Convex Optimization 10-725, Lecture 30: Optimization for Sampling.

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Today

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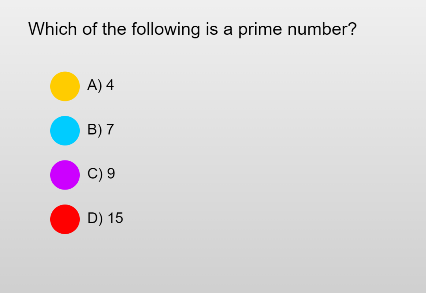
- 

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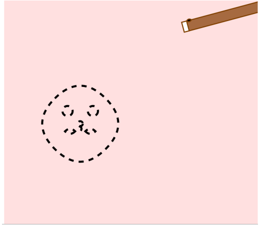
- 

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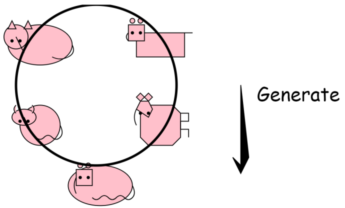

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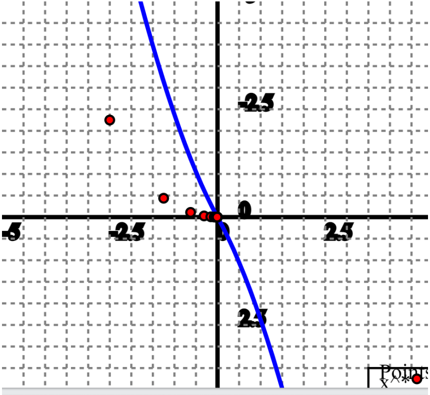

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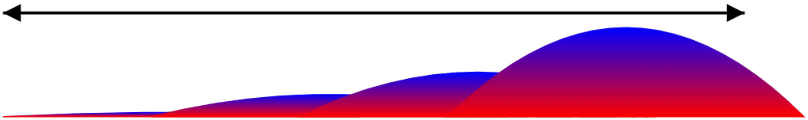

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- We will talk about the so-called diffusion approach.

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- We then reverse the process to go from Gaussian to q^*.

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- Theorem (classical result): As $t \to \infty$, the distribution of $X_t \to \mathcal{N}(0, I)$.
- In fact, we even know the explicit form: ($\gamma \sim \mathcal{N}(0, I)$ is the standard Gaussian)

$$X_t = e^{-t} X_0 + \sqrt{1 - e^{-2t}} \gamma$$

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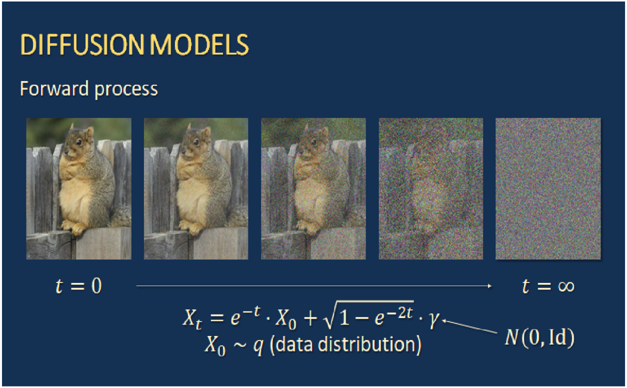
- 

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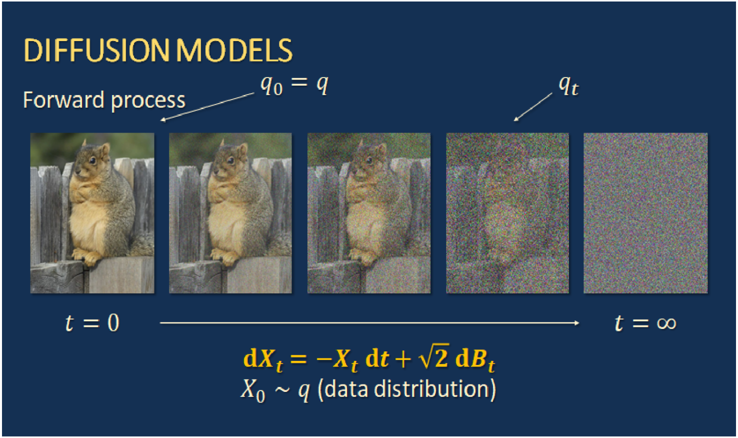
- 

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- By Fokker–Planck equation:

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where $\mathbf{X}_t$ and $\boldsymbol{\mu}(\mathbf{X}_t, t)$ are N-dimensional random vectors, $\boldsymbol{\sigma}(\mathbf{X}_t, t)$ is an $N \times M$ matrix and $\mathbf{W}_t$ is an M-dimensional standard Wiener process, the probability density $p(\mathbf{x}, t)$ for $\mathbf{X}_t$ satisfies the Fokker–Planck equation

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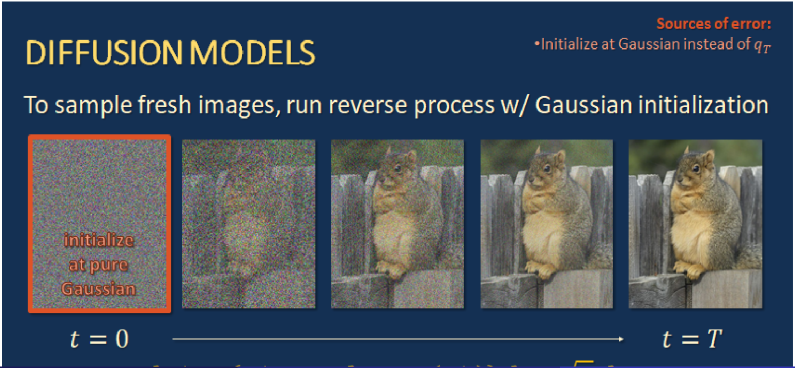
- 

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- This is due to

$$2\nabla \ln p_t(Y) = \frac{2\nabla p_t(Y)}{p_t(Y)}$$

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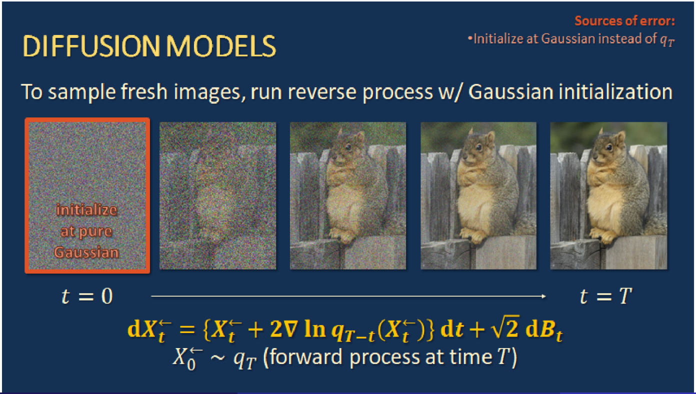
- 

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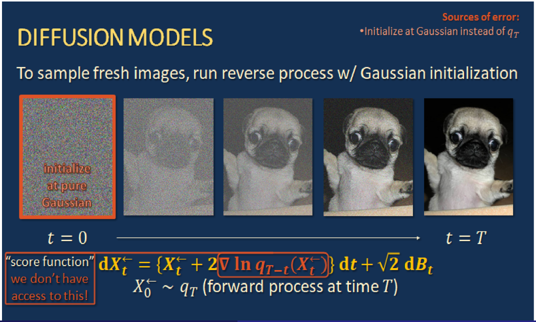
- 

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- However, we only have samples $X \sim q^* = q_0$, how do we compute $\nabla \ln q_t(X_t)$? – This is called score estimation.

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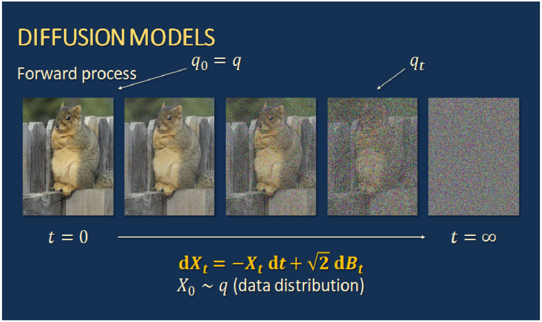
- 

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Statement of the lemma [edit]

Suppose X is a normally distributed random variable with expectation μ and variance σ^2. Further suppose g is a function for which the two expectations $E(g(X)(X-\mu))$ and $E(g'(X))$ both exist. (The existence of the expectation of any random variable is equivalent to the finiteness of the expectation of its absolute value.) Then

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- (2). Then use the backward diffusion process to sample Y_T: starting from $Y_0 \sim \mathcal{N}(0, I)$ and

$$dY_t = Y_t + 2s_{T-t}(Y_t)dt + \sqrt{2}dB_t$$

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