Solutions

NTU math 106

VERSION 1.0

1. **Answer** †

$$\begin{bmatrix} 1 & 0 & 1 \\ r & 1 & r \\ 3 & r & 2 \end{bmatrix} \tag{1}$$

2. **Answer** †

$$2 \tag{2}$$

Since $\boldsymbol{A}$ has eigenvalues 0 and 1, it's **idempotent**, $\boldsymbol{A}^2 = \boldsymbol{A}$.

3. Suppose

$$\boldsymbol{A} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

Then, we have projection

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^\intercal \boldsymbol{A})^{-1}\boldsymbol{A}^\intercal \tag{4}$$

Answer †

$$\frac{1}{x^2+y^2}\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \tag{5}$$

4. Suppose

$$\begin{cases} \beta & = \{(1,\ 0),\ (0,\ 1)\} \\ \gamma & = \{\boldsymbol{u} = (1,\ 0),\ \boldsymbol{v} = (a,\ b)\} \end{cases} \tag{6}$$

Then, we have

$$[x]_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ [y]_\beta = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ [x]_\gamma = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix},\ [y]_\gamma = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \tag{7}$$

Then ,we have transition matrix

$$\begin{aligned} [\boldsymbol{I}]_\gamma^\beta &= \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \\ \Rightarrow [x]_\beta &= \begin{bmatrix} s_1 + s_2 a \\ s_2 b \end{bmatrix}, \ [y]_\beta = \begin{bmatrix} t_1 + t_2 a \\ t_2 b \end{bmatrix} \end{aligned} \tag{8}$$

Then, we have

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &= (s_1 + s_2 a)(t_1 + t_2 a - t_2 b) + s_2 b \times (-t_1 - t_2 a + 4 \times t_2 b) \\ &= s_1 t_1 + s_2 t_2 \\ \Rightarrow a = b &= \pm \frac{1}{\sqrt{3}} \end{aligned} \tag{9}$$

Answer †

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \vee \begin{bmatrix} 1 & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} \end{bmatrix} \tag{10}$$

5. We have pseudo-inverse

$$\boldsymbol{A}^+ = (\boldsymbol{A}^\intercal \boldsymbol{A})^{-1} \boldsymbol{A}^\intercal \tag{11}$$

Answer †

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \tag{12}$$

6. **Answer** †

$$2^{m^2} - 2^{m^2-m} - 2^{m^2-m} \tag{13}$$

Since both reflexive and irreflexive are 2^{m^2-m}.

7. We have

$$\begin{aligned} & [(p \vee q) \wedge (\neg p \vee r)] \\ \Longleftrightarrow\ & (q \wedge \neg p) \vee (p \wedge r) \vee (q \wedge r) \\ & \text{(Draw the Venn diagram)} \\ \Longleftrightarrow\ & (p \wedge r) \vee (\neg p \wedge q) \end{aligned} \tag{14}$$

Answer †

$$(p \wedge r) \vee (\neg p \wedge q) \tag{15}$$

8. We have

$$\begin{aligned}
&\alpha^2 = \alpha + 1 \\
&\Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2} \\
&\Rightarrow a_n = c \times (\frac{1+\sqrt{5}}{2})^n + d \times (\frac{1-\sqrt{5}}{2})^n
\end{aligned} \tag{16}$$

And, we have

$$\begin{aligned}
&\begin{cases} a_0 = c + d \\ a_1 = c \times (\frac{1+\sqrt{5}}{2}) + d \times (\frac{1-\sqrt{5}}{2}) \end{cases} \\
&\Rightarrow c = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}},\ d = \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \\
&\Rightarrow a_n = \frac{2 \times a_1 - (1-\sqrt{5})a_0}{2\sqrt{5}} \times (\frac{1+\sqrt{5}}{2})^n + \frac{(1+\sqrt{5})a_0 - 2 \times a_1}{2\sqrt{5}} \times (\frac{1-\sqrt{5}}{2})^n
\end{aligned} \tag{17}$$

Answer †

$$A = 2 \times a_1 - (1-\sqrt{5})a_0,\ B = 1+\sqrt{5},\ C = (1+\sqrt{5})a_0 - 2 \times a_1,\ D = 1-\sqrt{5} \tag{18}$$

9. **Answer** † We have

$$\begin{aligned}
&\gcd(n, n-1) \\
&= \gcd(n-1,\ 1) \\
&= 1
\end{aligned} \tag{19}$$

10. **Answer** † bipartite

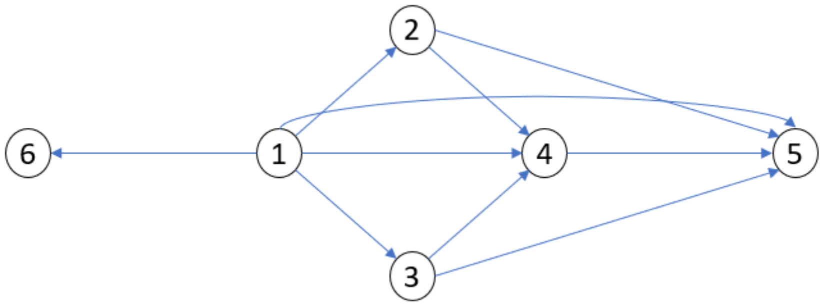
11. We have

$$\begin{aligned}
&2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} + \binom{n}{\frac{n+1}{2}} + \cdots + \binom{n}{n} \\
&\Rightarrow \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} = \frac{2^n}{2} = 2^{n-1}
\end{aligned} \tag{20}$$

Answer †

$$2^{n-1} \tag{21}$$

12. **Answer** †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