Solutions

NTU math 103

Version 1.0

1. **Answer** † There are 3 As, so we first **permute** other 4 characters, and then **insert** 3 As in the 5 spaces.
$$\frac{4!}{2!} \times \binom{5}{3} \tag{1}$$

2. We have
$$\begin{aligned} &x_1 + x_2 + \cdots + x_n = r, \ \forall\ x_i > 0, \ 1 \le i \le n \\ \Rightarrow\ &y_1 + y_2 + \cdots + y_n = r - n, \ \forall\ y_i \ge 0, \ 1 \le i \le n \end{aligned} \tag{2}$$

   Answer †
$$\binom{n + (r-n) - 1}{r-n} \tag{3}$$

3. **Answer** †
$$(2^2)^{(2^m)} = 4^{(2^m)} \tag{4}$$

4. We have
$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} x^n &= x + (1 + \frac{1}{2})x^2 + (1 + \frac{1}{2} + \frac{1}{3})x^3 + \cdots + (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})x^n + \cdots \\ &= 1 \times (x + x^2 + x^3 + \cdots + x^n + \cdots) + \frac{1}{2} \times (x^2 + x^3 + \cdots + x^n + \cdots) + \cdots \\ &+ \frac{1}{n} \times (x^n + x^{n+1} + \cdots) + \cdots \\ &= \frac{x}{1-x} + \frac{1}{2} \times \frac{x^2}{1-x} + \cdots + \frac{1}{n} \times \frac{x^n}{1-x} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-x} x^n = \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{1}{n} x^n \end{aligned} \tag{5}$$

And, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}x^n &\overset{\text{derivative}}{=} \sum_{n=1}^{\infty}x^{n-1} = \frac{1}{1-x}\\ \Rightarrow \frac{1}{1-x} &\overset{\text{integral}}{=} -\ln(1-x)\end{aligned} \tag{6}$$

Then, we have

$$\frac{1}{1-x}\sum_{n=1}^{\infty}\frac{1}{n}x^n = \frac{-\ln(1-x)}{1-x} \tag{7}$$

Answer †

$$\frac{-\ln(1-x)}{1-x} \tag{8}$$

5. We have

$$\begin{aligned}&\Rightarrow \alpha^2 = \alpha + 2\\ &\Rightarrow \alpha = 2 \vee \alpha = -1\\ &\Rightarrow a_n = c \times 2^n + d \times (-1)^n\end{aligned} \tag{9}$$

And, we have

$$\begin{aligned}&\begin{cases} a_0 = 0 = c + d\\ a_1 = 1 = 2 \times c - d\end{cases}\\ \Rightarrow &\begin{cases} c = \frac{1}{3}\\ d = -\frac{1}{3}\end{cases}\end{aligned} \tag{10}$$

Answer †

$$a_n = \frac{1}{3} \times 2^n - \frac{1}{3} \times (-1)^n \tag{11}$$

6. **Answer** †

$$cfjgda \tag{12}$$

7. **Answer** †

(a) If $S = \emptyset$, $\text{span}(S) = \{\mathbf{0}\} \subseteq V$.
Otherwise, if $S \neq \emptyset$, $\mathbf{0} \in \text{span}(S)$, and $\forall\ \boldsymbol{x},\ \boldsymbol{y} \in \text{span}(S)$, let

$$\begin{aligned}&\begin{cases}\text{span}(S) & = \text{span}\{v_1,\ v_2, \cdots,\ v_n\}\\ \boldsymbol{x} & = a_1v_1 + a_2v_2 + \cdots + a_nv_n\\ \boldsymbol{y} & = b_1v_1 + b_2v_2 + \cdots + b_nv_n\end{cases}\\ &\Rightarrow \forall\ \alpha,\ \beta \in \mathbb{R},\ \alpha\boldsymbol{x} + \beta\boldsymbol{y} =\\ &\quad (\alpha a_1 + \beta b_1)v_1 + (\alpha a_2 + \beta b_2)v_2 + \cdots + (\alpha a_n + \beta b_n)v_n \in \text{span}(S)\\ &\Rightarrow\ \text{span}(S) \subseteq V\end{aligned} \tag{13}$$

(b)

$$\begin{aligned} &S \subseteq U,\ \forall\, \boldsymbol{x} = \{x_1,\ x_2, \cdots,\ x_n\} \in S \\ \Rightarrow\ &\text{span}(S) = \{\alpha x_1 + \alpha x_2 + \cdots + \alpha x_n\} \subseteq U \end{aligned} \tag{14}$$

(c) Suppose

$$\exists\ T \subseteq V,\ \text{s.t.}\ T \subseteq U \tag{15}$$

And, we have

$$\begin{aligned} &S \subseteq \text{span}(S),\ \text{span}(S) \subseteq V \\ \Rightarrow\ &T \subseteq \text{span}(S) \end{aligned} \tag{16}$$

And, we have

$$\begin{aligned} &S \subseteq T,\ \text{span}(S) \subseteq T \\ \Rightarrow\ &T = \text{span}(S) \end{aligned} \tag{17}$$

8.

(a) **Answer** †

$$\text{nullity}(T) + \text{rank}(T) = \dim(V) \tag{18}$$

(b) We have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{\text{rref}}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{19}$$

Answer †

$$4 \times (2,\ 0,\ 1) + 1 \times (2,\ 1,\ -1) - 1 \times (2,\ -1,\ 0) = (8,\ 2,\ 3) \tag{20}$$

(c) **Answer** †

$$7 \times 4 = 28 \tag{21}$$

(d) **Answer** †

$$0,\ 1,\ 2,\ 3,\ 4,\ 5 \tag{22}$$

Since U and V are **distinct**, $U = V = W$ does NOT exist.

9. **Answer** † We have $\boldsymbol{A}^2 = \boldsymbol{I}$, so

$$\begin{aligned} \boldsymbol{A}^{-100} &= (\boldsymbol{A}^2)^{-50} = \boldsymbol{I} \\ \boldsymbol{A}^{101} &= (\boldsymbol{A}^2)^{50} \times \boldsymbol{A} = \boldsymbol{A} \end{aligned} \tag{23}$$

10. Find the minimal solution. We have

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{24}$$

Then, we have

$$\begin{aligned} &(\boldsymbol{A}\boldsymbol{A}^{\mathsf{H}})\boldsymbol{u} = \boldsymbol{b} \\ \Rightarrow &\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix} \\ \Rightarrow &\boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \end{aligned} \tag{25}$$

Then, we have

$$\boldsymbol{A}^{\mathsf{H}}\boldsymbol{u} = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{26}$$

Answer †

$$\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \tag{27}$$

11. **Answer** †

$$-1, \ 1, \ 2, \ 3 \tag{28}$$