Final 2019

Problem 1

- Left: the spread of residual increases as fitted value increases. Violation of constant variance assumption. WLS
- Middle: quadratic pattern. Add a quadratic term.
- Right: skewed and an outlier. Remove the outlier and may consider different distribution of ε.

Problem 2

(a) $R^2 = \frac{SSR}{SST}$, R-Squared is the proportion of the variation explained by the model.

(b) Since

$$F = \frac{SSR}{SSE} = \frac{1}{\frac{1}{R^2} - 1} = \frac{R^2}{1 - R^2} \sim F_{n-2,2}$$

A 0.95 CI for F would be

$$0 \leq F \leq F_{0.05,n-2,2}$$

Since,

$$F = \frac{R^2}{1 - R^2} \leq F_{0.05,n-2,2}$$

Solve this, we get a CI for R^2 is

$$0 \leq R^2 \leq \frac{F_{0.05,n-2,2}}{1 + F_{0.05,n-2,2}}$$

(c) pls see practice midterm.

Problem 3

(a) Let $D_i,\ i = 1, 2$ to be

$$D_i = \begin{cases} 1 & \text{if level } i \\ 0 & \text{if not level } i \end{cases}$$

The mean function for model 1 is

$$\mathrm{E}[Y|X] = \begin{cases} \beta_0 + \beta_1 X_1 + \beta_2 X_2 & \text{base level} \\ \beta_0 + \alpha_1 + \beta_1 X_1 + \beta_2 X_2 & \text{if level 1} \\ \beta_0 + \alpha_2 + \beta_1 X_1 + \beta_2 X_2 & \text{if level 2} \end{cases}$$

The mean function for model 2 is

$$\mathrm{E}[Y|X] = \begin{cases} \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 & \text{base level} \\ \beta_0 + \alpha_{1,1} + \beta_1 X_1 + \beta_2 X_2 + \alpha_{1,2} X_1 + \beta_3 X_1 X_2 & \text{if level 1} \\ \beta_0 + \alpha_{2,1} + \beta_1 X_1 + \beta_2 X_2 + \alpha_{2,2} X_1 + \beta_3 X_1 X_2 & \text{if level 2} \end{cases}$$

(b)

$$\alpha_i : \text{difference between base level and level } i$$

$$\beta_i : \text{increase 1 unit then increase by } \beta_i$$

(c)

$$AIC = 100 \log(\exp(1.1)) + 2 \times 3 = 116$$

$$AIC = 100 \log(\exp(1)) + 2 \times 5 = 110$$

$$BIC = 100 \log(\exp(1.1)) + \log(100) \times 3 = 123.8$$

$$BIC = 100 \log(\exp(1)) + \log(100) \times 5 = 123$$

Based on AIC, BIC, Model 2 is better.

(d) $df_{H_0} = 100 - (1+2+1+1) = 95$, $df_{H_A} = 100 - (1+2+1+1+2\times1+1\times1) = 92$, so $F_{3,92}$.

Problem 4

(a) Consider

$$\min_{\beta} \sum_i w_i (Y_i - \beta X_i)^2$$

$$\begin{aligned}
\frac{\partial}{\partial \beta} \sum_i w_i (Y_i - \beta X_i)^2 &= \frac{\partial}{\partial \beta} (Y - X\beta)^T W (Y - X\beta) \\
&= -2X^T W Y + 2X^T W X \beta \\
&= 0 \\
&\Rightarrow \hat{\beta} = (X^T W X)^{-1} X^T W Y
\end{aligned}$$

(b) $Y_i \sim N(X_i \beta, \sigma^2 / w_i)$.

$$P(Y_1, \ldots, Y_n) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma} \exp\{-\frac{(y_i - X_i\beta)^2}{2\sigma^2 / w_i}\} = \frac{1}{(2\pi)^{n/2}\sigma} \exp\{-\frac{\sum_{i=1}^{n} w_i (y_i - X_i \beta)^2}{2\sigma^2}\}$$

(c) MLE of β is WLS estimate.

$$\begin{aligned}
&\frac{\partial}{\partial \sigma^2} - \frac{n}{2} \log((2\pi\sigma^2)) - \frac{\sum_{i=1}^{n} w_i (y_i - X_i\beta)^2}{2\sigma^2} = \frac{\sum_{i=1}^{n} w_i (y_i - X_i\beta)^2}{2\sigma^4} - \frac{n}{2\sigma^2} = 0 \\
&\qquad \Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} w_i (y_i - X_i\beta)^2}{n}
\end{aligned}$$

(d) Ridge:

$$\min_{\beta} \sum_i w_i (Y_i - \beta X_i)^2 + \lambda \beta^T \beta$$

$$\begin{aligned}
\frac{\partial}{\partial \beta} \sum_i w_i (Y_i - \beta X_i)^2 + \lambda \beta^T \beta &= \frac{\partial}{\partial \beta} (Y - X\beta)^T W (Y - X\beta) + 2\lambda\beta \\
&= -2X^T W Y + 2(X^T W X + \lambda I)\beta \\
&= 0 \\
&\Rightarrow \hat{\beta} = (X^T W X + \lambda I)^{-1} X^T W Y
\end{aligned}$$