Solutions

NTU math 108

VERSION 1.0

1. **Answer** † 1, 3, 7, 9.

2. We have recurrence function

$$\begin{cases} a_n = 2 \times a_{n-2},\ n \geq 3 \\ a_1 = 2,\ a_2 = 2 \end{cases} \tag{1}$$

Then, we have

$$\begin{aligned} &\alpha^2 = 2 \\ &\Rightarrow \alpha = \pm\sqrt{2} \\ &\Rightarrow a_n = c \times (\sqrt{2})^n + d \times (-\sqrt{2})^n \end{aligned} \tag{2}$$

Then, we have

$$\begin{aligned} &\begin{cases} a_1 = 2 = \sqrt{2} \times c - \sqrt{2} \times d \\ a_2 = 2 = 2 \times c + 2 \times d \end{cases} \\ &\Rightarrow c = \frac{\sqrt{2}+2}{2\sqrt{2}},\ d = \frac{\sqrt{2}-2}{2\sqrt{2}} \end{aligned} \tag{3}$$

Answer †

$$a_n = \frac{\sqrt{2}+2}{2\sqrt{2}} \times (\sqrt{2})^n + \frac{\sqrt{2}-2}{2\sqrt{2}} \times (-\sqrt{2})^n \tag{4}$$

3. We have

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = 2 \times \binom{2n+2}{n+1} \tag{5}$$

Answer †

$$A = 2n+2,\ B = n+1 \tag{6}$$

4. We have

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n-(k-1)} = \binom{2n}{n+1} \tag{7}$$

Answer †

$$A = 2n,\ B = n + 1 \tag{8}$$

5. We have

$$\begin{aligned} &\alpha^2 = \alpha + 2 \\ \Rightarrow\ &\alpha = 2 \lor \alpha = -1 \\ \Rightarrow\ &a_n = c \times 2^n + d \times (-1)^n \end{aligned} \tag{9}$$

We have

$$\begin{aligned} &\begin{cases} a_0 = c + d \\ a_1 = 2 \times c - d \end{cases} \\ \Rightarrow\ &c = \frac{a_0 + a_1}{3},\ d = \frac{2 \times a_0 - a_1}{3} \\ \Rightarrow\ &a_n = \frac{2 \times a_0 - a_1}{3} \times (-1)^n + \frac{a_0 + a_1}{3} \times 2^n \end{aligned} \tag{10}$$

Answer †

$$A = \frac{2 \times a_0 - a_1}{3},\ B = \frac{a_0 + a_1}{3},\ X = 1,\ Y = 2 \tag{11}$$

6. We have

$$\begin{aligned} &x_1 + x_2 + \cdots + x_n = r,\ \forall\ x_i \geq n_i + 1,\ 1 \leq i \leq n \\ \Rightarrow\ &y_1 + y_2 + \cdots + y_n = r - ((\sum_{i=1}^{n} n_i) + n),\ \forall\ y_i \geq 0,\ 1 \leq i \leq n \end{aligned} \tag{12}$$

Answer †

$$\binom{n + r - (\sum_{i=1}^{n} n_i) - n - 1}{r - (\sum_{i=1}^{n} n_i) - n} \tag{13}$$

7. **Answer** †

$$\begin{aligned} &[(p \to q) \land \neg p] \to \neg q \\ \Longleftrightarrow\ &[(\neg p \lor q) \land \neg p] \to \neg q \\ \Longleftrightarrow\ &\neg[(\neg p \lor q) \land \neg p] \lor \neg q \\ \Longleftrightarrow\ &[(p \land \neg q) \lor p] \lor \neg q \\ \Longleftrightarrow\ &p \land (\neg q \lor p) \lor \neg q \\ \Longleftrightarrow\ &(p \lor \neg q) \land [(p \lor \neg q) \lor \neg q] \\ \Longleftrightarrow\ &(p \lor \neg q) \land [(p \lor \neg q) \lor \neg q] \\ \Longleftrightarrow\ &(p \lor \neg q) \end{aligned} \tag{14}$$

So, it's NOT tautology.

8. We have

- True. Let

$$
\begin{aligned}
&S = \{a,\ b\} \subset U \\
\Rightarrow\ &\mathrm{span}(s) = c_1 a + c_2 b \subset U
\end{aligned} \tag{15}
$$

- False. Counterexample:

$$
\begin{aligned}
&R = \{(1,\ 0),\ (0,\ 1),\ (0,\ 2)\} \\
\Rightarrow\ &(1,\ 0) \not\ni \mathrm{span}(R \setminus \{(1,\ 0)\})
\end{aligned} \tag{16}
$$

- True.
- False. Counterexample: $\mathrm{span}(\mathbf{0}) = \emptyset$, but $\emptyset$ is NOT orthonormal.
- True.

Answer †

$$3 \tag{17}$$

9. We have

- True.
- True. $\boldsymbol{A}$ is invertible $\iff \det(\boldsymbol{A}) \neq 0 \iff \det(\boldsymbol{A}^{\mathsf{H}}) \neq 0$
- True.
- True. $\boldsymbol{A}$ is invertible, so $\mathrm{rank}(\boldsymbol{A}) = m = n = \mathrm{rank}(\boldsymbol{A}^{-1})$.
- True. $\det(\boldsymbol{A}^{\mathsf{H}}) = \det(\overline{\boldsymbol{A}^{\mathsf{T}}}) = \overline{\det(\boldsymbol{A}^{\mathsf{T}})} = \overline{\det(\boldsymbol{A})}$

Answer †

$$5 \tag{18}$$

10. We have

- True. $\mathbb{Q}^n$ is the direct sum of eigenspace of $\boldsymbol{A} \iff$ there are n linearly independent eigenvectores of $\boldsymbol{A} \iff \boldsymbol{A}$ is diagonalizable
- True.
- False. $\boldsymbol{A}$ may NOT be split.
- False. If the $\boldsymbol{A}$ is **real** and symmetric, all of its eigenvalues are always real. It can NOT be ensured if $\boldsymbol{A}$ is **complex**.
- True.

Answer †

$$3 \tag{19}$$

11. We have inverse

$$\begin{bmatrix} \frac{529}{12167} & 0 & \frac{529}{12167} & \frac{529}{12167} \\ 0 & \frac{1587}{12167} & 0 & \frac{1058}{12167} \\ \frac{2116}{12167} & 0 & \frac{1587}{12167} & 0 \\ \frac{1058}{12167} & \frac{1058}{12167} & 0 & \frac{1587}{12167} \end{bmatrix} \tag{20}$$

Answer †

$$6 \tag{21}$$

12. **Answer** †

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$