Midterm 1

Problem 1

CI for $\hat{\beta}_1$: $[\hat{\beta}_1 \pm t_{\alpha/2} \times se(\hat{\beta})_1]$

Problem 2

(b)

$$\begin{aligned}
\mathrm{Var}(\hat{\beta}_0) = \sigma^2 (X^T X)^{-1}_{(1,1)} = \sigma^2 \frac{\sum_i x_i^2}{n \sum_i (x_i - \bar{x})^2} \quad &= \sigma^2 \frac{\sum_i (x_i^2 - \bar{X})^2 + n\bar{X}^2}{n \sum_i (x_i - \bar{x})^2} \\
= \sigma^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum_i (x_i - \bar{x})^2} \right)
\end{aligned}$$

$$(X^T X)^{-1} = \frac{1}{n \sum_i (x_i - \bar{x})^2} \begin{pmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{pmatrix}$$

(c)

$$\begin{aligned}
\hat{Y}_{n+1} &= \hat{\beta}_0 + \hat{\beta}_1 X_{n+1} \\
&= \bar{Y} + \hat{\beta}_1 (X_{n+1} - \bar{X}) \\
&= \bar{Y} + \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2} (X_{n+1} - \bar{X}) \\
&= \bar{Y} + \frac{\sum_{i=1}^n (X_i - \bar{X})(X_{n+1} - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} (Y_i - \bar{Y}) \\
&= \bar{Y} + \frac{\sum_{i=1}^n (X_i - \bar{X})(X_{n+1} - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} Y_i - \bar{Y}(X_{n+1} - \bar{X}) \frac{\sum_{i=1}^n X_i - n\bar{X}}{\sum_{i=1}^n (X_i - \bar{X})^2} \\
&= \sum_{i=1}^n \left[\frac{1}{n} + \frac{(X_i - \bar{X})(X_{n+1} - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \right] Y_i
\end{aligned}$$

(d) pls see solution

Problem 3

(a)

$$\begin{aligned}
Cov(\hat{e}, \hat{Y}) &= Cov((y - Hy), Hy) \\
&= Cov((I - H)y, Hy) \\
&= (I - H)Cov(y, y)H \\
&= (I - H)Cov(X\beta + e, X\beta + e)H \\
&= (I - H)(Var(X\beta) + 2Cov(X\beta, e) + Var(e))H \\
&= (I - H)Var(e)H \\
&= (I - H)\sigma^2 IH \\
&= \sigma^2(I - H)H \\
&= 0
\end{aligned}$$

$Cov(X\beta, e) = 0.$

(b) $Cov(\hat{e}, \hat{Y}) = E[\hat{e}\hat{Y}] - E[\hat{e}]E[\hat{Y}] \approx \frac{1}{n}\sum_{i=1}^{n} \hat{e}_i\hat{Y}_i - \frac{1}{(n)^2}\sum_i \hat{e}_i \sum_i \hat{Y}_i = \frac{1}{n}\hat{e}^T\hat{Y} - \frac{1}{(n)^2}\sum_i \hat{e}_i \sum_i \hat{Y}_i = \frac{1}{n}y^T(I - H)Hy - \frac{1}{(n)^2}\sum_i \hat{e}_i \sum_i \hat{Y}_i.$

$\frac{1}{(n)^2}\sum_i \hat{e}_i \sum_i \hat{Y}_i = \frac{1}{(n)^2}\sum_i \hat{Y}_i\hat{e}^T 1 = 0$

$$1 = (1, \ldots, 1).$$

$(I - H)^T X = 0,$

(c)

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_i (\hat{Y}_i - \bar{\hat{Y}})(\hat{e}_i - \bar{\hat{e}})}{\sum_i (\hat{Y}_i - \bar{\hat{Y}}_i)^2} \\
&= \frac{(\hat{Y} - \bar{\hat{Y}}1)^T(\hat{e} - \bar{\hat{e}}1)}{\sum_i (\hat{Y}_i - \bar{\hat{Y}}_i)^2} \\
&= \frac{\hat{Y}^T\hat{e} - \bar{\hat{Y}}1^T\hat{e} - \bar{\hat{e}}\hat{Y}^T 1 + \bar{\hat{Y}}\bar{\hat{e}}1^T 1}{\sum_i (\hat{Y}_i - \bar{\hat{Y}}_i)^2} \\
&= \frac{0 - 0 - \bar{\hat{e}}\sum_i \hat{Y}_i + n\bar{\hat{Y}}\bar{\hat{e}}}{\sum_i (\hat{Y}_i - \bar{\hat{Y}}_i)^2} \\
&= \frac{0 - 0 - \bar{\hat{e}}\sum_i \hat{Y}_i + n\bar{\hat{e}}\sum_i \hat{Y}_i}{\sum_i (\hat{Y}_i - \bar{\hat{Y}}_i)^2} \\
&= 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_i (Y_i - \bar{Y})(\hat{e}_i - \bar{\hat{e}})}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{(Y - \bar{Y}1)^T(\hat{e} - \bar{\hat{e}}1)}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{Y^T\hat{e} - \bar{Y}1^T\hat{e} - \bar{\hat{e}}Y^T1 + \bar{Y}\bar{\hat{e}}1^T1}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{Y^T\hat{e} - 0 - \bar{\hat{e}}\sum_i Y_i + n\bar{Y}\bar{\hat{e}}}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{Y^T\hat{e}}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{Y^T\hat{e} - \hat{Y}^T\hat{e}}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{(Y - \hat{Y})^T\hat{e}}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= \frac{\hat{e}^2}{\sum_i (Y_i - \bar{Y}_i)^2} \\
&= R^2
\end{aligned}
$$

because $\hat{Y}^T\hat{e} = 0$.

$\hat{\beta}_0$