Solutions

NTU math 107

Version 1.0

1. We have new question

$$
\begin{aligned}
&(x_1 + x_2 + \cdots + x_n \leq H) - (x_1 + x_2 + \cdots + x_n < L),\ \forall\, x_i \geq 0,\ 1 \leq i \leq n \\
&(\text{Let } x_{n+1} = H - (x_1 + x_2 + \cdots + x_n),\ x_{n+1} \geq 0, \\
&y_{n+1} = L - (x_1 + x_2 + \cdots + x_n),\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\, x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\quad - (x_1 + x_2 + \cdots + x_n + y_{n+1} = L,\ \forall\, x_i \geq 0,\ 1 \leq i \leq n,\ y_{n+1} > 0) \\
\Rightarrow\ &(x_1 + x_2 + \cdots + x_n + x_{n+1} = H,\ \forall\, x_i \geq 0,\ 1 \leq i \leq (n+1)) \\
&\quad - (x_1 + x_2 + \cdots + x_n + z_{n+1} = L - 1,\ \forall\, x_i \geq 0,\ 1 \leq i \leq n,\ z_{n+1} \geq 0)
\end{aligned} \tag{1}
$$

Answer †

$$
\binom{(n+1) + H - 1}{H} - \binom{(n+1) + (L-1) - 1}{L-1} \tag{2}
$$

2. We have

$$
\begin{aligned}
&\alpha^2 = 2 \times \alpha + 3 \\
\Rightarrow\ &\alpha = 3 \vee \alpha = -1 \\
\Rightarrow\ &a_n = c \times 3^n + d \times (-1)^n
\end{aligned} \tag{3}
$$

Then, we have

$$
\begin{aligned}
&\begin{cases} a_0 = 1 = c + d \\ a_1 = 1 = 3 \times c - d \end{cases} \\
\Rightarrow\ &c = \frac{1}{2},\ d = \frac{1}{2}
\end{aligned} \tag{4}
$$

Answer †

$$
a_n = \frac{1}{2} \times 3^n + \frac{1}{2} \times (-1)^n \tag{5}
$$

3. We have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)^2x^n\\ \overset{\text{integral}}{=}&\sum_{n=0}^{\infty}(n+1)x^{n+1}=x\sum_{n=0}^{\infty}(n+1)x\end{aligned}\tag{6}$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)x^n\\ \overset{\text{integral}}{=}&\sum_{n=0}^{\infty}x^{n+1}=\frac{x}{1-x}\end{aligned}\tag{7}$$

Then, we have

$$\begin{aligned}&\frac{x}{1-x}\overset{\text{derivative}}{=}\frac{1}{(1-x)^2}\\ &\Rightarrow\sum_{n=0}^{\infty}(n+1)x^{n+1}=\frac{x}{(1-x)^2}\end{aligned}\tag{8}$$

And, we have

$$\begin{aligned}&\frac{x}{(1-x)^2}\overset{\text{derivative}}{=}\frac{1+x}{(1-x)^3}\\ &\Rightarrow\sum_{n=0}^{\infty}(n+1)^2x^n=\frac{1+x}{(1-x)^3}\end{aligned}\tag{9}$$

Answer †

$$\frac{1+x}{(1-x)^3}\tag{10}$$

4. **Answer** †

$$2^{\binom{m}{2}}\tag{11}$$

5. **Answer** †

$$2^{\frac{n(n+1)}{2}},\ \binom{\binom{n}{2}}{m}\tag{12}$$

6. We have characteristic polynomial

$$p_{\boldsymbol{A}}(x)=x^2-5\times x+4\tag{13}$$

Then, we have

$$\begin{aligned}f(\boldsymbol{A})&=\boldsymbol{A}^4-3\times\boldsymbol{A}^3-6\times\boldsymbol{A}^2+7\times\boldsymbol{A}+2\times\boldsymbol{I}\\ &=(\boldsymbol{A}^2+2\times\boldsymbol{A})(\boldsymbol{A}^2-5\times\boldsymbol{A}+4\times\boldsymbol{I})+(-\boldsymbol{A}+2\times\boldsymbol{I})\\ &=(-\boldsymbol{A}+2\times\boldsymbol{I})\end{aligned}\tag{14}$$

Answer †

$$\begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix} \tag{15}$$

7. We have

$$\begin{aligned}\det(\boldsymbol{A}+t\boldsymbol{I}) &= \begin{bmatrix} t & 0 & 0 & \cdots & a_0 \\ -1 & t & 0 & \cdots & a_1 \\ 0 & -1 & t & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1}+t \end{bmatrix}_{n\times n} \\ &= t^{n-1}[(a_{n-1}+t)+\frac{1}{t}a_{n-2}+\frac{1}{t^2}a_{n-3}+\cdots+a_0] \\ &= t^n+t^{n-1}a_{n-1}+t^{n-2}a_{n-2}+\cdots+a_0 \\ &= t^n+\sum_{i=0}^{n-1}a_it^i\end{aligned} \tag{16}$$

Answer †

$$t^n+\sum_{i=0}^{n-1}a_it^i \tag{17}$$

8. By Gram-Schmidt process, we have

$$\begin{aligned} &u_1 = 1,\ ||u_1|| = \int_0^1 1\times 1dt = 1 \\ &u_2 = t - \frac{\int_0^1 1\times tdt}{1}\times 1 = t-\frac{1}{2},\ ||u_2|| = \int_0^1 (t-\frac{1}{2})^2dt = \frac{1}{12} \\ &u_3 = t^2 - \frac{\int_0^1 1\times t^2dt}{1}\times 1 - \frac{\int_0^1 (t-\frac{1}{2})\times t^2dt}{\frac{1}{12}}\times(t-\frac{1}{2}) = t^2-t+\frac{1}{6}, \\ &\qquad ||u_3|| = \int_0^1 (t^2-t+\frac{1}{6})^2dt = \frac{1}{180}\end{aligned} \tag{18}$$

We have projection

$$\begin{aligned} &\frac{\int_0^1 1\times t^3dt}{1}\times 1 + \frac{\int_0^1 (t-\frac{1}{2})\times t^3dt}{\frac{1}{12}}(t-\frac{1}{2}) + \frac{\int_0^1 (t^2-t+\frac{1}{6})\times t^3dt}{\frac{1}{180}}\times(t^2-t+\frac{1}{6}) \\ =&\frac{3}{2}\times t^2 - \frac{3}{5}\times t + \frac{1}{20}\end{aligned} \tag{19}$$

Answer †

$$\frac{3}{2}\times t^2 - \frac{3}{5}\times t + \frac{1}{20} \tag{20}$$

9. **Answer** † We have

- False.
- False. We have $\det(\boldsymbol{A}^\intercal) = \det(-\boldsymbol{A}) \iff \det(\boldsymbol{A}) = (-1)^n \times \det(\boldsymbol{A})$. ONLY if the dimension of $\boldsymbol{A}$, i.e. n, is odd, then $\boldsymbol{A}$ is singular; otherwise, it's non-singular.
- False. $(\boldsymbol{A}+\boldsymbol{I})^n = (2^n - 1) \times \boldsymbol{A}$.
- True, since symmetric matrix is **orthogonally diagonalizable**, and it's also **diagonalizable**.
- True. Suppose

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} \tag{21}$$

Then, we have

$$\begin{aligned}
&\Rightarrow \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{R} & \boldsymbol{S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \\
&\Rightarrow \begin{cases} \boldsymbol{BP} + \boldsymbol{CR} = \boldsymbol{I} \\ \boldsymbol{BQ} + \boldsymbol{CS} = \boldsymbol{O} \\ \boldsymbol{DR} = \boldsymbol{O} \to \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{DS} = \boldsymbol{I} \to \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \begin{cases} \boldsymbol{P} = \boldsymbol{B}^{-1} \\ \boldsymbol{Q} = -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{R} = \boldsymbol{O} \\ \boldsymbol{S} = \boldsymbol{D}^{-1} \end{cases} \\
&\Rightarrow \boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{C}\boldsymbol{D}^{-1} \\ \boldsymbol{O} & \boldsymbol{D}^{-1} \end{bmatrix}
\end{aligned} \tag{22}$$

10. **Answer** † We have

- True. We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{23}$$

And, we have

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\right) = 2 \tag{24}$$

is non-singular, so $\{u+v,\ v+w,\ w+u\}$ is linearly independent.

- True. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, so $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues.
- False. $\boldsymbol{A} \sim \boldsymbol{B} \to p_{\boldsymbol{A}} = p_{\boldsymbol{B}}$, but the eigenvectors may differ.
- False, since if $m \neq n$, $\boldsymbol{B}^\intercal \boldsymbol{A}$ may NOT exist.
- False, since

$$(U+W)^\perp = U^\perp \cap W^\perp \tag{25}$$