Early Indicators of Transversal Instability in Noisy Phase-Synchronized Systems

Carter Lentz
Oscie OOI

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Abstract

This note clarifies the formulation underlying a question on early indicators of instability in noisy phase-synchronized systems. We consider networks of coupled phase oscillators and focus on loss of transversal stability of synchronized or clustered states. The emphasis is on finite-time growth of transversal perturbations and phase-difference statistics, which may reveal degradation of stability before macroscopic order parameters change.

1 Reference model

We consider a network of N coupled phase oscillators with additive noise, in the standard form (cf. Pikovsky, Rosenblum, Kurths, 2001):

$$\dot{\phi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i) + \xi_i(t), \qquad i = 1, \ldots, N, \tag{1}$$

where $\phi_i \in \mathbb{R}/2\pi\mathbb{Z}$ are phases, ω_i intrinsic frequencies, K is the coupling strength, and $\xi_i(t)$ are mutually independent stochastic processes (e.g. Gaussian white noise with intensity σ^2).

The discussion extends directly to non–all-to-all coupling by replacing (K/N) with KA_{ij}.

2 Traditional order parameter

Global phase synchronization is commonly characterized by the Kuramoto order parameter

$$Re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}, \tag{2}$$

with $R \in [0, 1]$ measuring the degree of phase coherence.

Loss of synchronization is typically associated with a qualitative change in the long-time behavior or stationary distribution of R, for example when K decreases below a critical value or noise intensity increases.

3 Synchronized manifold and transversal dynamics

For identical oscillators ($\omega_i = \omega$), the fully synchronized solution $\phi_1 = \cdots = \phi_N$ defines the synchronization manifold

$$\mathcal{M} = \{\phi_1 = \phi_2 = \cdots = \phi_N\}. \tag{3}$$

More generally, clustered states define analogous invariant manifolds. Let $\bar{\phi}(t)$ denote the mean phase and define transversal deviations

$$\delta_i(t) = \phi_i(t) - \bar{\phi}(t), \qquad \sum_{i=1}^{N} \delta_i(t) = 0. \tag{4}$$

Linearization of the deterministic part of the dynamics around a synchronized (or clustered) trajectory yields transversal equations of the form

$$\dot{\boldsymbol{\delta}} = J_\perp(t)\boldsymbol{\delta} + \text{noise}, \tag{5}$$

where $J_\perp(t)$ is the Jacobian projected onto the transversal subspace.

4 Instability definition

In this work, instability refers to loss of transversal stability of the synchronized or clustered state. Formally, this corresponds to positive growth rates of transversal perturbations.

On finite time horizons T, this can be quantified via finite-time transversal Lyapunov exponents

$$\lambda_\perp^{(T)} = \frac{1}{T} \ln \frac{\|\boldsymbol{\delta}(t+T)\|}{\|\boldsymbol{\delta}(t)\|}. \tag{6}$$

Importantly, $\lambda_\perp^{(T)}$ may become positive for relevant finite T even when the global order parameter R remains close to unity.

5 Observable phase-based indicators

The following quantities are computable from observed phase trajectories and do not require perturbation of the system.

5.1 Phase-difference dispersion

Define pairwise phase differences

$$\Delta_{ij}(t) = \mathrm{wrap}_{(-\pi,\pi]}(\phi_i - \phi_j),$$

and the circular dispersion

$$D(t) = 1 - \left| \frac{2}{N(N-1)} \sum_{i<j} e^{i\Delta_{ij}(t)} \right|. \tag{7}$$

An increasing trend in $D(t)$ while R remains high indicates degradation of transversal coherence.

5.2 Transversal energy growth

Define transversal energy

$$E_{\perp}(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_i(t)^2. \tag{8}$$

Finite-time growth rates of $E_{\perp}(t)$ provide an output-level proxy for transversal instability.

5.3 Phase-locking persistence

For a tolerance $\varepsilon > 0$, define

$$P_{\varepsilon}(t) = \frac{2}{N(N-1)} \sum_{i<j} \mathbf{1}\{|\Delta_{ij}(t)| < \varepsilon\}. \tag{9}$$

A reduction in the persistence time of $P_{\varepsilon}(t)$ can precede visible changes in R.

6 Clarification of the original question

The question is whether, in noisy or weakly coupled regimes, output-derived indicators based on transversal dynamics can reliably signal degradation of stability before traditional order parameters such as R exhibit a qualitative transition.

7 Implementation context

In ongoing work, these quantities are evaluated on sliding time windows as observational diagnostics of oscillatoric stability. The analysis is strictly passive and does not modify the underlying dynamics.