Multivariable Calculus Notebook

Dr. Ice 4th Block

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I would like to dedicate this notebook to all of the entertainment that made this notebook possible.

I certify the work I am submitting is my original work.
I have not shared nor exchanged information or materials with anyone, n or will I do so in the future.
X Ryan Davis

Table of Contents

Vectors and the Geometry of Space

Three Dimensional Coordinate Systems

On a number line, each point is given a unique number and each number has a respective unique point. In a coordinate plane, a point is given a unique ordered pair. However, in three dimensions, a 3rd coordinate is needed to work in space, so a third axis is added. This is called the Z axis. Each coordinate plane is given by setting one of the coordinates equal to zero.

The *xy*-plane contains the *x* and *y* axes, so *z* = 0;

The *xz*-plane contains the *x* and *z* axes, so *y* = 0;

The *yz*-plane contains the *y* and *z* axes, so *x* = 0.

All of the points in 3D space (*a, b, c*) are contained by $\mathbf{R}^3$.

Distance Formula in $\mathbf{R}^3$

To find the distance between two points in 3D space, we merely use an adaption of the 2D distance formula. Given two points $P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$, the distance formula is:

$$|P - Q| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$$

For example, to find the distance from $P(5,1,3)$ to $Q(1,-15,11)$, the formula would be used as follows:

$$|P - Q| = \sqrt{(5-1)^2 + \left(1-(-15)\right)^2 + (3-1)^2} = \sqrt{(4)^2 + (16)^2 + \left((-8)\right)^2} = \sqrt{336}$$

Spheres

A sphere is the set of all points that are equidistant from the center of the sphere *C*. If the center of the sphere is located at a point (h, k, l), and the sphere has a radius of r, then the equation of a sphere is given as $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. For illustrative purposes, if one wanted to find the equation of a sphere with center $(6,5,-2)$ and radius $\sqrt{7}$, the sphere's equation would become $(x-6)^2 + (y-5)^2 + (z+2)^2 = 7$.

For a more advanced example of what can be done combining the equation of a sphere with previous formulas, one can find the equation of a sphere given the center and a point that the sphere passes through. Say that a sphere passes through the point $(4,3,-1)$ and has the center $(3,8,1)$. The first part of the equation is relatively trivial: substituting in the center point, it becomes $(x-3)^2 + (y-8)^2 + (z-1)^2 = r^2$. Since the distance from the center is the same for every point on a sphere, the radius can be found by applying the distance formula from the center point to the given point on the sphere. This way, the radius can be found as equaling $\sqrt{(4-3)^2 + (3-8)^2 + \left((-1)-1\right)^2} = \sqrt{1+25+4} = \sqrt{30}$, making the final equation of the sphere $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

Vectors in 3D

All of the definitions and properties from two-dimensional vectors also apply to vectors in 3D space. The vector $\mathbf{v} = PQ$ in **R³** is also defined by an initial point and a terminal point; given these points $P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$, the magnitude of the vector is given by the formula:

$$\|\mathbf{v}\| = \left\|\overline{PQ}\right\| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}$$

Dot Product of 3D Vectors

The dot product of $\overline{\mathrm{v}} = \langle a_1, b_1, c_1 \rangle$ and $\overline{\mathrm{u}} = \langle a_2, b_2, c_2 \rangle$ is given as

$$\overline{\mathrm{v}} \bullet \overline{\mathrm{u}} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Worth noting is the fact that a dot product yields a scalar, not a vector. For illustrative purposes, here are several examples:

1. $\overline{\mathrm{v}} \bullet \overline{\mathrm{u}}$ $\quad \overline{\mathrm{v}} = \langle 3, -1, 2 \rangle$ $\quad \overline{\mathrm{u}} = \langle -4, 0, 2 \rangle$

   $\overline{\mathrm{v}} \bullet \overline{\mathrm{u}} = (3)(-4) + (-1)(0) + (2)(2) = -8$

2. $\overline{\mathrm{u}} \bullet \overline{\mathrm{w}}$ $\quad \overline{\mathrm{u}} = \langle -4, 0, 2 \rangle$ $\quad \overline{\mathrm{w}} = \langle 1, -1, -2 \rangle$

   $\overline{\mathrm{u}} \bullet \overline{\mathrm{w}} = (-4)(1) + (0)(-1) + (2)(12) = -8$

Angle between Two Vectors

If θ is the angle between two nonzero vectors $\overline{\mathrm{u}}$ and $\overline{\mathrm{v}}$ and $0 \leq \theta \leq \pi$, then

$$\cos\theta = \frac{\overline{\mathrm{u}} \bullet \overline{\mathrm{v}}}{||\overline{\mathrm{u}}||\,||\overline{\mathrm{v}}||}$$

Simply rewriting this equation gives us another way to calculate the dot product through use of cosine, $\overline{\mathrm{u}} \bullet \overline{\mathrm{v}} = ||\overline{\mathrm{u}}||\,||\overline{\mathrm{v}}|| \cos\theta$.

Possible Orientations of Vectors

Two vectors are orthogonal if their dot product is equal to 0. Similarly, they're referred to as opposite vectors if the angle between them is equal to π radians, and parallel vectors if the angle between them is equal to zero.

Scalar and Vector Projections

If there exist two nonzero vectors $\overline{\mathrm{u}}$ and $\overline{\mathrm{v}}$, and $\overline{\mathrm{u}} = \overline{\mathrm{w}}_1 + \overline{\mathrm{w}}_2$ where $\overline{\mathrm{w}}_1$ is parallel to $\overline{\mathrm{v}}$ and $\overline{\mathrm{w}}_2$ is orthogonal to $\overline{\mathrm{v}}$, then three principles become true:

1. $\overline{\mathrm{w}}_1$ is the vector projection of $\overline{\mathrm{u}}$ onto $\overline{\mathrm{v}}$, and is represented by $\overline{\mathrm{w}}_1 = \mathrm{proj}_{\mathbf{v}}\,\overline{\mathrm{u}}$
2. $\overline{\mathrm{w}}_2 = \overline{\mathrm{u}} - \overline{\mathrm{w}}_1$, and $\overline{\mathrm{w}}_2$ is called the vector component of $\overline{\mathrm{u}}$ orthogonal to $\overline{\mathrm{v}}$.
3. The “scalar projection of $\overline{\mathrm{u}}$ onto $\overline{\mathrm{v}}$ ” (also known as the component of $\overline{\mathrm{u}}$ along $\overline{\mathrm{v}}$) is defined as the magnitude of the vector projection, $\left|\left|\overline{\mathrm{u}}\right|\right| \cos\theta$.

In summary, the vector projection of $\overline{\mathrm{u}}$ onto $\overline{\mathrm{v}}$ is represented as $\mathrm{proj}_{\mathbf{v}}\,\overline{\mathrm{u}}$, and the scalar projection is represented by $\mathrm{comp}_{\mathbf{v}}\,\overline{\mathrm{u}} = \frac{\overline{\mathrm{u}}\bullet\overline{\mathrm{v}}}{||\overline{\mathrm{u}}||\,||\overline{\mathrm{v}}||}$. Note that $\mathrm{comp}_{\mathbf{v}}\,\overline{\mathrm{u}}$ is simply the magnitude of the vector projection, so a simpler way to find the scalar projection becomes $\mathrm{comp}_{\mathbf{v}}\,\overline{\mathrm{u}} = \left|\left|\overline{\mathrm{u}}\right|\right| \cos\theta$.

Cross Product

The cross product, in comparison to the dot product, is a vector value and not a scalar, so it is commonly referred to as the vector product. It is defined as the determinants of two matrices, and the result of the cross product is orthogonal to the two original vectors.

If $\overline{\mathrm{v}} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\overline{\mathrm{w}} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors in three-dimensional space, then the cross product of $\overline{\mathrm{v}}$ and $\overline{\mathrm{w}}$ is defined as

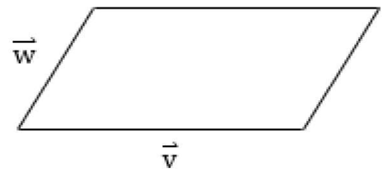
$$\overline{\mathrm{v}} \times \overline{\mathrm{w}} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = ||\overline{\mathrm{v}}||\left(||\overline{\mathrm{w}}|| \sin\theta\right)$$

In the latter case listed above, θ is the angle between $\overline{\mathrm{v}}$ and $\overline{\mathrm{w}}$ such that $0 \le \theta \le \pi$, giving us numerous applications for the cross product.

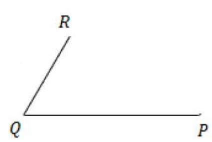
Area of a Parallelogram

One such application of a cross product is finding the area of a parallelogram. Since the formula above gives us an angle parameter, we can find the area of any parallelogram as follows:

$$A = \|\overline{\mathrm{v}}\|(\|\overline{\mathrm{w}}\| \sin\theta) = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|$$



This formula makes finding the area of a parallelogram as easy as finding the magnitude of the cross product of two of its vectors! For example, if one wanted to find the area of a parallelogram with vertices at $P(1,3,-2)$, $Q(2,1,4)$, $R(-3,1,6)$, it would be as simple as



$$A = \|\overrightarrow{QP} \times \overrightarrow{QR}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -6 \\ 5 & 0 & 2 \end{matrix} \right\| = \sqrt{(-4)^2 + (-32)^2 + (-10)^2} = \sqrt{1140}$$

Triple Scalar Product

Combining the dot product and cross product, we obtain what is known as the triple scalar product. The triple scalar product is simply the dot product of the result of a cross product, and it is given by the following:

Given:

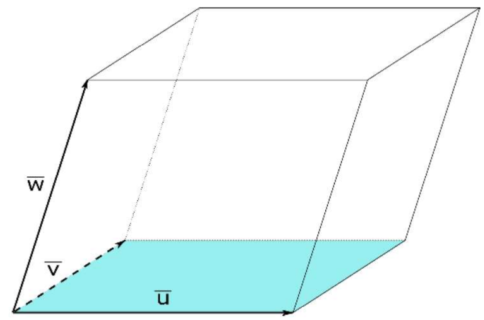
$\overline{\mathrm{u}} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$\overline{\mathrm{v}} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$\overline{\mathrm{w}} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$

$$\overline{\mathrm{u}} \bullet (\overline{\mathrm{v}} \times \overline{\mathrm{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Volume of a Parallelepiped



A triple scalar product allows us to apply the dot product and cross product more broadly, one such application being the volume of a parallelepiped. A parallelepiped is a polyhedron, all of whose faces are parallelograms. The volume of a parallelepiped whose adjacent edges are $\overline{\mathrm{u}}$, $\overline{\mathrm{v}}$ and $\overline{\mathrm{w}}$ (as shown in the graphic) is given by the formula

$$V = \|\overline{\mathrm{v}} \times \overline{\mathrm{w}}\| \bullet \|\overline{\mathrm{u}}\| \bullet \cos\theta = |\, \overline{\mathrm{u}} \bullet (\overline{\mathrm{v}} \times \overline{\mathrm{w}}) \,|$$

Worth noting is the fact that the volume is given by the absolute value of that formula, since volume cannot be negative. For example, to find the volume of a parallelepiped having the adjacent edges $\overline{\mathrm{u}} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\overline{\mathrm{v}} = 2\mathbf{j} + 2\mathbf{k}$, and $\overline{\mathrm{w}} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, the formula would be worked as follows:

$$V = \overline{\mathrm{u}} \bullet (\overline{\mathrm{v}} \times \overline{\mathrm{w}}) = \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 3[(2 \bullet 1) - (-2 \bullet 1)] + 5[-(-2 \bullet 3)] + \big(-(2 \bullet 3)\big) = 36$$

Equations of Lines in Space

A line in 3D space is determined by a point and a direction. Whereas in two-dimensional space a line had a slope, in three dimensions the direction is determined by a vector.

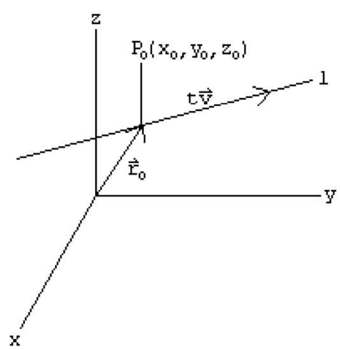


Image Source:

Oregon State University

In accordance with the graphic, the vector equation of a line is given as

$$\vec{r} = \vec{r_0} + t\vec{v}$$

In coordinate form, all components need one of these, making the equation

$$\langle x, y, z\rangle \; = \; \langle x_0 + ta,\; y_0 + tb,\; z_0 + tc\rangle \; = \; \langle x_0,\; y_0,\; z_0\rangle + t\langle a, b, c\rangle$$

Parametric Equations

Two vectors are equal if and only if their corresponding components are equal, therefore

$$x = \; x_0 + ta \qquad\qquad y = \; y_0 + tb \qquad\qquad z = \; z_0 + tc$$

The above equations are referred to as the parametric equations of a line through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathrm{v}\langle a, b, c\rangle$. Each value of the parameter t gives a point (x, y, z) on the line, hence the name.

Symmetric Equations

Solving each parametric equation to obtain the value of t yields the symmetric equations:

$$\frac{x - \; x_0}{a} = \frac{y \; - \; y_0}{b} = \frac{z \; - \; z_0}{c}$$

All of these forms can be used to find different qualities of the line more easily, and often the first step of any line problem is to find the other two forms (parametric and symmetric) from the vector form of the equation.

Intersecting Lines in Space

If a vector $\overline{\mathrm{v}}$ that gives the direction of a line L is written in the component form $\overline{\mathrm{v}} = \langle a, b, c \rangle$, then a, b, c are referred to as the direction numbers of the line L. Since any vector parallel to $\overline{\mathrm{v}}$ could also be used as the direction of the line, any three numbers proportional to a, b, c could be used as a set of direction numbers for L. Thus, if two lines have direction vectors that are simply scalar multiples of each other, then they are parallel.

In order for two lines to intersect, there must be values of t such that $\overline{r}(t_1) = \overline{r}(t_2)$. If the lines are neither parallel nor intersecting, they are referred to as skew. For example, given two lines

$$L_1{:}\;\; x = 1 + t_1 \quad y = -2 + 3t_1 \quad z = 4 - t_1$$

$$L_2{:}\;\; x = 2t_2 \quad y = 3 + t_2 \quad z = -3 + 4t_2$$

The equation would be set up as follows:

$$1 + t_1 = 2t_2$$

$$-2 + 3t_1 = 3 + t_2$$

$$4 - t_1 = -3 + 4t_2$$

$$t_1 - 2t_2 = -1$$

$$3t_1 - t_2 = 5$$

$$-3t_1 + 6t_2 = 3$$

$$3t_1 - t_2 = 5$$

$$5t_2 = 8 \therefore t_2 = \frac{8}{5}$$

$$1 + t_1 = 2t_2$$

$$1 + t_1 = 2\left(\frac{8}{5}\right)$$

$$1 + t_1 = \frac{16}{5} \therefore t_1 = \frac{11}{5}$$

$$4 - t_1 \stackrel{?}{=} -3 + 4t_2$$

$$4 - \frac{11}{5} \stackrel{?}{=} -3 + 4\left(\frac{8}{5}\right)$$

$$\frac{9}{5} \neq \frac{17}{5}$$

Since the two components are not equal the two lines are skew; they do not intersect.

Equations of Planes in 3D Space

Given a point $P(x_0, b_0, c_0)$ and a nonzero vector $\overline{\mathrm{N}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ starting at *P* that is normal to the plane, the equation for the plane that passes through point *P* and has the normal vector $\overline{\mathrm{N}}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$\langle a, b, c\rangle(x - x_0, y - y_0, z - z_0)$$

The linear equation for a plane in *x, y,* and *z* can also be written as $ax + by + cz + d = 0$.

Point where a Line Intersects a Plane

In order to find the point where a line intersects a plane, we merely substitute the line's equations into the plane equation and solve for t to get the point of intersection. Examples of this are given in this link from MIT OpenCourseWare.

Angle Between Two Planes

Two distinct planes in three-dimensional space are either parallel or they intersect in a line. If they intersect, the angle between them can be found from the angle between their normal vectors. The angle between two planes is given as

$$\cos\theta = \frac{|\overline{\mathrm{n}}_1 \bullet \overline{\mathrm{n}}_2|}{||\overline{\mathrm{n}}_1||\ ||\overline{\mathrm{n}}_2||}$$

Accordingly, two planes are perpendicular if $\overline{\mathrm{n}}_1 \bullet \overline{\mathrm{n}}_2 = 0$. The two planes are parallel if $\overline{\mathrm{n}}_1$ is a scalar multiple of $\overline{\mathrm{n}}_2$, just like with parallel line equations.

Line of Intersection of Two Planes

In order to find the line of intersection of two planes, simply take the cross product of their normal vectors to get a vector parallel to the line, and solve the systems of equations of the planes to get a point on the line. An example of this technique is given in this link from WVU.

Distance from a Point to a Plane

The distance between a plane and a point $Q(x_0, y_0, z_0)$ not in the plane and the plane given by $ax + by + cz + d = 0$ is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Where $P(x_1, y_1, z_1)$ is a point in the plane and $d = -(ax_1 + by_1 + cz_1)$. For example, consider finding the distance between point $Q(1,5,-4)$ and plane $3x - y + 2z - 6 = 0$. The distance equation would be set up as follows:

$$D = \frac{|3(1) - 5 + 2(-4) - 6|}{\sqrt{9 + 1 + 4}} = \frac{|3 - 5 - 8 - 6|}{\sqrt{9 + 1 + 4}} = \frac{16}{\sqrt{14}}$$

Distance between Two Parallel Planes

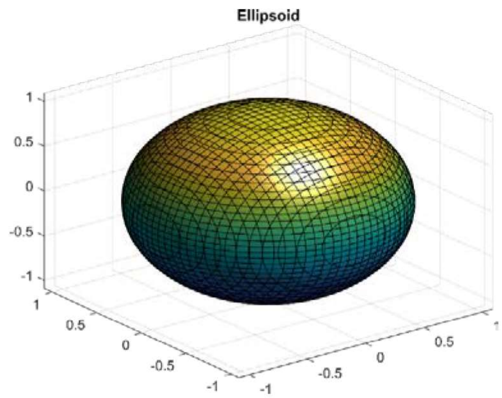
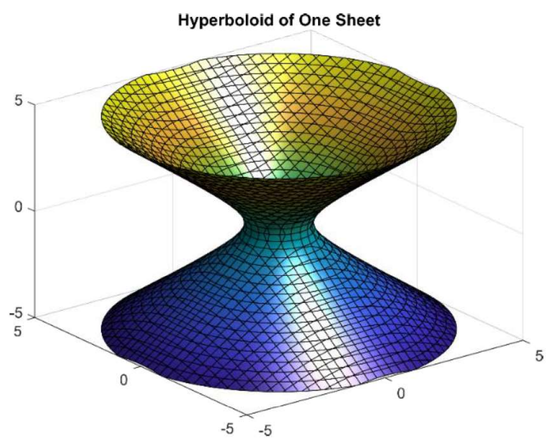
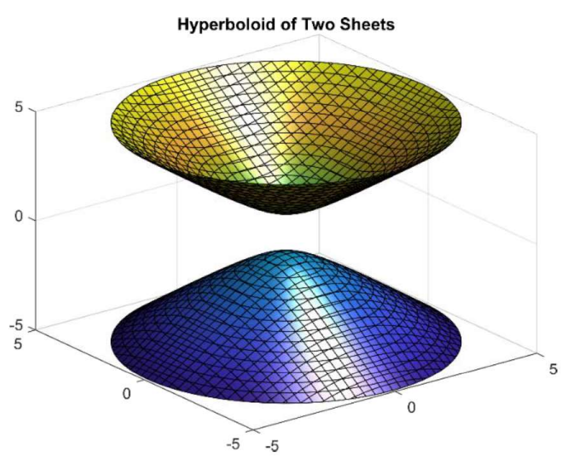
Finding the distance between two parallel planes is very similar to the above equation: merely pick a point on one of the planes, and find the distance from that point to the other plane. (It's more convenient if you pick a point on an axis.)

Distance between Two Lines in Space

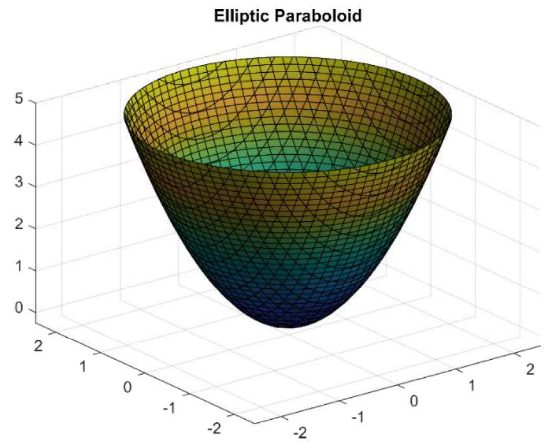
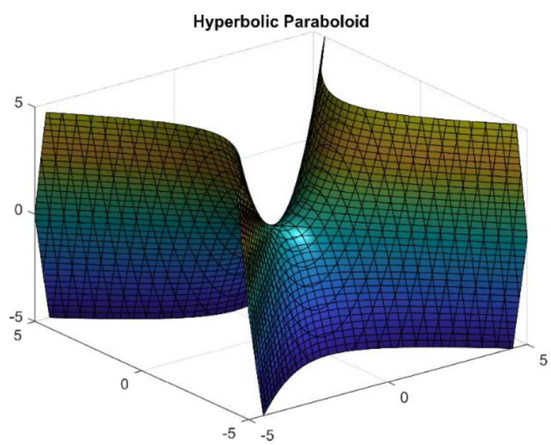
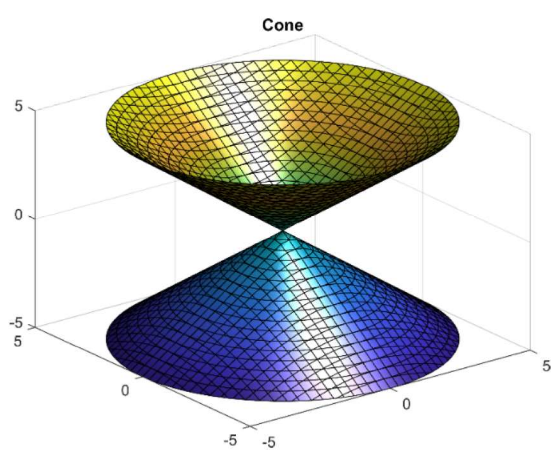
If L is a line represented by the equation $\vec{r}(t) = Q + t\vec{u}$ and M is another line represented by the equation $\vec{s}(t) = P + t\vec{v}$, then the distance between the two lines is given as

$$D = \frac{\left|\left(\overrightarrow{PQ}\right) \bullet (\overline{u} \times \overline{v})\right|}{|\overline{u} \times \overline{v}|}$$

Quadric Surfaces

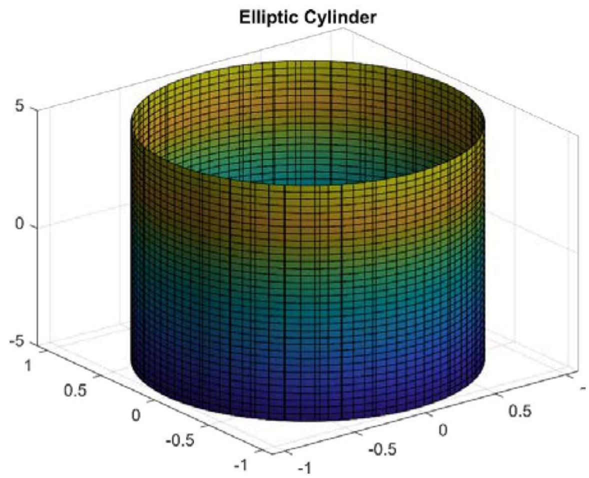
Surface	Equation	Traces
	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$	All traces are ellipses.
	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$	Horizontal traces are ellipses. Vertical traces are hyperbolas.
	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$	Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas.

Quadric Surfaces (Continued)

Surface	Equation	Traces
	$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$	Horizontal traces are ellipses. Vertical traces are parabolas.
	$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$	Horizontal traces are hyperbolas. Vertical traces are parabolas.
	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$	Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and y $= k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

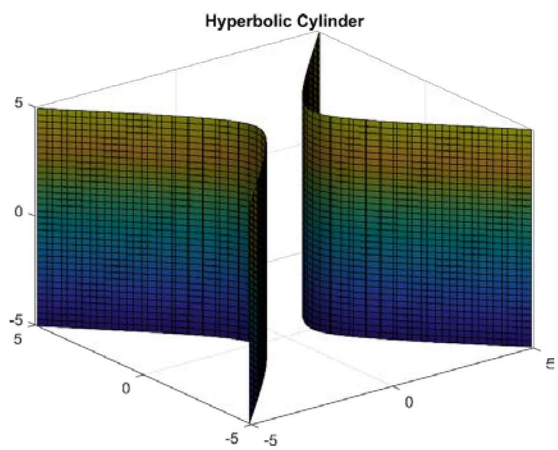
Cylinders

Surface | Equation | Rulings



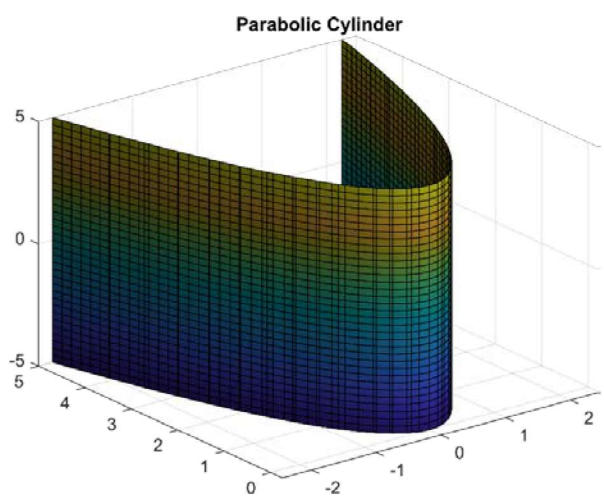
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Rulings are ellipses parallel to the xy plane.



$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

Rulings are hyperbolas parallel to the xy plane.



$$y = ax^2$$

Rulings are parabolas parallel to the xy plane.

Vector Functions

A function with the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ in a plane, or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ in space, is called a vector-valued function because it maps real numbers to vectors. The component functions f, g, and h are the real-valued functions of the parameter t. Generally, a vector-valued function is used to trace the graph of a curve; if t is set equal to time, then it can represent motion along a curve. The domain of a vector valued function is the set of all possible values of t that belong to the function's components; the range is the set of generated position vectors.

Continuity of Vector Functions

A vector function $\mathbf{r}(t)$ is continuous at appoint given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$. It follows that such a function is continuous on an interval only if it is continuous at every point in the interval. For example, to determine the continuity of $\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k}$ at $t = 0$, simply check if the limit exists as follows:

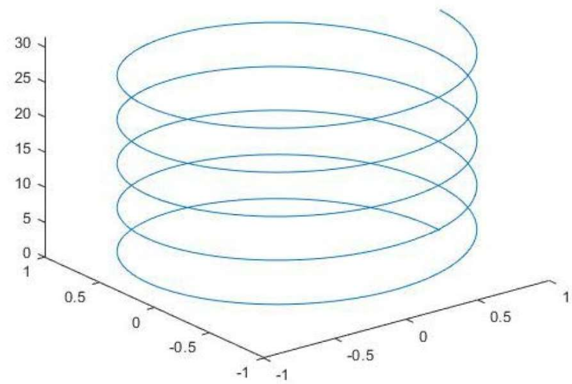
$$\lim_{t\to 0} \mathbf{r}(t) = \lim_{t\to 0}(t)\mathbf{i} + \lim_{t\to 0}(a)\mathbf{j} + \lim_{t\to 0}(a^2 - t^2)\mathbf{k}$$

$$\lim_{t\to 0} \mathbf{r}(t) = 0\mathbf{i} + a\mathbf{j} + a^2\mathbf{k}$$

$\therefore$ since the limit exists $\mathbf{r}(t)$ is continuous at $t = 0$, $\mathbf{r}(0) = a\mathbf{j} + a^2\mathbf{k}$

Space Curves

A space curve is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ where f, g, and h are continuous functions of t. If the components of $\mathbf{r}(t)$ are continuous functions, then the terminal point traces a path in space and $\mathbf{r}(t)$ is known as a vector parameterization of the path. To assist in visualization, a graph of a space curve has been provided, using the equation $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$.



Differentiation of Vector Functions

Limit Usage

The derivative of a vector function is defined by:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

This definition applies for all values of t for which the limit exists. If $\mathbf{r}'^{(c)}$ exists, then $\mathbf{r}$ is differentiable at c ; it follows that if $\mathbf{r}'^{(c)}$exists on an open interval, then $\mathbf{r}$ is differentiable on the interval.

Component Usage

Just like the original derivative definition, it is not easy or convenient to use the limit equivalency in every case. Thus, one can derive a vector function by simply deriving its components. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

As expected, higher order derivatives of vector functions can be determined by successive differentiation of each component function. For example, if $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 2t\mathbf{k}$, then $\mathbf{r}'(t) = \langle -\sin t, \cos t, 2\rangle$ and $\mathbf{r}''(t) = \langle -\cos t, -\sin t, 2\rangle$.

Tangent Vector to Space Curve

If C is a smooth curve, the derivative of $\mathbf{r}(t_0)$, $\mathbf{r}'(\mathrm{t}_0)$, is the tangent vector or velocity vector to the curve at $\mathbf{r}(t_0)$. If $\mathbf{r}'(t_0) \neq 0$, then the direction of $\mathbf{r}'(t)$ gives the direction of motion at time t, and the vector parameterization of the tangent line at t_0 is given as

$$\mathbf{L}(t) = \mathbf{r}(t_0) + t\,\mathbf{r}'(t_0)$$

Equation of Tangent Line to Space Curve at a Point

To find the equation of the tangent line at a point, we must merely substitute the point in question into our above equation. For example, to find the vector equation for the tangent line to the curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin(t), t\rangle$ at the point $t = \frac{\pi}{4}$, the equation would be set up as follows:

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin(t), t\rangle \qquad \mathbf{r}'(t) = \langle 2\sin(t), 2\cos t, 1\rangle$$

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4}\rangle \qquad \mathbf{r}'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 1\rangle$$

$$\mathbf{L}(t) = \langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4}\rangle + t\langle -\sqrt{2}, \sqrt{2}, 1\rangle$$

Integration of Vector Functions

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, and h are continuous on the range $[a, b]$, then

$$\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}$$

The definite integral over the interval $a \leq t \leq b$ is therefore

$$\int_a^b \mathbf{r}(t)dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}$$

The antiderivative of a vector function results in three constants of integration, one from each equation. These three scalar constants all combine to produce one vector constant of integration: $\mathbf{C} = \langle c_1, c_2, c_3\rangle$.

For example, consider the problem $\int(2t\mathbf{i} + \mathbf{j} + \sqrt{t}\mathbf{k})dt$. It is necessary to integrate each separately, resulting in $(t^2 + c_1)\mathbf{i} + (t + c_2)\mathbf{j} + \left(\frac{2}{3}t^{\frac{3}{2}} + c_3\right)\mathbf{k}$. Combining the three scalar constants, we end up with a final answer of $\langle t^2, t, \frac{2}{3}t^{\frac{3}{2}}\rangle + \mathbf{C}$.

Arc Length of Space Curve

Finding the arc length of a space curve is fairly simple: given the equation of the arc $\mathbf{r}(t)$, plug it into the formula:

$$\int_a^b \|\mathbf{r}'(t)\| dt$$

Consider finding the arc length of the circular helix with the vector equation $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ from the point (1,0,0) to the point (1,0,2π).

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + 1\mathbf{k}$$

$$L = \int_0^{2\pi} \left[\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\right] dt$$

$$L = \int_0^{2\pi} \sqrt{2}\ dt = 2\pi\sqrt{2}$$

Importantly, since this formula is taking the integral of the magnitude of the derivative, the final value is a scalar (as is expected of an arc length).

Smooth Curve

The parameterization of the curve represented by the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is smooth on an open interval when f', g', and h' are continuous on the interval and $\mathbf{r}'(t) \neq 0$ for any value of t on the interval.

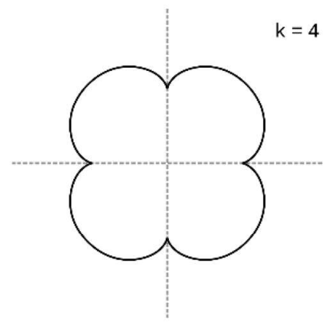


Image Source: Notes

This epicycloid shape is not smooth where it makes abrupt change in direction (its cusps).

For example, to find the intervals on which the epicycloid given by $\mathbf{r}(t) = (5\cos t - \cos 5t)\,\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}$ is smooth in the range $0 \leq t \leq 2\pi$, one would first have to find the derivative of $\mathbf{r}(t)$. After deriving this, $\mathbf{r}'(t) = (-5\sin t + \sin 5t)\,\mathbf{i} + (5\cos t - \cos 5t)\,\mathbf{j}$. The next step is to find the locations where $\mathbf{r}'(t) \neq 0$ on $[0,2\pi]$. To achieve this, one must set each component of $\mathbf{r}'(t)$ equal to 0 and solve for t; at the t values where both components are equal to zero, the curve is smooth. In the case of this equation, it is smooth on the intervals

$\left(0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right) \cup \left(\pi,\frac{3\pi}{2}\right) \cup \frac{3\pi}{2}, 2\pi.$

Curvature

Curvature, a scalar value represented by K, is a measure of how sharply a curve bends. It is calculated as the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to the arc length s. If C is a smooth curve given by $\mathbf{r}(s)$ then the curvature is equal to

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \ \mathbf{T}'(s)$$

where $\mathbf{T}$ is the unit tangent vector defined as $\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ (where $\mathbf{r}'(t) \neq 0$)

To find the curvature at a point t, where the curve is given by the vector function $\mathbf{r}(t)$ use the formula:

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

where K is a scalar representing the curvature of the line. For example, to find the curvature of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{4}\mathbf{k}$ at the point $P(2,4,2)$:

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{4}\mathbf{k}$$

$$\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + \left(\frac{3}{4}t^2\right)\mathbf{k}$$

$$\mathbf{r}''(t) = 0\mathbf{i} + 2\mathbf{j} + \left(\frac{3}{2}t\right)\mathbf{k}$$

$$t = 2$$

$$\mathbf{r}'(2) = \langle 1,4,3 \rangle$$

$$\mathbf{r}''(2) = \langle 0,2,3 \rangle$$

$$\|\mathbf{r}'(2) \times \mathbf{r}''(2)\| = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 0 & 2 & 3 \end{bmatrix} = \langle 6, -3, 2 \rangle$$

$$K = \frac{\sqrt{6^2 + (-3)^2 + 2^2}}{\left(\sqrt{1^2 + 4^2 + 3^2}\right)^3} = \frac{\sqrt{49}}{\sqrt{26}^3} = \frac{7}{\sqrt{26}^3}$$

Similarly, if C is the graph of a function given by $y = f(x)$ then the curvature at the point (x, y) is equal to

$$K = \frac{\|y''\|}{\sqrt{[1 + (y')^2]^3}}$$

Unit Tangent, Unit Normal, and Binormal Vectors

Unit Tangent Vector and Unit Normal Vector

As briefly mentioned in the previous section, the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ and is orthogonal to its derivative $\mathbf{T}'(t)$. Assuming that this derivative is not zero, the unit vector in the direction $\mathbf{T}'(t)$ is called the unit normal vector $\mathbf{N}(t)$. $\mathbf{N}$ indicates the direction in which the curve is turning and points to the “inside” of the curve. The unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.

Binormal Vector

The binormal vector $\mathbf{B}(t)$ is perpendicular to $\mathbf{T}$ and $\mathbf{N}$**.** It is important in analysis of 3D motion, and it is calculated using the following:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{\mathbf{r}' \times \boldsymbol{r}''}{\|\mathbf{r}' \times \boldsymbol{r}''\|}$$

Example: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for $\mathbf{r}(t) = (e^t)\mathbf{i} + (e^t \sin t)\mathbf{j} + (e^t \cos t)\mathbf{k}$ at the point $(1,0,1)$.

$$\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t\rangle$$

$$\mathbf{r}'(t) = \langle e^t, e^t \sin t + e^t \cos t, -e^t \sin t + e^t \cos t\rangle$$

$$\mathbf{r}'(0) = \langle 1, \sin 0 + \cos 0, \cos 0 - \sin 0\rangle$$

$$\mathbf{r}'(0) = \langle 1,1,1\rangle$$

$$\mathbf{T}(t) = \frac{\langle e^t, e^t \sin t + e^t \cos t, -e^t \sin t + e^t \cos t\rangle}{\|\langle e^t, e^t \sin t + e^t \cos t, -e^t \sin t + e^t \cos t\rangle\|} = \frac{e^t\langle 1, \sin t + \cos t, -\sin t + \cos t\rangle}{e^t\sqrt{1^2 + (\sin t + \cos t)^2 + (-\sin t + \cos t)^2}}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{3}}\langle 1, \cos t + \sin t, \cos t - \sin t\rangle$$

$$\mathbf{T}(0) = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}}\langle 0, \cos t - \sin t, -\sin t - \cos t\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

$$\mathbf{N}(t) = \frac{\frac{1}{\sqrt{3}}\langle 0, \cos t - \sin t, -\sin t - \cos t\rangle}{\frac{1}{\sqrt{3}}\sqrt{0^2 + (\cos t - \sin t)^2, (-\sin t - \cos t)^2}}$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}\langle 0, \cos t - \sin t, -\sin t - \cos t\rangle$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{2}}\langle 0,1,-1\rangle$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0)$$

$$\mathbf{B}(0) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$\mathbf{B}(0) = \frac{1}{\sqrt{6}}\langle -2, 1, 1\rangle$$

Motion in Space: Velocity and Acceleration

For a path that is equal to $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ in 2d space, or $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in 3D space, the following must be true:

Velocity $\mathbf{v}(t) = \mathbf{r}'(t)$

Acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$

Speed $s(t) = \|\mathbf{v}(t)\|$

Using these, one can find the velocity vector, acceleration vector, and speed at any given time given a parameterization of the path. Please refer to this link for numerous applications and worked examples of these formulas.

Partial Derivatives

Functions of Two or More Variables

For functions of two or more variables, there are multiple ways that they can be written. For example, $z = f(x, y) = x^2 + y^2$ or $w = f(x, y, z) = x + 2y - 3z$ are both acceptable notations.

Domain & Range (of Functions of Two or More Variables)

The graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f.To each point (x, y) in D there corresponds a point (x, y, z) on the surface. Conversely, for each point (x, y, z) on the surface, there is a corresponding point (x, y) in D. The domain and range can also be found; to find the domain and range of

$$f(x, y) = \sqrt{9 - x^2 - y^2}$$

$$9 - x^2 - y^2 \geq 0$$

Domain: $x^2 + y^2 \leq 9$

Range: $0 \leq x \leq 3, 0 \leq y \leq 3$

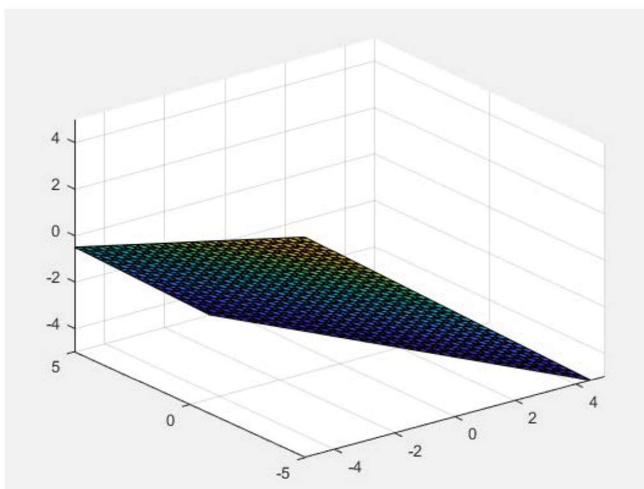
Sketching a Function of Two or More Variables

Sketching a Plane

If a plane in space intersects a coordinate axis, the line of intersection is the trace of the plane in the coordinate plane.

- For an *xy*-trace, set z = 0
- For a *yz*-trace, set y = 0
- For an *xz*-trace, set y = 0

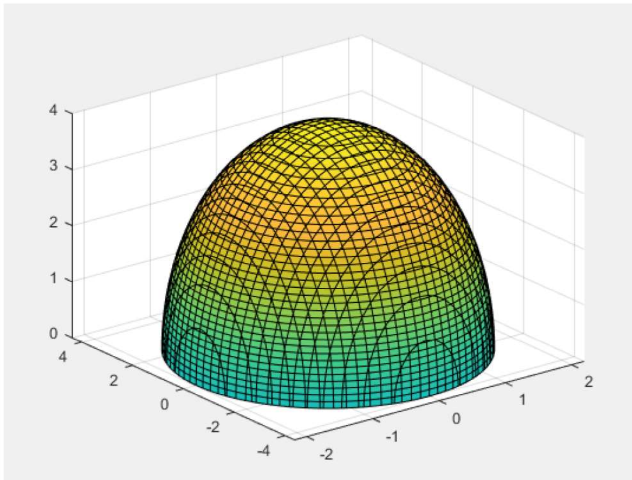
Here is an example of a plane with the equation $3x + 2y + 4z = 12$:



Sketching a Function

Use traces parallel to the coordinate planes to sketch the surface. In general, a vertical trace in the plane $y = b$ is the intersection of the graph with the vertical plane $y = b$. The same is true for $x = a$, as they are merely intersection with the plane $x = a$ instead. In contrast, a horizontal trace at height c is the intersection of the graph with the horizontal plane $z = c$. The main difference here is that this projects down to the curve $f(x, y) = c$ in the *xy*-plane.

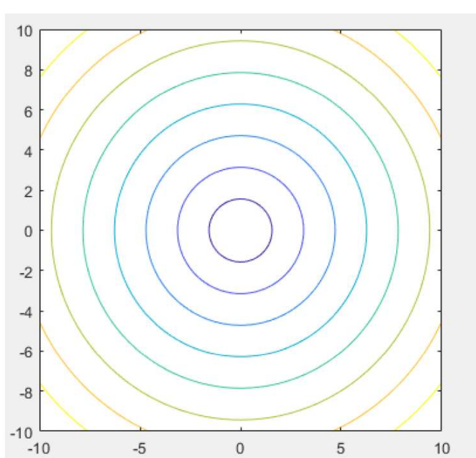
To graph a function of three variables, merely find the domain and range and use the traces accordingly. Here is an example of a graph of the equation $f(x, y) = \sqrt{16 - 4x^2 - y^2}$.



Level Curves

A level curve is the set of all the points in the domain of f where f takes on a given value k such that $f(x, y) = k$. To sketch level curves, the function $f(x, y)$ is set equal to a few different values of k , and these equations are all graphed on the same plot.

For example, to sketch the level curves for the function $f(x, y) = \sqrt{64 - x^2 - y^2}$ for $c = 0 \dots 8$, simply solve for z and set it equal to c. In this case, it is already solved for $f(x, y)$, meaning all that is left to do is graph it:



Limits and Continuity in Several Variables

Limits

The definition of a limit in three variables is $\lim_{(x,y)\to(a,b)} f(x,y) = L$. In clearer terms, this means that we can make the values of $f(x, y)$ as close to L as we by taking the point (x, y) close to, but not equal to, the point (a, b). For functions in two variables, (x, y) can approach (a, b) from an infinite number of directions. If the limit exists, $f(x, y)$ must approach the same limit disregarding which direction (x, y) approaches (a, b). If $f(x, y) \to L_1$ along one path and $f(x, y) \to L_2$ from another path, L_1 must be equal to L_2, otherwise the limit does not exist. Luckily, all of the limit laws from single variable functions extend to functions of two variables, and the Squeeze Theorem becomes particularly useful (in fact, it is the only way to check limits without trying every possibility). The Squeeze Theorem, adapted for functions of two variables, is as follows:

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\lim_{(x,y)\to(a,b)} c = c$$

Exhaustive examples of how to find limits in functions of two variables are included in [this link] from Lamar University.

Continuity

One of the most important applications for limits in functions of two variables is continuity of these functions. A function f of two variables is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Continuous functions include Polynomial Functions, Rational Functions (on their own domain), and composite functions of continuous functions, just like with continuous single-variable functions. To expand on what exactly is meant by continuous functions, if c is a real number and functions f and g are continuous at (a, b), then the following composites are also continuous at (a, b).

- Scalar multiple: kf
- Sum and difference: $f \pm g$
- Product: fg
- Quotient: $\frac{f}{g}$ if $g(a, b) \neq 0$

Partial Derivatives

A partial derivative is the rate of change of a function f with respect to *one* of its variables. Essentially, to find the partial derivative f_x, consider y to be a constant and differentiate with respect to x. Similarly to find the partial derivative f_y, consider x to be a constant and differentiate with respect to y. If $x = f(x, y)$, then the first partial derivatives of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad f_x(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided, of course, that the limit exists. The much more common notation follows; the limit definition is very rarely used in any meaningful application.

$$\frac{\delta}{\delta x} f(x, y) = f_x(x, y) = z_x = \frac{\delta z}{\delta x} \qquad \frac{\delta}{\delta y} f(x, y) = f_y(x, y) = z_y = \frac{\delta z}{\delta y}$$

As far as how to interpret the results of these partial derivatives, if $y = y_0$ then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$. Then, $f_x(x_0, y_0)$ is the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$. Also, $f_x(x_0, y_0)$ represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at the point $(x_0, y_0, f(x_0, y_0))$ as well. This allows finding the instantaneous slopes of a function in the x and y directions for a surface. For example, to find these slopes for the equation $f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$ at the point $\left(\frac{1}{2}, 1, 2\right)$, one must merely take the partials and evaluate them at the given point. In this case, $f_x(x, y) = -x$ and $f_y(x, y) = -2y$, which of course become $-\frac{1}{2}$ and 2 when evaluated (respectively). This also allows us to take higher order partial derivatives, as seen below.

Higher Order Partial Derivatives

These higher order derivatives are denoted by the order in which the differentiation occurs.

There are four second order partial derivatives for a given function $z = f(x, y)$:

$$f_{xx} = \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta x^2} \qquad f_{xy} = \frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta y \delta x}$$

$$f_{yy} = \frac{\delta}{\delta y}\left(\frac{\delta f}{\delta y}\right) = \frac{\delta^2 f}{\delta y^2} \qquad f_{yx} = \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta y}\right) = \frac{\delta^2 f}{\delta x \delta y}$$

These last two are known as mixed partial derivatives, because they are first taken with respect to one variable, and then with respect to another. For all notations, differentiate first with the variable closest to f.

Clairaut's Theorem

One of the applications of second order partial derivatives is Clairaut's Theorem, which states:

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x, y) in R, $f_{xy}(x, y) = f_{yx}(x, y)$.

More generally, if f, f_{xy}, and f_{yx} are continuous for all real numbers, then $f_{xy} = f_{yx}$. This webpage not only goes into more detail about what is meant by an open disk, but also provides several examples of verifying Clairaut's theorem for all manner of functions.

Tangent Planes

If $z = f(x, y)$ represents a surface where f has continuous first partial derivatives, C_1 and C_2 are the curves that result from intersecting the vertical planes $y = y_0$ and $x = x_0$, $P(x_0, y_0)$ is a point on the surface that lies on C_1 and C_2, and T_1 and T_2 are the tangent lines to both C_1 and C_2, then

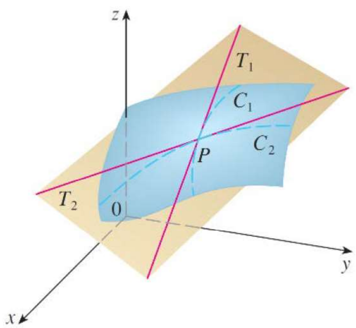


Image Source:

Calculus Early Transcendentals

the tangent plane to the surface at point P is defined to be the plane that contains both tangent lines. A graphical representation of this is below.

More succinctly, the tangent plane at point P is the plane that most closely represents the surface of f near P. This tangent plane is found through an equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

To find the equation of a tangent plane at a point, merely evaluate the partials for the surface and plug in the points for x_0 and y_0. For example, to find the equation of a tangent plane to the surface $z = 4x^2 - y + 2y$ at the point $(-1,2,4)$:

$$f_x(x_0, y_0) = 8x \,|_{x=-1} = -8$$

$$f_y(x_0, y_0) = -2y + 2 \,|_{y=2} = -2$$

$$z = 4 - 8(x + 1) - 2(y - 2)$$

$$z = -8 - 2y$$

Linear Approximations

Points can be approximated using the tangent planes in the last section, similar to linear approximations in 2d. The linearization of $f(x, y)$ at (a, b) is defined as:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

This function's graph is the linear function whose graph is the tangent plane at the point $(a, b, f(a, b))$. This can be used to form the linear approximation (also known as the tangent plane approximation) of $f(x, y)$ at (a, b), which is defined by the equation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Chain Rules for Functions of Several Variables

One Independent Variable

Let $w = f(x, y)$ where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\delta w}{\delta x}\frac{dx}{dt} + \frac{\delta w}{\delta y}\frac{dy}{dt}$$

Example 1:

Let $w = x^2y - y^2$, where $x = \sin(t)$ and $y = e^t$. Find $\frac{dw}{dt}$ when $t = 0$.

$$\frac{dw}{dt} = \frac{\delta w}{\delta x}\frac{dx}{dt} + \frac{\delta w}{\delta y}\frac{dy}{dt}$$

$$= (2xy)(\cos(t)) + (x^2 - 2y)(e^t)$$

$$\frac{dw}{dt} = 2(\sin(t)(e^t)(\cos(t) + \left(\sin(t)^2 - 2(e^t)(e^t)\right)|_{t=0} = -2$$

Example 2:

Let $f(x, y) = x^2 - y^3$, where $x = \cos(2t)$ and $y = \cos(t)$. Find $\frac{df}{dt}$.

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= 2x \cdot -2\sin(2t) + 3y^2 \cdot -\sin(t)$$

$$\frac{df}{dt} = 2\cos(2t) \cdot -2\sin(2t) + 3\cos^2(t) \cdot -\sin(t)$$

Two Independent Variables

Let $w = f(x, y)$ where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\frac{\delta x}{\delta s}, \frac{\delta x}{\delta t},$ and $\frac{\delta y}{\delta t}$ all exist, then $\frac{\delta w}{\delta s}$ and $\frac{\delta w}{\delta t}$ exist and are given by

$$\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta w}{\delta y}\frac{\delta y}{\delta s}$$

$$\frac{\delta w}{\delta t} = \frac{\delta w}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta w}{\delta y}\frac{\delta y}{\delta t}$$

Example 1:

Find $\frac{\delta w}{\delta s}$ and $\frac{\delta w}{\delta t}$ for $w = 2xy$ where $x = s^2 + t^2$ and $y = \frac{s}{t}$.

$$\frac{\delta w}{\delta s} = \frac{\delta w}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta w}{\delta y}\frac{\delta y}{\delta s}$$

$$= (2y)(2s) + (2x)\left(\frac{1}{t}\right)$$

$$= \left(\frac{2s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right)$$

$$\frac{\delta w}{\delta s} = \frac{6s^2 + 2t^2}{t}$$

$$\frac{\delta w}{\delta t} = \frac{\delta w}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta w}{\delta y}\frac{\delta y}{\delta t}$$

$$(2y)(2t) + (2x)\left(-\frac{s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right)$$

$$\frac{\delta w}{\delta t} = \frac{6st^2 + 2s^3}{t^2}$$

Implicit Partial Differentiation

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}, \quad F_y(x,y) \neq 0$$

If the equation $F(x, y) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\delta z}{\delta x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \qquad \frac{\delta z}{\delta y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

$$F_z(x,y,z) \neq 0$$

Example 1:

Find $\frac{dy}{dx}$ for $y^3 + y^2 - 5y - x^2 + 4 = 0$.

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{2x}{3y^2 + 2y - 5}$$

Example 2:

Find $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ for $3x^2 z - x^2 y^2 + 2z^3 + 3yz - 5 = 0$.

$$\frac{\delta z}{\delta x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{6xz - 2xy^2}{3x^2 + 6z^2 + 3y} \qquad \frac{\delta z}{\delta y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \frac{-2x^2 y + 3z}{3x^2 + 6z^2 + 3y}$$

Directional Derivatives

A directional derivative is the rate of change of a function of two or more variables in the direction of any vector emanating from the point of tangency. In more concise terms, it is the slope of a surface in a given direction indicated by a vector. As an illustration, suppose $f(x, y)$ is defined throughout a region in the xy-plane, $P(x_0, y_0)$ is a point in the domain of f, and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. The equations $x = x_0 + su_1$ and $y = y_0 + su_2$ parametrize the line through the point P parallel to $\mathbf{u}$. s is the parameter that measures the arc length of P in the direction of $\mathbf{u}$. In this case, the rate of change of f at P in the direction of $\mathbf{u}$ is $\frac{df}{ds}$ at P is represented as:

$$D_{\mathbf{u}}f(P) = \lim_{s \to 0} \frac{f(x_0 + su_1, y + su_2) - f(x_0, y_0)}{s}$$

assuming, of course, that the limit exists.

Interpreting this formula is not immediately intuitive: $x = f(x, y)$ represents a surface in space, and the point $P(x_0, y_0)$ lies on the surface. The vertical plane that passes through P parallel to $\mathbf{u}$ intersects the surface in a curve C. The rate of change of f in the direction of $\mathbf{u}$ is the slope of the tangent to C at P. When $\mathbf{u} = \mathbf{i}$, the directional derivative at P is $\frac{\delta f}{\delta x}$ evaluated at P. Conversely, when $\mathbf{u} = \mathbf{j}$, the directional derivative at P is $\frac{\delta f}{\delta y}$ evaluated at P. Rather than calculate the derivative using the limit definition, two more concise formulas can be used (depending on the type of equation you have):

$$\left(\frac{df}{ds}\right)_{\mathbf{u}\,P_0} = \left(\frac{\delta \mathrm{f}}{\delta x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\delta \mathrm{f}}{\delta y}\right)_{P_0} \frac{dy}{ds} \qquad D_{\mathbf{u}P_0} = \left(\frac{\delta \mathrm{f}}{\delta x}\right)_{P_0} \cdot u_1 + \left(\frac{\delta \mathrm{f}}{\delta y}\right)_{P_0} \cdot u_2$$

For functions of three variables, the directional derivative in the direction of $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by $D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_{y(x,yz)} + cf_z(x, y, z)$.

Gradient Vector

The gradient vector is, as one would expect, a vector-valued function that contains the partial derivatives with respect to all variables. Also known as the gradient, $f(x, y)$ at a point P_0 is the vector in the xy plane. The formula is written as

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

OR

$$\nabla f(x, y) = \left\langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y} \right\rangle$$

Sometimes, the $\nabla f(x, y)$ must be in a different format, so it is read as "del f" or written as $\text{grad } f(x, y)$. As far as functions of three variables are concerned, the formula can be written as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_x(x, y, z)\mathbf{k}$$

OR

$$\nabla f(x, y, z) = \left\langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z} \right\rangle$$

For more information on how exactly the gradient vector is used and interactive examples, refer to this link from Math Insight.

Extrema and Critical Points

Just like their two-dimensional counterparts, three dimensional functions also have critical points and relative extrema; the criteria are modified for three dimensions accordingly.

Relative and Absolute Extrema

If f is a function defined on an open region R containing the point (x_0, y_0), then f has a relative minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0). Conversely, f has a relative maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0). Just like in single variable calculus, the relative maximum/minimum that produces the largest/smallest function value is called the absolute maximum or absolute minimum.

Critical Points

If f is a function defined on an open region R containing the point (x_0, y_0), then (x_0, y_0) is considered to be a critical point if one of the following conditions are met:

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0 \qquad \textbf{OR} \qquad f_x(x_0, y_0) \text{ or } f_y(x_0, y_0) \text{ does not exist}$$

If f has a relative minimum or maximum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f. However, the converse of this is <u>not</u> true.

Saddle Points

Saddle points are a new concept in multivariable, but they are easy enough to understand: they are a critical point at which there is neither a relative minima nor a relative maxima. Graphically, they occur at a point where there is a minimum in one variable and a maximum in the other; this why they are called saddle points, as the shape that creates them resembles a saddle. The saddle point on an example curve is highlighted in the image to the right.

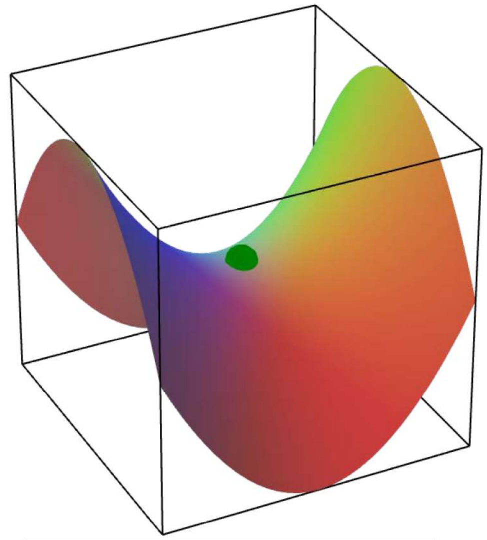


Image Source: Math Insight

Second Partial Derivatives Test

The second partial derivatives test lets you more easily determine whether a point is a local minimum, local maximum, or saddle point. To perform this test, a function f must have continuous first and second partial derivatives on an open region containing a point (a, b) for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$. If this occurs, then

- f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a, b)
- f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at (a, b)
- f has a saddle point at (a, b) if $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at (a, b)
- The test is inconclusive at (a, b) if $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at (a, b)

Lagrange Multipliers

When making use of the many applications of extrema (as we will do in the next section), the problems often have constraints that make determining the optimal solution a bit easier. However, sometimes the optimal solution can occur at the boundary point of a domain, increasing the difficulty. Luckily, the Method of Lagrange Multipliers makes use of Lagrange's Theorem to find extreme values of a function, as long as the function is subject to a constraint. Lagrange's Theorem states that if two functions f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$, and if $\nabla g(x_0, y_0) \neq 0$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

The method of Lagrange's multipliers makes use of this as previously stated. If two functions f and g satisfy Lagrange's Theorem, and f has a minimum or maximum subject to the constraint $g(x, y) = c$, then the minimum or maximum can be found by using these steps:

1. Simultaneously solve the equations $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ and $g(x, y) = c$ by solving the following system of equations.

$$\nabla f_x(x, y) = \lambda \nabla g_x(x, y)$$

$$\nabla f_y(x, y) = \lambda \nabla g_y(x, y)$$

$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The greatest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the least value yields the minimum of f subject to the constraint $g(x, y) = c$.

Applications of Extrema

Maximum Revenue

A company manufactures running shoes and basketball shoes. The total revenue from x units of running shoes and y units of basketball shoes is $R(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$ where x and y are in thousands of units. Determine x and y so as to maximize the revenue.

$R_x = -10x - 2y + 42$

$R_y = -16y - 2x + 102$

$5x + y = 21$

$x + 8y = 51$

$-5x - 40y = -255$

$-39y = -234$

$y = 6$

$x = 51 - 8y$

$x = 3$

$x = 3, y = 6$ (thousands of units)

Minimum Cost

A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. The bottom will cost \$5 per square foot to construct and the sides will cost \$3 per square foot to construct. Determine the dimensions of the container of this size that has minimum cost.

$C(x, y, z) = 3(xy + 2xz + 2yz) + 5xy$

$C(x, y, z) = 8xy + 6yz + 6xz$

$C_x = 8y + 6z$

$C_y = 8x + 6z$

$C_z = 6y + 6x$

$8y + 6z = \lambda yz$

$8x + 6z = \lambda xz$

$8x + 6y = \lambda xy$

$8y - 8x + \lambda xz - \lambda yz = 0$

$(y - x)(8 - \lambda z) = 0$

$y = x, z = \dfrac{8}{\lambda}$

$6y + 6y = \lambda yy$

$y(2y - 12) = 0$

$y = \dfrac{12}{\lambda}, \lambda = \dfrac{12}{y}$

$8y + 6z = \lambda yz$

$8y = 6z$

$z = \dfrac{8y}{6}$

$xyz = 480;\ x = 7.114, y = 7.114, z = 9.485$

Linear Algebra

Systems of Linear Equations

A linear equation using the variables $x_1, \ldots, x_n$ may be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

In this form, b and the coefficients $a_1, \ldots, a_n$ represent any number, usually known in advance for the formulation of the problem. A system of these equations is defined as multiple equations in this form involving the same variables. The solution of such a system is a list of numbers that cause each equation to be a true statement. Therefore, equivalent systems have the same set of solutions. A system can have either one solution, no solutions, or infinitely many solutions. A system is called consistent if it has one or infinitely many solutions, and inconsistent if it has no solution.

Matrix Notation

Given the system

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 + 8x_3 = 8$$

$$-4x_1 + 5x_2 + 5x_3 = -9$$

The coefficient matrix is defined as $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ and the augmented matrix is defined as

$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$.

Solving a Linear System

Three basic operations can be performed while solving a linear system, namely:

1. Replacement – Replacing one equation with the sum of itself and the multiple of another equation.
2. Interchange – Exchanging the position of two rows in the matrix
3. Scaling – Multiplying all entries in a row by a nonzero constant

Row Echelon Form (Echelon Form)

Consider the phrase "non-zero row or column" to refer to a row or column with more than one nonzero entry, and "leading entry" to mean the leftmost nonzero entry in a nonzero row. A matrix is then in row-echelon form if it satisfies the following three conditions:

1. All nonzero rows are above any rows with all zeros
2. Each leading entry is in a column to the right of the leading entry of the above row
3. All entries below a leading entry in the same column are zeros

For example, the following matrix is in row echelon form: $\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Reduced Row Echelon Form (Reduced Echelon Form)

If a matrix in row-echelon form and also fulfills the following conditions, it is considered to be in reduced row-echelon form:

1. The leading entry in each nonzero row is 1
2. Each leading 1 is the *only* nonzero entry in its column

For example, the following matrix is in reduced row-echelon form: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Any nonzero matrix may be row reduced into more than one matrix in ordinary row-echelon form. However, the reduced row-echelon form of a matrix is unique, which is to say that each matrix is equivalent to one and only one reduced row-echelon matrix.

Pivot Position and Pivot Column

A pivot is a nonzero entry that is either used within a pivot position in the matrix to create zeroes, or a position that is changed into a leading 1. A pivot position, therefore, is a position in the matrix that becomes a leading entry in reduced row echelon form, and a pivot column is a column that contains a pivot position.

In the following RREF matrix, the pivot positions are highlighted in red: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basic Variables and Free Variables

A variable in a system of equations is called a basic variable if it corresponds to a pivot column within a matrix. Otherwise, it is called a free variable. In order to discover which variables are basic and which are free, the matrix must be row reduced to echelon form.

For example, the system of equations

$$x_1 + 2x_2 - x_3 = 4$$

$$2x_1 + 4x_2 = 5$$

can be given the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & -4 & 0 & 5 \end{bmatrix}$$

which can be row-reduced to

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -8 & 2 & -3 \end{bmatrix}$$

Once that matrix has been row reduced to row-echelon form, the pivot positions can be identified. The first two columns have pivot positions, so those columns correspond to x_1 and x_2, making them basic variables. The third column is not a pivot column, so x_3 is referred to as a free variable.

Vector Equations

Column Vector

A column vector is a matrix with only one column that represents the vector from the origin to the point (a, b, c), which is to say that the vector from $(0,0,0)$ to (a, b, c) is given by $\langle a, b, c\rangle$. The vector $\langle a, b, c\rangle$ is the matrix with only one column $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Linear Combinations

The vector $\mathbf{w}$ is a linear combination if it can be written in the form $\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + \cdots$. This form is referred to as a linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \cdots, \mathbf{v_p}$ with given scalars $c_1, c_2, c_3, \cdots, c_p$, also known as weights.

For example, let $\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a_1}$ and $\mathbf{a_2}$? If so, determine the proper coefficients of $\mathbf{a_1}$ and $\mathbf{a_2}$.

$$c_1\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ 5 & 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(1)c_2 = 2 \ \therefore \ c_2 = 2$$

$$c_1 + 2(2) = 7 \ \therefore \ c_1 = 3$$

Thus, $3\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ is the linear combination of $\mathbf{b}$.

Vector Spans

Span $\{\mathbf{v}\}$ is the set of all multiples of $\mathbf{v}.$ Similarly, span $\{\mathbf{u}, \mathbf{v}\}$ is the set of all vectors of the form $a\mathbf{u} + b\mathbf{v}.$ Determining if a vector is included in a span is determining whether a linear system with an augmented matrix has a solution. In effect, this is tantamount to asking if the vector equation has a solution in the first place, just in a different notation. To make use of the example above, vector $\mathbf{b}$ is in span $\{\mathbf{a_1}, \mathbf{a_2}\}$, because a linear combination of those two vectors can equal $\mathbf{b}$.

The Matrix Equation *A*x = b

An underlying principle of linear algebra as a whole is the ability to view a linear combination of vectors as the product of a matrix and a vector. If A is an $m \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ is the product of the matrix such that $A\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights, as follows.

$$A\mathbf{x} = [\mathbf{a_1}\ \mathbf{a_2} \dots \mathbf{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} + \cdots + x_n\mathbf{a_n}$$

The Homogenous Equation Ax = 0

A system of linear equations is said to be homogenous if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

The homogenous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable. Nontrivial is specified because it always has a solution where $\mathbf{x} = \mathbf{0}$, which is typically called the trivial solution, but this is typically disregarded most of the time.

For example, to verify that the following system is homogenous:

$$3x_1 + 5x_2 - 4x_3 = 0$$

$$-3x_1 - 2x_2 + 4x_3 = 0$$

$$6x_1 + x_2 - 8x_3 = 0$$

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -3 \end{bmatrix} \bullet \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{4}{3}x_3 = 0 \ \therefore \ x_1 = \frac{4}{3}x_3$$

$$x_2 = 0$$

$$x_3 = x_3$$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Applications of Linear Systems

Linear systems have a myriad of available applications, most of which relate to input and output or flow through a system. The former of these is often applied in economic applications like the following:

Suppose that a given economy consists of Coal, Electric, and Steel sectors, and the output is distributed among each sector as shown in this table.

Distribution of Output From:			
Coal	**Electric**	**Steel**	**Purchased By:**
0%	40%	60%	**Coal**
60%	10%	20%	**Electric**
40%	50%	20%	**Steel**

Denote the prices of the total annual outputs of Coal. Electric, and Steel sectors by P_C, P_E, and P_S, respectively.

$$\mathrm{P_c} = 0.4\mathrm{P_E} + 0.6\mathrm{P_S}$$

$$P_E = 0.6P_C + 0.1P_E + 0.2P_S$$

$$P_S = 0.4P_c + 0.5P_E + 0.2P_S$$

$$1\mathrm{P_c} - 0.4\mathrm{P_E} - 0.6\mathrm{P_S} = 0$$

$$-0.6P_C + 0.9P_E - 0.2P_S = 0$$

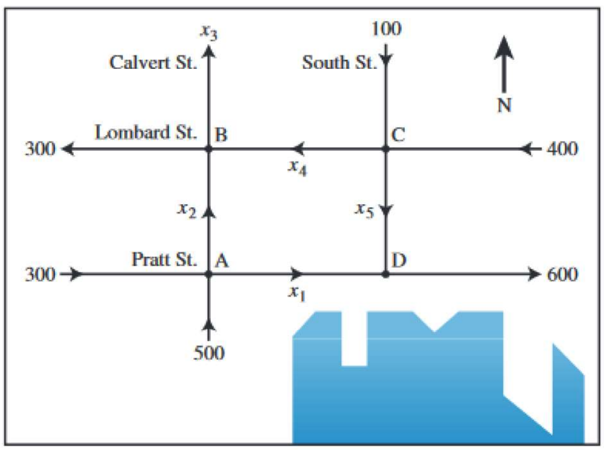
$$-0.4P_c - 0.5P_E + 0.8P_S = 0$$

$$-0.4P_c - 0.5P_E + 0.8P_S = 0$$

$$\begin{bmatrix} 1 & -0.4 & -0.6 & 0 \\ -0.6 & 0.9 & -0.2 & 0 \\ -0.4 & -0.5 & 0.8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -0.94 & 0 \\ 0 & 1 & -0.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P_C \\ P_E \\ P_S \end{bmatrix} = P_S \begin{bmatrix} 0.9 \\ 0.85 \\ 1 \end{bmatrix} \quad \therefore \quad \begin{matrix} P_S = P_S \\ P_E = 0.85\,P_S \\ P_C = 0.94\,P_S \end{matrix}$$

As for linear applications covering flow through a system, this specific application is very important in varying disciplines like traffic patterns, electrical circuits, and economic networks. A network consists of a set of junctions with lines connecting some of the junctions. The direction of flow in each branch is indicated, and the flow rate in each branch is denoted by a variable. The assumption that allows solution of networks is that the total flow into a junction is equal to the total flow out of a junction. This can be applied in situations like the one below.



This network shows the traffic flow (in vehicles per hour) through one way streets. Determine the general flow pattern.

$$\begin{array}{ll} Node\ A & 500 + 300 = x_1 + x_2 \\ Node\ B & x_2 + x_4 = x_3 + 300 \\ Node\ C & 100 + 400 = x_4 + x_5 \\ Node\ D & x_1 + x_5 = 600 \\ Total & 100 + 400 + 500 + 300 = 300 + x_3 + 600 \end{array}$$

$$x_1 + x_2 = 800$$
$$x_2 - x_3 + x_4 = 300$$
$$x_4 + x_5 = 500$$
$$x_1 + x_5 = 600$$
$$x_3 = 400$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{bmatrix}$$

Inverse of a Matrix

If A is a square matrix, and there exists a matrix B so that $AB = BA = I$, (I being the identity matrix), then matrix B is the inverse of matrix A, and can also be denoted A^{-1}. In order to find the inverse of a 2x2 matrix, one should use the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

For example, in order to find the inverse of the given matrix $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, it equals $\frac{1}{3(2)-1(4)} \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$, or $\begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$. Sometimes, the inverse of a 3x3 (or larger) matrix must be found instead, making the process a bit more complicated; such an inverse can be found by using row reduction. In order to find A^{-1} of a 3x3 matrix, simply augment the matrix with its identity matrix to the right, and manipulate it to obtain the identity matrix on the left. (An example is included on the next page.)

For instance, to get the inverse of the matrix *A*:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$ Swap 1 and 2

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$ -4 * row 1 + row 3 = new row 3

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$ 3 * row 2 + row 3 = new row 3

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$ 1/3 * row 3

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$ -3 * row 3 + row 1 = new row 1

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$ -2 * row 3 + row 2 = new row 2

$$\therefore\ A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & 0 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

Cramer's Rule

Cramer's Rule is useful for solving systems of equations with as many equations as unknowns, and so it serves as a "shortcut" of sorts from typical row reduction. It expresses a solution in terms of determinants of the original matrix through replacing rows. For example, to solve the system of equations:

$$3x_1 - 2x_2 = 6$$

$$-5x_1 + 4x_2 = 8$$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\det(A) = 3(4) - (-2)(-5) = 12 - 10 = 2$$

Replace first column of A with b and find A_1(b) determinant

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} \qquad \det A_1(b) = 24 - (-16) = 40$$

Replace second column of A with b and find A_2(b) determinant

$$A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix} \qquad \det A_2(b) = 24 - (-30) = 54$$

Now…

$$x_1 = \frac{\det A_1(b)}{\det(A)} = \frac{40}{2} = 20$$

$$x_2 = \frac{\det A_2(b)}{\det(A)} = \frac{54}{2} = 27$$

$$\therefore \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 27 \end{bmatrix}$$

Cramer's rule works exactly the same with larger matrices, except finding the determinant of the larger matrices takes more time. Nevertheless, it is still an invaluable tool for solving systems of equations without using row reduction.

Eigenvalues and Eigenvectors

As far as these are concerned, we just want to find a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$. A main application of eigenvalues is the solving of linear differential equations. As an illustration, let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since $A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3\mathbf{x}$, then $\lambda = 3$ is an eigenvalue of A and $\mathbf{x}$ is an eigenvector belonging to A. More examples are provided on this webpage from MIT.

Multiple Integrals

Iterated Integrals

Iterated integrals are actually quite simple: merely taking the integral more than once with respect to different variables. They are written accordingly:

$\int_c^d \int_a^b f(x,y)dydx$ **OR** $\int_c^d \int_a^b f(x,y)dxdy$

While integrating, think of all the x's as constants and integrate with respect to y or think of all y's as constants and integrate with respect to x. To solve, you take the result of the innermost integral, and integrate it again with respect to the outer. Simplifying the notation, we can get

$$\int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

which more clearly states the way that iterated integrals are computed. An example would be $\int_0^3 \int_1^2 (x^2y)dy\,dx$: taking the result of $\int_1^2 (x^2y)dy$, which is $\frac{3}{2}x^2$, we then compute $\int_0^3 (\frac{3}{2}x^2)dx$ to get the final result $\frac{27}{2}$. This is precisely why they are called iterated integrals, as the functions are integrated in iterations from innermost to outermost. There can be more than two nested integrals, which will become important later.

Double Integrals and Fubini's Theorem

Double integrals are almost exactly the same as integrated integrals, written in the notation

$$\iint_R f(x,y)\,dA$$

The only difference between double integrals and integrated integrals is that double integrals are done over a region R instead of being given discrete limits. Now, the question becomes how to go about solving them? This is where Fubini's Theorem comes into play. Fubini's Theorem states that the order of integration may be swapped!

Given a region $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dydx$$

It is important to note that it does not matter which order you take the integrals in; you can take the integral with respect to y and then with respect to x, or vice versa. As long as the limits of integration are consistent, these will produce the same result. Using Fubini's theorem, we can set up double integrals over a region as iterated integrals and solve them in the same fashion.

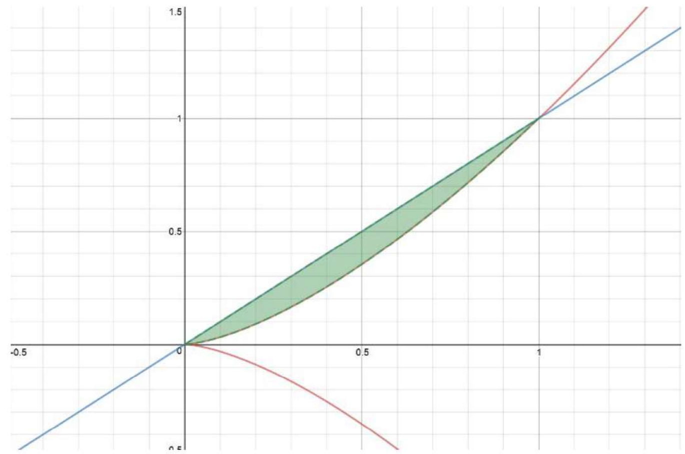
For example, to evaluate

$$\iint_R (x - 3y^2)\,dA \qquad R = \{(x,y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$$

the iterated integral would be set up and solved as follows:

$$\int_0^2 \int_1^2 (x - 3y^2)dydx = \int_0^2 [xy - y^3]_1^2\,dx = \int_0^2 (x - 7)dx = -12$$

Area of a Region with Iterated Integrals



One of the common applications for iterated integrals is the area of two-dimensional regions. If given an area bounded by one or more regions, it is a more elegant solution to simply use nested integrals instead of subtraction. Most of the difficulty in this method lies in getting the limits to use, however. For example, to find the area of $\iint_R xy\,dA$ bounded by the functions $y^2 = x^3$ and $y = x$, the first step is to determine the limits, since they are not immediately apparent. It is important to note that we can have functions in the limits of inner integrals, as the aforementioned functions can be resolved into numbers by the outer integrals.

$$y^2 = x^3 \qquad\qquad y = x$$

$$x^2 = x^3$$

$$0 = x^3 - x^2$$

$$0 = x^2(x-1)$$

$$x = 0, x = 1$$

$$y = x, y = \pm\sqrt{x^3}$$

$$\int_0^1 \int_{x^{\frac{3}{2}}}^{x} xy\,dy\,dx$$

Now that we have obtained our integral to find the area of the region, it is relatively trivial to solve as demonstrated below:

$$\int_0^1 \int_{x^{\frac{3}{2}}}^{x} xy\,dy\,dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_{x^{\frac{3}{2}}}^{x} dx = \frac{1}{2}\int_0^1 (x^3 - x^4)dx = \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{40}$$

Volume of a Solid Region

Finding the volume of a solid region is exactly like finding the area of a region, only with a third variable. For example, to find the volume of a tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$, the procedure is as follows:

$$x = 2y \therefore y = \frac{1}{2}x$$

$$x + 2y + z = 2$$

$$x + 2y = 2 - z$$

$$y = \frac{1}{2}(2 - x)$$

$$\frac{1}{2}x = 1 - \frac{1}{2}x$$

$$\text{intersects at } \left(1, \frac{1}{2}\right)$$

Finding the limits like this, the integration is set up as follows:

$$\int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2 - x - 2y)dy\,dx$$

$$= \int_0^1 [2y - xy - y^2]_{\frac{1}{2}x}^{1-\frac{1}{2}x}\,dx$$

$$= \int_0^1 (1 - 2x + x^2)dx$$

$$= \left[x - x^2 + \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

Average Value of a Function over a Region

In addition to finding the area of a region over an integral, double integrals can also be used to find the average value of a function over a given region. If $A(R)$ is the area of the region, use the equation

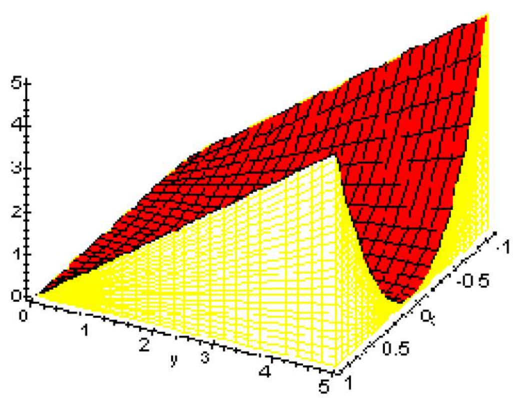
$$\frac{1}{A(R)}\iint\limits_R f(x,y)dA$$

For example, to find the average value of $f(x,y) = x^2y$ over the rectangle R if R has vertices $(-1,0), (-1,5), (1,5)$, and $(1,0)$, first find the area of R: $[1-(-1)] * [5-0] = 10$. Then, integrate as follows:

$$\frac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dxdy = \frac{1}{10}\int_0^5\left[y\frac{x^3}{3}\right]_{-1}^1 dy = \frac{1}{15}\int_0^5 ydy = \frac{1}{15}\left[\frac{y^2}{2}\right]_0^5 = \frac{5}{6}$$

As a visual reference, a graph depicting this process is provided below. This example was sourced from the UCLA Calculus website.

f := (x,y) ->x^2*y



Double Integrals in Polar Coordinates

Double integrals in polar coordinates are solved exactly the same as regular double integrals covered previously, except instead of rectangular coordinates, it is typically swapped to polar coordinates for ease of integration. As is expected the three polar conversions carry over from previous classes: $r^2 = x^2 + y^2$ $\quad x = r\,cos(\theta)$ $\quad y = rsin(\theta)$

Using these, we can obtain the polar double integral notation over a general region:

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta)\,, r\sin(\theta))r\, dr\, d\theta$$

As an example of a problem where it becomes easier to swap to polar coordinates, consider evaluating $\iint_R f(3x + 4y^2)\, dA$ where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Since the boundaries are circles, it is preferred to find the limits and swap to a polar integral as follows:

$R = \{(x, y) \mid y \geq 0 \; and \; 1 \leq x^2 + y^2 \leq 4\}$

$1 \leq x^2 + y^2 \leq 4$

$1 \leq r^2 \leq 4$

$1 \leq r \leq 2$

$y \geq 0$

$r\sin(\theta) \geq 0$

$\sin(\theta) \geq 0$

$\theta \geq \sin^{-1}(0)$

$0 \leq \theta \leq \pi$

Using these limits, and swapping the integral to polar coordinates, the integral becomes

$$\int_0^\pi \int_1^2 3(r\cos(\theta)) \; + \; 4(r\sin(\theta))^{\,2} \; r\, dr\, d\theta = \int_0^\pi \int_1^2 3r^2\cos(\theta) \; + \; 4r^3\sin(\theta)^{\,2} \; dr\, d\theta$$

$$= \int_0^\pi [3r^2\cos\theta + r^4\sin^2\theta]_1^2 \; d\theta = \int_0^\pi 7\cos\theta + 15\sin^2\theta \; d\theta = \int_0^\pi 7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta\; d\theta$$

$$= \int_0^\pi 7\cos\theta d\theta + \int_0^\pi \frac{15}{2} d\theta + \int_0^\pi -\frac{15}{2}\cos 2\theta\; d\theta = 0 + \frac{15\pi}{2} + 0 = \frac{15\pi}{2}$$

Mass, Moments, and Center of Mass of a Lamina

Another application of double integrals other than finding area or average value is calculating mass, center of mass and moment of inertia. Imagine a lamina, a thin plate of material, with a varying density. Suppose that such a lamina occupies a region D and has a density function $\rho(x, y)$. The moment of a particle about an axis is the product of its mass and its directed distance from the axis; this means that the moments of a lamina about the x-axis and y-axis (respectively) are defined as

$$M_x = \iint_D y\ \rho(x, y) dA \qquad\qquad M_y = \iint_D x\ \rho(x, y) dA$$

Using these moments, we can then calculate the exact coordinates of the lamina's center of mass, $(\bar{x}, \bar{y})$. These coordinates for a lamina occupying a region D and having a density function $\rho(x, y)$ and a mass $m = \iint_D \rho(x, y) dA$ are defined as $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$. As this problem requires a substantial number of steps that all require the previous actions to be correct, additional practice may be required; extensive examples of this formula's usage are provided in this webpage from Whitman College.

Triple Integrals

Triple integrals, like their predecessors double integrals, are simply nested integrals evaluated in order from inside to outside. The only difference here is that there are three nested integrals instead of just two, which makes keeping track of integration limits and constants of the utmost importance. They are given the general formula

$$\iiint_R f(x, y, z) dV = \int_b^a \int_d^c \int_\beta^\alpha f(x, y, z)\ dx\ dy\ dz$$

As with double integrals, Fubini's Theorem applies, allowing them to be rearranged in any order; as one might expect, where double integrals compute area, triple integrals can compute volume.

Vector Calculus

Vector Fields

Vector fields are simple: they assign a vector to appoint in the plane or a point in space, and they're very useful for representing various types of force and velocity fields. In general, a vector field is a function whose domain is a set of points in $\mathbf{R}^2$ $(\boldsymbol{or}\ \mathbf{R}^3)$ and whose range is a set of vectors in $\mathbf{V}^2$ $(\boldsymbol{or}\ \mathbf{V}^3)$. A vector field over a plane region R is a function F that assigns a vector $\mathbf{F}(x, y)$ to each point in R. By contrast, a vector field over a solid region Q in space is a function F that assigns a vector $\mathbf{F}(x, y, z)$ to each point in Q. Aa vector field $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(\mathrm{x, y, z})\mathbf{k}$ is continuous if and only if each of its component functions M, N, and P is continuous at that point. Because vector fields consist of infinitely many vectors, it is not possible to create a sketch of the entire field. Instead, sketch representative vectors to help visualize the field. To do this, choose a representative set of domain points, and sketch the vector attached to each point. For a more extended example, refer to this video on Khan Academy for more information.

Conservative Vector Field

Some vector fields can be represented as the gradients of differentiable functions and some cannot – those that can are called conservative vector fields. A vector field $\mathbf{F}$ is called conservative when there exists a differentiable function such that $\mathbf{F} = \nabla f$. The function f is called a potential function for $\mathbf{F}$. If M and N have continuous first partial derivatives, a given vector field $\mathbf{F}(x, y,) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if $\frac{\delta N}{\delta x} = \frac{\delta M}{\delta y}$. Examples of testing if a vector field is conservative and finding its potential function can be found on this site from Lamar University.

Line Integrals…

…with Respect to Arc Length

Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \leq t \leq b$, then

$$\int_C f(x,y)ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leq t \leq b$, then

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

…of Vector Fields

An important application of line integrals is finding the work done on an object moving in a force field. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t), a \leq t \leq b$. The line integral of $\mathbf{F}$ on C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big) \cdot \mathbf{r}'(t)\, dt$$

…in Differential Form

If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r}$ is often written as $Mdx + Ndy$.

Fundamental Theorem of Line Integrals

Recall that the Fundamental Theorem of Calculus states that $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$. Similarly, the Fundamental Theorem of Line Integrals states that if the vector field $\mathbf{F}$ is conservative, then the line integral between any two points is simply the difference in the values of the potential function at these points. Formally, it is defined as follows: Let C is a piecewise smooth curve lying in an open region R and given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where $a \leq t \leq b$. If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\big(x(b), y(b)\big) - f\big(x(a), y(a)\big)$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

Before using the Fundamental Theorem of Line Integrals, one should note that line integrals of conservative vector fields only use the initial and terminal point of a curve; therefore, they are independent of path, but a vector field must be conservative in order to use the Fundamental Theorem of Line Integrals.

Green’s Theorem

This theorem states that the value of the double integral over a simply connected plane region R is determined by the value of the line integral around the boundary of R. To use Green’s Theorem, let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise. If M and N have continuous first partial derivatives in an open region containing R, then

$$\int_C M\,dx + N\,dy = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dA$$

Because these two are equal, we are able to calculate line integrals using nothing but the double integrals we already know. For more graphics, examples, and information on Green’s theorem, take a look at this site provided by Lamar University.

Curl of a Vector Field

Curl is a vector that describes the rotation of a vector field. One can think of curl as a from of differentiation for vector fields; it gives the direction and magnitude of the vector field's rotation at a point.

The curl of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \left(\frac{\delta P}{\delta y} - \frac{\delta N}{\delta z}\right)\mathbf{i} + \left(\frac{\delta P}{\delta x} - \frac{\delta M}{\delta z}\right)\mathbf{j} + \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)\mathbf{k}$$

If curl $\mathbf{F} = \mathbf{0},$ then $\mathbf{F}$ is called irrotational. Curl can also be written in its determinant from, which is easier to remember and thus more commonly used when calculating it by hand.

$$\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ M & N & P \end{vmatrix}$$

For an intuitive explanation of applications of curl and how to visualize it in a three-dimensional space, watch [this video].

Divergence of a Vector Field

Divergence is a scalar function that, similar to curl, is performed on vector fields. Put simply, if you have a vector field representing velocities of moving particles, the divergence measures the rate of particle flow per unit volume. The divergence of $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is

$$\text{div}\,\mathbf{F}(x, y, z) = \nabla\cdot\mathbf{F}(x, y, z) = \left(\frac{\delta M}{\delta x}\right) + \left(\frac{\delta N}{\delta y}\right)$$

Likewise, the divergence of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{div}\,\mathbf{F}(x, y, z) = \nabla\cdot\mathbf{F}(x, y, z) = \left(\frac{\delta M}{\delta x}\right) + \left(\frac{\delta N}{\delta y}\right) + \left(\frac{\delta P}{\delta z}\right)$$

If div $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be divergence free.

For more information on the applications of divergence, this video is very helpful.

Stokes' Theorem

Recall that Green's theorem states that the double integral over a plane region R is determined by the value of a line integral around the boundary. Similarly, Stokes' Theorem gives the relationship between a surface integral over an oriented surface S and a line integral along a closed space curve C forming the boundary of S; it is a more general form of Green's Theorem that applies to space curves. Let S be an oriented surface with the unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple curve C with a positive orientation. If F is a vector field whose component functions have continuous first partial derivatives on an open region containing S and C, then Stokes' Theorem states that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_C \text{curl}\,\mathbf{F}\cdot\mathbf{N}\,dS$$

Interestingly, many of the previously-covered identities can be derived from Stokes' Theorem because of its generality – but that is outside the scope of this course.

Resources

Notes and In-Class Demonstrations

Calculus Early Transcendentals, 7th and 8th Ed. By James Stewart

Ron Larson, Bruce Edwards - Calculus, 11th Edition

MIT OpenCourseWare:

https://ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/1.-vectors-and-matrices/

West Virginia University Math:

https://math.wvu.edu/~hjlai/Teaching/Tip-Pdf/

UCLA Calculus:

http://www.math.ucla.edu/~ronmiech/Calculus_Problems/32B/chap13/section2/843d35/843_35.html

MathInsight

Khan Academy

Lamar University Multivariable:

http://tutorial.math.lamar.edu/Classes/CalcIII/CalcIII.aspx

All graphs and figures have been created by me unless otherwise noted.