DispersalNim: A Combinatorial Game on Graphs

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HON 495
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May 2025

Abstract

A combinatorial game is a two-player game of perfect information, in which players alternate moves with no element of chance. One of the most fundamental combinatorial games is Nim, where players remove tokens from a set of heaps of tokens. There are variations of Nim that allow players to split heaps, as well as ones in which heaps are positioned on the vertices of a graph (a network). This project explores the intersection of these two variations with a new type of game called DispersalNim. In DispersalNim, tokens are placed on the vertices of a graph. Players are allowed to remove tokens from a vertex, and optionally split the remaining tokens between the adjacent vertices. The initial analysis will involve computational methods, including calculation of nimbers (Grundy numbers), as well as a fake probabilistic method, in which randomized gameplay is simulated and progressively de-randomized to extract strategic behavior. This research aims to add to the literature in several topics, including games on graphs, dispersal models (such as chip-firing games), and graph pebbling, by exploring a novel game that unites these frameworks.

1 Introduction

A combinatorial game is a game between two players who alternate turns, where each of them has perfect information. This means that they have full knowledge of the game state and possible moves. Accordingly, such games avoid any elements of random chance; there is no card shuffling and no dice. This definition includes many familiar games, such as Chess, Tic-Tac-Toe, and Dots-and-Boxes. This familiarity is a great motivator for research into a general theory of these games. Combinatorial game theory (CGT), while having some early roots, including 1930's works by Sprague and Grundy, largely exists to the extent that it does today due to work in the 1970's and 1980's by Conway, Berlekamp, and Guy [Sie13]. Their work, consolidated in the four-volume *Winning Ways for Your Mathematical Plays*, contains a thorough introduction to the subject [BCG05]. In comparison to the fields of algebra, geometry, and analysis, which have had centuries to develop, it is a newer field, and remains

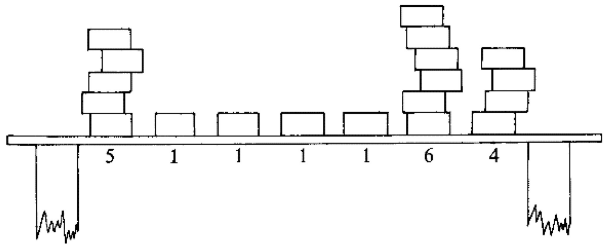

Figure 1: An example of a Nim position (from [BCG05]).

an active area of research. Games tend to be natural human thought experiments and pasttimes, and in fact much of the field originates from recreational mathematics. However, CGT as it stands today has found applications in logic, complexity theory, biology, and artificial intelligence [BC01; LC16].

A very important example of a combinatorial game is that of **Nim**. In Nim, there are several heaps of tokens (also called sticks or pieces). Each player, on their turn, is able to take as many tokens from a heap as they want, but cannot take tokens from more than a single heap [Con06]; see figure 1. The problem of the existence of a winning strategy was solved completely over a century ago. That is, from any starting position, we are able to determine the best possible moves and ultimately who will win the game [Bou01]. We note two major aspects of Nim which give it its utility. First, it is **impartial**, meaning both players have access to the same set of moves. Second, it is carried out in **normal-play**, meaning the win-condition for a players is that the next player has no possible moves. It was found by Sprague and Grundy independently that any impartial game in normal-play is equivalent to a game of Nim, making it profoundly useful in theoretical analysis. For example, because we can evaluate every game of Nim as either an $\mathcal{N}$-position (the first player can always win) or a $\mathcal{P}$-position (the second player can always win), we are able to do the same for any impartial game in normal-play [Con06].

Naturally, variations of Nim have been developed and studied. Some have combined it with graphs, for example. A **graph** $G = (V, E)$ is a heavily-studied object in discrete mathematics, and consists of a vertex set V along with an edge set E, with endpoints of edges in the vertex set. In essence, a graph is a mathematical way to show connections between objects. In the context of CGT, there are many different games that can be played on and with graphs, as well as game theory that can be developed using graphs. In particular, graphs and Nim have already been studied together before. For example, consider the game **Vertex NimG**, in which tokens are removed from vertices, and the adjacency of the graph determines where players may move on successive turns. This has yielded some interesting results. For example, the complexity of determining whether the first or second player wins a game of Vertex NimG has been looked at and shown to be PSPACE-complete [BG14].

Other variations of Nim have studied the option of splitting heaps, recognizing how this behavior might help to understand a wider scope of games. In **Kayles**, for example, the game starts with a number of bowling pins lined up in a row. On a player's turn, they roll a bowling ball and can knock down either one or two pins; see figure 2. The winner is the

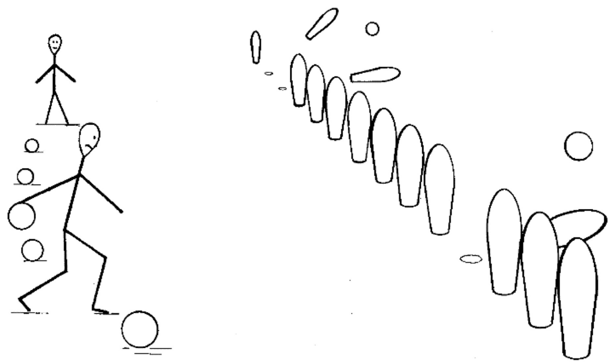

Figure 2: Kayles (from [BCG05])

player to knock down the final pin. Because this is an impartial game in normal-play, by Sprague-Grundy, we can equate the initial line of pins with a heap of Nim tokens. When a player hits pins, (notably when they hit a pin in the middle of the line,) the line splits; the players will no longer be able to hit pins from both parts of the line. It can be shown that the equivalent heap of Nim tokens is also split in two [BCG05].

With Nim and its application to games like Kayles, we see the effects of splitting heaps in two. However, we could also ask about the possibility of splitting heaps into more than two parts. Moreover, we can wonder how the structure of a graph can help to confine such a behavior. One instance of tokens being split around a graph is **chip-firing**, where chips (i.e. tokens) are placed on the vertices of a graph, and are then "fired" between adjacent vertices. The chips of a chip-firing game do not behave like the tokens in a game of Nim, as chip-firing is not combinatorial per se. There is therefore room to investigate such a game.

In this project, we further explore the relationship between Nim and graphs by looking at how assigning heaps to vertices and permitting moves that split heaps between vertices affect winning strategies, and how this relationship can be related to typical Nim. This generalization of Nim-type moves on graphs has had little to no previous research done on it. More precisely, we consider the following game.

Definition 1.1. Let $G = (V, E)$ be a graph. A *(token) configuration* is a function $T : V \to \mathbb{N}$ that indicates the number of tokens on each vertex.

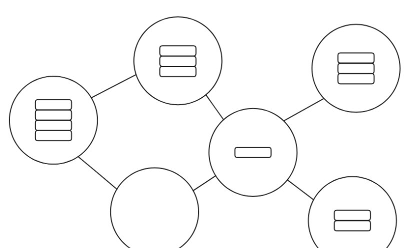

Figure 3: An example of a token configuration on a graph.

Definition 1.2 (DispersalNim)**.** The game board is a graph $G = (V, E)$ with a token configuration T. On a player's turn, they pick a vertex $v \in V$, and an amount $n \in \mathbb{N}$. Let k be the degree of v and let $u_1, u_2, \ldots, u_k$ be the neighbors of v. The player also picks nonnegative integers $a_{u_1}, a_{u_2}, \ldots, a_{u_k}$ such that $n + \sum_{i=1}^{k} a_{u_i} \leq T(v)$ and at least one of $n, a_{u_1}, a_{u_2}, \ldots, a_{u_k}$ is positive. We remove n tokens from v and then send a_{u_i} tokens from v to u_i. This yields a new configuration T' where

$$T'(u) = \begin{cases} T(v) - n - \sum_{i=1}^{k} a_{u_i}, & u = v \\ T(u) + a_u, & uv \in E \\ T(u), & uv \notin E \end{cases}$$

The next player moves on the game board G with the configuration T'. If $T'(v) = 0$ for all $v \in G$, then the next player loses.

Remark 1.3. DispersalNim can be viewed as a generalization of some of the aforementioned games.

1. Nim is the special case of DispersalNim in which $G = (V, \varnothing)$.
2. Kayles is the case of DispersalNim in which G is a binary tree, and the initial configuration T has a single heap of tokens at the root.

Because this is a finite impartial game in normal play, one player must eventually win. Hence, the major questions in the analysis of this game are the following.

Question 1.4. Given a graph G with configuration T,

1. What is the optimal move for the player to make?
2. With both players playing optimally, who is the eventual winner of the game?

Accordingly, we will investigate how changes in G and changes in T affect the outcome of these questions. We may also look further into variations of the game rules and ask the same questions.

Example 1.5 (Variations of DispersalNim)**.** We may vary the rules for making a move:

1. (Complete DispersalNim). A player makes a move as in DispersalNim, but when picking n and $a_{u_1}, \ldots a_{u_k}$, we require that $n + \sum_{i=1}^{k} a_{u_i} = T(v)$.
2. (Constant DispersalNim). A player makes a move as in DispersalNim, but when picking $a_{u_1}, \ldots a_{u_k}$, we require that $a_{u_1} = a_{u_2} = \cdots = a_{u_k}$.

Or we may vary the win-condition:

3. (Scored DispersalNim). Each player has a score which starts at zero. The players move as in DispersalNim, but after their move, n is added to their score. When $T'(v) = 0$ for all v, the winner is the player with the higher score.

2 Literature Review

In this survey of the current theory, we look at three areas: impartial combinatorial games, graph-based games, and dispersal mechanisms on graphs. We find that while each of these areas has been studied individually, there remains room to explore games which combine all three—in particular, games where tokens are removed and redistributed across neighboring vertices.

To build context for these games, we first review the key developments in combinatorial game theory. This includes an examination of classical impartial games such as Nim, which form the foundation of much of the theory. From there, we consider how graph structures have been incorporated into gameplay, followed by a look at systems that involve redistribution or dispersal of objects on graphs. Surveying these areas will demonstrate how existing work informs (and leaves space for) new game variants that combine these elements.

2.1 Impartial Games

Before looking at games on graphs, a natural starting point is classical combinatorial games. One of the most fundamental impartial games is Nim, first analyzed mathematically by Bouton in 1901. The key insight of Bouton was to consider the value of each heap in binary, and then apply bit-wise XOR. For example, if we have three piles of 5, 9, and 12 tokens, then we first consider that

$$5 = 101_2, \quad 9 = 1001_2, \quad 12 = 1100_2$$

Taking the bit-wise XOR across these values, we get

$$101_2 \oplus 1001_2 \oplus 1100_2 = 0000_2,$$

so our final value for the game is 0. If the value of a game is 0, then the next player to move will lose; otherwise, the next player has a winning strategy [Bou01]. This bit-wise XOR operation is referred to as taking the **nim sum** of the heaps.

Question 2.1 (Sums vs Nim Sums)**.** How do the game strategies and outcomes change when re-distribution follow the rule $T'(u_i) = T(u_i) \oplus a_{u_i}$, using nim sums rather than normal sums?

In a game of nim, the resulting nim sum, called the **nimber** (or **Grundy number**) of the game, not only tells us which player can win, but also how the game outcome would change if other heaps were to be added to the game.

Despite its simple rules, Nim holds great significance in the theory of impartial games, particularly those which observe normal-play. Sprague and Grundy both independently proved the following theorem in the mid-1930s.

Theorem 2.2 (Sprague-Grundy)**.** Every impartial combinatorial game in normal-play is equivalent to a Nim heap of some size.

Essentially, this means that we are able to assign any game a nimber [BCG05]. This is particularly useful when combined with nim sums. The **disjunctive sum** of two combinatorial games is the game created by placing the two games side-by-side, in which a player can choose to make their move in one of the two games. This is analogous to how players must make a single heap in Nim; that is, a game of Nim is the disjunctive sum of several heaps. A complex game can occasionally be broken down and considered as the disjunctive sum of several simpler games, each of which can be assigned a nimber. Taking the nim sum then allows us to assign the more whole game a nimber, and provides insight into strategies for the more complex game.

Within classical combinatorial games, there are other variants of Nim. **Octal games** are a generalization of Nim and help to categorize Nim-like games; every octal game has a corresponding octal code $d_0.d_1d_2d_3\ldots$. Each d_i is a digit from 0 to 7 and dictates what types of moves are allowed to be performed after taking i tokens away from a pile. We note in particular that when some d_i is 4 or more, the player is allowed to split the remaining tokens in a heap into two piles. Phrased another way, we are sometimes allowed to disperse tokens into two piles [BCG05]. In this sense, because octal games and their nimbers are relatively well-studied, they will likely be instrumental in the analysis of DispersalNim.

2.2 Vertex-Based Games

Another broad and well-studied class of combinatorial games involves playing on the vertices of a graph. The edges of the graph then constrain players' options and add new levels of strategic complexity. One particular type within this class is **pursuit-evasion games**, where we find something like **Cops and Robbers**. In Cops and Robbers, the base playing field is a finite undirected graph (we can traverse its edges in either direction). There are one or more cops and a robber who occupy the vertices of the graph and move along the edges of the graph. The cops aim to catch the robber by landing on the same vertex as them, while the robber attempts to avoid this fate indefinitely.

A major focus in studying Cops and Robbers is determining the **cop number** of a graph–the minimum number of cops needed to guarantee a win for the cops regardless of the robber's strategy. This heavily involves structural graph theory–we find some types of graphs (like trees and planar graphs) have known cop numbers, while other types are open problems. For our game of DispersalNim, we might take inspiration from the graph-related invariants in Cops and Robbers, such as the cop number to construct our own invariants which are independent of the configuration of tokens on the graph. Other pursuit-evasion variants explore changes in movement rules, probabilistic elements, and limited visibility of the players. In general, we find a rich interplay of graph topology and game dynamics, showing that graph structure directly affects strategies and outcomes [BN11].

Pursuit-evasion games are generally partisan rather than impartial, as each player has a distinct role. However, impartial games on graphs have also been studied before. A family of games within this category is graph-based Nim, wherein heaps of tokens are placed on vertices of a graph, and players take turns removing tokens from a vertex, but have moves constrained by the structure of the graph. One such game is **Vertex Nim** or **Vertex NimG**, where

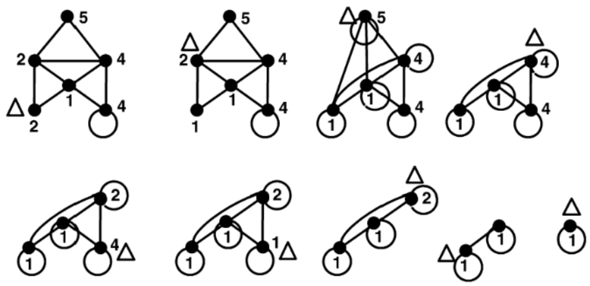

Figure 4: Playing VertexNim (from [DR14]).

players move a special indicator along the edges of a graph, (shown in Figure 4 as a triangle,) dictating which vertex they can remove nim tokens from. On successive moves, players can only move the indicator to an adjacent vertex. This makes the graph's connectivity essential to analyze [Duc+17; Sto04]. Generally, graph-based Nim blends the algebraic/arithmetic theory of Nim with the more spatial and combinatorial aspects of graphs. This naturally builds a bridge to the study of dispersal mechanisms, where tokens may be redistributed to neighboring vertices.

By examining pursuit-evasion games and impartial Nim-like games on graphs, we see that graph structure plays a heavy role in governing player movement, decision-making, and optimal strategies. As we look at DispersalNim, we will find both impartial gameplay and graph-based dynamics crucial to understand.

2.3 Dispersal Mechanisms

To be more precise, we define **dispersal** as the redistribution of objects within a network, in contrast to the deletion or addition of said objects. In this project, the dispersal of tokens between vertices of a graph is the particular game mechanism of interest. While dispersal on its own is a well-studied phenomenon, its potential role in impartial combinatorial games has not been developed.

A closely related and well-studied example of dispersal on graphs is the **abelian sandpile model**. In this model, each vertex of a graph holds grains of sand. Sand is added to the model, and when the number of grains at a vertex reaches a certain threshold, it topples, and sends a grain of sand to each of its neighboring vertices. An initial toppling may cause a chain reaction that cascades throughout the graph. These models are known for their connections to other areas including dynamical systems, statistical physics, and even group theory [Dha90; Woo16].

Sandpile models may help to create a framework for dispersal within games. Rather than solely considering local redistributions of tokens, an optimal strategy might have a player deliberately cause redistributions throughout the graph that take several moves to play out. This could add an interesting layer of complexity, where players must consider both the ongoing depletion of tokens as well as the spread of tokens across the game field. A strategy may try to ensure that certain areas of the graph are difficult to deplete, for example.

Often considered alongside sandpile models are **chip-firing games**, which involve pushing whole numbers ("firing chips") from a vertex to all its neighboring vertices, according to fixed rules. Research has focused on reachability–if certain chip configurations may be transformed into one another–as well as periodicity–if certain firings cause repeated behavior and how often [BL92].

2.4 Summary and Research Gap

Between these many theories and analyses of games and models, there has been relatively little research done on impartial games that use dispersal mechanisms.

Question 2.3. Incorporating such a mechanism opens the door to several new strategic questions.

1. How does the redistribution of tokens affect nimber calculation? Can players manipulate dispersal patterns of tokens and control future game states?

2. What effects and constraints does the structure and shape of the underlying graph pose on strategies?

Such questions are so far unanswered, as the intersection of impartial games, structural graph theory, and dispersal mechanisms is still open for exploration. Considering the connections to processes such as sandpiles, the development of dispersal games could ultimately be an avenue to extend combinatorial game theory into more dynamic areas.

To summarize, combinatorial game theory has had a rich history, with the analysis of classical impartial games like Nim providing deep insight into game strategy, using tools such as nimbers and the Sprague-Grundy theorem. One direction in which combinatorial games have evolved is the analysis of games on graphs. This introduces spatial and structural constraints that shape gameplay. Pursuit games such as Cops and Robbers demonstrate how a graph's shape can define outcomes and strategy, while adaptations of Nim to graphs have explored how heaps can interact across a network. Separate but related to graph-based games is the study of dispersal mechanisms and processes, including abelian sandpile models and chip-firing games. These may help to illustrate how resources can flow throughout the global structure of a graph given simple local rules.

Despite the overlap in themes, there remains a gap in the intersection of impartial combinatorial games, graph-based games, and dispersal dynamics. Few works have explored games where players interact through both subtraction and redistribution of tokens, blending elements of Nim with the toppling and firing behaviors seen in sandpiles and chip-firing models. This presents an opportunity to develop and analyze new impartial games on graphs where dispersal is an integral part of game strategy. This could reveal novel strategic behaviors and deepen the connections between combinatorial game theory and dynamic processes on graphs.

3 Methodology

In order to analyze the strategies and win-conditions for my Nim-like games, I will use a joint theoretical-computational approach, combining mathematical proofs, manual and computer-assisted computations, and simulated gameplay. The theoretical aspect will involve defining key terminology and game rules, extending my literature review to explore related topics, formulating conjectures, and proving theorems. The computational aspect will consist of nimber computations and simulated random gameplay, with potential refinements using derandomization techniques. The overall goals are to view computational results as suggestions for formal mathematical results, and to find how ideas from similar games/topics can be incorporated into my analysis. Because of this, my project will have exploratory, descriptive, and explanatory aspects, with computations and observations being more exploratory and descriptive, while the subsequent formal proofs will be explanatory.

3.1 Minimum Excludant Method

The first stage will involve manually computing nimbers for small cases using the "minimum excludant" (mex) method, which is a standard method for computing nimbers [Con06]. These early computations will focus on small graphs of basic types such as lines/paths, cycles, trees, and complete graphs, where the structure is relatively manageable. For each graph, various configurations of Nim heaps will be placed and the corresponding nimber will be calculated. Results will indicate which player can win, and will provide insight into initial conjectures about the interplay between graph structure and Nim piece configurations. These calculations will then be expanded using computational methods, allowing for larger graphs and more complex scenarios. This will be carried out using an algorithmic approach in a suitable programming language, such as Python, and run on my personal computer. Results will be preliminarily stored in a standard spreadsheet format using software such as Google Sheets. To ensure accuracy, the manual results for small cases and computer-assisted results will be cross-verified.

3.2 Fake Probabilistic Method

Beyond these static calculations, I will also conduct random gameplay simulations to identify possible strategies and patterns in play. This will be executed in the same programming language as previously used for nimbers. In viewing randomized gameplay, I may find emergent behaviors that are otherwise not immediately apparent. For example, I will track whether or not dispersal generally benefits or hinders a player. By learning more about these tendencies, they can be reincorporated into simulations; the simulated players will be programmed to make certain moves when faced with certain patterns. This follows an existing paradigm for combinatorial game analysis, called the "fake probabilistic method." [Bec08] As this process of derandomization continues, the simulation will transition towards both players having knowledge of their optimal strategies. Each step in this process will be logged in order to inform subsequent theoretical analysis.

3.3 Resources and Organization

Regarding the theoretical aspect, my primary source of information will be academic literature, including any relevant textbooks and research papers on combinatorial game theory, graph games, and dynamical processes on graphs. I will access these materials though online scholarly databases such as UH OneSearch and arXiv.org, as well as tangibly through the UH Hamilton Library. This will ensure that my research is grounded in existing mathematical frameworks while also extending them in a novel direction. Additionally, as I plan to learn to code, I will use online references and previous knowledge of programming in order to build and run my programs.

Once all data has been collected, whether through computations, simulation, or proof, I will organize my findings systematically. Theoretical findings, including key conjectures and theorems, will build the overall structure of the thesis, while nimber calculations and simulation results may be placed into tables according to graph type and Nim token configuration, serving as supporting examples and evidence. By integrating these different approaches, I aim to develop an understanding of dispersal game behavior and where it fits within the broader landscape of combinatorial game theory.

3.4 Estimated Timeline

See Table 1 for an estimated timeline for this project.

Date	Goals
July-Sep '25	Compute nimbers by hand for small graphs
	Settle on a programming language and learn to compute nimbers and simulate games on graphs.
	Identify early patterns/preliminary conjectures.
Oct-Nov '25	Begin formal proof attempts for small graphs.
	Scale up computational nimber calculations on more complex graphs.
	Create formal conjectures based on observations.
Dec '25-Jan'26	Strengthen theoretical proofs; considering families of graphs, for example.
	Systematically consider different variations in game rules and parameters.
	Refine research direction based on findings.
Feb-Mar '26	Organize core theoretical results.
	Reflect on open questions.
Apr-May '26	Compose, edit, receive feedback, and submit the final paper.

Table 1: Outline of Project

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