liebreadth

Breadth in finite dimensional algebras

0.2

13 August 2025

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Chapter 1

Preface

In this package we compute the breadth of Lie algebras. This package also include functions to compute covered Lie algebras of maximal class using a proces called inflation, this is described by Caranti, Mattarei, and Newman [CMN97].

1.1 Introduction

Given a Lie algebra L, we define its lower central series as $L = L^1 > L^2 > \ldots$, where $L^{i+1} = L^iL$. The algebra L is nilpotent if there exists $c \in$ such that $L^{c+1} = 0$, and the minimal such c is called the class $\mathrm{cl}(L)$ of L. For a nilpotent Lie algebra L, the type of L is the vector $(d_1, \ldots, d_c)$, where $d_i = \dim L^i$. A nilpotent Lie algebra L is said to be of maximal class if the type of L is $(2, 1, \ldots, 1)$. The centralizer of $x \in L$ is the subspace of L defined by $C_L(x) = \{a \in L \mid ax = 0\}$. For an algebra L, we define

$$\mathrm{br}(L) = \max\{\mathrm{br}(x) \mid x \in L\}, \quad \text{where} \quad \mathrm{br}(x) = \dim(L) - \dim(C_L(x)).$$

The class-breadth conjecture asserts that $\mathrm{cl}(L) \leq \mathrm{br}(L) + 1$ for an algebra L. This holds for nilpotent Lie algebras over infinite fields and for nilpotent associative algebras over arbitrary fields.

Let L be a Lie algebra over a field K, and let $B = \{b_1, \ldots, b_n\}$ be a basis of L. The multiplication in L is described by structure constants

$$b_i \cdot b_j = \sum_{k=1}^{n} c_{ijk} b_k \quad 1 \leq i, j \leq n.$$

For each k, write $C_k = (c_{ijk})_{1 \leq i,j \leq n}$ for the $n \times n$ matrix over K with entries c_{ijk}, these matrices are the structure matrices of L. For $x = x_1b_1 + \ldots + x_nb_n \in L$, let $\bar{x} = (x_1, \ldots, x_n) \in K^n$ denote the coefficient vector of x. Then $C_k\bar{x}^{tr}$ is a column vector over K. We write $M_B(x)$ for the $n \times n$ matrix over K whose k-th column is $C_k\bar{x}^{tr}$, this is the adjoint matrix of L with respect to the basis B.

Chapter 2

Lie algebras and breadth

2.1 Attributes

2.1.1 LieClass (for IsLieNilpotent)

▷ `LieClass(`*L*`)` (attribute)

Computes the class of the nilpotent Lie algebra *L*.

2.1.2 LieType (for IsLieNilpotent)

▷ `LieType(`*L*`)` (attribute)

Computes the type of the nilpotent Lie algebra *L*.

2.1.3 IsOfMaximalClass (for IsLieNilpotent)

▷ `IsOfMaximalClass(`*L*`)` (property)

Returns: `true` or `false`

Returns whether the nilpotent Lie algebra *L* is of maximal class or not.

2.1.4 StructureMatrices (for IsLieAlgebra)

▷ `StructureMatrices(`*L*`)` (attribute)

Computes the structure matrices of the nilpotent Lie algebra *L*.

2.1.5 BasisLieCenter (for IsLieAlgebra)

▷ `BasisLieCenter(`*L*`)` (attribute)

Returns the elements of the basis Lie algebra *L* that are contained in the center of *L*.

2.1.6 BasisLieDerived (for IsLieAlgebra)

▷ `BasisLieDerived(L)` (attribute)

Returns the elements of the basis Lie algebra *L* that are contained in the derived subalgebra of *L*.

2.1.7 LieAdjointMatrix (for IsLieAlgebra)

▷ `LieAdjointMatrix(L)` (attribute)

Computes the adjoint matrix of the Lie algebra *L*.

2.1.8 PrintLiePresentation (for IsLieAlgebra)

▷ `PrintLiePresentation(L)` (attribute)

Prints the Lie presentation of the Lie algebra *L*.

2.2 Breadth

2.2.1 InfoLieBreadth

▷ `InfoLieBreadth` (info class)

Info class for the functions of the breadth of Lie algebras.

2.2.2 LieBreadth (for IsLieNilpotent)

▷ `LieBreadth(L)` (attribute)

Computes the breadth of the Lie algebra *L*.

2.2.3 IsTrueClassBreadth (for IsLieNilpotent)

▷ `IsTrueClassBreadth(L)` (property)

Returns: `true` or `false`

Returns whether the Lie algebra *L* holds the class-breadth conjecture or not.

2.2.4 TGroupBreadth

▷ `TGroupBreadth(G)` (function)

Computes the breadth of the T-groupo *G*.

2.3 Inflation of Lie algebras

A grading for a Lie algebra L is a decomposition $\mathrm{L} = \bigoplus_{i=1}^{n} L_i$ that respects the Lie bracket, *i.e.* $[L_i, L_j] \subseteq L_{i+j}.\}$. Any nilpotent Lie algebras can be graded by taking $L_i = \gamma_i(L)/\gamma_{i+1}(L)$. Let L be a nilpotent Lie algebra of maximal class. The two-step centralizers are the sets $C_i = C_{L_1}(L_i) = \{x \in L_1 \mid [x, L_i] = 0\}$ for all $2 \leq i \leq c$. Let $\mathscr{C} = \{C_i\} \setminus L_1$, we say that a Lie algebra of maximal class is covered if the set $\mathscr{C}$ consist of all one-dimensional subspaces of L_1.

Let L be a graded Lie algebra $L = \bigoplus_{i=1}^{n} L_i$ over $K =_q$ for some prime power $q = p^n$. Consider the field extension $A = K[\varepsilon]/\langle \varepsilon^p \rangle$ which is a vector space over K of dimension p. The algebra A is an associative, commutative and has a unit. The algebra $L \otimes A$ over K defined by $[x \otimes a, y \otimes b] = [x,y] \otimes ab$ is a graded Lie algebra. Let M be a maximal ideal of L and consider the Lie subalgebra $M^{\uparrow} = M \otimes A$. Let D be a derivation of L of degree 1. Define $D^{\uparrow}$ via: $D^{\uparrow}(x \otimes \varepsilon^i) = D(x) \otimes \varepsilon^i$, this is a derivation of $M^{\uparrow}$ of degree 1. Define the derivation $E \in M^{\uparrow}$ by $E(x \otimes \varepsilon^i) = D^{\uparrow}(x \otimes \varepsilon^i \cdot \varepsilon^{p-1}) + 1 \otimes \partial_{\varepsilon}(\varepsilon^i)$. Let $s \in L_1 \setminus M$, take $D = \mathrm{ad}_s$, and extend it naturally to $M^{\uparrow}$. Denote $E_{s'}$ as previously. The *inflation* ${}^{M}L$ of L at M by $s \in L_1 \setminus M$ is the graded Lie algebra obtained as an extension of $M^{\uparrow}$ by an element s', which is the extension of s that induces the derivation $E_{s'}$, that is,

$$[x \otimes a, s'] = E_{s'}(x \otimes a) = \mathrm{ad}_{s'}(x \otimes \varepsilon^i) + x \otimes \partial_{\varepsilon}(\varepsilon^i).$$

2.3.1 InfoInflation

▷ `InfoInflation` (info class)

Info class for the functions of the inflation of Lie algebras.

2.3.2 LieNilpotentGrading (for IsLieNilpotent)

▷ `LieNilpotentGrading(L)` (attribute)

Computes a grading of the nilpotent Lie algebra L using the lower central series.

2.3.3 LieTwoStepCentralizers (for IsLieNilpotent)

▷ `LieTwoStepCentralizers(L)` (attribute)

Computes the two step centralizers of the Lie algebra L.

2.3.4 IsLieCovered (for IsLieNilpotent)

▷ `IsLieCovered(L)` (property)

Returns: `true` or `false`

Returns whether the Lie algebra L is covered or not.

2.3.5 PolynomialAlgebra (for IsField and IsFinite)

▷ `PolynomialAlgebra(F)` (attribute)

For a finite field F computes the polynomial algebra $F[x]$.

2.3.6 LieCoveredInflated (for IsInt)

▷ `LieCoveredInflated(n)` (attribute)

For $n = 2,3,4$ computes the covered Lie algebras of maximal class using the polynomial algebra of dimension n. For n=3 one can directly load the Lie algebra by reading the file "inflation_3.g".

2.3.7 LieMinimalQuotientClassBreadth (for IsLieAlgebra)

▷ `LieMinimalQuotientClassBreadth(L)` (attribute)

Given a covered Lie algebras of maximal class L computes the minimal quotient that not holds the class-breadth conjecture.

References

[CMN97] A. Caranti, S. Mattarei, and M. F. Newman. Graded lie algebras of maximal class. *Transactions of the American Mathematical Society*, 349:4021--4051, 1997. 3

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