Solutions

NTU math 101

Version 1.0

1. **Answer** †

$$\begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$3 \tag{2}$$

3. We have

$$\boldsymbol{B} \overset{\text{rref}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 \end{bmatrix} \tag{3}$$

   Answer †

$$3 \tag{4}$$

4. We have

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{5}$$

   $\boldsymbol{A}$ is a rotation matrix, which rotates $\frac{\pi}{3}$ **clockwisely**. Then, we have

$$\boldsymbol{A}^{300} = \begin{bmatrix} \cos(100\pi) & \sin(100\pi) \\ -\sin(100\pi) & \cos(100\pi) \end{bmatrix} \tag{6}$$

   Answer †

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

5. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x) = -x^3 + x^2 - 3 \times x + 2 \tag{8}$$

Suppose $\lambda_{\boldsymbol{A}} = a,\ b,\ c$, then we have,

$$\begin{aligned}
&\begin{cases} a+b+c = 1 \\ ab+bc+ac = 3 \\ abc = 2 \end{cases} \\
\Rightarrow &\begin{cases} a^2+b^2+c^2 = (a+b+c)^2 - 2 \times (ab+bc+ac) = -5 \\ a^2b^2+b^2c^2+a^2c^2 = (ab+bc+ac)^2 - 2 \times (abc) \times (a+b+c) = 5 \\ a^2b^2c^2 = (abc)^2 = 4 \end{cases}
\end{aligned} \tag{9}$$

Then, we have characteristic polynomial

$$p_{\boldsymbol{A}^2} = \det(\boldsymbol{A}^2 - \boldsymbol{I}\boldsymbol{x}) = -x^3 + (-5) \times x^2 - (5) \times x + 4 \tag{10}$$

Answer †

$$\det(\boldsymbol{x}\boldsymbol{I} - \boldsymbol{A}^2) = -(-x^3 + (-5) \times x^2 - (5) \times x + 4) = x^3 + 5 \times x^2 + 5 \times x - 4 \tag{11}$$

6. **Answer** †

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \tag{12}$$

7. We have

$$\begin{aligned}
&-\boldsymbol{H}^3 + \alpha \boldsymbol{H}^2 + \beta \boldsymbol{H} + \gamma \boldsymbol{I} = \boldsymbol{O} \\
\Rightarrow &\begin{cases} \alpha = \operatorname{tr}(\boldsymbol{H}) = 34 \\ \beta = -\operatorname{tr}_2(\boldsymbol{H}) \\ \quad = -(\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 9 & 11 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 13 & 16 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 10 & 11 \end{vmatrix} + \begin{vmatrix} 6 & 8 \\ 14 & 16 \end{vmatrix} + \begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix}) = 80 \\ \gamma = \det(\boldsymbol{H}) = 0 \end{cases}
\end{aligned} \tag{13}$$

Answer †

$$\begin{bmatrix} 34 \\ 80 \\ 0 \end{bmatrix} \tag{14}$$

8. We have

$$\begin{aligned}
&\overset{r_{12}^{1},\ r_{23}^{1},\ \cdots,\ r_{56}^{1}}{=} \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 & 3\times 2^4 \\ 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 & 3\times 2^3 \\ 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 & 3\times 2^2 \\ 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 & 3\times 2 \\ 3\times 2^4 & 3\times 2^3 & 3\times 2^2 & 3\times 2 & 3 & 3 \end{vmatrix} \\
&\overset{r_{2}^{\frac{1}{3}},\ r_{3}^{\frac{1}{3}},\ \cdots,r_{6}^{\frac{1}{3}}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 1 & 1 & 2 & 2^2 & 2^3 & 2^4 \\ 2 & 1 & 1 & 2 & 2^2 & 2^3 \\ 2^2 & 2 & 1 & 1 & 2 & 2^2 \\ 2^3 & 2^2 & 2 & 1 & 1 & 2 \\ 2^4 & 2^3 & 2^2 & 2 & 1 & 1 \end{vmatrix} \\
&\overset{r_{12}^{-1},\ r_{23}^{-1},\ \cdots,\ r_{56}^{-1}}{=} 3^5 \times \begin{vmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 0 & -1 & -2 & -2^2 & -2^3 & -2^4 \\ 1 & 0 & -1 & -2 & -2^2 & -2^3 \\ 2 & 1 & 0 & -1 & -2 & -2^2 \\ 2^2 & 2 & 1 & 0 & -1 & -2 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix} \\
&\overset{r_{65}^{-2},\ r_{54}^{-2},\ \cdots,\ r_{21}^{-2}}{=} 3^5 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 & 0 & 0 \\ -6 & -3 & -2 & -1 & 0 & 0 \\ -12 & -6 & -3 & -2 & -1 & 0 \\ 2^3 & 2^2 & 2 & 1 & 0 & -1 \end{vmatrix}
\end{aligned} \tag{15}$$

Answer †

$$-\,3^5 \tag{16}$$

9. Suppose

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{100} \end{bmatrix}, \ \boldsymbol{A}_{100} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}_{100\times 100} \tag{17}$$

$$\Rightarrow q(x_1,\ x_2,\ \cdots,\ x_{100}) = \sum_{k=1}^{99} x_k x_{k+1} = \boldsymbol{x}^\intercal \boldsymbol{A}_{100}\boldsymbol{x}$$

By Rayleigh principle, $\max_{||\boldsymbol{x}||=1} \boldsymbol{x}^\intercal \boldsymbol{A}_n \boldsymbol{x} = \lambda_{\max}(\boldsymbol{A}_n)$. And, we have general tridiagnoal matrix

$$B_n = \begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & a & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & a & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & b & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b & a & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & a \end{bmatrix}_{n\times n} \tag{18}$$

By solving the recurrence function, we have general eigenvalues formula for $\boldsymbol{B}_n$

$$\lambda_k = a + 2\times b\cos(\frac{k\times\pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{19}$$

Then, we have $a = 0,\ b = \frac{1}{2}$, and get $\boldsymbol{A}_n$'s eigenvalues

$$\lambda_k = \cos(\frac{k\times\pi}{n+1}),\ k = 1,\ 2,\ \cdots,\ n \tag{20}$$

And, we have

$$\cos(\frac{1\times\pi}{100+1}) \tag{21}$$

as largest eigenvalue of $\boldsymbol{A}_{100}$.

Answer †

$$\cos(\frac{\pi}{101}) \tag{22}$$

10. (a) has 2 degree-3 vertice and 3 degree-2 vertices, when (b), (c), and (d) have same 4 degree-3 vertice and 1 degree-2 vertices, so (a) can NOT be an isomorphism of others.

And, we have correspondence

(b)	(c)	(d)
1	4	5
2	1	2
3	5	1
4	3	4
5	2	3

So, $(b)(c)(d)$ are isomorphic.

Answer †

$$(b)(c)(d) \tag{23}$$

11. (a) **Answer** †

$$s_n = s_{n-1} + \frac{n(n-1)}{2} \tag{24}$$

(b) **Answer** †

$$a_0 + a_1 + a_2 + a_3 + \cdots = s_1 = 1 \tag{25}$$

12. Suppose

$$\Rightarrow \begin{cases} x_1 = a - 1 \geq 0 \\ x_2 = b - a \geq 2 \\ x_3 = c - b \geq 2 \\ x_4 = d - c \geq 2 \\ x_5 = 12 - d \geq 0 \end{cases}, \sum_{i=1}^{5} x_i = 11$$
$$\Rightarrow \begin{cases} y_1 = x_1 \geq 0 \\ y_2 = x_2 - 2 \geq 0 \\ y_3 = x_3 - 2 \geq 0 \\ y_4 = x_4 - 2 \geq 0 \\ y_5 = x_5 \geq 0 \end{cases}, \sum_{i=1}^{5} y_i = 11 - 3 \times 2 \tag{26}$$

Answer †

$$\binom{5 + (11-6) - 1}{(11-6)} = 126 \tag{27}$$

13. (a) We have constraints: m and n must be **even** (≥ 1 Euler circuits), and $m \neq n$ (NO Hamilton cycle).

Answer †

$$2,\ 8 \tag{28}$$

(b) **Answer** †

$$m \text{ and } n \text{ is even, and } m \neq n \tag{29}$$

14. **Answer** △

$$\begin{aligned}
&(\Rightarrow)\ \because\ (S,\ +,\ \cdot) \text{ is a ring} \\
&\qquad \therefore\ \forall\ a,\ b \in S,\ a+b \in S,\ a \cdot b \in S \\
&(\Leftarrow)\ \forall\ a \in S,\ -a \in S \\
&\qquad \because (S,\ +,\ \cdot) \text{ is closed},\ \forall\ a \in S,\ a,\ 2a,\ \cdots \in S \\
&\qquad \exists\ i < j,\ ia = ja \to (j-i)a = 0 \to a + (j-i-1)a = 0 \\
&\qquad \therefore\ -a = \begin{cases} 0(=a) \in S & ,\ j = i+1 \\ (j-i-1)a \in S & ,\ j > i+1 \end{cases}
\end{aligned} \tag{30}$$