1+0,\ 1+(-20),\ 1+90 \tag{20}$$

Answer †

$$1,\ 1,\ 1,\ -19,\ 91 \tag{21}$$

9. We have

$$
\begin{aligned}
&(\because\ r_{n(n-1)}^{-1},\ r_{n(n-2)}^{-1},\ \cdots,\ r_{n1}^{-1})\\
&=\begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-1}{3\times(n+2)} & \cdots & \frac{n-1}{2n\times(n+1)}\\
\frac{n-2}{3\times(n+1)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{n-2}{2n\times(n+2)}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n\times(n+1)} & \frac{1}{(n+1)\times(n+2)} & \cdots & \frac{1}{2n\times(2n-1)}\\
\frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n}
\end{vmatrix}_{n\times n}\\
&(\because\ r_1^{\frac{1}{n-1}},\ r_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ c_1^{n+1},\ c_2^{n+2},\ \cdots,\ c_n^{2n})\\
&=(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1}\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1}\\
1 & 1 & \cdots & 1
\end{vmatrix}_{n\times n}\\
&(\because\ c_{n(n-1)}^{-1},\ c_{n(n-2)}^{-1},\ \cdots,\ c_{n1}^{-1})\\
&=\frac{n!(n-1)!}{(2n)!}\begin{vmatrix}
\frac{n-1}{2\times(n+1)} & \frac{n-2}{3\times(n+1)} & \cdots & \frac{1}{n\times(n+1)} & \frac{1}{n+1}\\
\frac{n-1}{3\times(n+2)} & \frac{n-2}{4\times(n+2)} & \cdots & \frac{1}{(n+1)\times(n+2)} & \frac{1}{n+2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{n-1}{n\times(2n-1)} & \frac{n-2}{(n+1)\times(2n-1)} & \cdots & \frac{1}{(2n-2)\times(2n-1)} & \frac{1}{2n-1}\\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{n\times n} \qquad (22)\\
&(\because\ c_1^{\frac{1}{n-1}},\ c_2^{\frac{1}{n-2}},\ \cdots,\ r_{n-2}^{\frac{1}{2}},\ r_1^{n+1},\ r_2^{n+2},\ \cdots,\ r_{n-1}^{2n-1})\\
&=\frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & 1\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} & 1\\
0 & 0 & \cdots & 0 & 1
\end{vmatrix}_{(n-1)\times(n-1)}\\
&=\frac{n!(n-1)!}{(2n)!}(\prod_{i=1}^{n-1} i)(\prod_{j=1}^{n}\frac{1}{n+j})\begin{vmatrix}
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n}\\
\frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2}
\end{vmatrix}_{(n-1)\times(n-1)}\\
&=\frac{(n!)^2[(n-1)!]^2\times 2n}{[(2n)!]^2}\times P_{n-1}
\end{aligned}
$$

And, we have

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{23}$$

Answer †

$$\frac{\det(P_{n+1})}{\det(P_n)} = \frac{[(n+1)!]^2(n!)^2 \times (2n+2)}{[(2n+2)!]^2} \tag{24}$$

10. We have

$$\begin{cases} \boldsymbol{u}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{v}^{\mathsf{H}}\boldsymbol{v} \\ \boldsymbol{u} \neq \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} = \boldsymbol{I} \\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{v} \end{cases} \tag{25}$$

Let $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, and we assume

$$\boldsymbol{A} = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{26}$$

Then, we have

$$\begin{aligned} \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{I}\boldsymbol{u} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{w} = \boldsymbol{v} \\ \boldsymbol{A}^{\mathsf{H}}\boldsymbol{A} &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})^{\mathsf{H}}(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= (\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}})(\boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}) \\ &= \boldsymbol{I} - \frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} + \frac{1}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - [\frac{1}{\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}} + \frac{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}]\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{\boldsymbol{w}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{w}) + \boldsymbol{u}^{\mathsf{H}}\boldsymbol{w}}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} - \frac{(\boldsymbol{u} - \boldsymbol{v})^{\mathsf{H}}\boldsymbol{v} + \boldsymbol{u}^{\mathsf{H}}(\boldsymbol{u} - \boldsymbol{v})}{(\boldsymbol{u}^{\mathsf{H}}\boldsymbol{w})(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u})}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \\ &= \boldsymbol{I} \end{aligned} \tag{27}$$

Answer †

$$\boldsymbol{A}(\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{I} - \frac{1}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{u}}\boldsymbol{w}\boldsymbol{w}^{\mathsf{H}} \tag{28}$$

11. We have

   (a) **Answer** †

$$6 \times 6 = 36 \tag{29}$$

   Since $K_{6,6}$ has the maximal edges.

(b) **Answer** †

$$e \leq 3 \times 5 - 6 = 9 \tag{30}$$

12. We have

$$\begin{aligned} &\Rightarrow (\alpha - 2)(\alpha - 3) = \alpha^2 - 5 \times \alpha + 6 = 0 \\ &\Rightarrow a_{n+2} - 5 \times a_{n+1} + 6 \times a_n = q_1 \times n + q_2 \end{aligned} \tag{31}$$

And, we have

$$(n + 2 - 7) - 5 \times (n + 1 - 7) + 6 \times (n - 7) = 2 \times n - 17 \tag{32}$$

(a) **Answer** †

$$p_1 = -5, \ p_2 = 6 \tag{33}$$

(b) **Answer** †

$$q_1 = 2, \ q_2 = -17 \tag{34}$$

13. (a) **Answer** †

$$2^{4^2-4} = 4096 \tag{35}$$

(b) **Answer** †

$$2^{\frac{4^2-4}{2}-1} \times 2^4 = 512 \tag{36}$$

14. **Answer** † Suppose

$$x < 50 \wedge y < 50 \rightarrow x + y < 100 \tag{37}$$

contradiction, so

$$x \geq 50 \wedge y \geq 50 \tag{38}$$

15. **Answer** † Let $*$ be operator of G and $\cdot$ be operator of H.

$$\begin{aligned} &\because \ f \text{ is onto} \\ &\therefore \ \forall \, y_1, \ y_2 \in H, \ \exists \, x_1, \ x_2, \ \text{s.t. } f(x_1) = y_1, \ f(x_2) = y_2 \\ &\Rightarrow \ y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 * x_2) = f(x_2 * x_1) = f(x_2) \cdot f(x_1) = y_2 \cdot y_1 \end{aligned} \tag{39}$$

Then, if G is abelian, then H is abelian.