Computing the breadths of algebras and $\mathcal{T}$-groups

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Joint work with B. Eick.

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Leedham-Green, Neumann, and Wiegold (1969) defined the breadth $\mathrm{br}(g)$ of an element g, for a finite p-group G, as the size of its conjugacy class;

$$p^{\mathrm{br}(g)} = |g^G| = [G : C_G(g)].$$

The breadth $\mathrm{br}(G)$ of G is

$$\mathrm{br}(G) = \max\{\mathrm{br}(g) \mid g \in G\}.$$

Theorem 1

Let G be a finite p-group, then it holds:

$$\mathrm{cl}(G) \leq \frac{p}{p-1}\,\mathrm{br}(G)$$

CLASS-BREADTH CONJECTURE

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- Felsch, Neubüser, and Plesken (1981) found a family of 2-groups that break the conjecture.
- The conjecture remains open for p-groups of odd order.

ALGEBRAS

Let L be a finite dimensional algebra over a field K. The breadth of $x \in L$ is the integer

$$\mathrm{br}(x) = \dim(L) - \dim(C_L(x)),$$

The breadth of L is then defined as,

$$\mathrm{br}(L) = \max\{\mathrm{br}(x) \mid x \in L\}.$$

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- Eick, Newman, and O'Brien (2006), there are nilpotent Lie algebras over $\mathbb{F}_q$ for all prime powers $q = p^e$ that serve as counterexamples of the class-breadth conjecture.

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- L be an algebra over a field K.
- $B = \{b_1, \ldots, b_n\}$ is a basis of L.
- $\mathcal{L}_x : L \to L$ defined by $\mathcal{L}_x(y) = y \cdot x$.
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Lemma 1

If L is a finite dimensional algebra with basis B and $x \in L$, then $\mathrm{br}(x) = \mathrm{rank}(M_B(x))$.

THE ALGORITHM

Denote by $X = X_1 b_1 + \ldots + X_n b_n$ a linear combination for indeterminates $X_1, \ldots, X_n$ over K, then $M_B(X)$ is a matrix whose entries are linear polynomials in $K[X_1, \ldots, X_n]$.

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Definition 1

A polynomial $f(X_1, \ldots, X_n)$ vanishes over K if $f(k_1, \ldots, k_n) = 0$ for all $k_1, \ldots, k_n \in K$.

Theorem 2

The set of polynomials in $K[X_1, \ldots, X_n]$ that vanish over K forms an ideal with Gröbner basis $\{X_i^q - X_i \mid 1 \leq i \leq n\}$ under the lexicographical order.

Theorem 3

Let L be a finite dimensional algebra over the field K with basis B. Then $\mathrm{br}(L) = m$ if and only if for each $l > m$ all $l \times l$ minors of $M_B(X)$ vanish over K and there is an $m \times m$ minor that does not vanish over K.

Theorem 4

Let L be a nilpotent algebra over a field K of class $c > 1$ and dimension n with basis B. Then $M_B(X)$ has a $(c-1) \times (c-1)$ minor $f(X_1, \ldots, X_n)$ which is non-zero and,

(a) *If K is infinite, then the class-breadth conjecture holds for L.*

(b) *If K is finite and there exists a $(c-1) \times (c-1)$ matrix minor f with $|K| > \mathrm{md}(f)$, then the class-breadth conjecture holds for L.*

THE ALGORITHM

Algorithm Breadth algorithm

Require: L a finite dimensional algebra with basis $B = \{b_1, \ldots, b_n\}$ over a field K.

if L is abelian **then**
    return 0.
else
    Let $X_1, \ldots, X_n$ be indeterminates over K and write $X = X_1 b_1 + \ldots + X_n b_n$.
    Determine the $n \times n$ matrix $M_B(X)$ with entries in $K[X_1, \ldots, X_n]$.
    for $i = \dim L$ **to** 1 **do**
        Let S denote the set of subsets of size i in $\{1, \ldots, n\}$.
        for $(s_1, s_2) \in S \times S$ **do**
            Let t be the minor of $M_L(X)$ with rows in s_1 and columns in s_2.
            if t does not vanish **then**
                return i.
            end if
        end for
    end for
end if

Results

Class	Breadths
1	0
2	1,2,3
3	2,3,4
4	3,4,5
5	4,5
6	5
7	5,6

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1	0
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5	4,5,6
6	5,6
7	5,6
8	7

Table. Classes and breadths of nilpotent Lie algebras over $\mathbb{F}_2$ of dimension 8 (left) and those of dimension 9 (right)

Polycyclic groups

For polycyclic groups the breadth is defined as:

$$b(G) = \max\{h(G) - h(C(g)) \mid g \in G\}.$$

Mann and Segal (2007), showed that the class breadth-conjecture holds for finitely generated torsion-free nilpotent groups ($\mathcal{T}$-groups).

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Theorem 5

Let G be a $\mathcal{T}$-group, let $\mathbb{Q}G$ be its $\mathbb{Q}$-powered hull, and let $\Lambda(G)$ be the nilpotent Lie algebra over $\mathbb{Q}$ associated with $\mathbb{Q}G$ via the Mal'tsev correspondence. It holds,

$$\mathrm{br}(G) = \mathrm{br}(\Lambda(G)).$$

Corollary 1

If G is a $\mathcal{T}$-group, then the class-breadth conjecture holds for G.

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