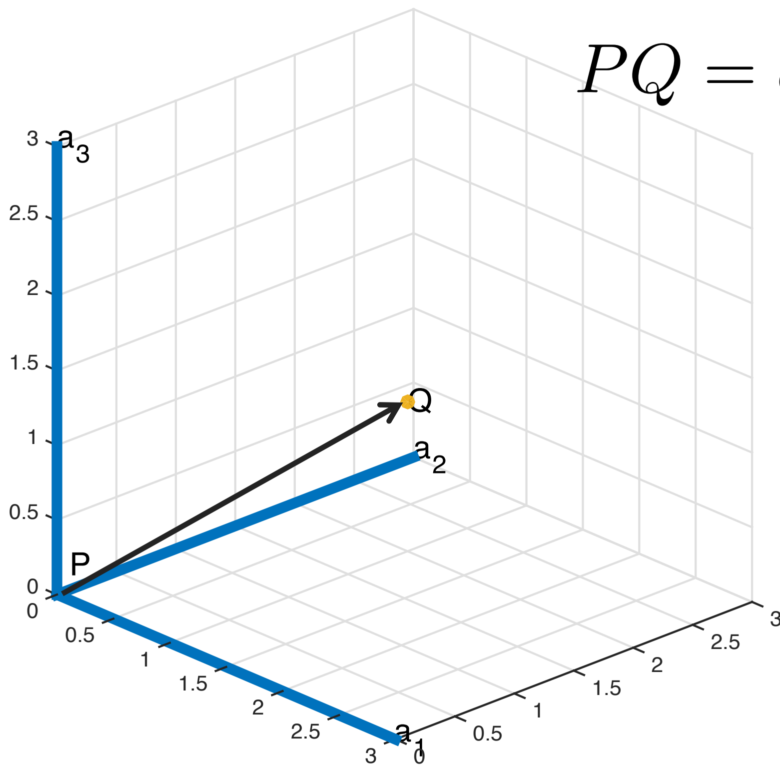
Rigid-Body Displacements

Penn Engineering

Rigid-Body Displacement

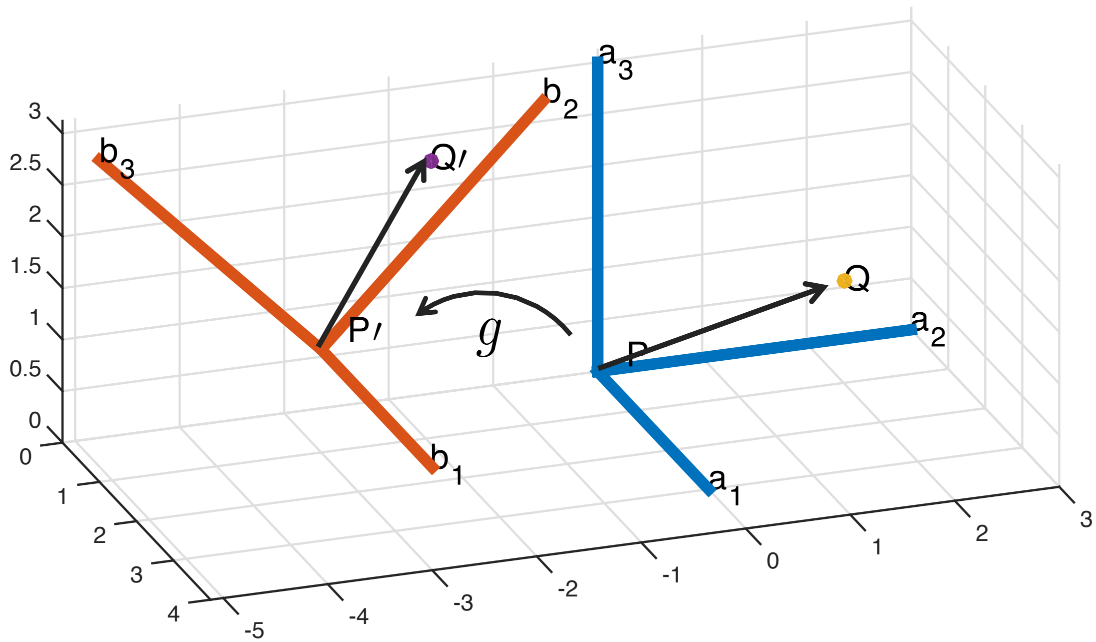
Consider Frame A and vector PQ.

$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$



Rigid-Body Displacement

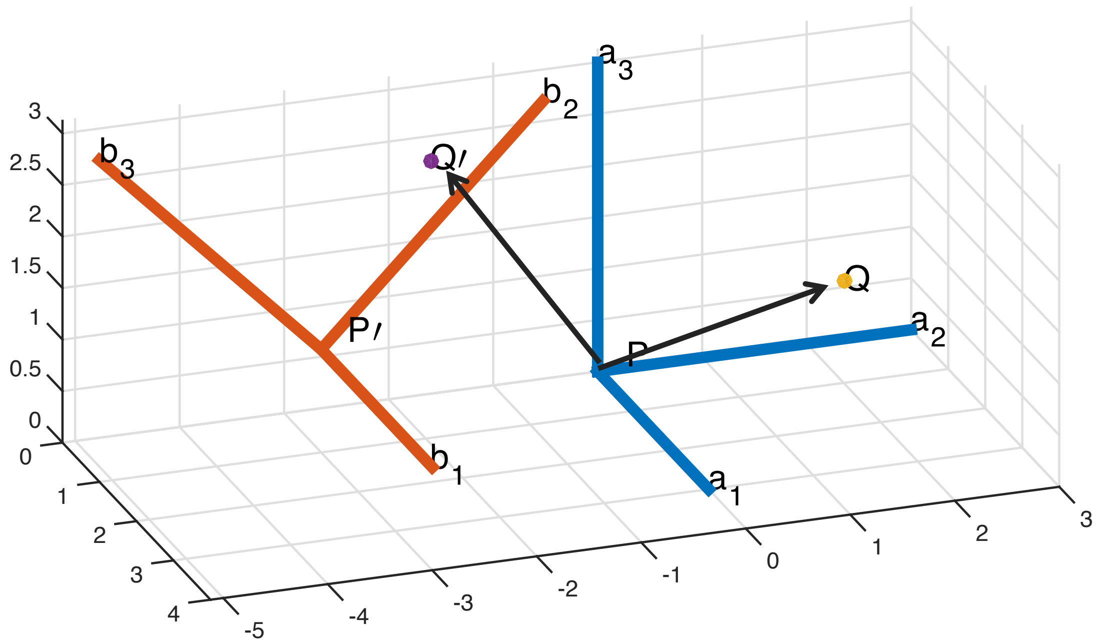
Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$
$$P'Q' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



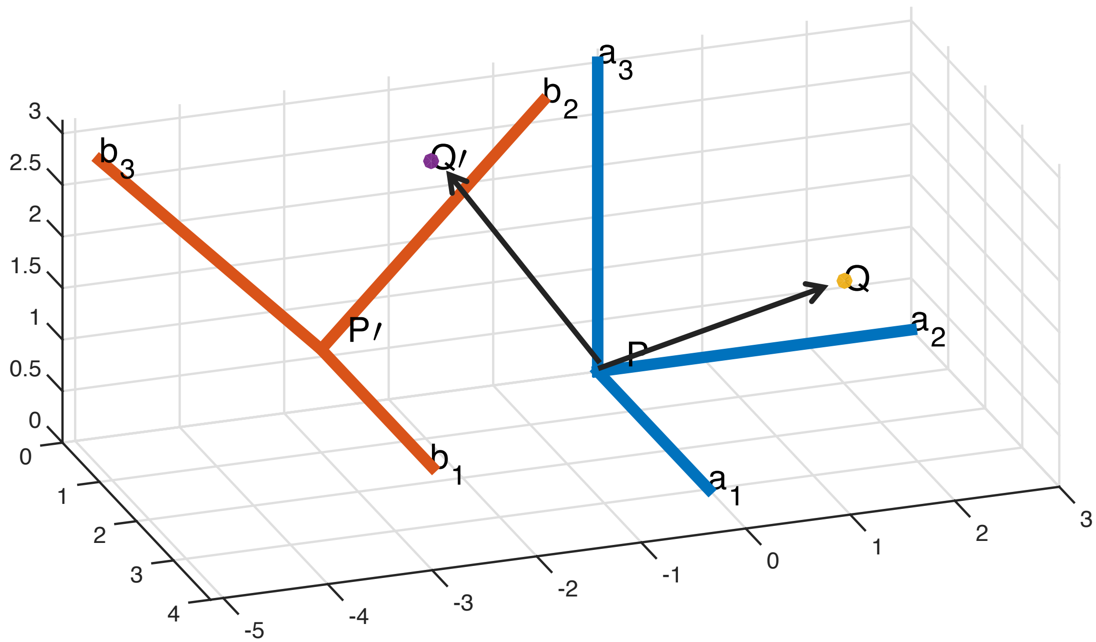
$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

Penn Engineering

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



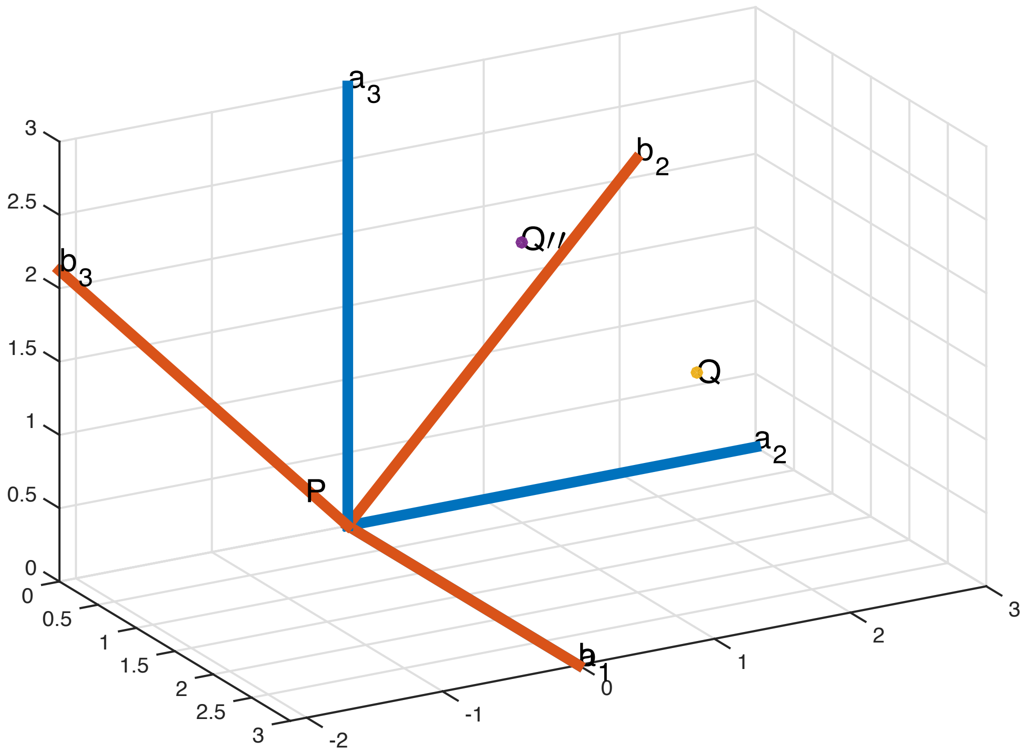
$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\boxed{\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = R \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}} + \mathbf{d}$$

Rotation

Translate frame B so reference frames share an origin.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

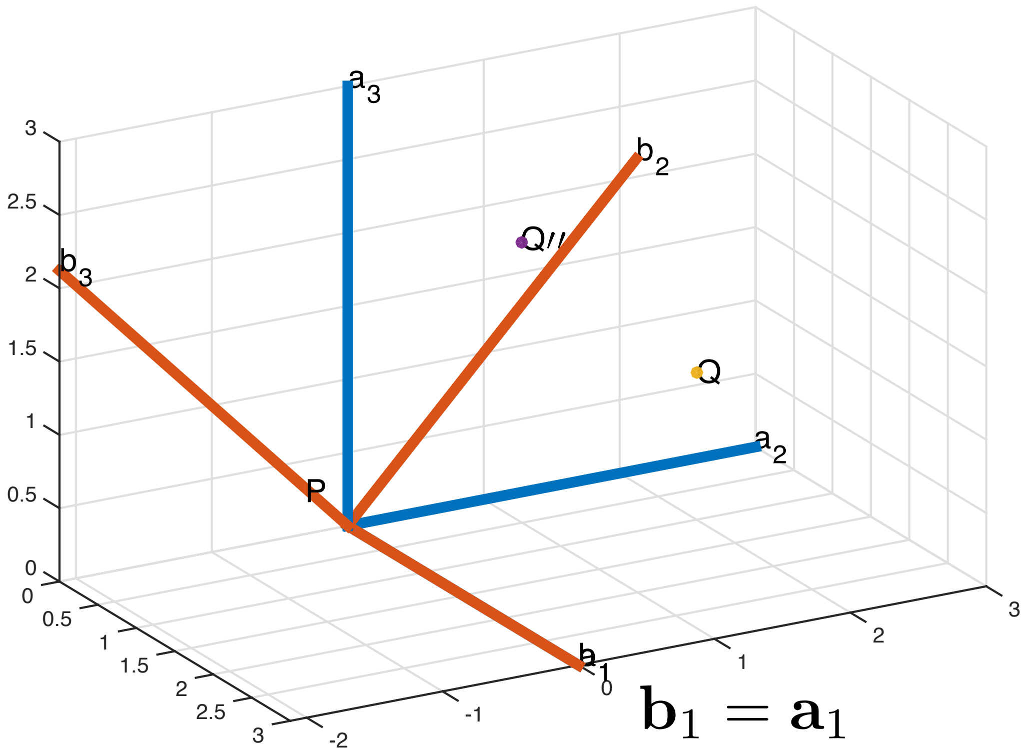
$$PQ'' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

$$= q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Rotation

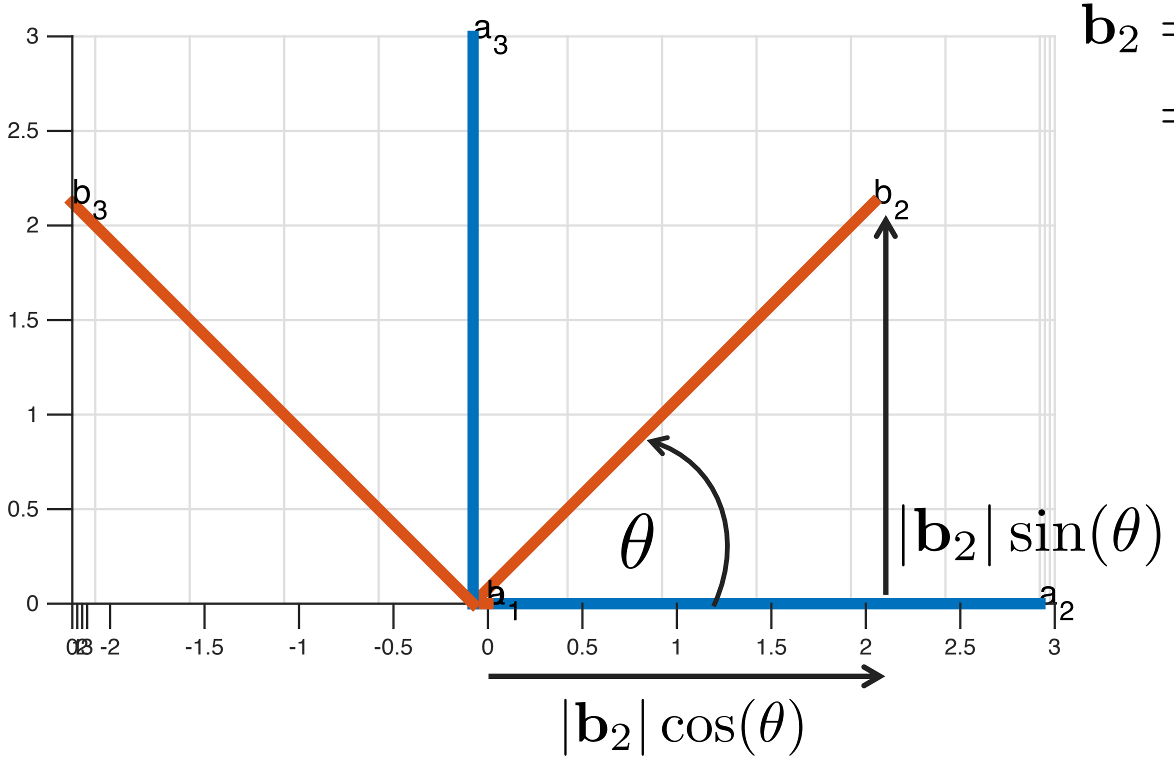
Translate Frame B so reference frames share an origin.



$$\mathbf{b}_1 = \mathbf{a}_1$$

Rotation

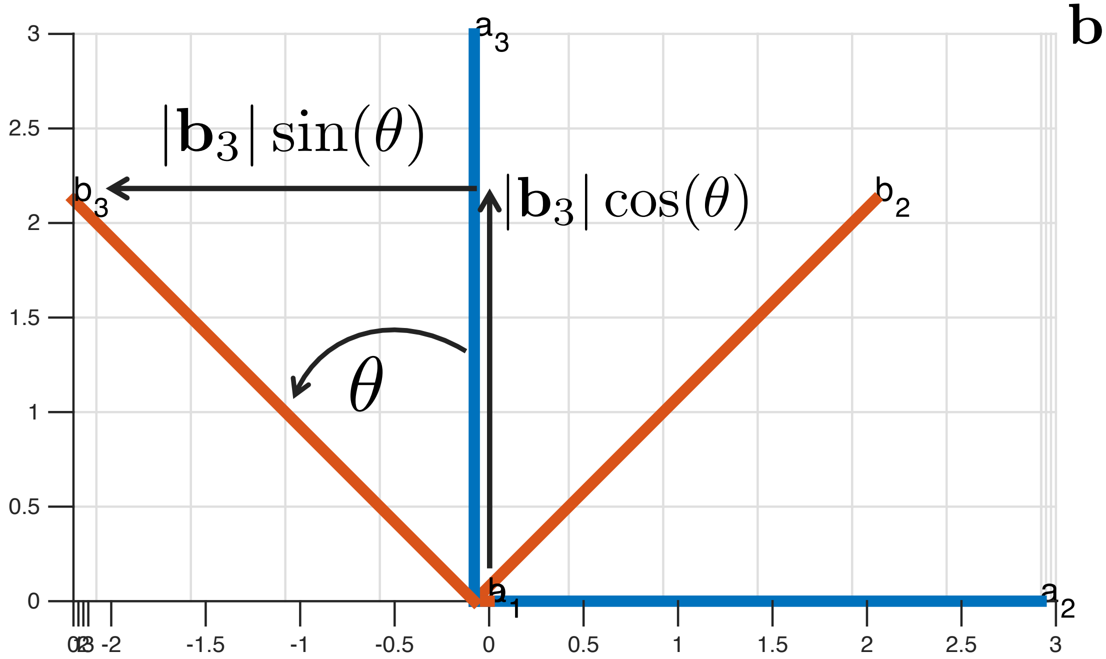
Translate Frame B so reference frames share an origin.



$$\mathbf{b}_2 = |\mathbf{b}_2|\cos(\theta)\mathbf{a}_2 + |\mathbf{b}_2|\sin(\theta)\mathbf{a}_3$$
$$= \cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3$$

Rotation

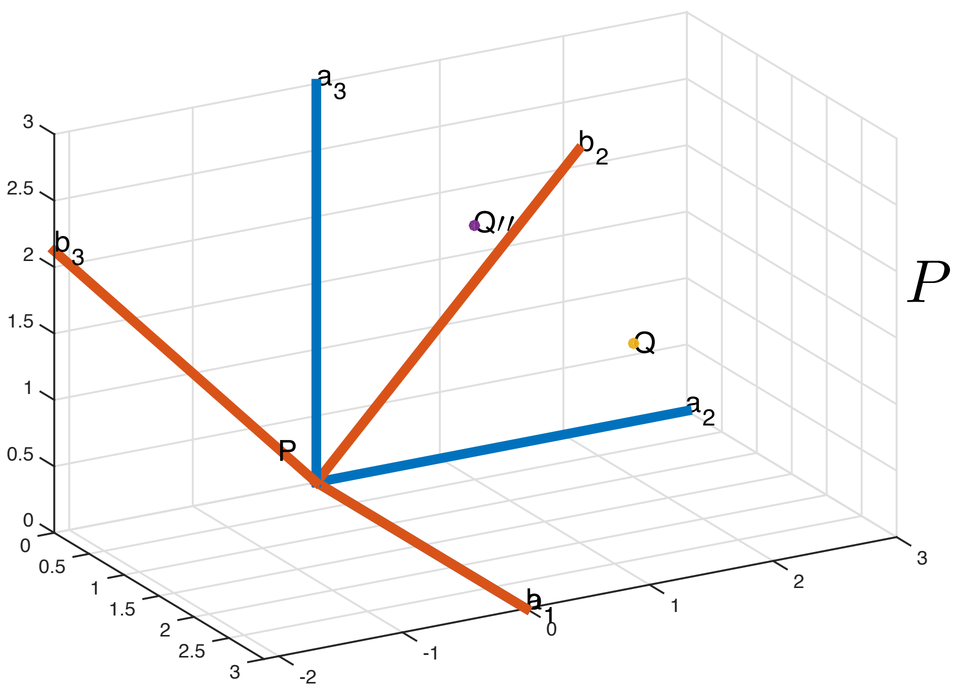
Frame B so the two frames share an origin.



$$
\begin{aligned}
\mathbf{b}_3 &= -|\mathbf{b}_3|\sin(\theta)\mathbf{a}_2 + |\mathbf{b}_3|\cos(\theta)\mathbf{a}_3 \\
&= -\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3
\end{aligned}
$$

Rotation

Translate frame B so reference frames share an origin.



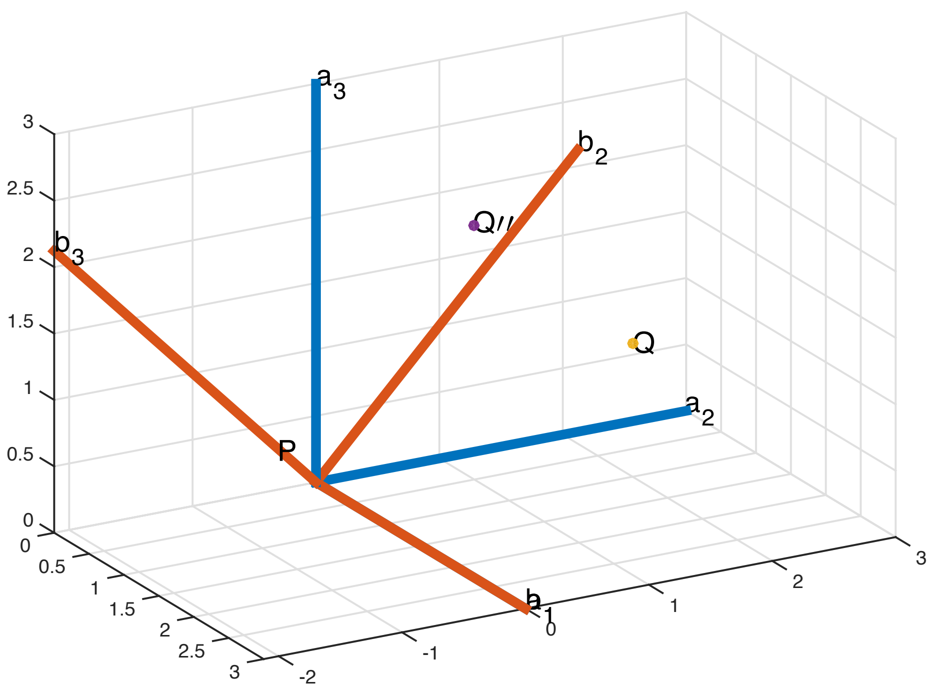
$$PQ'' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$
$$= q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$PQ'' = q_1(\mathbf{a}_1) + q_2(\cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3) + q_3(-\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3)$$

$$= q_1\mathbf{a}_1 + (q_2\cos(\theta) - q_3\sin(\theta))\mathbf{a}_2 + (q_2\sin(\theta) + q_3\cos(\theta))\mathbf{a}_3$$

Rotation

Translate frame B so reference frames share an origin.



$$PQ'' = q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$PQ'' = q_1\mathbf{a}_1 + (q_2\cos(\theta) - q_3\sin(\theta))\mathbf{a}_2 + (q_2\sin(\theta) + q_3\cos(\theta))\mathbf{a}_3$$

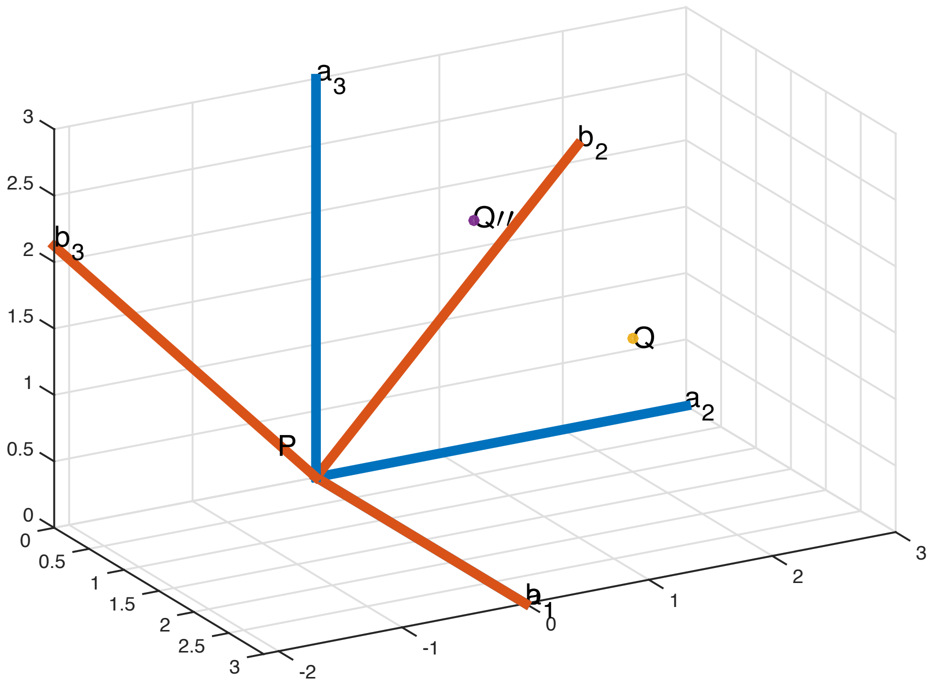
$$q_1'' = q_1$$

$$q_2'' = q_2\cos(\theta) - q_3\sin(\theta)$$

$$q_3'' = q_2\sin(\theta) + q_3\cos(\theta)$$

Rotation

Translate frame B so reference frames share an origin.



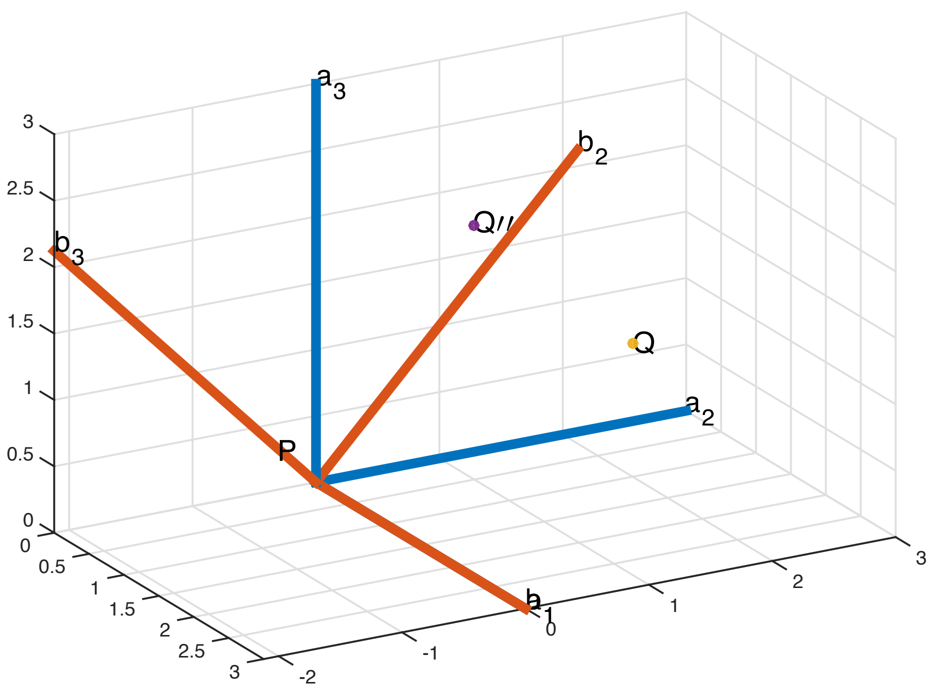
$$PQ'' = q_1(\mathbf{a}_1) + q_2(\cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3) + q_3(-\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3)$$

$$PQ'' = q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Rotation

Translate frame B so reference frames share an origin.



$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = Rot(x, \theta)$$

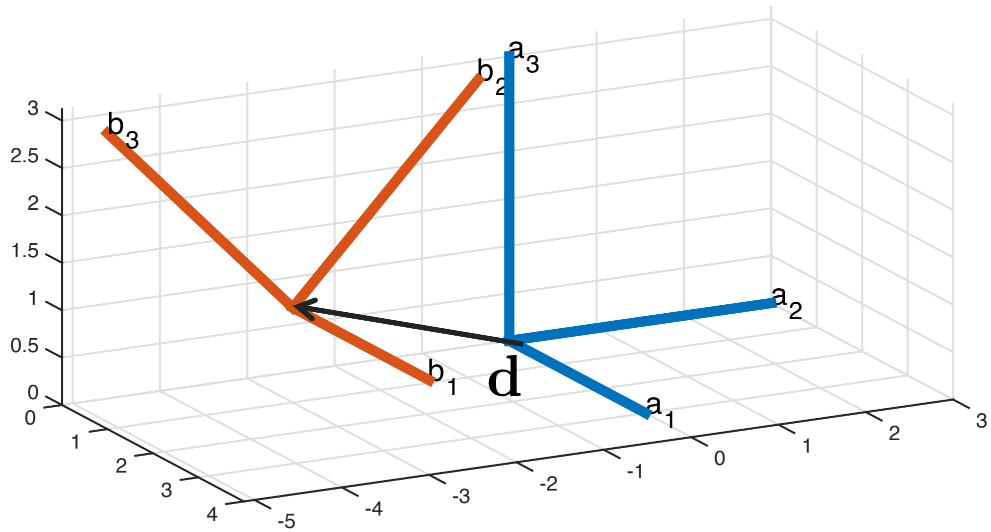
$$\theta = \frac{\pi}{4} \rightarrow$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Penn Engineering

Translation

Let **d** be the vector from the origin of Frame A to the origin of Frame B, expressed in terms of Frame A.

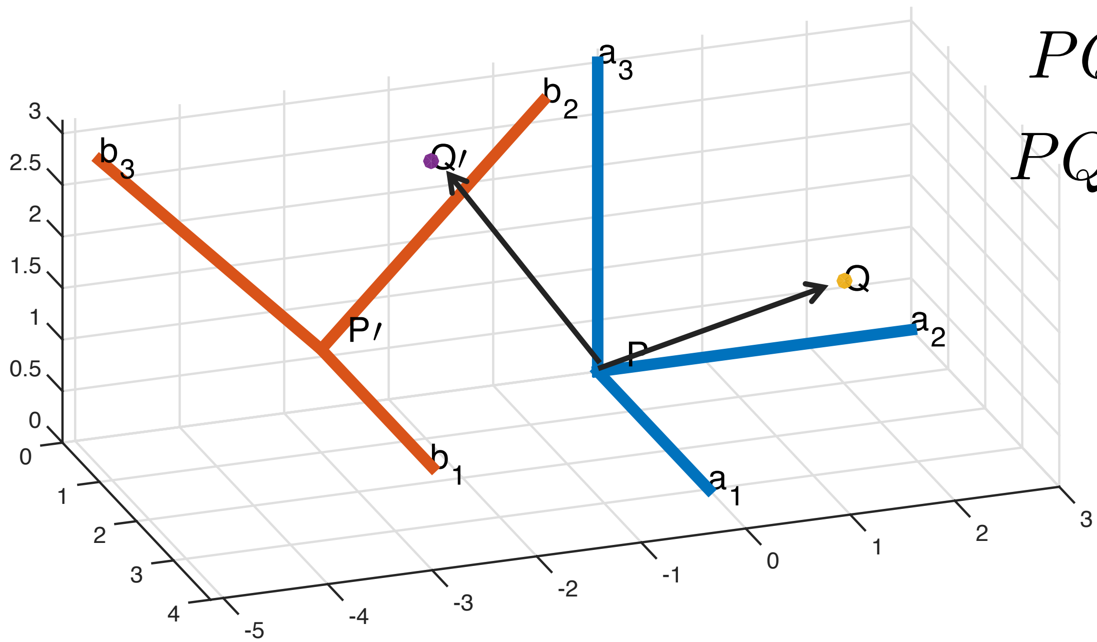


$$\mathbf{d} = 1\mathbf{a}_1 - 3\mathbf{a}_2 + 1\mathbf{a}_3$$

We can characterize a rigid-body displacement with a rotation matrix and translation vector.

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



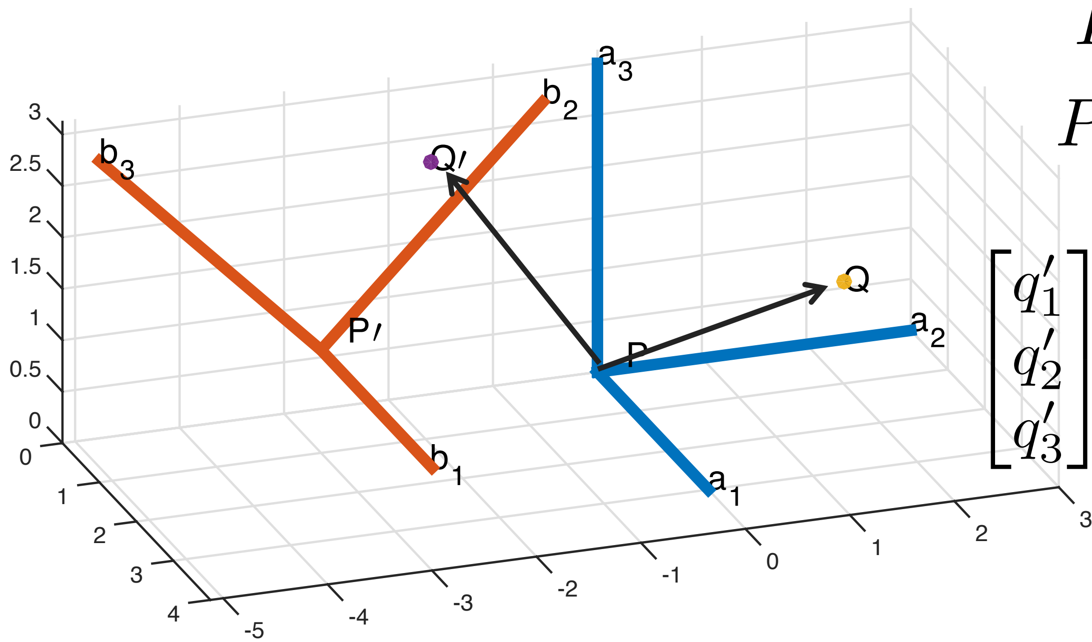
$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = R \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \mathbf{d}$$

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

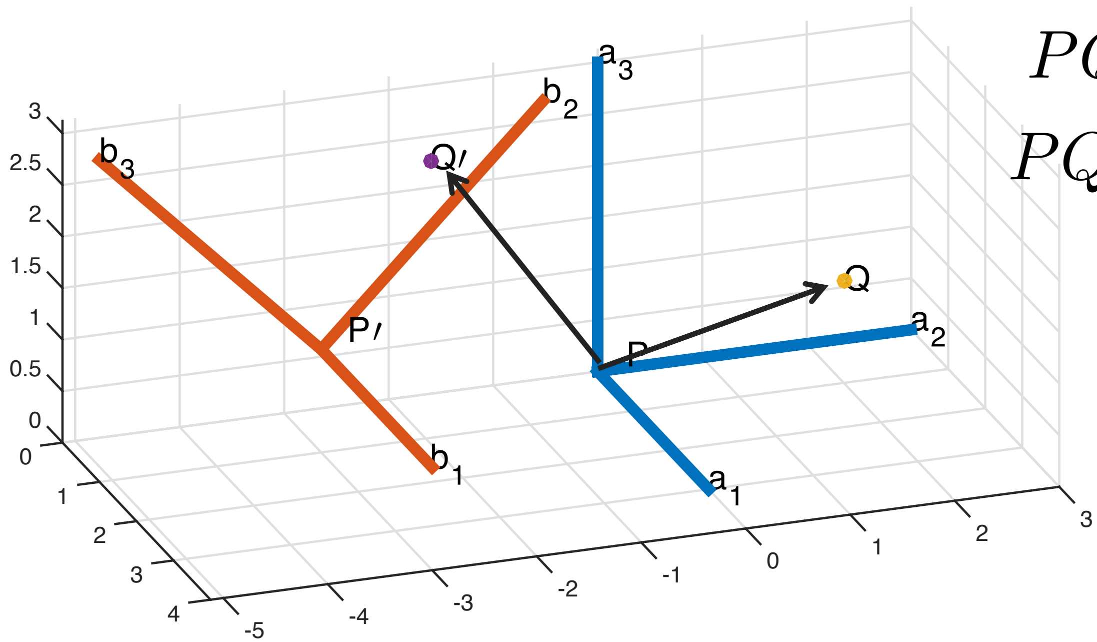
$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2.29 \\ 3.12 \end{bmatrix}$$

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = R^T \left(\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} - \mathbf{d} \right)$$