Signals and Systems

Learn tools necessary to analyse signals and systems.

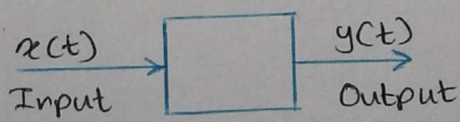
- A signal is a variation of a physical property over time.

	Signal	Physical Property
Direct outcome of a physical process	Speech	air pressure
	Current through a wire	current
	Light	Electromagnetic field
Information extracted from a physical process	Dialy rainfall over a time	
	Average daily temperature	
	Dialy light hours per day	

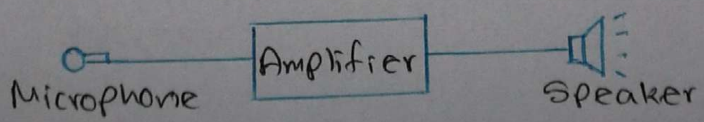
- For any signal, independent variable is time.
- Dependent variable can be number of physical parameters.

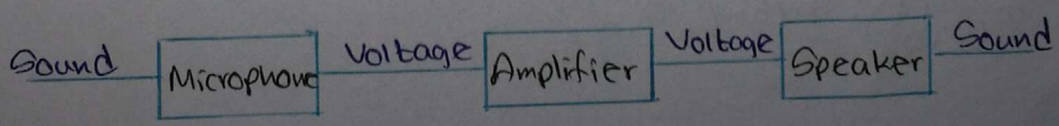
Ex:- Voltage, Current, Electric field, etc

- A system is an interconnection of components, devices, subsystems.



- Transducer converts one physical parameter to another.





- Signals and system interact to give a productive outcome.

* Classifications

01) Continuous time vs Descrete time signals

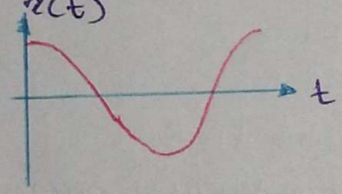
- A continuous time signal is defined for all instances of time.

  Example:- $A\sin\omega t$, $Ae^{j\omega t}$

- A discrete time signal is defined for certain intances of time.

  The time difference between those time instances need not be equal.

$x(t) = A\cos\omega t \quad ; t \geq 0$



$\omega = 2\pi f$

$f = \frac{1}{T}$

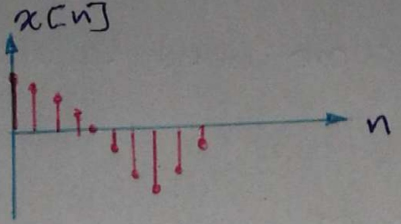
$\omega = \frac{2\pi}{T}$

$x(t) = A\cos\left(\frac{2\pi t}{T}\right)$

we measure $x(t)$ at every $T/10$

$t = 0, T/10, 2T/10, \ldots, 9T/10$

$$\left.\begin{aligned}
x(0) &= A\cos 0 &&= x[0] \\
x\left(\tfrac{T}{10}\right) &= A\cos\left(\tfrac{2\pi}{10}\right) &&= x[1] \\
x\left(\tfrac{2T}{10}\right) &= A\cos\left(\tfrac{2\pi.2}{10}\right) &&= x[2] \\
&\ \vdots && \\
x\left(\tfrac{9T}{10}\right) &= A\cos\left(\tfrac{2\pi.9}{10}\right) &&= x[9]
\end{aligned}\right\}\ \text{Discrete time signal}$$



By selecting values of a continuous time signal at discrete time instances (Sampling) a discrete time signal can be generated.

Time Series data comes as discrete time signals.

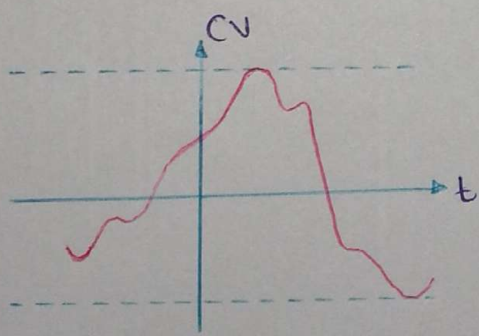
02) Continuous valued signals vs Discrete valued signals

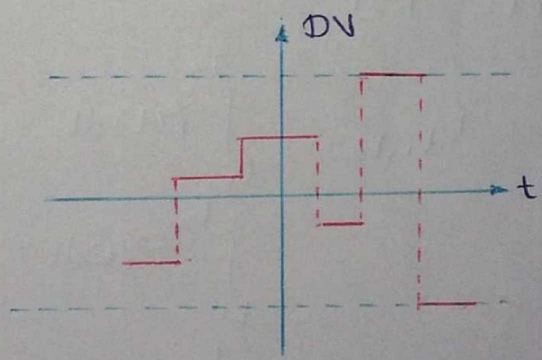
- A continuous valued signal can take any value within a given range

  Ex:- $x(t) = A\sin\omega t \in [-A, A]$

  $f(t) \in [a, b]$; $a, b \in (-\infty, \infty)$

- Discrete valued signal is a signal that takes values from a finite set.





03) Deterministic vs Random signal

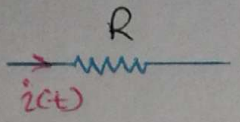
- Deterministic signals can be uniquely expressed by an explicit mathematical expression.

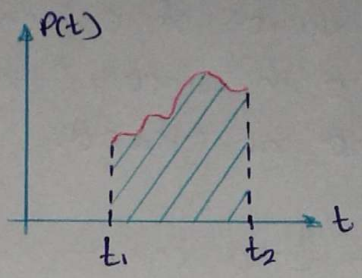
  Ex:- $x(t) = A\sin\omega t$

- Random signal is a signal that can not be expressed to any reasonable degree of accuracy using explicit equation

Probability and random variables are used to analyse random signals.

* Signal energy and signal power





$$p(t) = i^2(t) \cdot R$$

$$E_{t_1,t_2} = \int_{t_1}^{t_2} p(t)\,dt$$

$$P_{t_1,t_2\,\text{average}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)\,dt$$

* Total energy of a signal

let $R = 1$

$$E_{t_1,t_2} = \int_{t_1}^{t_2} p(t)\,dt$$

$$= \int_{t_1}^{t_2} i^2(t)\,dt$$

$$= \int_{t_1}^{t_2} |i(t)|^2\,dt$$

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |i(t)|^2\,dt = \int_{-\infty}^{\infty} |i(t)|^2\,dt$$

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2\,dt$$

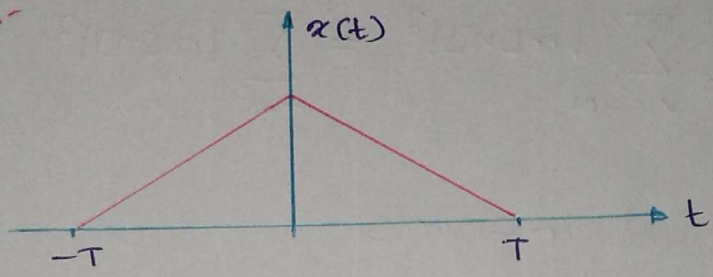
- A signal is an energy signal if it has finite energy

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \leq C \quad ; \quad C \text{ - finite constant}$$

for a complex signal

$$|x(t)|^2 = x(t) \cdot x^*(t)$$

Example:-



$$x(t) = \begin{cases} 0 & ; \ t < -T \\ \frac{t}{T} + 1 & ; \ -T \leq t < 0 \\ -\frac{t}{T} + 1 & ; \ 0 \leq t < T \\ 0 & ; \ t > T \end{cases}$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-T}^{T} |x(t)|^2 dt$$

$$= \int_{-T}^{0} \left(\frac{t}{T} + 1 \right)^2 dt + \int_{0}^{T} \left(-\frac{t}{T} + 1 \right)^2 dt$$

$$= \int_{-T}^{0} \left\{ \left(\frac{t}{T} \right)^2 + \frac{2t}{T} + 1 \right\} dt + \int_{0}^{T} \left\{ \left(\frac{t}{T} \right)^2 - \frac{2t}{T} + 1 \right\} dt$$

$$= \left[\frac{t^3}{3T^2} + \frac{2t^2}{2T} + t \right]_{-T}^{0} + \left[\frac{t^3}{3T^2} - \frac{2t^2}{2T} + t \right]_{0}^{T}$$

$$= \left\{ 0 - \left(-\frac{T}{3} + T - T \right) \right\} + \left(\frac{T}{3} - T + T \right)$$

$$= \frac{2T}{3}$$

Energy signal has finite support.

For a continuous time signals

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

For a discrete time signals

$$E \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{-\infty}^{\infty} |x[n]|^2$$

For a continuous time signals

$$P_{average} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

For a discrete time signals

$$P_{average} \triangleq \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x[n]|^2$$

- A signal with a finite power is called a power signal.
- An energy signal has zero averge power.

$x(t) = A\sin\omega t$

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \sin^2\omega t \, dt$$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$e^{-j\omega t} = \cos\omega t - j\sin\omega t$

$$e^{j\omega t} - e^{-j\omega t} = 2j \sin\omega t$$

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$x(t) = A\sin\omega t$

$$x(t) = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \qquad x^*(t) = \left[\frac{A}{2}\left(j e^{-j\omega t} - j e^{j\omega t}\right)\right]^*$$

$$x(t) = \frac{A}{2}\left(j e^{-j\omega t} - j e^{j\omega t}\right) \qquad x^*(t) = \frac{A}{2}\left(j e^{-j\omega t} - j e^{j\omega t}\right)$$

* Periodic Signals

- A signal is periodic if there exists T such that
$x(t) = x(t+T) \quad \forall t$
where T is the period of the signal

Example:-
$$x(t) = e^{j\omega_0 t}$$

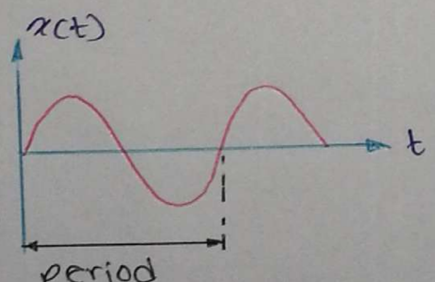
Find T s.t. $x(t) = x(t+T) \quad \forall t$

$$x(t) = x(t+T)$$
$$e^{j\omega_0 t} = e^{j\omega_0(t+T)}$$
$$e^{j\omega_0 t} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$
$$1 = e^{j\omega_0 T}$$
$$1 = \cos\omega_0 T + j\sin\omega_0 T$$

Equating Real Part
$$1 = \cos\omega_0 T$$
$$\cos 0 = \cos\omega_0 T$$
$$2n\pi + 0 = \omega_0 T$$
$$\frac{2n\pi}{\omega_0} = T \quad ; n \in \mathbb{Z}$$

$$\therefore \quad T = \frac{2\pi}{\omega_0}$$

- Periodic signals are power signals



$$P_{avg} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_{avg} = \frac{1}{\text{period}} \int_{\text{over period}} |x(t)|^2 dt$$

Example :-

$x(t) = e^{j\omega_0 t}$

$$E_{period} = \int_{period} |e^{j\omega_0 t}|^2 dt$$

$$= \int_{period} e^{j\omega_0 t} \cdot e^{-j\omega_0 t} dt$$

$$= \int_{period} 1 \, dt$$

$$= [t]_0^T$$

$$= T$$

$$P_{avg} = \frac{E_{period}}{period}$$

$$= \frac{T}{T}$$

$$= 1$$

* Complex exponential signals

* Continuous time signals

$x(t) = Ce^{at}$ where C and a are complex numbers

$C = C_R + jC_I \quad , \quad a = a_R + ja_I$

$$x(t) = (C_R + jC_I) e^{(a_R + ja_I)t}$$

$$x(t) = C_R e^{(a_R + ja_I)t} + jC_I e^{(a_R + ja_I)t}$$

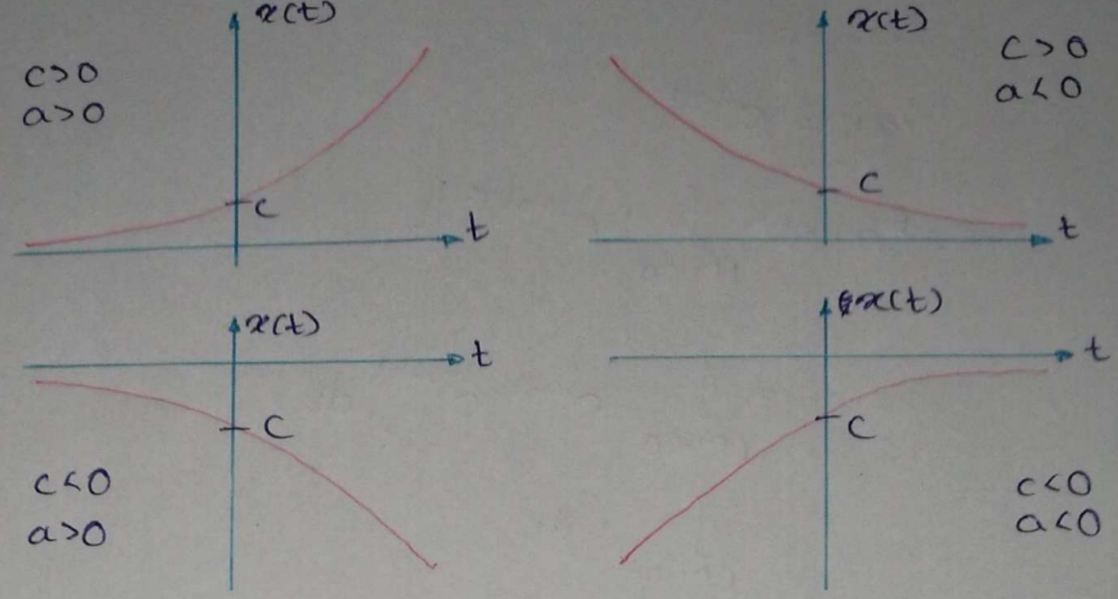
If c and a are real sketch $x(t)$ for

(i) $c > 0 \, , \, a > 0$

(ii) $c > 0 \, , \, a < 0$

(iii) $c < 0 \, , \, a > 0$

(iv) $c < 0 \, , \, a < 0$



* Discrete time Signals

$x[n] = C\alpha^{n}$; C and α are complex numbers

$x[n] = Ce^{\beta n}$; $\alpha = e^{\beta}$

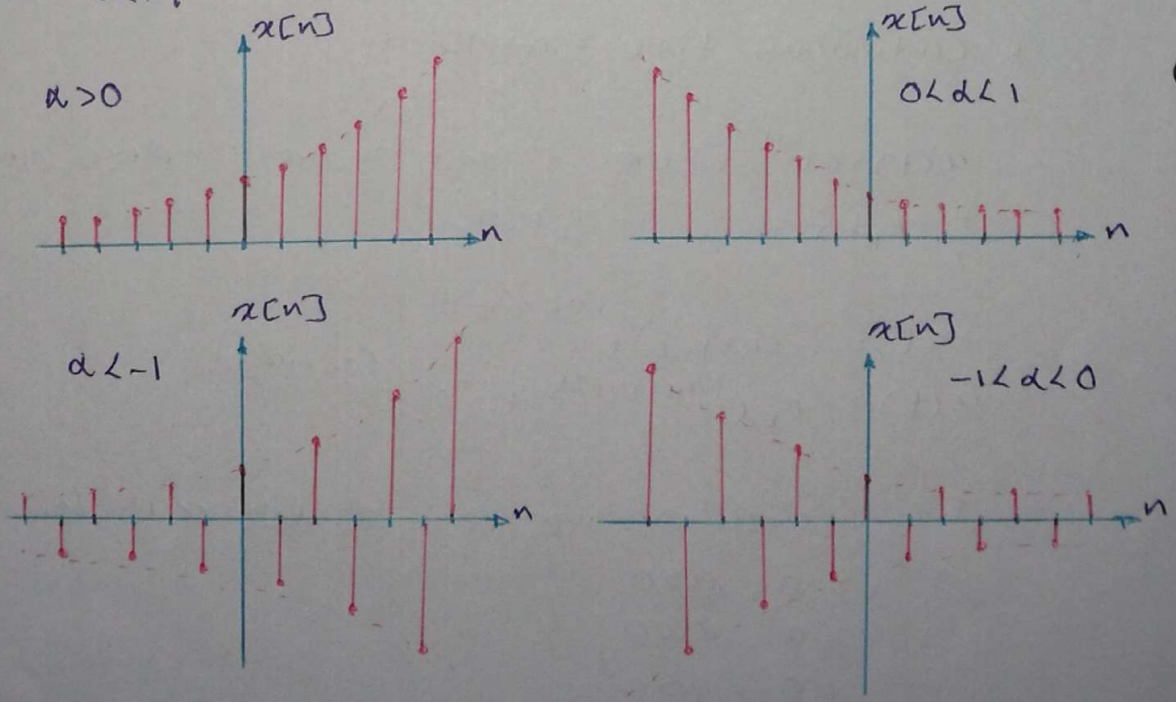
If c and α are real numbers sketch $x[n]$ for

(I) $\alpha > 1$ let $C>0$

(II) $0 < \alpha < 1$

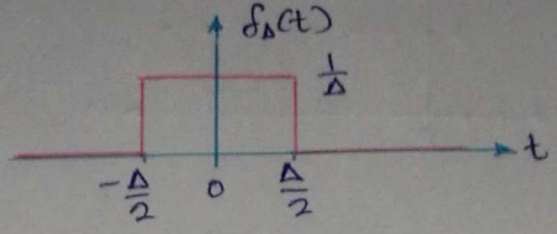
(III) $-1 < \alpha < 0$

(IV) $\alpha < -1$



* Impulse

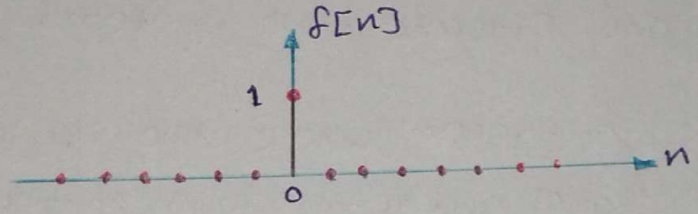
* Continuous time impulse (Dirac Delta)



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

Area under $\delta(t) = 1$

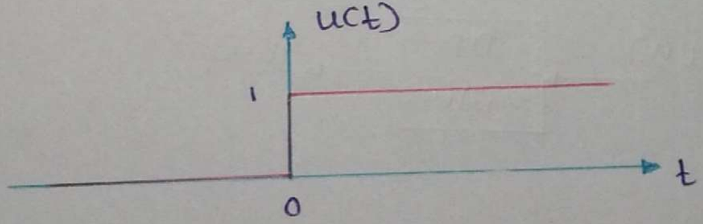
* Discrete time impulse (Kronecker delta)



$$\delta[n] = \begin{cases} 1 & ; n=0 \\ 0 & ; \text{elsewhere} \end{cases}$$

* Unit Step

* Continuous time unit step

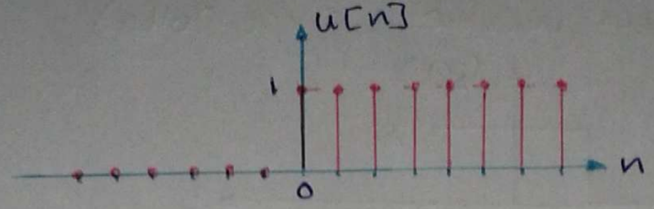


$$u(t) = \begin{cases} 1 & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$u(t)$ is a discontinuous function

* Discrete time unit step

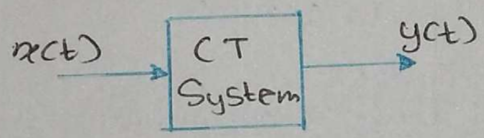


$$u[n] = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

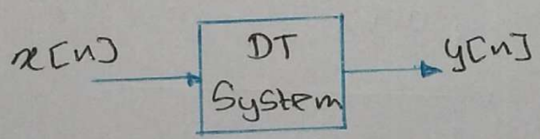
$$u[n] = \sum_{N=-\infty}^{n} \delta[N]$$

* Continuous time and Discrete time systems

- A CT System is a system where the i/p is a continuous time signal and results in a continuous o/p.

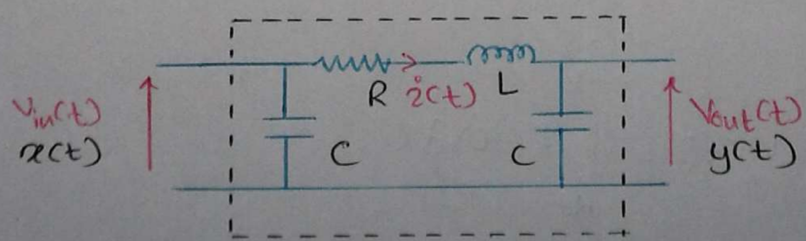


- A DT System is a system where i/p is DT signal and results in a DT o/p.



Example :-

- A CT system behaviour can be represented by a differential equation.



$$v_{in}(t) = R \cdot i(t) + L \frac{di(t)}{dt} + V_{out}(t)$$

$$V_{out}(t) = \frac{1}{C} \int_{-\infty}^{t} i(t)\,dt$$

$$i(t) = C \cdot \frac{d V_{out}(t)}{dt}$$

$$V_{in}(t) = RC \frac{d V_{out}(t)}{dt} + LC \frac{d^2 V_{out}(t)}{dt^2} + V_{out}(t)$$

$$x(t) = RC \frac{d y(t)}{dt} + LC \frac{d^2 y(t)}{dt^2} + y(t)$$

Example:-

- A DT System behaviour can be represented by a difference equation

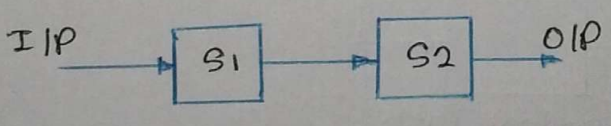
Bank balance $= y[n]$
Net deposit $= x[n]$ = Deposit − Withdrawl

$$y[n] = y[n-1] + \alpha\, y[n-1] + x[n]$$

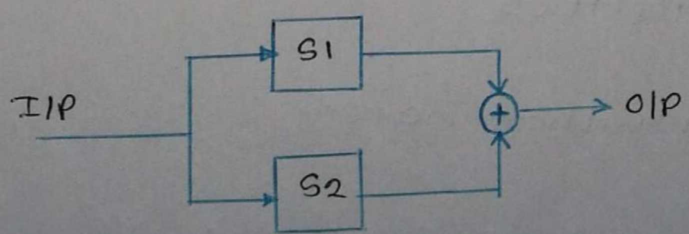
$\alpha = f$ (interest rate)

* Interconnection between Systems

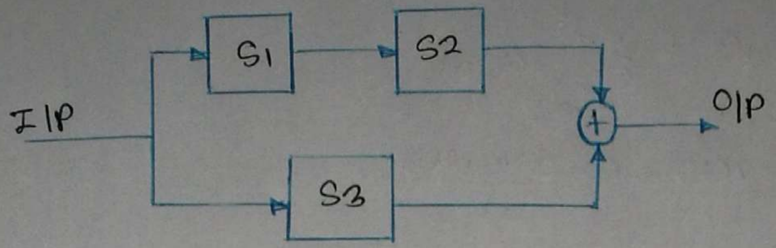
* Cascade (Series) interconnection

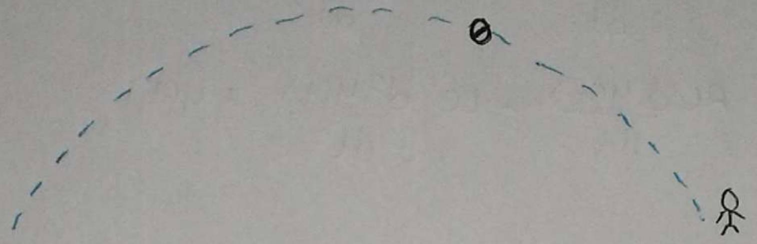


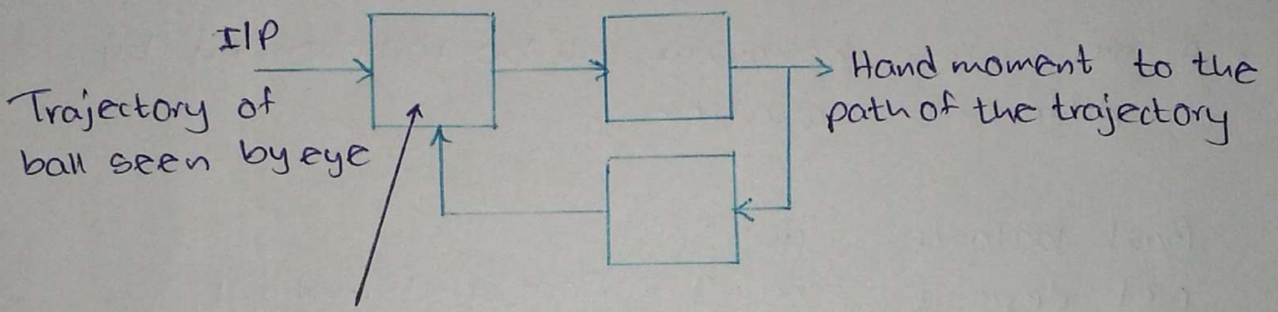
* Parallel interconnection



* Series /parallel interconnection

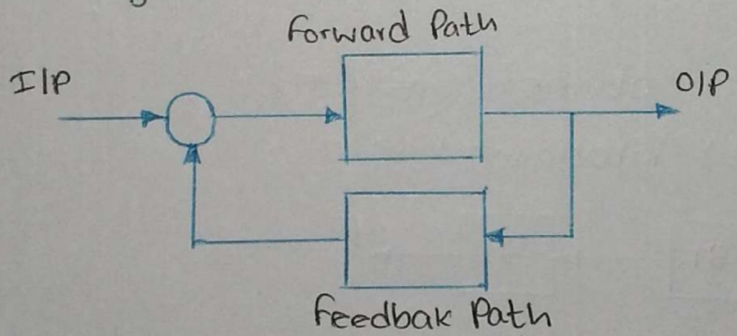






- compare the updated trajectory and hand location
- Adjest the hand according to that

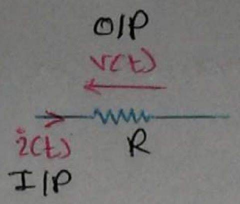
* Feedback Systems

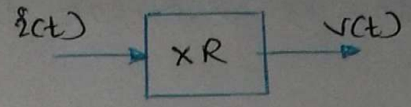


* Basic System Properties

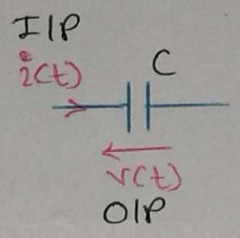
01) System with or without memory

- A system is without memory if the O/P at a certain time instant only depends on the i/p at the same time instant



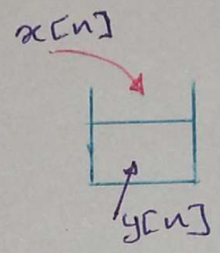


- In a system with memory O/P at the certain time instant depends on the input at the same time instant and previous time instances



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

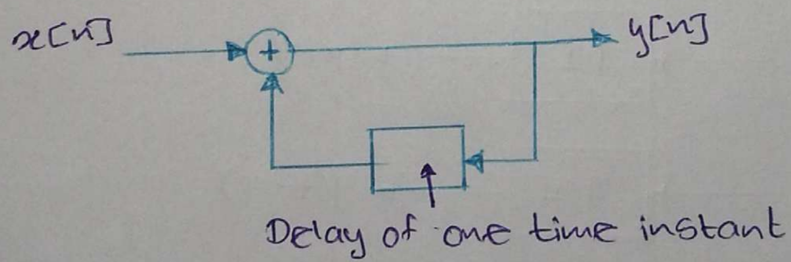
Accumilator



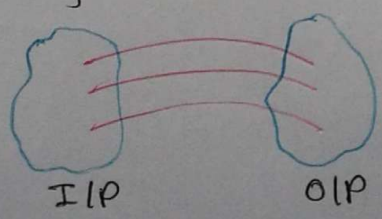
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

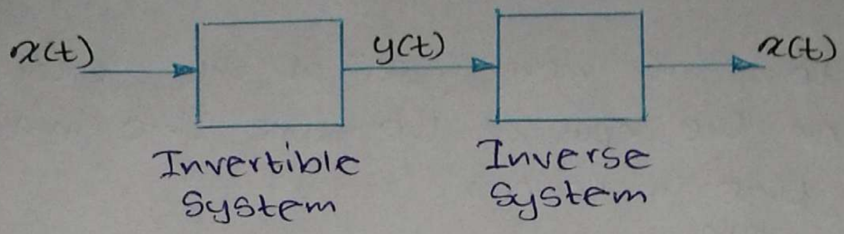


02) Invertibility and inverse system

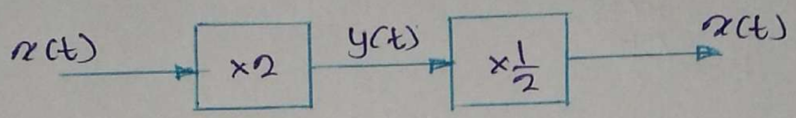


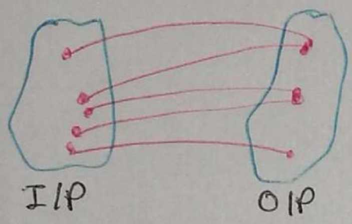
A system is invertible if there is one-to-one correspondance between i/p and o/p

If a system is invertible there exists an inverse system.



Example :- $y(t) = 2x(t)$

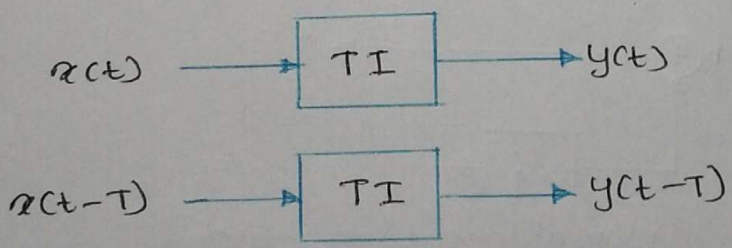




Non-Invertible system

In non-invertible system there is no one-to-one mapping between the i/p and o/p

03) Time Invariance



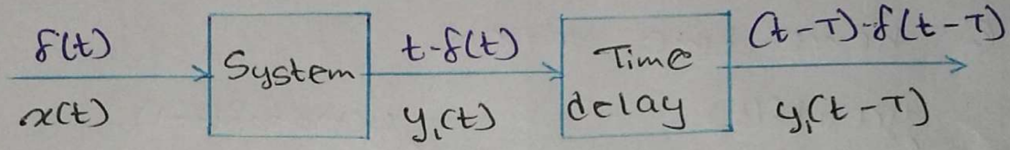
System behaviour does not depend on the time.

1) $x(t) = t\,x(t)$ where $x(t) = \delta(t)$

$$x(t) = \delta(t)$$

$$x(t-T) = \delta(t-T)$$

Configuration (1)

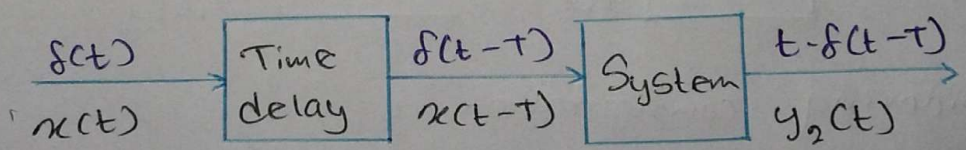


considering configuration 1

$$y_1(t) = t\cdot\delta(t)$$

$$y_1(t-T) = (t-T)\cdot\delta(t-T) \text{ ——— (A)}$$

Configuration (2)



considering configuration

$$y_2(t) = t\cdot\delta(t-T) \text{ ——— (B)}$$

Hence, result Ⓐ ≠ Ⓑ violates the condition to be time invariant

So that,

$y(t) = t \cdot x(t)$ time variant.

- $y(t) = t \cdot x(t)$

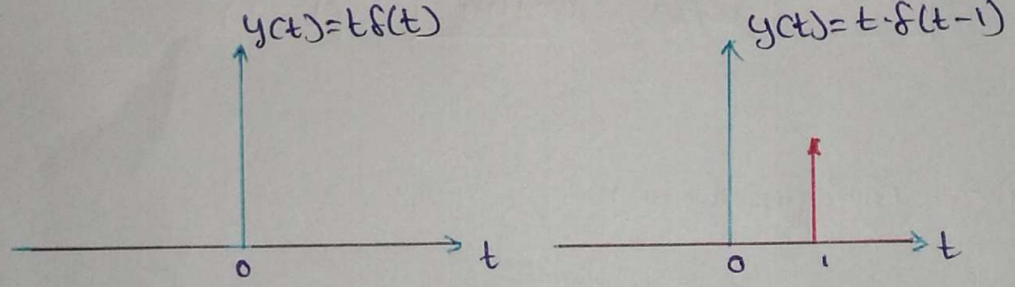
when $x(t) = \delta(t)$

$$y(t) = t \cdot \delta(t) \quad -①$$

when $x(t) = \delta(t-1)$

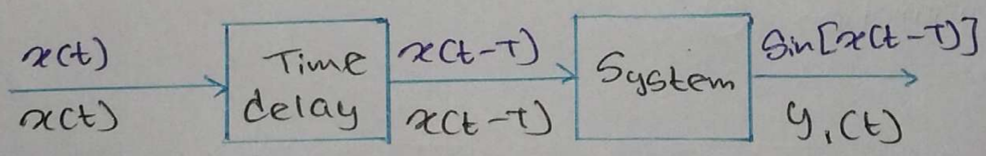
$$y(t) = t \cdot \delta(t-1) \quad -②$$

① ≠ ② hence time variant



ii) $y(t) = \sin[x(t)]$

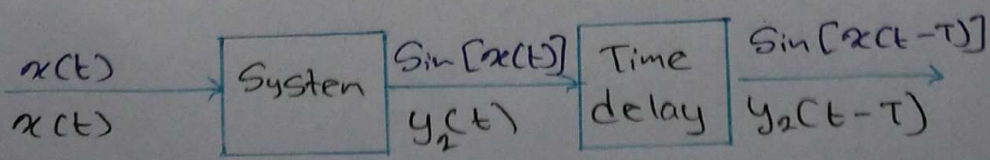
Configuration ①



considering configuration 1

$$y_1(t) = \sin[x(t-T)] \quad — Ⓐ$$

configuration ②



Considering configuration 2

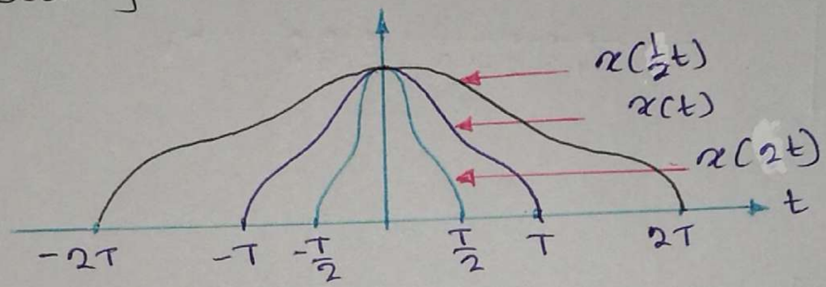
$$y_2(t) = \sin[x(t)]$$

$$y_2(t-T) = \sin[x(t-T)] \quad \text{—— (B)}$$

hence (A) = (B)

$y(t) = \sin(x(t))$ is time invariant

* Scaling in time



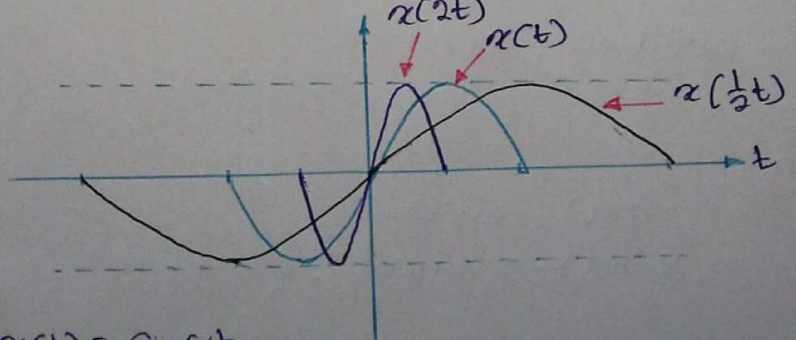
$x(t) = 0 \quad t > T$
$x(t) = 0 \quad t < -T$

$x(T) = 0$
$x(2t_1) = 0$
$2t_1 = T$
$t_1 = \frac{T}{2}$

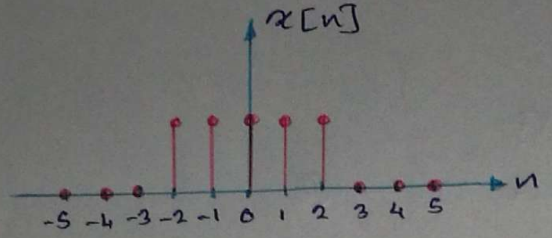
$x(-T) = 0$
$x(2t_2) = 0$
$2t_2 = -T$
$t_2 = -\frac{T}{2}$

$x(T) = 0$
$x(\frac{1}{2}t_3) = 0$
$x(\frac{1}{2}t_3) = x(T) = 0$
$\frac{1}{2}t_3 = T$
$t_3 = 2T$

$x(-T) = 0$
$x(\frac{1}{2}t_4) = 0$
$x(\frac{1}{2}t_4) = x(-T) = 0$
$\frac{1}{2}t_4 = -T$
$t_4 = -2T$

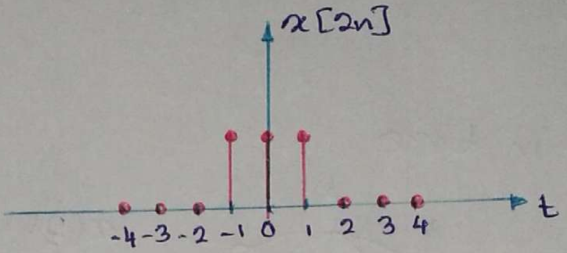


$x(t) = \sin\omega t$
$x(2t) = \sin(2\omega t)$
$x(\frac{1}{2}t) = \sin(\frac{1}{2}\omega t)$



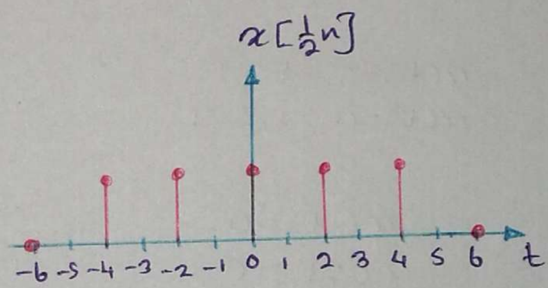
$x[2n]$

n	$x[2n]$
0	$x[0]$
1	$x[2]$
2	$x[4]$
⋮	⋮



$x[\frac{1}{2}n]$

n	$x[\frac{1}{2}n]$
0	$x[0]$
1	$x[\frac{1}{2}]$ ← not defined
2	$x[1]$
⋮	⋮



04) Linearity

* Additivity

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$

$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$$

additive system

* Scaling

$$x(t) \longrightarrow y(t)$$
$$\alpha \cdot x(t) \longrightarrow \alpha \cdot y(t) \quad ; \alpha - \text{Scalar}$$

- In a linear System

$$x_1(t) \rightarrow y_1(t)$$
$$x_2(t) \rightarrow y_2(t)$$

$$\alpha_1 \cdot x_1(t) + \alpha_2 \cdot x_2(t) \rightarrow \alpha_1 \cdot y_1(t) + \alpha_2 \cdot y_2(t)$$

-Example:-

$$y(t) = ax(t) + b$$

$$x_1(t) \rightarrow a \cdot x_1(t) + b$$
$$x_2(t) \rightarrow a \cdot x_2(t) + b$$
$$x_3(t) = x_1(t) + x_2(t) \rightarrow a(x_1(t) + x_2(t)) + b = y_3(t)$$

Condition for linearity: $y_1(t) + y_2(t) = y_3(t)$

$$y_1(t) + y_2(t) = a(x_1(t) + x_2(t)) + 2b \neq y_3(t)$$

$\therefore$ System is not linear

Example:-

$$y[n] = n \cdot x[n]$$

$$x_1[n] \rightarrow n \cdot x_1[n] = y_1[n]$$
$$x_2[n] \rightarrow n \cdot x_2[n] = y_2[n]$$
$$x_3[n] = \alpha \cdot x_1[n] + \beta x_2[n] \rightarrow n\{\alpha x_1[n] + \beta \cdot x_2[n]\} = y_3[n]$$

$$\alpha \cdot y_1[n] + \beta y_2[n] = y_3[n]$$

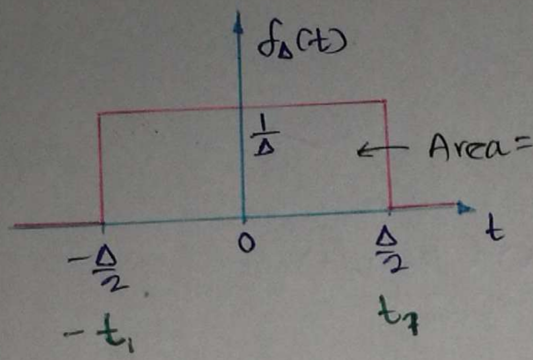
$\therefore$ Linear time varying system

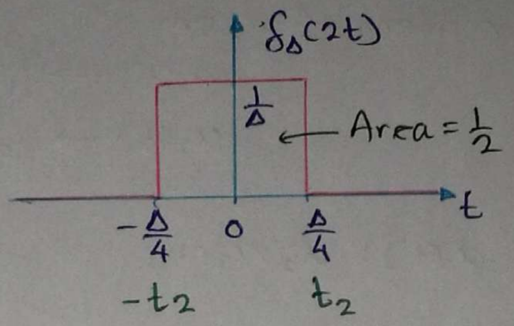
Example:- $y[n] = x^2[n]$

$$x_1[n] \rightarrow x_1^2[n] = y_1[n]$$
$$x_2[n] \rightarrow x_2^2[n] = y_2[n]$$
$$x_3[n] = x_1[n] + x_2[n] \rightarrow (x_1[n] + x_2[n])^2 = y_3[n]$$

$$y_3[n] \neq y_1[n] + y_2[n]$$

$\therefore$ System is not linear





$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$\delta(2t) = \lim_{\Delta \to 0} \delta_\Delta(2t)$$

$$\delta_\Delta(t_1) = \frac{1}{\Delta}$$

$$\delta_\Delta(t_1) = \delta_\Delta\left(\frac{\Delta}{2}\right)$$

$$t_1 = \frac{\Delta}{2}$$

$$\delta_\Delta(2t_2) = \frac{1}{\Delta}$$

$$\delta_\Delta(2t_2) = \delta_\Delta\left(\frac{\Delta}{2}\right)$$

$$2t_2 = \frac{\Delta}{2}$$

$$t_2 = \frac{\Delta}{4}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\infty}^{\infty} \left\{ \lim_{\Delta \to 0} \delta_\Delta(t) \right\} dt$$

$$= \lim_{\Delta \to 0} \left\{ \int_{-\infty}^{\infty} \delta_\Delta(t)\,dt \right\}$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \times \left(\frac{\Delta}{2} - \left(-\frac{\Delta}{2}\right)\right)\right\}$$

$$= \lim_{\Delta \to 0} 1$$

$$= 1$$

$$\int_{-\infty}^{\infty} \delta(2t)\,dt = \int_{-\infty}^{\infty} \left\{ \lim_{\Delta \to 0} \delta_\Delta(2t) \right\} dt$$

$$= \lim_{\Delta \to 0} \left\{ \int_{-\infty}^{\infty} \delta_\Delta(2t)\,dt \right\}$$

$$= \lim_{\Delta \to 0} \left\{ \frac{1}{\Delta} \times \left[\frac{\Delta}{4} - \left(-\frac{\Delta}{4}\right)\right]\right\}$$

$$= \lim_{\Delta \to 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 2\times\int_{-\infty}^{\infty} \delta(2t)dt$$

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\infty}^{\infty} 2\delta(2t)dt$$

$$\delta(t) = 2\delta(2t)$$

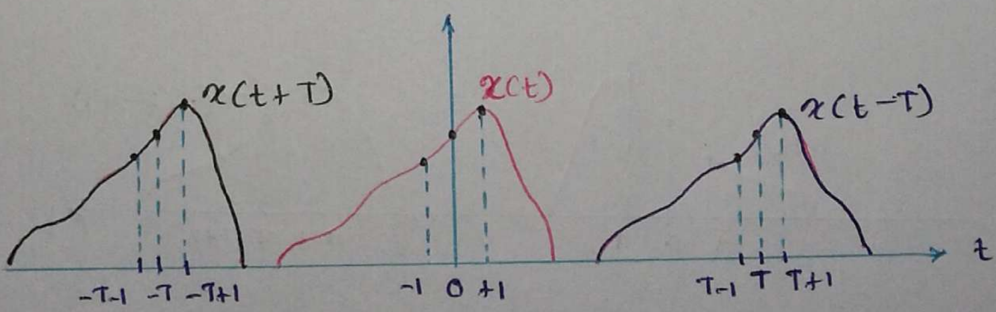
* Causility

In a causal system o/p at any time instant depends on the I/P at the same time instant and previous time instances.

Ex:- $y[n] = x[n] + 2x[n-1]$ ← Causal System

* $x'(t) = x(t-T)$

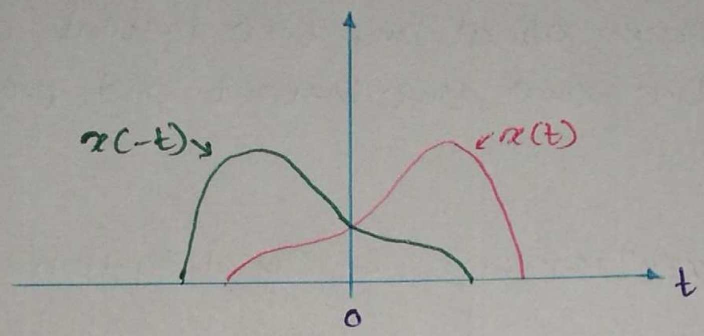
t	$x(t)$	$x'(t)$
$T-1$	$x(T-1)$	$x(-1)$
T	$x(T)$	$x(0)$
$T+1$	$x(T+1)$	$x(1)$



$x(t)$ is shifted ~~b~~ in time by T

- $x'(t) = x(-t)$

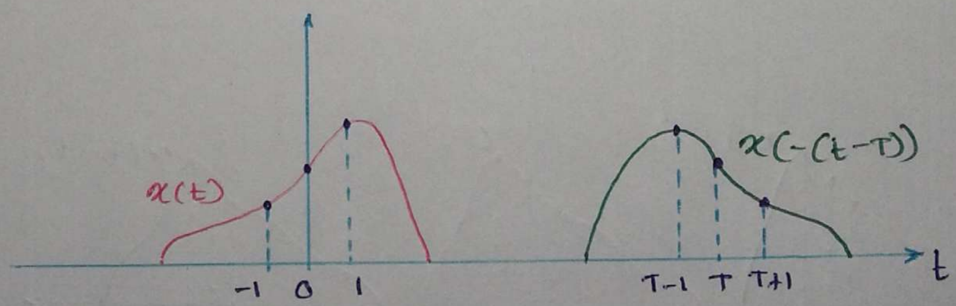
t	$x(t)$	$x'(t)$
-2	$x(-2)$	$x(2)$
-1	$x(-1)$	$x(1)$
0	$x(0)$	$x(0)$
1	$x(1)$	$x(-1)$
2	$x(2)$	$x(-2)$



$x(-t)$ is the mirror image of $x(t)$ around $t=0$

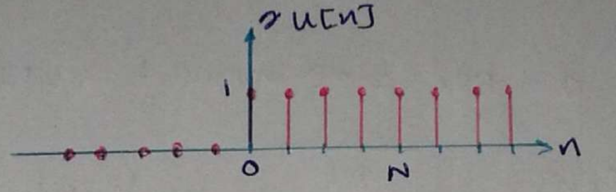
$x'(t) = x(-(t-T))$

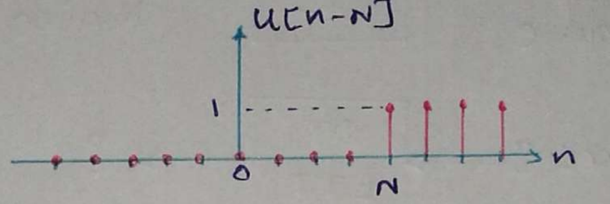
t	$x(t)$	$x'(t)$
$T-1$	$x(T-1)$	$x(1)$
T	$x(T)$	$x(0)$
$T+1$	$x(T+1)$	$x(-1)$

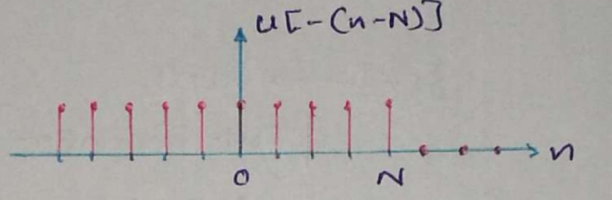


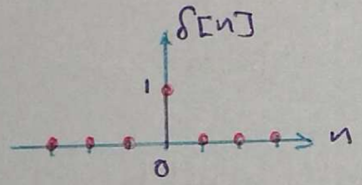
$x'(t) = x(-(t-T))$ is $x(t)$ shifted in time by T and mirror imaged about $t=T$.

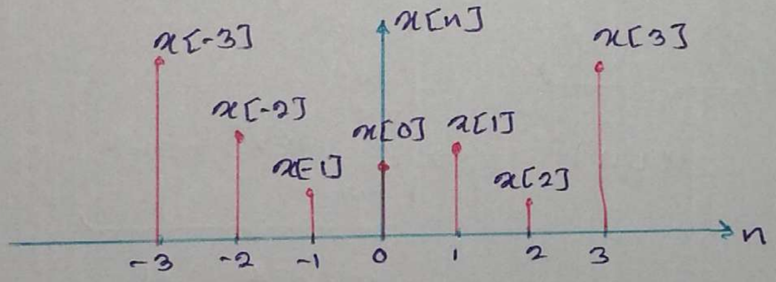
• $x[n] = u[-(n-N)]$











n	$x[n]$
1	$x[1] = x[1]\cdot\delta[n-1]$
0	$x[0] = x[0]\cdot\delta[n]$
-1	$x[-1] = x[-1]\cdot\delta[n+1]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]$$

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* Linear Time Invariant Systems

Let's consider a linear time invariant (LTI) system

$$\underset{\uparrow \atop \text{impulse}}{\delta[n]} \longrightarrow \boxed{\begin{matrix}\text{DT}\\ \text{LTI}\end{matrix}} \longrightarrow \underset{\uparrow \atop \text{impulse response}}{h[n]}$$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{DT}\\ \text{LTI}\end{matrix}} \longrightarrow ?$$

$$\delta[n] \longrightarrow h[n]$$

$$\Big\downarrow \text{ Time invariance}$$

$$\delta[n-k] \longrightarrow h[n-k]$$

$$\Big\downarrow \text{ Scaling } (\because \text{Linear})$$

$$x[k]\cdot\delta[n-k] \longrightarrow x[k]\cdot h[n-k]$$

$$\Big\downarrow \text{ Additivity } (\because \text{linear})$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\cdot\delta[n-k]}_{x[n]} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

$$x[n] \longrightarrow \underbrace{\underbrace{\sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]}_{\substack{x[n] * h[n] \\ \text{Convolution}}}}_{y[n]}$$

$$y[n] = x[n] * h[n]$$

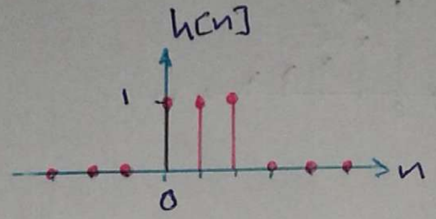
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

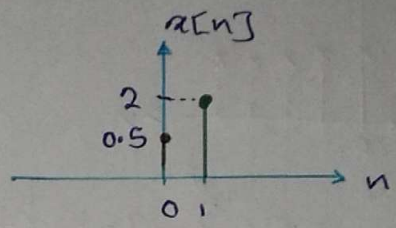
Example :-

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

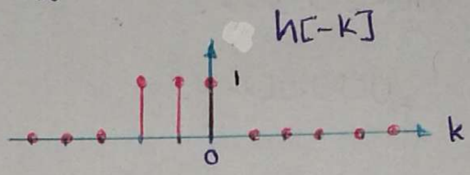
$$h[n] = \{\ldots, 0, 1, 1, 1, 0, \ldots\}$$

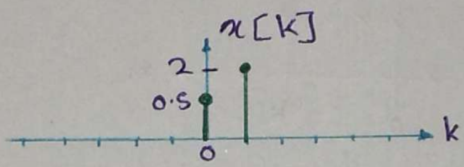
$$x[n] = \{0.5, 2\}$$

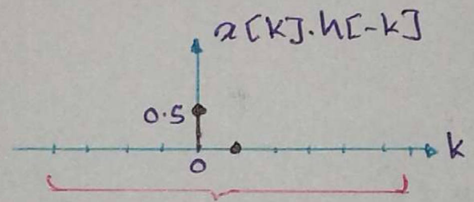




$$y[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[-k]$$







$0.5 + 0$

$y[0] = 0.5$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k] = 0.5 + 2 = 2.5$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[2-k] = 0.5 + 2 = 2.5$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[3-k] = 0 + 2 = 2$$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[4-k] = 0 + 0 = 0$$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[-1-k] = 0 + 0 = 0$$

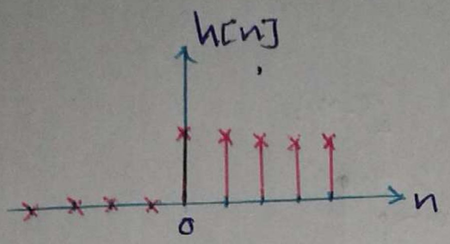
$h[1-k] = h[-(k-1)]$ $\quad$ $\overrightarrow{\text{Shift}}$ $h[k]$ by 1 and mirror about $k=1$

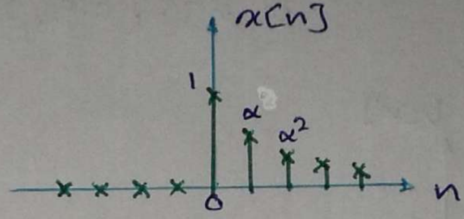
$h[-1-k] = h[-(k+1)]$ $\quad$ $\overleftarrow{\text{Shift}}$ $h[k]$ by 1 and mirror about $k=-1$

Example :-

$$h[n] = u[n]$$

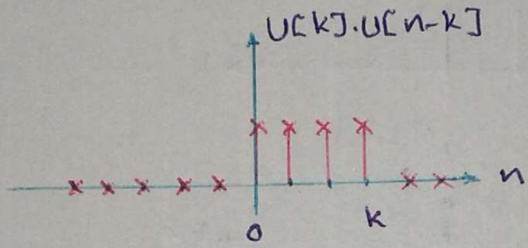
$$x[n] = \alpha^2 \cdot u[n] \quad ; \; 0 < \alpha < 1$$





$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k \cdot u[k] \cdot u[n-k]$$



$$y[0] = 1$$

$$y[1] = 1 + \alpha$$

$$y[2] = 1 + \alpha + \alpha^2$$

$$y[3] = 1 + \alpha + \alpha^2 + \alpha^3$$

$$\vdots \qquad \vdots$$

$$y[n] = 1 + \alpha + \dots + \alpha^n$$

$$y[n] = \sum_{k=0}^{n} \alpha^k \quad —① $$

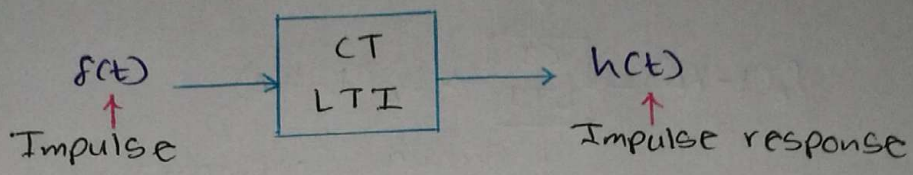
$$\alpha y[n] = \sum_{k=0}^{n} \alpha^{k+1} \quad —② $$

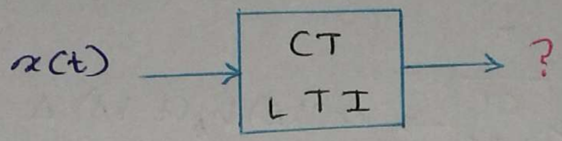
① − ② ⇒

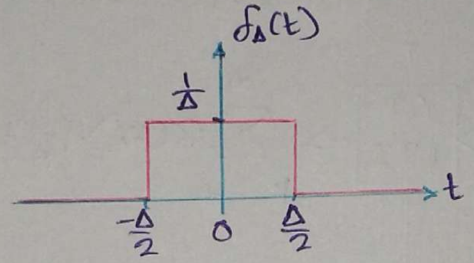
$$(1-\alpha) y[n] = 1 - \alpha^{n+1}$$

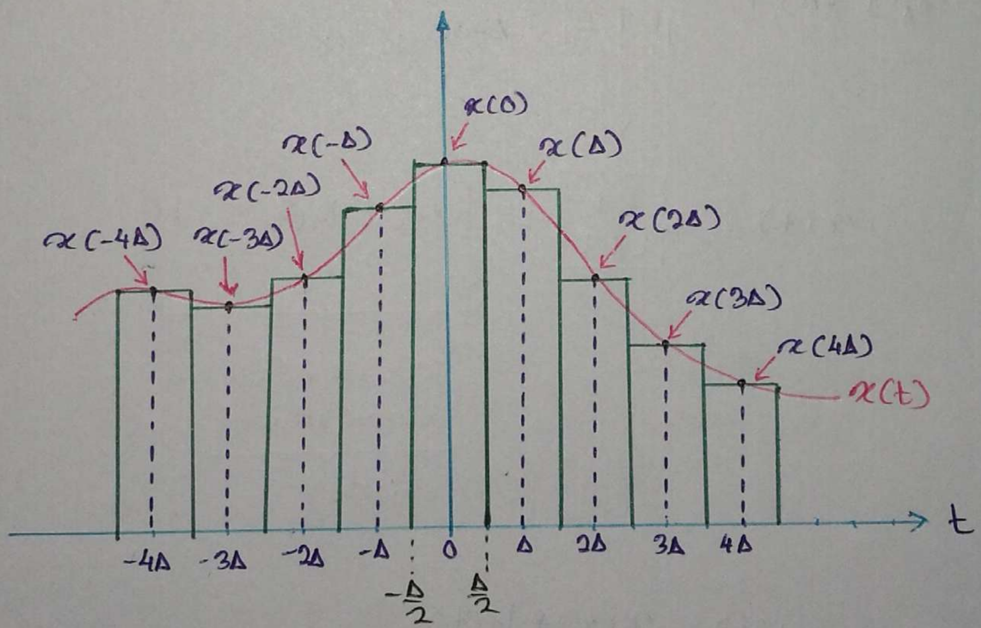
$$y[n] = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

Let's consider a linear time invariant (LTI) system









$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\cdot\Delta)\cdot\delta_{\Delta}(t-k\cdot\Delta)\cdot\Delta$$

$$x(t) = \lim_{\Delta\to 0}\hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau$$

$$x(t) = x(t) * \delta(t)$$

$$\delta_\Delta(t) \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow h_\Delta(t)$$

$\downarrow$ Time invariant

$$\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow h_\Delta(t-k\cdot\Delta)$$

$\downarrow$ Scaling ($\because$ Linear)

$$x(k\cdot\Delta)\cdot\delta_\Delta(t-k\Delta)\,\Delta \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow x(k\Delta)\cdot h_\Delta(t-k\Delta)\cdot\Delta$$

$\downarrow$ Additivity ($\because$ Linear)

$$\sum_{k=-\infty}^{\infty} x(k\Delta)\cdot\delta_\Delta(t-k\Delta)\cdot\Delta \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)\cdot h_\Delta(t-k\Delta)\cdot\Delta$$

$\downarrow$

$$\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\cdot\delta_\Delta(t-k\Delta)\cdot\Delta \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\cdot h_\Delta(t-k\Delta)\cdot\Delta$$

$\downarrow$

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{CT}\\ \text{LTI}\end{array}} \longrightarrow \underbrace{\underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}_{x(t)*h(t)\ \text{convolution}}}_{y(t)}$$

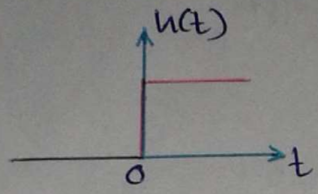
$$y(t) = x(t)*h(t)$$

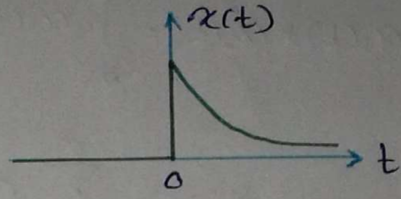
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Example :-

$$h(t) = u(t)$$

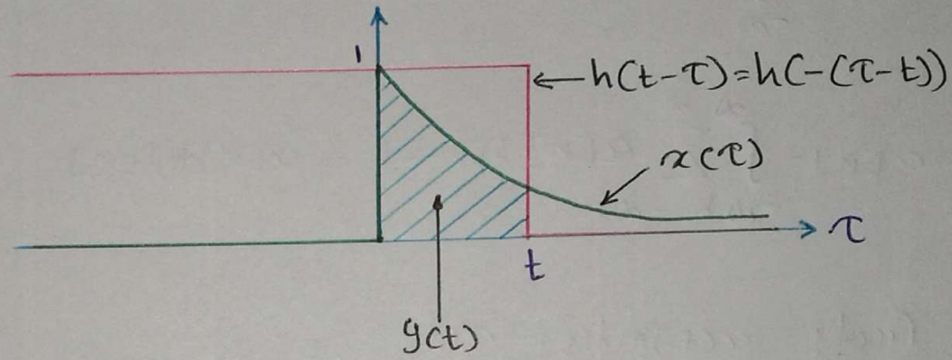
$$x(t) = e^{-at}\, u(t)$$





$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$



for $t < 0$, $y(t) = 0$

for $t \geq 0$,

$$y(t) = \int_{0}^{t} x(\tau) h(t-\tau)\, d\tau$$

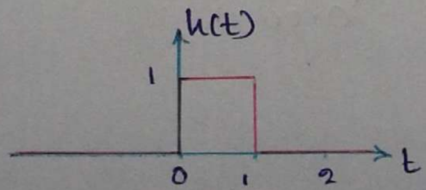
$$= \int_{0}^{t} e^{-a\tau}\, d\tau$$

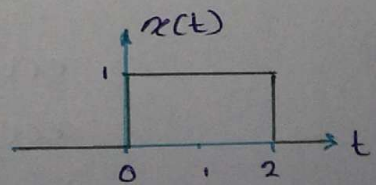
$$= \left[\frac{e^{-a\tau}}{-a} \right]_0^t$$

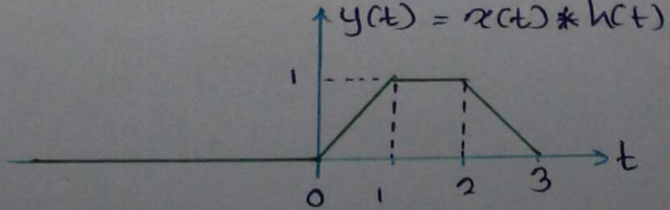
$$= \frac{e^{-at}}{(-a)} - \left(\frac{1}{-a} \right)$$

$$= \frac{1}{a} \left(1 - e^{-at} \right)$$

Example :-







* Properties of convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

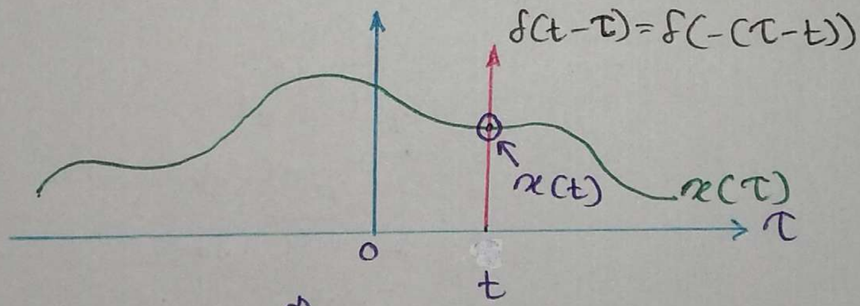
$$x_1[n] * x_2[n] = \sum_{k\,-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau = x(t) * \delta(t)$$

$$x[n] = \sum_{k\,-\infty}^{\infty} x[n] \cdot \delta[n-k] = x[n] * \delta[n]$$

Proof: $x(t) * \delta(t) = x(t)$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau$$



$$= \int_{-\infty}^{\infty} x(t) \cdot \delta(t-\tau)\, d\tau$$

$$= x(t) \int_{-\infty}^{\infty} \underbrace{\delta(t-\tau)}_{\text{area}=1}\, d\tau$$

$$= x(t) \times 1$$

$$= x(t)$$

$$x(t) * \delta(t) = x(t)$$

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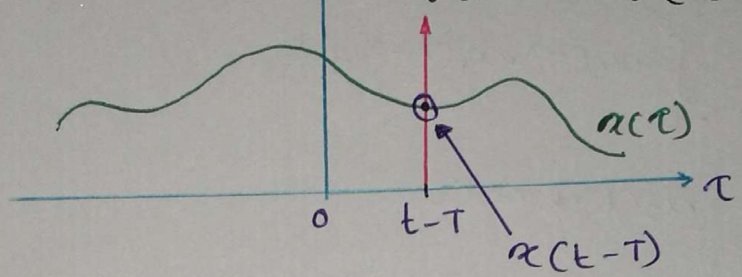
$$x(t) * \delta(t-T) = x(t-T)$$

$$x[n] * \delta[n-N] = x[n-N]$$

Proof :- $x(t) * \delta(t-T) = x(t-T)$

$$x(t) * \delta(t-T) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{x(t),\ t \Rightarrow \tau} \cdot \underbrace{\delta(t-T-\tau)}_{\delta(t-T),\ t \Rightarrow t-\tau} \, d\tau$$

$$\delta(t-T-\tau) = \delta(-(\tau-(t-T)))$$



$$= \int_{-\infty}^{\infty} x(t-T)\, \delta(t-T-\tau)\, d\tau$$

$$= x(t-T) \cdot \int_{-\infty}^{\infty} \underbrace{\delta(t-T-\tau)\, d\tau}_{\text{area}=1}$$

$$= x(t-T) \times 1$$

$$= x(t-T)$$

$$x(t) * \delta(t-T) = x(t-T)$$

* Commutativity

- $x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$

let $m = n-k$

$\therefore k = n-m$

when $k \to -\infty$; $m \to \infty$

$k \to +\infty$; $m \to -\infty$

$$= \sum_{m=\infty}^{-\infty} x_1[n-m] \cdot x_2[m]$$

$$= \sum_{m=-\infty}^{\infty} x_2[m]\, x_1[n-m]$$

$$= x_2[n] * x_1[n]$$

DT convolution is commutative

$x_1[n] * x_2[n] = x_2[n] * x_1[n]$

- $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

let $T = t - \tau$ when $\tau \to -\infty$; $T \to \infty$

$dT = -d\tau$ $\tau \to \infty$; $T \to -\infty$

$$= \int_{\infty}^{-\infty} x_1(t-T) \cdot x_2(T)\,(-dT)$$

$$= \int_{-\infty}^{\infty} x_2(T) \cdot x_1(t-T)\, dT$$

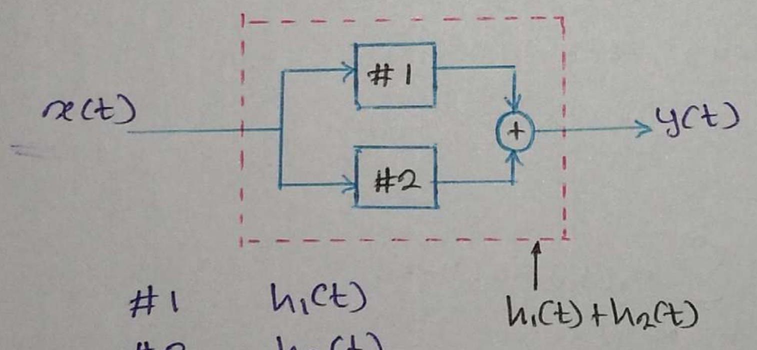
$$= x_2(t) * x_1(t)$$

CT convolution is commutative

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

* Distributivity

- $x_1(t) * (x_2(t) + x_3(t)) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$

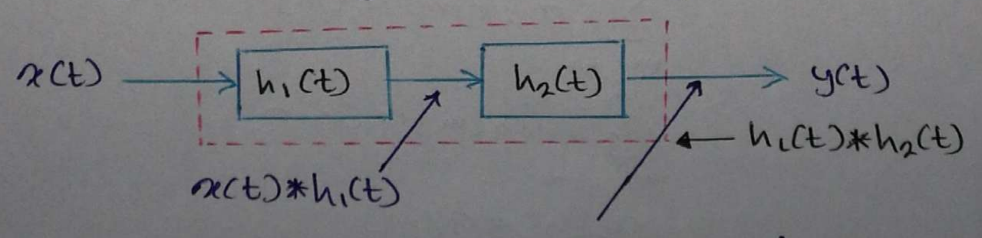


#1 $h_1(t)$

#2 $h_2(t)$

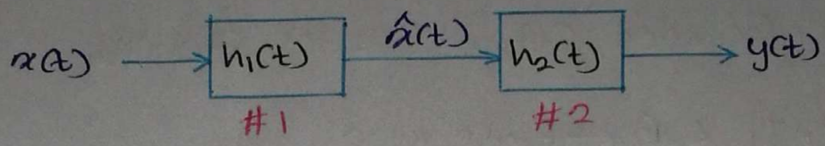
$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$
$$= x(t) * [h_1(t) + h_2(t)]$$

- $x_1(t) * \{x_2(t) * x_3(t)\} = \{x_1(t) * x_2(t)\} * x_3(t)$



$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$

*



System #2 is the inverse system of system #1

$$\therefore y(t) = x(t) \quad -(1)$$

but,

$$y(t) = x(t) * h_1(t) * h_2(t) \quad -(2)$$

$$x(t) = x(t) * \underbrace{h_1(t) * h_2(t)}_{\delta(t)}$$

also $\quad x(t) = x(t) * \delta(t)$

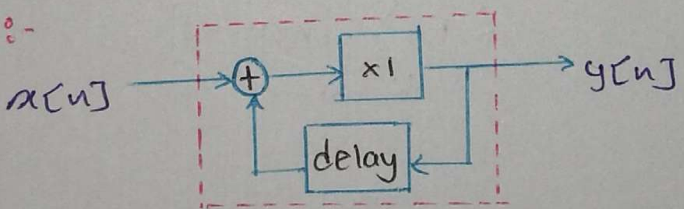
Therefore,

$$h_1(t) * h_2(t) = \delta(t)$$

* Stability

- If a system output is bounded for a bounded input, such a system is stable.

Example:-



n	$x[n]$	$y[n]$
0	Δ	Δ
1	Δ	2Δ
2	Δ	3Δ
3	Δ	4Δ
⋮	⋮	⋮

$$\lim_{n \to \infty} y[n] \to \infty$$

* Bounded Input Bounded Output (BIBO) Stability

$$|x[n]| < B \leftarrow \text{Finite value}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |x[k] \cdot h[n-k]|$$

$$\leq \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]|$$

Since $|x[k]| < B$

$$\sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| < \sum_{k=-\infty}^{\infty} B \cdot |h[n-k]|$$

$$< B \sum_{k=-\infty}^{\infty} |h[n-k]|$$

Therefore

$$|y[n]| < B \sum_{k=-\infty}^{\infty} |h[n-k]|$$

For BIBO stability $|y[n]| < C \leftarrow$ Finite value

For that $\displaystyle\sum_{k=-\infty}^{\infty} |h[n-k]| < D \leftarrow$ Finite value

For BIBO stability of a DT System,
Impulse response should be absolutely summable.

$|x(t)| \leq B \leftarrow$ Finite value

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right| \leq \int_{-\infty}^{\infty} |x(\tau) h(t-\tau)| d\tau$$

$$\leq \int_{-\infty}^{\infty} |x(\tau)| |h(t-\tau)| d\tau$$

Since $|x(t)| \leq B$

$$\int_{-\infty}^{\infty} |x(\tau)| |h(t-\tau)| d\tau \leq \int_{-\infty}^{\infty} B \cdot |h(t-\tau)| d\tau$$

$$\leq B \cdot \int_{-\infty}^{\infty} |h(t-\tau)| d\tau$$

Therefore,

$$|y(t)| \leq B \int_{-\infty}^{\infty} |h(t-\tau)| d\tau$$

For BIBO Stability $|y(t)| \leq C \leftarrow$ Finite value

For that $\int_{-\infty}^{\infty} |h(t-\tau)| d\tau \leq D \leftarrow$ Finite value

For the BIBO stability of a CT system, impulse response should be absolutely integrable

Example:- $y[n] = x[n] + y[n-1]$

I What is the impulse response

ii Check the stability

I Impulse response

Apply an impulse $x[n] = \delta[n]$

n	$x[n]$	$y[n-1]$	$y[n]$
-2	0	0	0
-1	0	0	0
0	1	0	1
1	0	1	1
2	0	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$y[n] = x[n] + y[n-1]$$
$$y[n] = x[n] + x[n-1] + y[n-2]$$
$$y[n] = x[n] + x[n-1] + x[n-2] + \dots$$
$$y[n] = \sum_{N\ -\infty}^{n} x[N]$$

Apply an impulse $x[n] = \delta[n]$

Then $$y[n] = \sum_{N\ -\infty}^{n} \delta[N]$$

$$y[n] = u[n]$$

Therefore,

Impulse response : $h[n] = u[n]$

ii $$u[n] = \sum_{N\ -\infty}^{n} \delta[N]$$

$$|y[n]| < \sum_{k\ -\infty}^{\infty} |h[n-k]|$$

$$|y[n]| < \sum_{k-\infty}^{\infty} |u[n-k]|$$

$$\rightarrow \infty$$

then unstable system

* Laplace transform of convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

$x_1(t) = 0 \quad ; t < 0$

$x_2(t) = 0 \quad ; t < 0$

$$L[x_1(t) * x_2(t)] = \int_{0^-}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau \right) e^{-St}\, dt$$

$$= \int_{0^-}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, e^{-St}\, d\tau\, dt$$

$$= \int_{-\infty}^{\infty} \int_{0^-}^{\infty} x_1(\tau)\, x_2(t-\tau)\, e^{-St}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \underbrace{\int_{0^-}^{\infty} x_2(t-\tau)\, e^{-St}\, dt}_{\substack{L[x_2(t-\tau)] \\ \downarrow \\ X_2(S) e^{-S\tau}}}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot X_2(S) \cdot e^{-S\tau}\, d\tau$$

$$= X_2(S) \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-S\tau}\, d\tau$$

Since $x_1(t) = 0 \quad ; t < 0$

$$= X_2(S) \underbrace{\int_{0^-}^{\infty} x_1(\tau)\, e^{-S\tau}\, d\tau}_{L[x_1(t)]}$$

$$= X_2(S) \cdot X_1(S)$$

$$= X_1(S) \cdot X_2(S)$$

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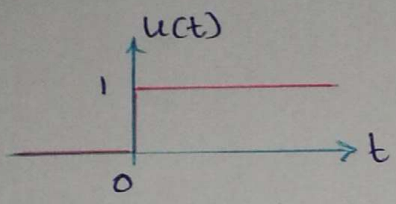
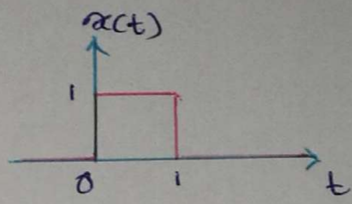
$$
\begin{tikzcd}
x_1(t) * x_2(t) \arrow[d, "L"'] \arrow[d, shift left=8, "L"] \\
X_1(S) \times X_2(S) \arrow[u, bend right=60, "L^{-1}"']
\end{tikzcd}
$$

Example :-





$$x(t) = \begin{cases} 1 & ; 0 \le t \le 1 \\ 0 & ; \text{o.w.} \end{cases}$$

- $L[u(t)] = \frac{1}{S}$

$$L[x(t)] = \int_{0^-}^{\infty} x(t) e^{-St} dt$$

$$= \int_{0}^{1} x(t) e^{-St} dt$$

$$= \int_{0}^{1} e^{-St} dt$$

$$= \left[\frac{e^{-St}}{-S} \right]_0^1$$

$$= \frac{1}{S}(1 - e^{-S})$$

- $L[x(t) * u(t)] = L[x(t)] \cdot L[u(t)]$

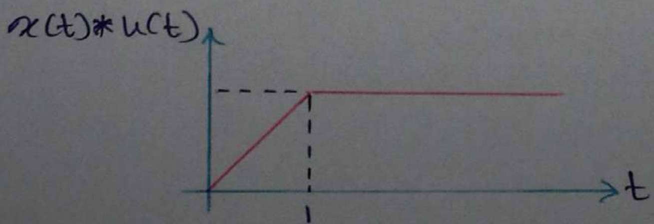
$$= \frac{1}{S}(1 - e^{-S}) \times \frac{1}{S}$$

$$= \frac{1}{S^2}(1 - e^{-S})$$

$$x(t) * u(t) = L^{-1}\left[\frac{1}{S^2}(1 - e^{-S}) \right]$$

$$= L^{-1}\left[\frac{1}{S^2} - \frac{1}{S^2} e^{-S} \right]$$

$$= t \cdot u(t) - (t-1) u(t-1)$$

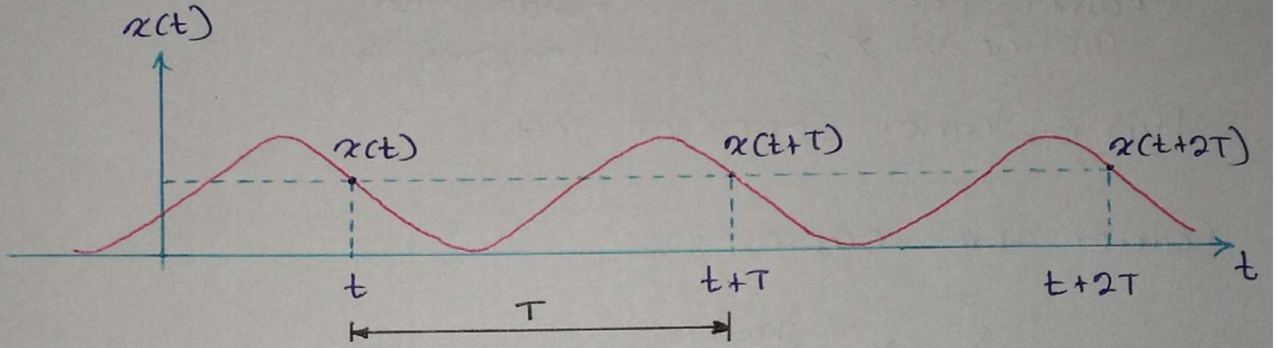


Inter Domain Analysis

* Periodic Signal

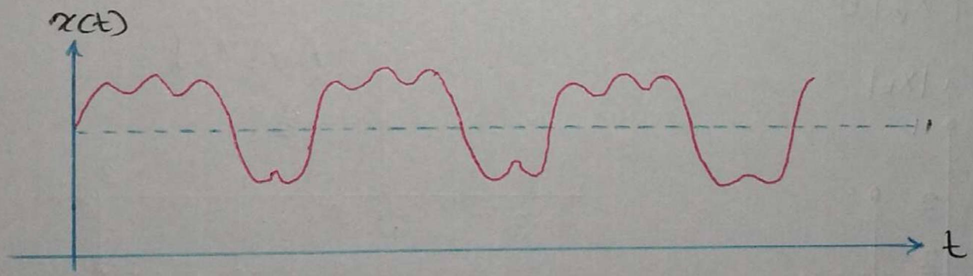
$$x(t) = x(t + mT) \quad ; m \in \mathbb{Z}, T > 0$$

The smallest T for which the above is satisfied is called the fundemental period of pediodic signal.



* Fourier Series

Example:- $x(t) = 11 + 4\sin(5t) + \frac{4}{3}\sin(15t)$



fundemental frequency = frequency of the lowest frequency component

$\omega_0 = 5$

$T = \frac{2\pi}{5}$

$f_0 = \frac{5}{2\pi}$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

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$$x(t) = 11 + 4\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right) + \frac{4}{3}\left(\frac{e^{j15t} - e^{-j15t}}{2j}\right)$$

$$x(t) = \underbrace{\left[\tfrac{2}{3}e^{j\frac{\pi}{2}}\right]}_{\text{Fourier series coeficients } X_{-3}} e^{(-3)j5t} + \left[2e^{j\frac{\pi}{2}}\right] e^{(-1)j5t} + [11]\, e^{(0)j5t}$$

$$+ \left[2e^{-j\frac{\pi}{2}}\right] e^{(+1)j5t} + \left[\tfrac{2}{3}e^{-j\frac{\pi}{2}}\right] e^{(+3)j5t}$$

Fourier series coeficients X_{-3} Fundemental frequency ω_0

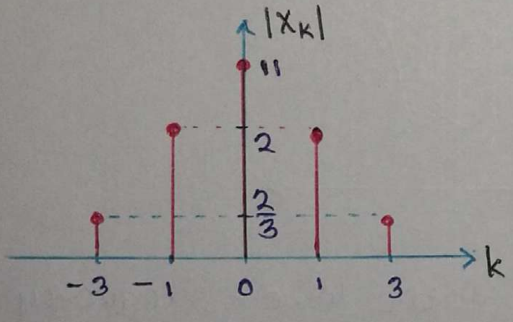
This is fourier series of $x(t)$

Fourier series expression

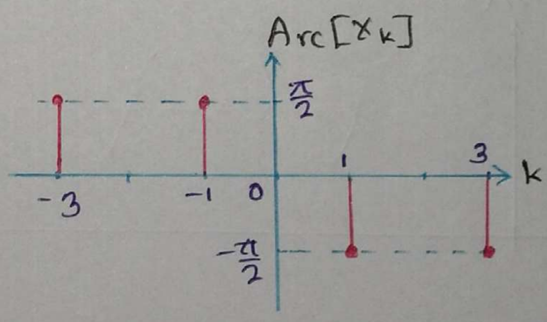
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad ; \quad X_k \in \mathbb{C}$$

$$X_k = Re[X_k] + j\, Im[X_k]$$

$$X_k = |X_k| e^{j \cdot Arc[X_k]}$$



Magnitude plot



Phase plot

* Discontinuous signals can be express as a collection of continuous signals using fourier series.

$$x(t) = \sum_{m} X_m e^{jm\omega_0 t}$$

$$x(t) = \dots + X_{-2} e^{j(-2)\omega_0 t} + X_{-1} e^{j(-1)\omega_0 t} + X_0 + X_1 e^{j1\omega_0 t} + X_2 e^{j2\omega_0 t} + \dots$$

Apply to LHS

$$\frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

Apply to m^{th} term of R.H.S

$$\frac{1}{T_0} \int_{T_0} X_m e^{jm\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

If $k = m$

$$\frac{1}{T_0} \int_{T_0} X_k dt = X_k = X_m$$

If $k \neq m$

$$\frac{1}{T_0} \int_{T_0} X_m e^{j(m-k)\omega_0 t} dt = 0$$

* Fourier Series

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$X_0 = \frac{1}{T_0} \int_{T_0} x(t) dt \quad \text{— Average value/DC value}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

Note.

From vector space

Inner product

$$\langle \underline{e}_i, \underline{e}_j \rangle = 0 \quad ; i \neq j$$

$\underline{e}_i$ and $\underline{e}_j$ are orthogonal

$$\langle \underline{e}_i, \underline{e}_j \rangle = \begin{cases} 1 ; & i = j \\ 0 ; & \text{otherwise} \end{cases}$$

For a periodic function

$$\langle f(t), g(t) \rangle = \frac{1}{T} \int_0^T f(t) \cdot g^*(t)\, dt$$

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = \frac{1}{T} \int_0^T e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_0^T e^{j(n-m)\omega_0 t}\, dt$$

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = \frac{1}{T} \int_0^T e^{j(n-m)\omega_0 t}\, dt = \begin{cases} 1 & ; n = m \\ 0 & ; \text{otherwise} \end{cases}$$

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = 0 \quad ; \quad n \neq m$$

$$\langle e^{jn\omega_0 t}, e^{jn\omega_0 t} \rangle = \| e^{jn\omega_0 t} \|^2 = 1$$

Therefore they are orthonormal.

Projection of $f(t)$ on $e^{jn\omega_0 t}$

$$\langle f(t), e^{jn\omega_0 t} \rangle = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t}\, dt = F(jn\omega_0)$$

$$\underline{v} = \sum_{i=1}^{n} v_i \underline{e}_i \quad \text{where} \quad v_i = \langle \underline{v}, \underline{e}_i \rangle$$

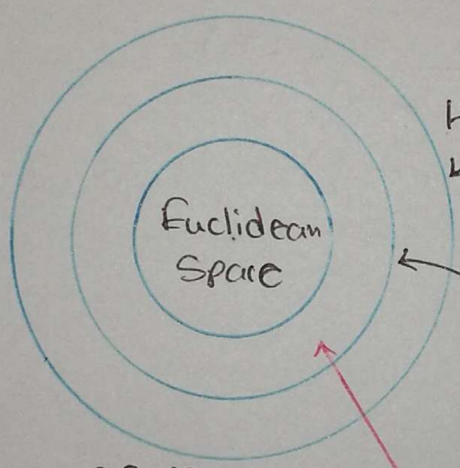
Therefore

$$f(t) = \sum_{n=-\infty}^{\infty} F(jn\omega_0)\, e^{jn\omega_0 t}$$

Power of the signal

Since $\langle \underline{a}, \underline{a} \rangle = \|\underline{a}\|^2$

$$\langle f(t), f(t) \rangle = \frac{1}{T}\int_0^T f(t) \cdot f^*(t)\,dt = \frac{1}{T}\int_0^T |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} F^2(jn\omega_0)$$



Hilbert space

linear norm space
- Infinite dimension
- linear
- norm

Euclidean space
- Finite dimension
- Linear
- norm

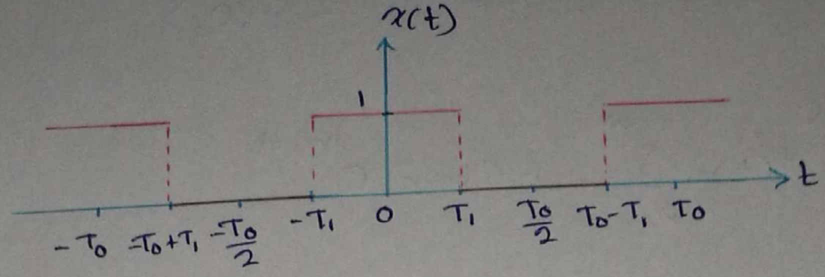
Fourier Series in here

If $\frac{d}{dt} x(t) = \lambda \cdot x(t)$ $\quad x(t)$ is a eigen function

$$\frac{d}{dt} e^{jk\omega_0 t} = jk\omega_0 \cdot e^{jk\omega_0 t}$$

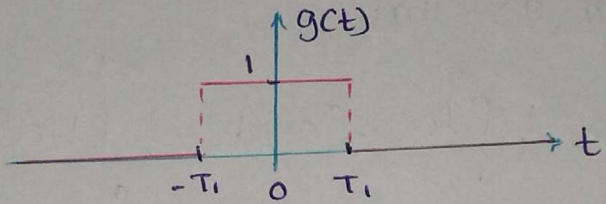
Therefore $e^{jk\omega_0 t}$ is a eigen function

Example :-



$2T_1 < T_0$

$$g(t) = \begin{cases} 1 ; & t \leq |T_1| \\ 0 ; & \text{elsewhere} \end{cases}$$



$$x(t) = \sum_{k=-\infty}^{\infty} g(t - kT_0)$$

Find the f.S. coeficients of above waveform

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

↓

Since signal is even, even captured window selected

↓

$$X_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \times \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{jk\omega_0}$$

$$= \frac{2j}{jk\omega_0 T_0} \times \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 t_1}}{2j}$$

Since, $\omega_0 = \frac{2\pi}{T_0}$

$$X_k = \left[\frac{1}{k\pi} \sin(k\omega_0 T_1) \right]$$

$X_k = \frac{1}{k\pi} \sin(k\omega_0 T_1)$

Therefore $X_k \in \mathbb{R}$

Since $X_k \in \mathbb{R}$ phase plot has 0 or π

Also this is infinite series.
because $x(t)$ is discontinuous.

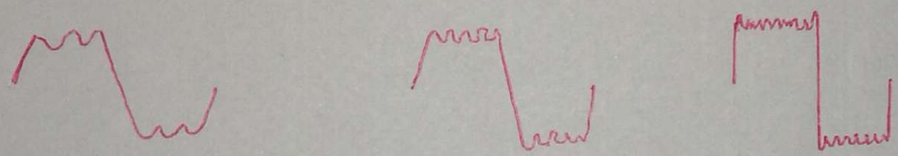
When $k \to \infty$ $|X_k| \to 0$

hence,

low frequency components have high strength
high frequency components have low strength

Therefore

this is low pass



Sharp edge can be formed by high frequency components

Thats why $k \to \infty$ harmonic $f \to \infty$

$$X_k = \frac{1}{k\pi} \sin(k\omega_0 T_1) \quad ; k \neq 0$$

When $k=0$

$$X_0 = \frac{1}{T_0} \int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T_0}$$

Therefore,

$$x(t) = \frac{2T_1}{T_0} + \sum_{k \neq 0} \left[\frac{1}{k\pi} \sin(k\omega_0 T_1) \right] e^{jk\omega_0 t}$$

* Properties of the fourier series

* Linearity

Consider a periodic Signal $x(t)$ with a fundemental frequency ω_0 and k^{th} F.S. coeficient given by X_k

Consider another periodic signal $y(t)$ with the same fundemental frequency ω_0 and k^{th} F.S. coeficient given by Y_k

$$x(t) \longrightarrow X_k$$

$$y(t) \longrightarrow Y_k$$

$$Ax(t) + By(t) \longrightarrow AX_k + BY_k$$

* Time shifting

If F.S. coeficient of the periodic signal $x(t)$ is X_k

$$x(t) \longrightarrow X_k$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t}$$

Apply $t \Rightarrow t - t_0$

$$x(t-t_0) = \sum_k X_k e^{jk\omega_0(t-t_0)}$$

$$x(t-t_0) = \sum_k \underbrace{\left(X_k e^{-jk\omega_0 t_0}\right)}_{X_k'} e^{jk\omega_0 t}$$

$$X_k' = X_k e^{-jk\omega_0 t_0}$$

$$|X_k'| = |X_k|$$

$$\angle X_k' = \angle X_k - \angle k\omega_0 t_0$$

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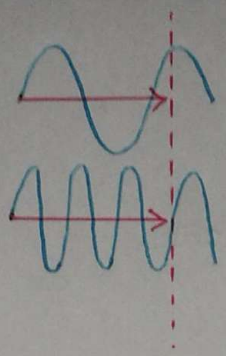
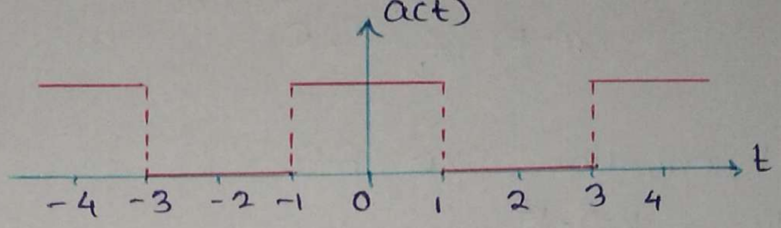
* Degree of change of phase angle of higher harmonics is higher



Example:-

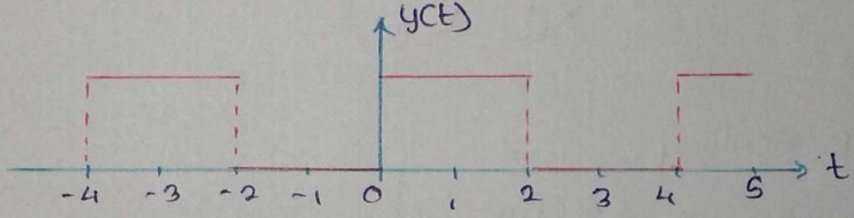


$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$

$T_1 = 1$

$a_0 = \frac{2T_1}{T_0} = \frac{1}{2}$

$a_K = \frac{1}{K\pi} \sin\left(K\frac{\pi}{2}\right)$



$y(t) = a(t-1)$

$Y_K = a_K \, e^{-jK\omega_0 t_0}$

$Y_K = a_K \, e^{-jK\frac{\pi}{2}}$

$Y_K = \left[\frac{1}{K\pi} \sin\left(\frac{K\pi}{2}\right)\right] e^{-jK\frac{\pi}{2}}$

$Y_0 = \frac{1}{2}$

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* Time Reverse

$$x(t) \longrightarrow X_k$$

$$x(-t) \longrightarrow X_k'$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t}$$

Apply $t \Rightarrow -t$

$$x(-t) = \sum_k X_k e^{-jk\omega_0 t}$$

To get equation like F.S.

Apply $k \Rightarrow -k$

$$x(-t) = \sum_k X_{-k} e^{jk\omega_0 t}$$

$$X_k' = X_{-k}$$

* Time scaling

$$x(t) \longrightarrow X_k$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t}$$

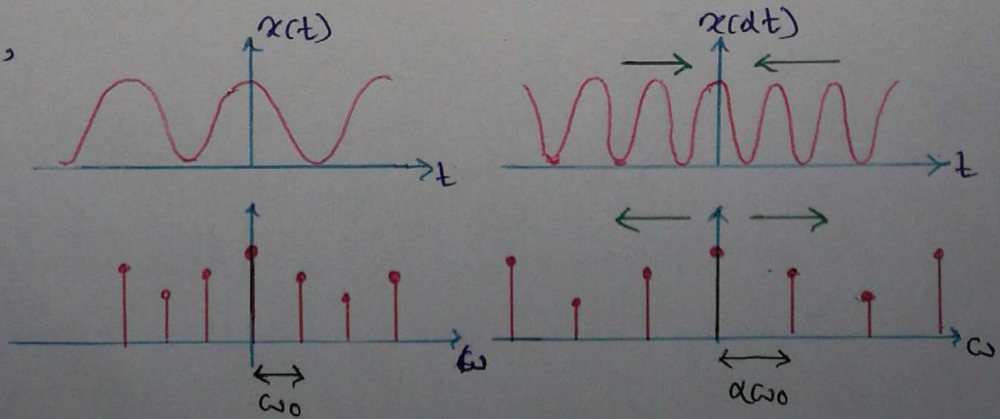
$t \Rightarrow \alpha t$

$$x(\alpha t) = \sum_k X_k e^{jk\boxed{\omega_0 \alpha} t}$$

New fundemental frequency

$\omega_0' = \alpha \omega_0$

If $\alpha > 1$,



* F.S. Properties for real signals

$$x(t) = \sum_k X_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_k X_k^* e^{-jk\omega_0 t}$$

Apply $k \Rightarrow -k$

$$x^*(t) = \sum_k X_{-k}^* e^{+jk\omega_0 t}$$

Since $x(t) = x^*(t)$

$$\sum_k X_k e^{jk\omega_0 t} = \sum_k X_{-k}^* e^{jk\omega_0 t}$$

Therefore,

$$X_k = X_{-k}^*$$

Conjugate symmetry

$$X_{-k} = X_k^*$$

Consequence of this

$$|X_k| = |X_{-k}|$$

$$\angle X_k = -\angle X_{-k}$$

* Real and even signals

As $x(t)$ is real

$$X_k = X_{-k}^*$$

However, as $x(t)$ is even

$$x(t) = x(-t)$$

Therefore from the time reversal property

$$X_k = X_{-k}$$

Then,

$$X_k = X_{-k} = X_k^*$$

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$$X_k = X_k^*$$

Therefore if $x(t)$ is real and even, X_k is real and even

* Real and odd signals

As $x(t)$ is real

$$X_k = X_{-k}^*$$

However, as $x(t)$ is odd

$$x(t) = -x(-t)$$

therefore, from time reversal property

$$X_k = -X_{-k}$$

then,

$$X_k = -X_{-k} = -X_k^*$$

$$X_k = -X_k^*$$

Therefore, if $x(t)$ is real and odd, X_k is imaginary and odd

* The differentiation/Integration property of F.S.

$$x(t) = X_0 + \sum_{\forall k \neq 0} \boxed{X_k} e^{jk\omega_0 t}$$

$$x'(t) = \frac{d x(t)}{dt} = \sum_{\forall k \neq 0} \boxed{X_k (jk\omega_0)} e^{jk\omega_0 t}$$

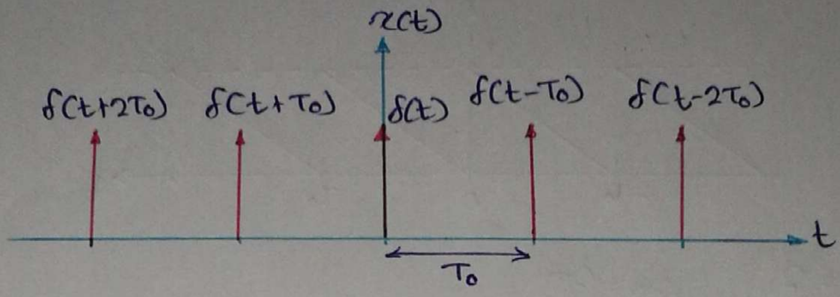
$$X_k' = jk\omega_0 X_k \quad ; k \neq 0$$

Differentiation is high pass process / sharpning process

$$X_k = \frac{X_k'}{jk\omega_0} \quad ; k \neq 0$$

Integration is low pass process / smoothing process

* The Fourier Series of an Impulse



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

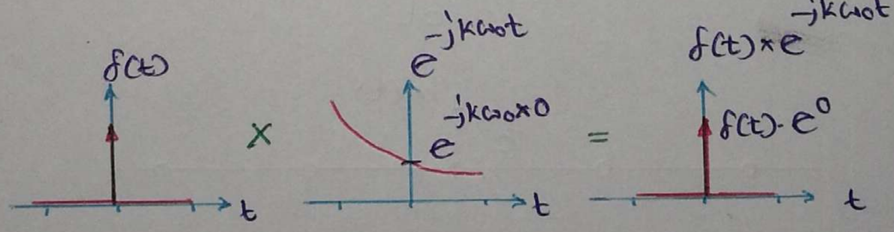
$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\,dt$$

$$= \frac{1}{T_0}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\cdot e^{-jk\omega_0 t}\,dt$$

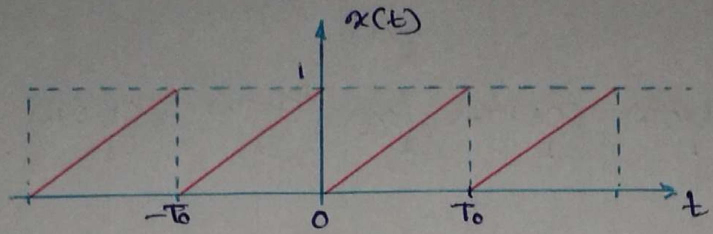


$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\cdot e^{-jk\omega_0 \times 0}\,dt$$

$$= \frac{1}{T_0}$$

$$x(t) = \sum_{\forall k} \delta(t - kT_0) = \sum_{\forall k} \frac{1}{T_0} e^{jk\omega_0 t}$$

Example :- Detemine the F.S. coeficients of the below Signal



$T_0 = 2 \qquad \omega_0 = \frac{2\pi}{T_0} = \pi$

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{T_0}\int_0^2 \left(\frac{t}{2}\right)dt$$

$$= \frac{1}{2}$$

for $k \neq 0$,

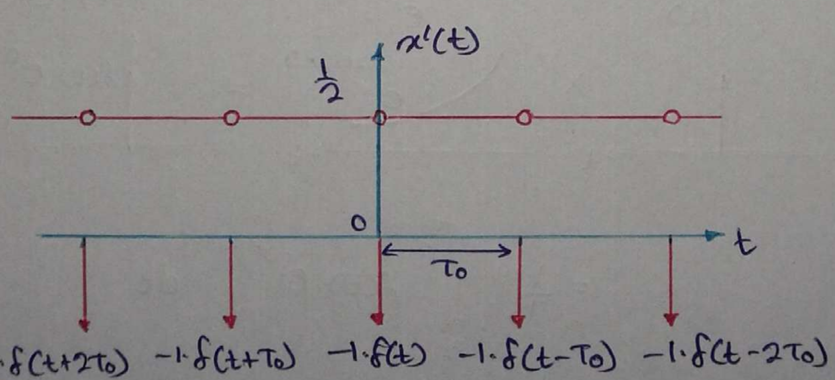
$$X_k = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^2 \frac{t}{2}\cdot e^{-jk\omega_0 t}\,dt$$

Difficult to integrate

Therefore consider $x'(t)$ (easier to integrate)

$$x'(t) = \frac{d\,x(t)}{dt}$$



$$X_k' = \frac{1}{T_0}\int_0^{T_0} x'(t)\,e^{-jk\omega_0 t}\,dt$$

$$X_k' = \frac{1}{T_0}\int_{0^-}^{0^+} [-1\cdot\delta(t)]\cdot e^{-jk\omega_0 t}\,dt + \frac{1}{T_0}\int_{0^+}^{T_0^-} \frac{1}{2} e^{-jk\omega_0 t}\,dt$$

$$= \frac{-1}{T_0}\int_{0^-}^{0^+} \delta(t)\cdot e^{-jk\omega_0 \times 0}\,dt + \frac{1}{2T_0}\int_{0^+}^{T_0^-} \underbrace{e^{-jk\omega_0 t}}_{0}\,dt$$

Integration over period

$$= \frac{-1}{T_0}\int_{0^-}^{0^+} \delta(t)\,dt + \frac{1}{2T_0}\times 0$$

$$= \frac{-1}{T_0}$$

$$= \frac{-1}{2}$$

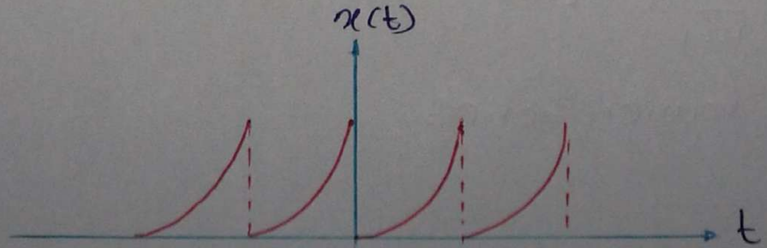
Therefore,

$$X_k = \frac{X_k'}{jk\omega_0} \quad ; k \neq 0$$

$$X_k = \frac{-1/2}{jk\pi} \quad ; k \neq 0$$

$$x(t) = \frac{1}{2} + \sum_{\forall k \neq 0} \left(\frac{j}{2\pi k}\right) e^{jk\omega_0 t}$$

likewise if we have higher order functions like x^n we can easily get F.S. by differentiating n times and apply this



* The trignometric Fourier Series

$$x(t) = X_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} X_k e^{jk\omega_0 t}$$

Consider the k^{th} term

$$X_k e^{jk\omega_0 t} = X_k [\cos(k\omega_0 t) + j\sin(k\omega_0 t)] \quad \text{—①}$$

$$X_{-k} e^{-jk\omega_0 t} = X_{-k} [\cos(k\omega_0 t) - j\sin(k\omega_0 t)] \quad \text{—②}$$

$$X_k e^{jk\omega_0 t} + X_{-k} e^{-jk\omega_0 t} = \underbrace{(X_k + X_{-k})}_{A_k}\cos(k\omega_0 t) + \underbrace{j(X_k - X_{-k})}_{B_k}\sin(k\omega_0 t)$$

note.

$$\sum_{k=-\infty}^{\infty} x_k = \sum_{k=-\infty}^{-1} x_k + x_0 + \sum_{k=1}^{\infty} x_k$$

$$\Big\downarrow k \Rightarrow -k$$

$$= \sum_{k=1}^{\infty} x_{-k} + x_0 + \sum_{k=1} x_k$$

$$= x_0 + \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{\infty} x_{-k}$$

$$= x_0 + \sum_{k=1}^{\infty} (x_k + x_{-k})$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} [A_k\cos(k\omega_0 t) + B_k\sin(k\omega_0 t)]$$

Trignometric fourier series.

$$A_k = X_k + X_{-k}$$

$$= \frac{1}{T_0}\int_{T_0} x(t)[e^{jk\omega_0 t} + e^{-jk\omega_0 t}]dt$$

$$= \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)dt$$

$$B_k = j(X_k - X_{-k})$$

$$= \frac{j}{T_0}\int_{T_0} x(t)[e^{jk\omega_0 t} - e^{-jk\omega_0 t}]dt$$

$$= \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)dt$$

If signal is odd $A_k = 0$
If signal is even $B_k = 0$

* Orthogonality

If exponential fourier series $x(t)$ is given by

$$x(t) = \sum_{\forall k} X_k e^{jk\omega_0 t}$$

$$x(t) = \sum_{\forall k} X_k \cdot \phi_k(t)$$

where,

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$\langle \phi_k(t), \phi_m(t) \rangle = \int_{T_0} \phi_k(t) \cdot \phi_m^*(t)dt$$

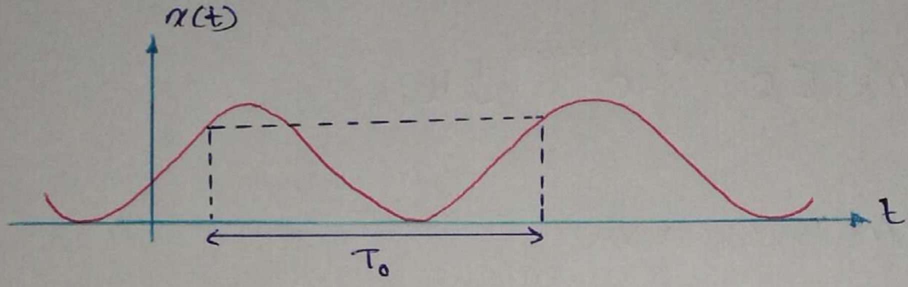
$$= \int_{T_0} e^{jk\omega_0 t} \cdot e^{-jm\omega_0 t} dt$$

$$= \int_{T_0} e^{j(k-m)\omega_0 t} dt$$

$$\langle \phi_k(t), \phi_m(t) \rangle = \begin{cases} 1 \text{ ; If } k=m \\ 0 \text{ ; If } k \neq m \end{cases}$$

This implies that $\phi_k(t)$ and $\phi_m(t)$ are orthogonal to each other.

* Power evaluation of a periodic signal



$$P_{avg} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Parseval's theorem

$$P_{avg} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

$$= \frac{1}{T_0} \int_{T_0} x(t) . x^*(t) dt$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_k X_k e^{jk\omega_0 t} . \sum_m X_m^* e^{-jm\omega_0 t} \right) dt$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_k \sum_m X_k e^{jk\omega_0 t} . X_m^* e^{-jm\omega_0 t} \right) dt$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_k \sum_m X_k X_m^* . e^{j(k-m)\omega_0 t} \right) dt$$

$$= \sum_k \sum_m \left(\frac{1}{T_0} \int_{T_0} X_k X_m^* e^{j(k-m)\omega_0 t} dt \right)$$

$$= \sum_k \sum_m \underbrace{\left(X_k X_m^* \cdot \frac{1}{T_0} \int_{T_0} e^{-j(k-m)\omega_0 t} dt \right)}$$

$$\frac{1}{T_0} \int_{T_0} e^{-j(k-m)\omega_0 t} dt = \begin{cases} 1 ; k=m \\ 0 ; k \neq m \end{cases}$$

$$X_k \cdot X_k^*$$

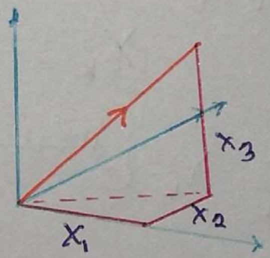
$$= \sum_k X_k \cdot X_k^*$$

$$= \sum_k |X_k|^2$$

Therefore

$$P_{avg} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_k |X_k|^2$$

" Parsevals theorem "



$\int |x(t)|^2 dt$ 2nd norm in time domain

$\sum_k |X_k|^2$ 2nd norm in fre domain

* Periodic Signal transmission through LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x(t) = \sum_k X_k e^{+jk\omega_0 t}$$

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau) d\tau$$

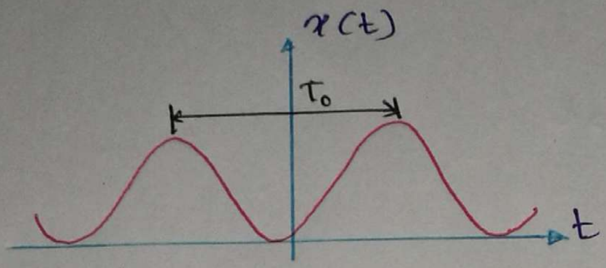
$$= \int_{-\infty}^{\infty} h(\tau) \cdot \sum_k X_k \cdot e^{jk\omega_0 (t-\tau)} d\tau$$

$$= \sum_k X_k e^{jk\omega_0 t} \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) \cdot e^{-jk\omega_0 \tau} d\tau \right]}_{H(jk\omega_0)}$$

$$= \sum_k \underbrace{X_k \cdot H(jk\omega_0)}_{Y_k} \cdot e^{jk\omega_0 t}$$

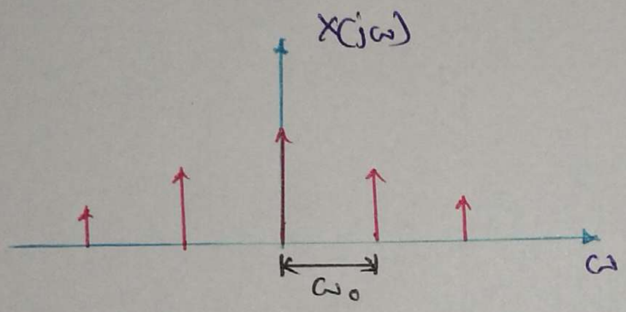
$$= \sum_k Y_k \cdot e^{jk\omega_0 t}$$

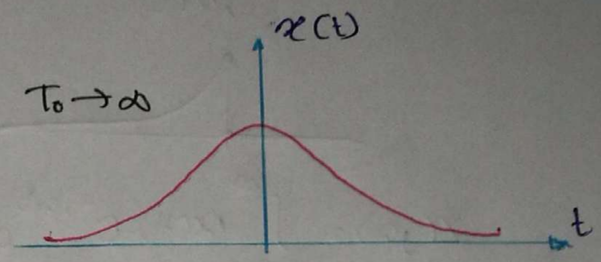
* Fourier Transform



Power Signal

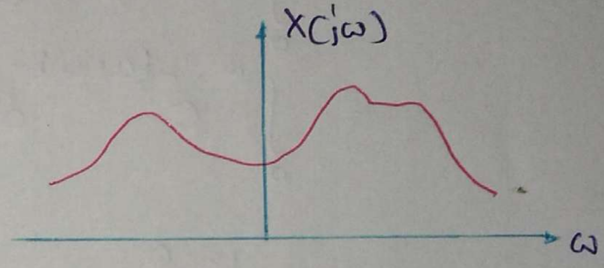
↓ Fourier Series





Energy Signal

↓ Fourier Transform



* Fourier Transform

$$X(j\omega) = \mathcal{F}[x(t)]$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j \cdot 2\pi f t} \, dt$$

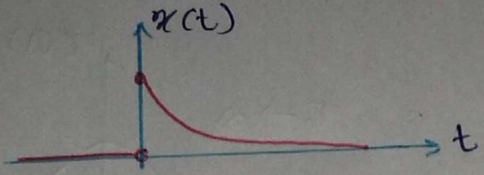
* Inverse Fourier Transform

$$x(t) = \mathcal{F}^{-1}[X(j\omega)]$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{+j\omega t} \, d\omega$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{+j \cdot 2\pi f \cdot t} \, df$$

* Example :- $x(t) = e^{-at} u(t)$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cdot u(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt$$

$$= \frac{1-0}{a+j\omega}$$

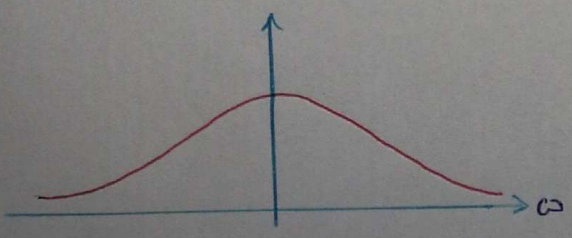
$$= \frac{1}{a+j\omega}$$

$$\mathcal{F}[e^{-at} . u(t)] = \frac{1}{a+j\omega}$$

$$= \frac{a}{a^2+\omega^2} - j \frac{\omega}{a^2+\omega^2}$$

Magnitude spectrum

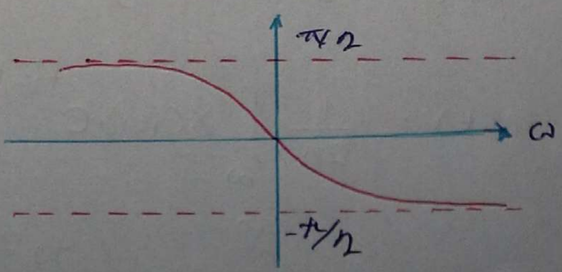
$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$



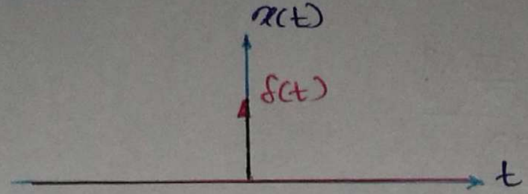
Phase spectrum

$$\angle X(j\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right)$$

$$= -\tan^{-1}\left(\frac{\omega}{a}\right)$$



* Delta Function



$x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} \, dt$$

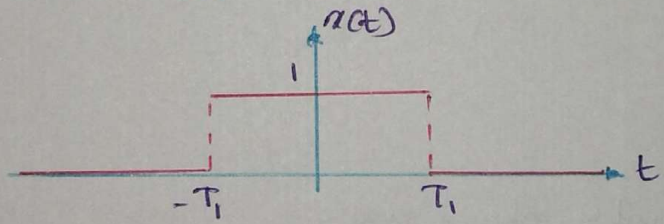
$$= \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega \times 0} \, dt$$

$$= \int_{-\infty}^{\infty} \delta(t) dt$$

$$= 1$$

$$\mathcal{F}[\delta(t)] = 1$$

* Square Pulse



$x(t) = \Pi\left(\frac{1}{2T_1}\right)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-T_1}^{+T_1} 1 \cdot e^{-j\omega t} \, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T_1}^{T_1}$$

$$= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

$$= \frac{2}{\omega} \left(\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j} \right)$$

$$= \frac{2}{\omega} \cdot \sin(\omega T_1)$$

$$= 2T_1 \times \frac{\sin\left[\pi\left(\frac{\omega T_1}{\pi}\right)\right]}{\pi\left(\frac{\omega T_1}{\pi}\right)}$$

$$= 2T_1 \cdot \mathrm{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\mathcal{F}\left[\pi\left(\frac{t}{2T_1}\right)\right] = 2T_1\, \mathrm{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$

Note.

$$\mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

At $x=0$,

$$\lim_{x\to 0} \mathrm{Sinc}(x) = \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x}$$

$$= \lim_{x\to 0} \frac{\pi\cos(\pi x)}{\pi}$$

$$= 1$$

$$\mathrm{Sinc}(0) = 1$$

Intersection points of x axis

$$\mathrm{Sinc}(x) = 0$$

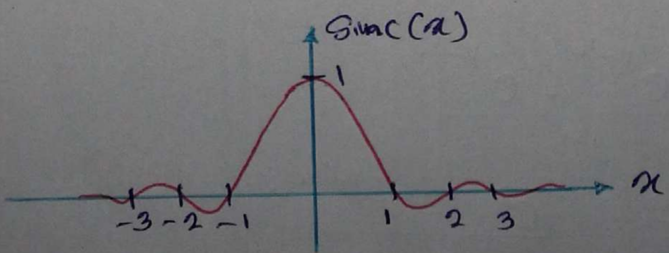
$$\frac{\sin(\pi x)}{\pi x} = 0$$

$$\sin(\pi x) = 0$$

$$\sin(\pi x) = \sin(0)$$

$$\pi x = n\pi \quad ; n \in \mathbb{Z}$$

$$x = n$$



$$\mathcal{F}[\Pi(\tfrac{t}{2T_1})] = 2T_1 \,\text{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$

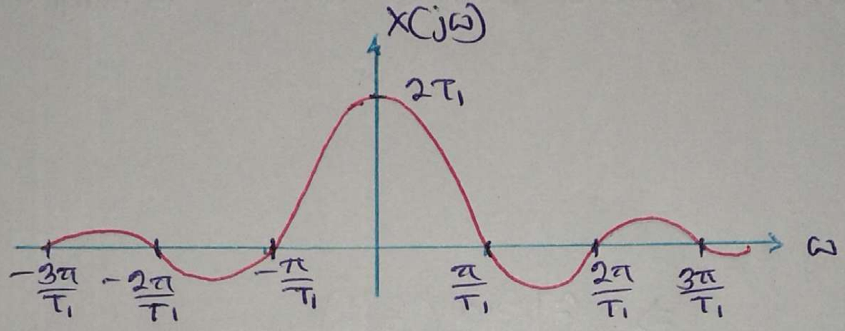
Intersection points on ω axis

$$X(j\omega) = 0$$

$$\text{Sinc}\left(\frac{\omega T_1}{\pi}\right) = 0$$

$$\frac{\omega T_1}{\pi} = n \quad ; n \in \mathbb{Z}$$

$$\omega = \frac{n\pi}{T_1}$$



* Properties

* Linearity

$$\mathcal{F}[x_1(t)] = X_1(j\omega)$$

$$\mathcal{F}[x_2(t)] = X_2(j\omega)$$

$$\mathcal{F}[A x_1(t) \pm B x_2(t)] = A X_1(j\omega) \pm B X_2(j\omega)$$

* Time shifting

$$\mathcal{F}[x(t)] = X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, dt$$

$t \Rightarrow t - t_0$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega(t-t_0)}\, dt$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{X(j\omega)\cdot e^{-j\omega t_0}}\, e^{+j\omega t}\, dt$$

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$$X'(j\omega) = \mathcal{F}[x(t-t_0)] = X(j\omega) \cdot e^{-j\omega t_0}$$

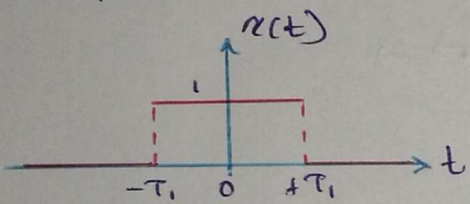
$$|X'(j\omega)| = |X(j\omega)|$$

Time shift will not affect the magnitude spectrum

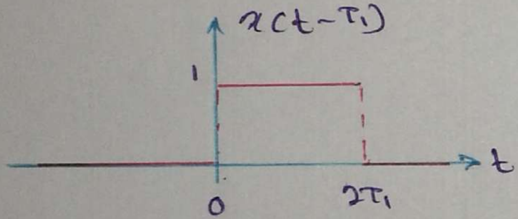
$$\angle X'(j\omega) = \angle X(j\omega) - \omega t_0$$

Phase spectrum is affected in a linear manner where phase shift is larger in large ω

* Example:-



$$\Rightarrow \quad 2T_1 \text{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$



$$\Rightarrow \quad 2T_1 \text{Sinc}\left(\frac{\omega T_1}{\pi}\right) \cdot e^{-j\omega T_1}$$

* Modulation / Shifting in frequency

$$\mathcal{F}[x(t)] = X(j\omega)$$

$$\mathcal{F}[x(t)e^{+j\omega_0 t}] = \int_{-\infty}^{\infty} x(t) \cdot e^{j\omega_0 t} \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j\boxed{(\omega-\omega_0)}t} \, dt$$

$$= X[j(\omega-\omega_0)]$$

$$\mathcal{F}[x(t)e^{+j\omega_0 t}] = X[j(\omega-\omega_0)]$$

$$\mathcal{F}[x(t-t_0)] = X(j\omega) \cdot e^{-j\omega_0 t}$$

t Shiting in time domain is called delaying

ω Shifting in frequency domain is called upconversion

* Invers fourier transform of $\delta(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\cdot e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\cdot e^{+j\times 0\times t}\, d\omega$$

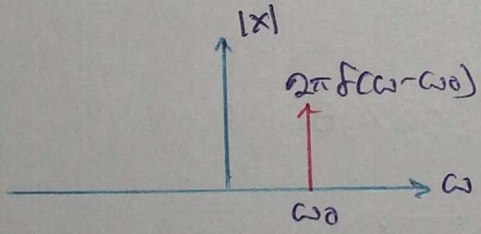
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, d\omega$$

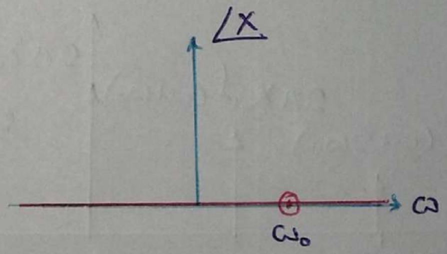
$$= \frac{1}{2\pi}$$

$$\mathcal{F}[1] = 2\pi\cdot\delta(\omega)$$

* The fourier transform of periodic signals

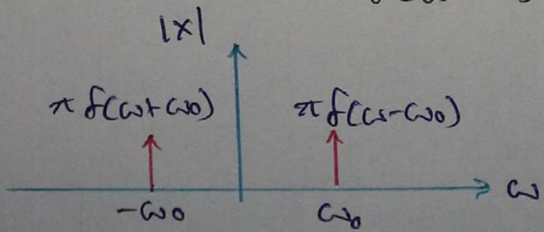
$$\mathcal{F}[e^{j\omega_0 t}] = \mathcal{F}[1\times e^{j\omega_0 t}]$$
$$= 2\pi\,\delta(\omega-\omega_0)$$

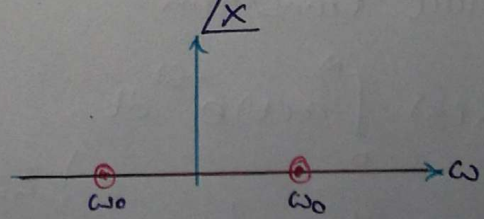




$$\mathcal{F}[\cos(\omega_0 t)] = \mathcal{F}\left[\frac{e^{+j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$





$$\mathcal{F}[\sin(\omega_0 t)] = \mathcal{F}\left[\frac{e^{+j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right]$$

$$= \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$$

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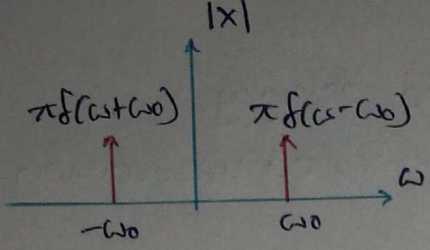
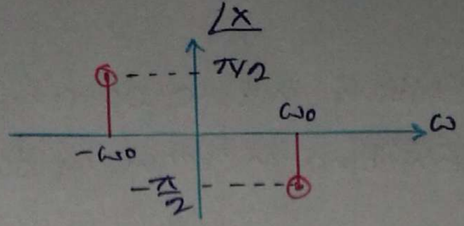




Consider a periodic signal with fundemental frequency ω_0

$$x(t) = \sum_k X_k e^{+jk\omega_0 t}$$

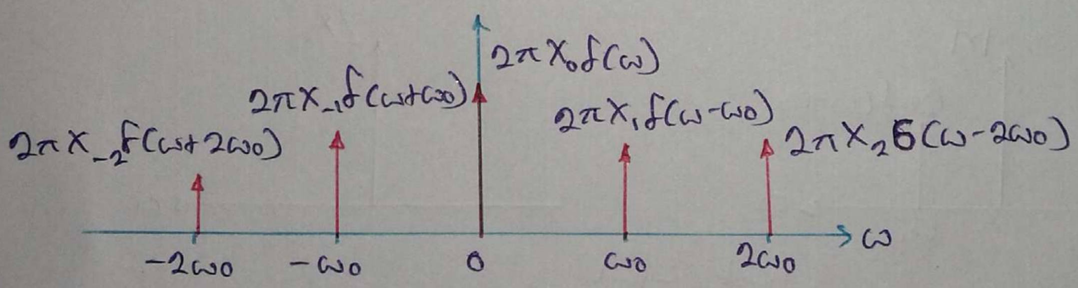
$$F[x(t)] = F\left[\sum_k X_k e^{+jk\omega_0 t}\right]$$

$$= \sum_k F[X_k e^{+jk\omega_0 t}]$$

$$= \sum_k X_k \cdot F[e^{+jk\omega_0 t}]$$

$$= \sum_k X_k \cdot 2\pi\delta(\omega - k\omega_0)$$

$$= 2\pi \sum_k X_k \cdot \delta(\omega - k\omega_0)$$



- Laplace transform

$$X(s) = \int_0^\infty x(t) e^{-st} dt$$

$s = j\omega$

- Fourier transform

$$X(j\omega) = \int_0^\infty x(t) \cdot e^{-j\omega t} dt$$

The fourier transform is the laplace transform evaluated on the imajinary axis for one sided signals

* Differentiation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x'(t) = \frac{dx(t)}{dt}$$

$$= \frac{d}{dt}\left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left[X(j\omega)\cdot e^{+j\omega t} \right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot \frac{d}{dt}\left(e^{+j\omega t} \right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{(j\omega)\cdot X(j\omega)}\, e^{+j\omega t}\, d\omega$$

$$X'(j\omega) = (j\omega)\cdot X(j\omega)$$

$$X^{n}(j\omega) = (j\omega)^{n}\cdot X(j\omega)$$

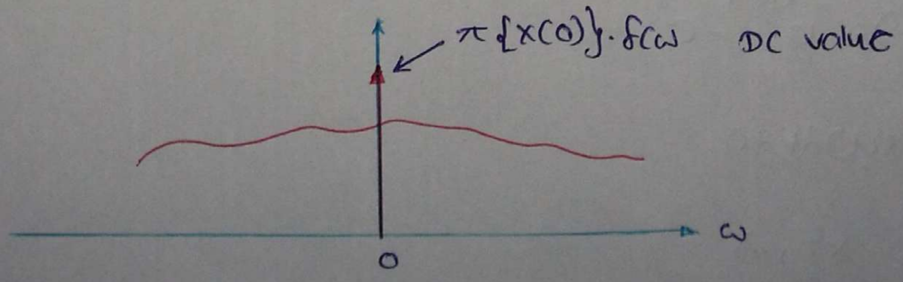
* Integration

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{X(j\omega)}{j\omega} + \pi\cdot\{X(0)\}\cdot\delta(\omega)$$

where,

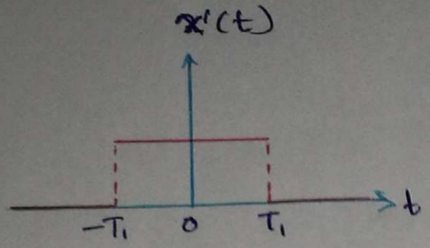
$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

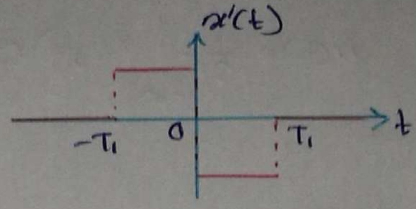
Area under $x(t)$ curve



$$F[x(t)] = \frac{X'(j\omega)}{j\omega} + \pi\{X'(0)\}\cdot\delta(\omega)$$

*Example :-



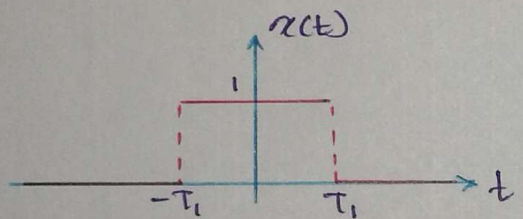


$$\int_{-\infty}^{\infty} x'(t)\,dt \neq 0 \qquad \int_{-\infty}^{\infty} x'(t)\,dt = 0$$

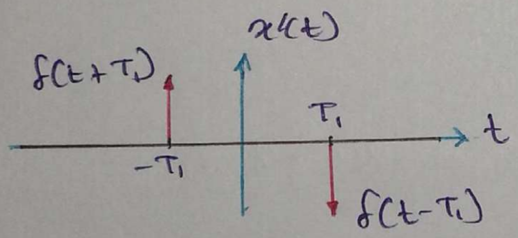
If $\int_{-\infty}^{\infty} x'(t)\,dt = 0$

$$\mathcal{F}[x(t)] = \frac{X'(j\omega)}{j\omega}$$

* Example :-



$$X(j\omega) = 2T_1 \text{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$



$$\mathcal{F}[\delta(t)] = 1$$

$$\mathcal{F}[\delta(t).e] = 1.e^{-j\omega t_0}$$

Therefore

$$X'(j\omega) = e^{+j\omega T_1} - e^{-j\omega T_1}$$

also

$$\int_{-\infty}^{\infty} x'(t)\,dt = 0$$

$$\therefore X(j\omega) = \frac{X'(j\omega)}{j\omega}$$

$$X(j\omega) = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

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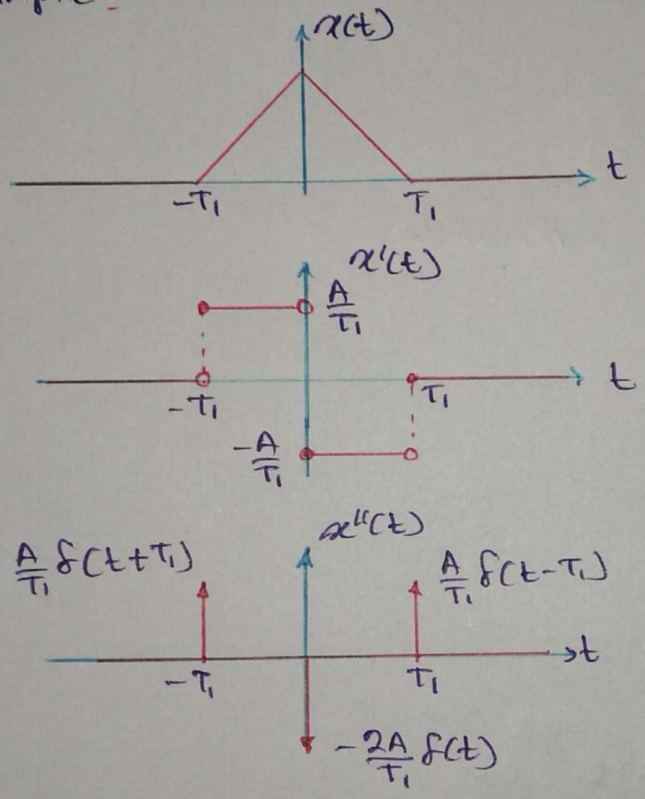
$$x(j\omega) = \frac{2}{\omega} \cdot \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j}$$

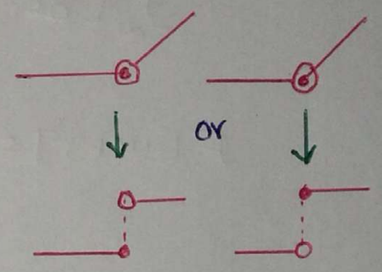
$$x(j\omega) = \frac{2}{\omega} \sin(\omega T_1)$$

$$x(j\omega) = 2T_1 \times \frac{\sin[\pi(\frac{\omega T_1}{\pi})]}{\pi(\frac{\omega T_1}{\pi})}$$

$$x(j\omega) = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

* Example :-





$$x''(j\omega) = \frac{A}{T_1} e^{+j\omega T_1} - \frac{2A}{T_1} + \frac{A}{T_1} e^{-j\omega T_1}$$

$$= \frac{A}{T_1}\left(e^{j\omega T_1} - 2 + e^{-j\omega T_1}\right)$$

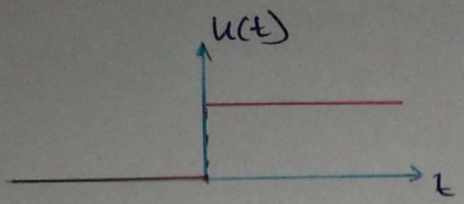
$$\int_{-\infty}^{\infty} x''(t)\, dt = 0$$

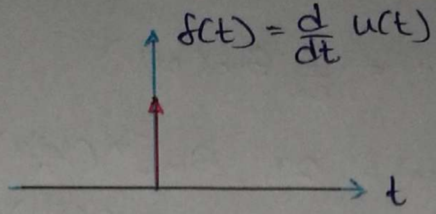
$$\therefore \quad x'(j\omega) = \frac{\frac{A}{T_1}\left(e^{j\omega T_1} - 2 + e^{-j\omega T_1}\right)}{j\omega}$$

$$\int_{-\infty}^{\infty} x'(t) = 0$$

$$\therefore \quad x(j\omega) = \frac{\frac{A}{T_1}\left(e^{j\omega T_1} - 2 + e^{-j\omega T_1}\right)}{(j\omega)^2}$$

* Fourier transform of unit step function





$$\mathcal{F}[\delta(t)] = 1$$

$$\mathcal{F}\left[\frac{d}{dt} u(t)\right] = 1$$

also $$\int_{-\infty}^{\infty} \left[\frac{d}{dt} u(t)\right] dt = 1$$

$$\therefore \quad \mathcal{F}[u(t)] = \frac{1}{j\omega} + \pi \times 1 \times \delta(\omega)$$

* Time Scaling

$$\mathcal{F}[x(t)] = X(j\omega)$$

- Consider $a > 0$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

let $at = \lambda$

$$= \int_{-\infty}^{\infty} \frac{x(\lambda)\, e^{-j\left(\frac{\omega}{a}\right)\cdot\lambda}}{a}\, d\lambda$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) \cdot e^{-j\left(\frac{\omega}{a}\right)\lambda}\, d\lambda$$

$$= \frac{1}{a} X\left(j\frac{\omega}{a}\right)$$

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \quad ; \; a > 0$$

- Consider $a<0$ case

$$a = -|a|$$

$$F[x(at)] = \int_{-\infty}^{\infty} x(-|a|t)\, e^{-j\omega t}\, dt$$

let $-|a|t = \lambda$

When, $t \to +\infty \quad \lambda \to -\infty$
$t \to -\infty \quad \lambda \to +\infty$

$$= \int_{+\infty}^{-\infty} \frac{x(\lambda)\, e^{-j\left(\frac{-\omega}{|a|}\right)\lambda}}{-|a|}\, d\lambda$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\lambda)\, e^{-j\left(-\frac{\omega}{|a|}\right)\lambda}\, d\lambda$$

$$= \frac{1}{|a|} X\left(+j\frac{\omega}{-|a|}\right)$$

$$= \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

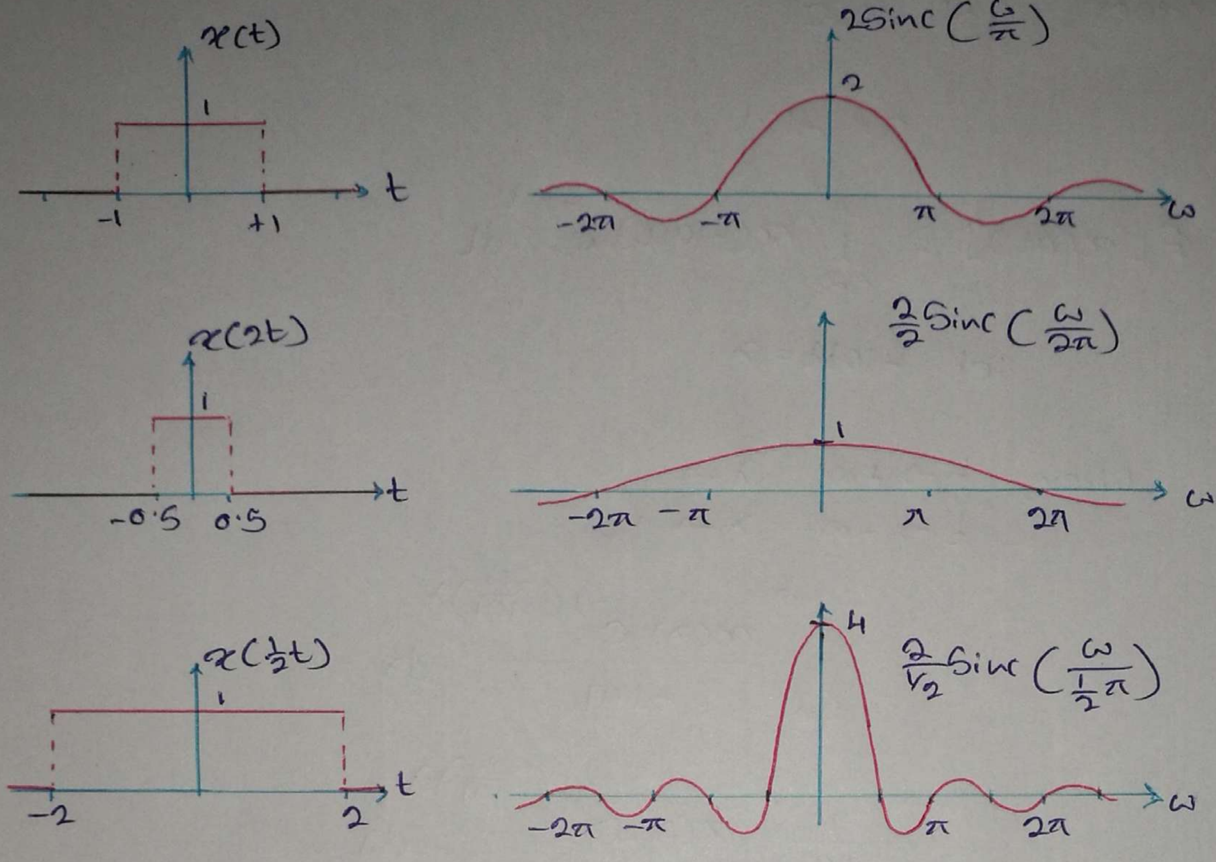
$$F[x(at)] = \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \quad ; \quad a<0$$

* Time reversal

When $a=-1$

$$F[x(-t)] = X(-j\omega)$$

Example :-



Signals are fluctuating rapidly have more high frequency components

* The Fourier transform properties of real signals

Since it is real signal

$$x(t) = x^*(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{+j\omega t}\, dt$$

Since $x(t) = x^*(t)$

$$= \int_{-\infty}^{\infty} x(t)\, e^{+j\omega t}\, dt$$

$\omega \Rightarrow -\omega$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$X^*(-j\omega) = X(j\omega)$$

$$[X^*(-j\omega)]^* = [X(j\omega)]^*$$

$$X(-j\omega) = X^*(j\omega)$$

$$X^*(j\omega) = X(-j\omega)$$ Conjugate symmetry

Consequence of this

$$|X(-j\omega)| = |X^*(j\omega)| = |X(j\omega)|$$

$$|X(-j\omega)| = |X(j\omega)|$$

- Magnitude spectrum is even symmetric

$$\angle X(-j\omega) = \angle X^*(j\omega)$$

$$\angle X(-j\omega) = -\angle X(j\omega)$$

- Phase spectrum is odd symmetric

* If $x(t)$ is real and even

As $x(t)$ is real

$$x(t) = x^*(t)$$

$$X^*(j\omega) = X(-j\omega)$$

However, as $x(t)$ is even

$$x(t) = x(-t)$$

$$X(j\omega) = X(-j\omega)$$

Therefore,

$$X(j\omega) = X(-j\omega) = X^*(j\omega)$$

$$X(j\omega) = X^*(j\omega)$$

Therefore $X(j\omega)$ is real and even

* If $x(t)$ is real and odd

As $x(t)$ is real

$$x(t) = x^*(t)$$
$$X^*(j\omega) = X(-j\omega)$$

However, as $x(t)$ is odd

$$x(t) = -x(-t)$$
$$X(j\omega) = -X(-j\omega)$$

Therefore,

$$X(j\omega) = -X(-j\omega) = -X^*(j\omega)$$

$$X(j\omega) = -X^*(j\omega)$$

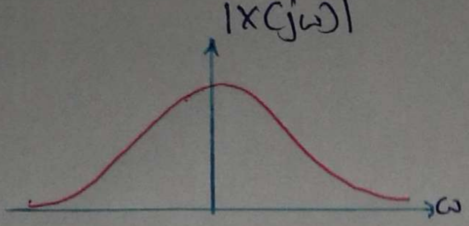
Therefore $X(j\omega)$ is imajinary and odd

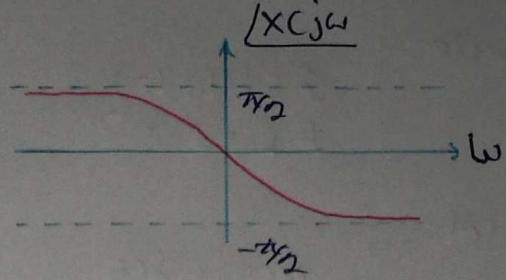
* Example:- $x(t) = e^{-at}u(t)$

$$X(j\omega) = \frac{1}{a+j\omega}$$
$$= \frac{a-j\omega}{a^2+\omega^2}$$
$$= \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(j\omega) = \tan^{-1}\left(-\frac{\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\therefore \quad X(j\omega) = \frac{1}{\sqrt{a^2+\omega^2}} \cdot e^{-j\cdot\tan^{-1}\left(-\frac{\omega}{a}\right)}$$





* Duality

$$\mathcal{F}[x(t)] = X(j\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$t \Rightarrow \omega$, $\omega \Rightarrow t$

$$X(t) = \int_{-\infty}^{\infty} x(\omega) e^{-jt\omega} \, d\omega$$

$$X(t) = \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} \, d\omega$$

$t \Rightarrow -t$

$$X(-t) = \int_{-\infty}^{\infty} x(\omega) e^{+j\omega t} \, d\omega$$

$$\frac{X(-t)}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{+j\omega t} \, d\omega$$

$$\frac{X(-t)}{2\pi} = \mathcal{F}^{-1}[x(\omega)]$$

$$\mathcal{F}[X(-t)] = 2\pi x(\omega)$$

$t \Rightarrow -t$

$$\mathcal{F}[X(t)] = 2\pi x(-\omega)$$

" Duality property "

* Example :-

$$\mathcal{F}[\delta(t)] = 1$$
$$\mathcal{F}[1] = ?$$

let ,
$$x(t) = \delta(t)$$

then
$$X(\omega) = 1$$
$$X(t) = 1$$

From duality,

$$\mathcal{F}[X(t)] = \mathcal{F}[1] = 2\pi x(-\omega)$$
$$= 2\pi \delta(-\omega)$$
$$= 2\pi \delta(\omega)$$

$$\mathcal{F}[1] = 2\pi\delta(\omega)$$

* Energy of the signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} d\omega \cdot x^*(t)\, dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \int_{-\infty}^{\infty} x^*(t) e^{+j\omega t} dt\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \underbrace{\left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt\right]^*}_{X(j\omega)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot X^{*}(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$$

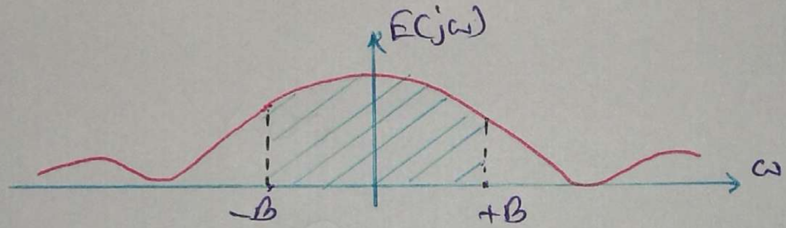
Energy $E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|\,d\omega$

2^{nd} norm in time domain

2^{nd} norm in frequency domain

*Example :-

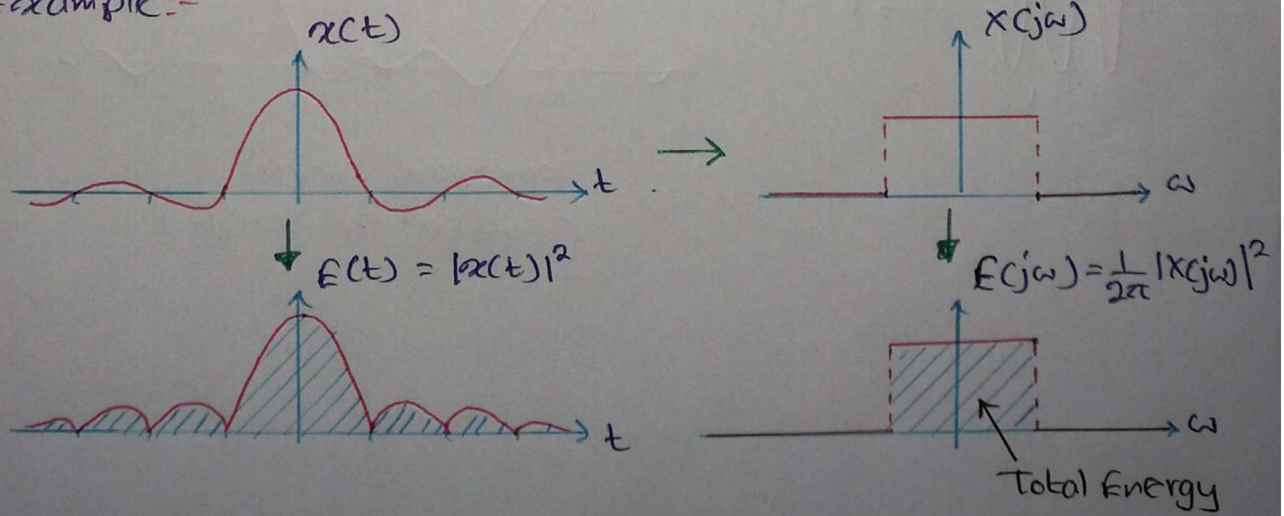
$$E(j\omega) = \frac{1}{2\pi}|X(j\omega)|^2$$



$$E_{-B\to B} = \int_{-B}^{B} E(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-B}^{B} |X(j\omega)|^2\,d\omega$$

*Example :-



Total Energy $= \int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega$

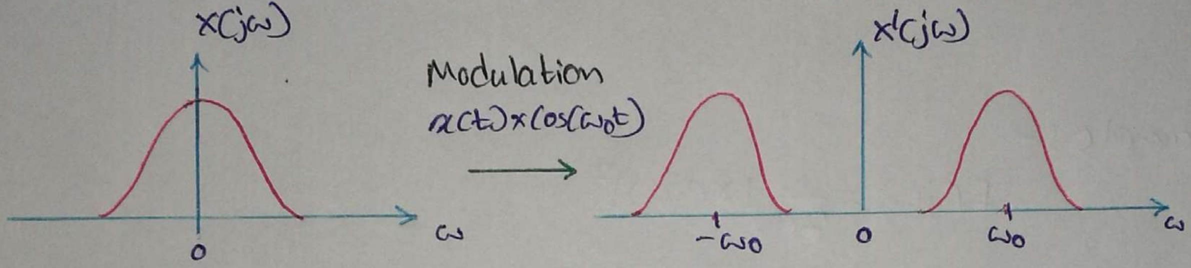
* Modulation

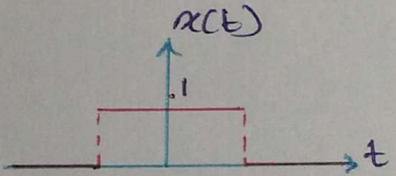
$$\mathcal{F}[x(t)e^{+j\omega t}] = X[j(\omega-\omega_0)]$$

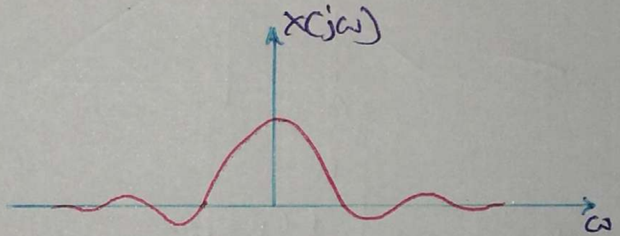
$$\mathcal{F}[x(t)\cos(\omega_0 t)] = \mathcal{F}\left[x(t)\cdot\left(\frac{e^{+j\omega_0 t}+e^{-j\omega_0 t}}{2}\right)\right]$$

$$= \frac{\mathcal{F}[x(t)e^{j\omega_0 t}]}{2} + \frac{\mathcal{F}[x(t)e^{-j\omega_0 t}]}{2}$$

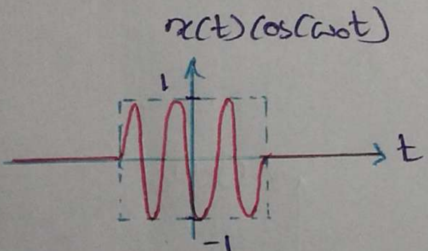
$$= \frac{X[j(\omega-\omega_0)]}{2} + \frac{X[j(\omega+\omega_0)]}{2}$$

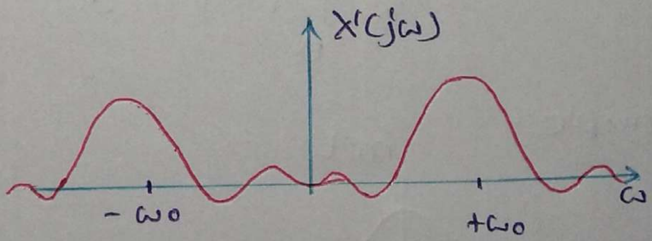






low pass signal
(near origin)





bandpass signal
(shifted from origin)

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* LTI Systems and the Fourier transform

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$F[y(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t}\, dt}_{F[h(t-\tau)]} \cdot d\tau$$

$$F[h(t-\tau)] \downarrow H(j\omega) \cdot e^{-j\omega \tau}$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot H(j\omega) \cdot e^{-j\omega \tau}\, d\tau$$

$$= H(j\omega) \underbrace{\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega \tau}\, d\tau}_{X(j\omega)}$$

$$= H(j\omega) \cdot X(j\omega)$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

Output signal spectrum

Frequency respons of the system

Input signal spectrum

If a signal $x(t)$ passes through a LTI System with an impulse respons $h(t)$, the output signal $y(t)$ has the following relationship.

The spectrum of the input signal $X(j\omega)$ multiplied by the frequency respons of the System $H(j\omega)$ gives the output signal spectrum $Y(j\omega)$

* Convolution Property

$$F[x(t) * y(t)] = X(j\omega) \cdot Y(j\omega)$$

$$F[x(t) * y(t)] = X(\omega) \cdot Y(\omega)$$

from the duality theorem

$$F[X(t) \cdot Y(t)] = 2\pi \cdot x(-\omega) * y(-\omega) \text{ --- } ①$$

but,

$$F[x(t)] = X(\omega) \qquad F[y(t)] = Y(\omega)$$

also from the dulity theorem

$$F[X(t)] = 2\pi \cdot x(-\omega) \qquad F[Y(t)] = 2\pi \cdot y(-\omega)$$

$$x(-\omega) = \frac{F[X(t)]}{2\pi} \qquad y(-\omega) = \frac{F[Y(t)]}{2\pi}$$

① ⇒

$$F[X(t) \cdot Y(t)] = 2\pi \cdot \left(\frac{F[X(t)]}{2\pi}\right) * \left(\frac{F[Y(t)]}{2\pi}\right)$$

$$F[X(t) \cdot Y(t)] = \frac{1}{2\pi} \cdot F[X(t)] * F[Y(t)]$$

Consequence of this

$$F[x(t) \cdot y(t)] = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

* Differentiation Property

$$F\left[\frac{d\,x(t)}{dt}\right] = (j\omega) \cdot X(j\omega)$$

$$F\left[\frac{dx(t)}{dt}\right] = j\omega \cdot X(\omega)$$

from the duality theorem

$$F[jt\,X(t)] = 2\pi \cdot \frac{d\,x(-\omega)}{d(-\omega)} \text{ --- } ①$$

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but

$$\mathcal{F}[x(t)] = X(\omega)$$

then, from the duality theorem

$$\mathcal{F}[X(t)] = 2\pi \cdot x(-\omega)$$

$$\frac{d(\mathcal{F}[X(t)])}{d\omega} = 2\pi \cdot \frac{d x(-\omega)}{d\omega}$$

① ⇒

$$\mathcal{F}[jt\,x(t)] = \frac{d(\mathcal{F}[x(t)])}{-d\omega}$$

$$\mathcal{F}[t\,x(t)] = -j\,\frac{d(\mathcal{F}[x(t)])}{-d\omega}$$

Consequence of this

$$\mathcal{F}[t\,x(t)] = +j\,\frac{dX(j\omega)}{d\omega}$$

Example :-

$$h(t) = e^{-3t}u(t)$$

$$x(t) = e^{-t}u(t)$$

$$H(j\omega) = \frac{1}{3+j\omega}, \quad X(j\omega) = \frac{1}{1+j\omega}$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

$$= \frac{1}{(1+j\omega)} \cdot \frac{1}{(3+j\omega)}$$

$$= \frac{1/2}{(1+j\omega)} + \frac{-1/2}{(3+j\omega)}$$

$$y(t) = \frac{1}{2}(e^{-t} - e^{-3t})u(t)$$

$F[\delta(t)] = 1$

If $x(t) = \delta(t)$

$X(j\omega) = 1$

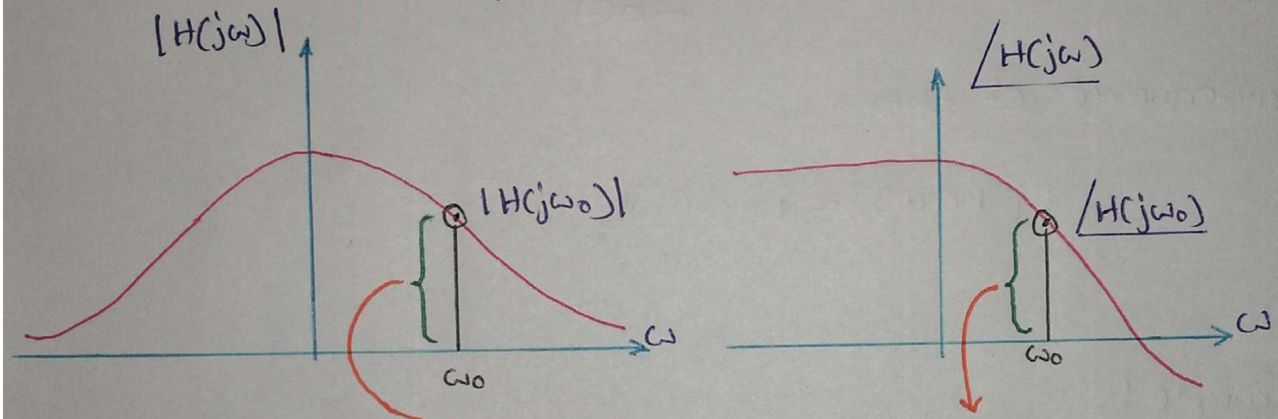
then

$Y(j\omega) = H(j\omega)$

$|H(j\omega)|$ - Magnitude frequency respons of the LTI System

$\angle H(j\omega)$ - Phas respons of the LTI System

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$



$$Y(j\omega) = |H(j\omega_0)| \, e^{j\omega_0 t + \angle H(j\omega_0)}$$

where;

$$X(j\omega) = e^{j\omega t}$$

Output spectrum is the intersection of the spectrum of the impulse respons.

* LTI systems can be characterize by differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{d^m x(t)}{dt} + \dots + b_0 x(t)$$

by applying fourier transform on both sides

$$a_n (j\omega)^n \cdot Y(j\omega) + a_{n-1} (j\omega)^{n-1} \cdot Y(j\omega) + \dots + a_0 Y(j\omega) = b_m (j\omega)^m \cdot X(j\omega) + \dots + b_0 X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_m (j\omega)^m + b_{m-1} (j\omega)^{m-1} + \dots + b_0}{a_n (j\omega)^n + a_{n-1} (j\omega)^{n-1} + \dots + a_0}$$

* Example :-

$$\frac{d^2 y(t)}{dt^2} + \frac{4 dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$H(j\omega) = \frac{(j\omega) + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

$$= \frac{(j\omega) + 2}{(j\omega + 3)(j\omega + 1)}$$

$$= \frac{1/2}{(j\omega + 3)} + \frac{1/2}{(j\omega + 1)}$$

$$h(t) = \frac{1}{2} (e^{-3t} + e^{-t}) u(t)$$