

CSC411: Assignment 3

Due on Sunday, December 3rd, 2017

Student Name: **Gokul K. Kaushik**

Student Number: **999878191**

Table of Contents

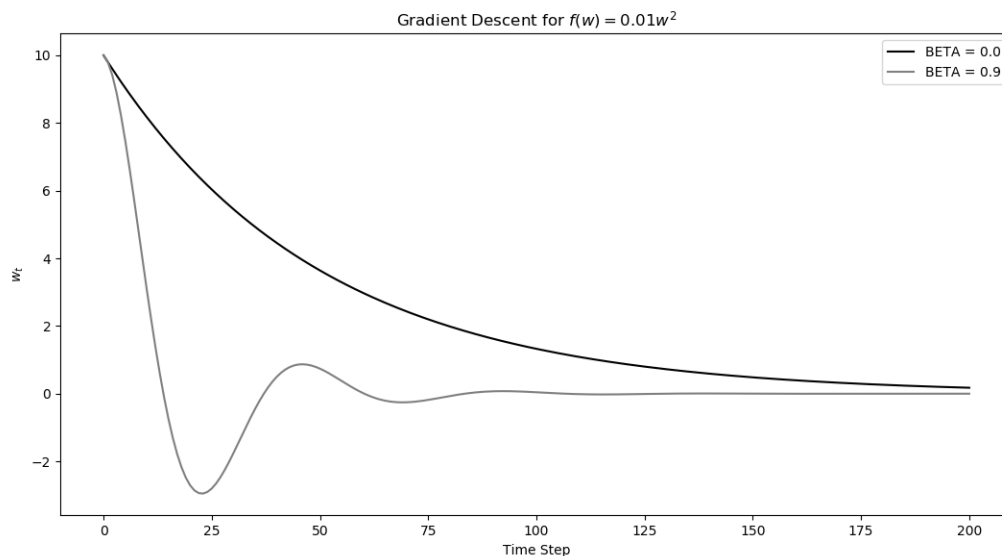
1 - 20 Newsgroup Predictions	1
2 - Training SVM with SGD	1
2.1 - SGD with Momentum	1
2.2 - Training SVM	2
2.3 - Apply 4-vs-9 Digits on MNIST	2
2.3.1 - Training Loss	2
2.3.2 - Test Loss	2
2.3.3 - Classification Accuracy on the Training Set	2
2.3.4 - Classification Accuracy on the Test Set	2
2.3.5 - Plot w as a 28×28 image	2
3 - Kernels	2
3.1 - Positive Semi definite and Quadratic Form	2
3.2 - Kernel Properties	3
3.2.1 - Prove Property $k(\mathbf{x}, \mathbf{y}) = \alpha$ is a kernel for $\alpha > 0$	3
3.2.2 - Prove Property $k(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) \cdot f(\mathbf{y})$ is a kernel for $f : \mathbb{R}^d \rightarrow \mathbb{R}$	3
3.2.3 - Prove Property If $k_1(\mathbf{x}, \mathbf{y})$ and $k_2(\mathbf{x}, \mathbf{y})$ are kernels then $k(\mathbf{x}, \mathbf{y}) = a \cdot k_1(\mathbf{x}, \mathbf{y}) + b \cdot k_2(\mathbf{x}, \mathbf{y})$ for $a, b > 0$ is a kernel	3
3.2.4 - Prove Property If $k_1(\mathbf{x}, \mathbf{y})$ is a kernel then $k(\mathbf{x}, \mathbf{y}) = \frac{k_1(\mathbf{x}, \mathbf{y})}{\sqrt{k_1(\mathbf{x}, \mathbf{x})} \sqrt{k_1(\mathbf{y}, \mathbf{y})}}$ is a kernel	4

1 - 20 Newsgroup Predictions

2 - Training SVM with SGD

2.1 - SGD with Momentum

Plotting w_t for each time step t by applying the iterative stochastic gradient-descent on $f(w) = 0.01w^2$, we get the following graph (for $\beta = 0.1$ and $\beta = 0.9$ for upto 200 time-steps):



2.2 - Training SVM

2.3 - Apply 4-vs-9 Digits on MNIST

2.3.1 - Training Loss

2.3.2 - Test Loss

2.3.3 - Classification Accuracy on the Training Set

2.3.4 - Classification Accuracy on the Test Set

2.3.5 - Plot w as a 28×28 image

3 - Kernels

3.1 - Positive Semi definite and Quadratic Form

Prove that a symmetric matrix $K \in \mathbb{R}^{d \times d}$ is a positive semi definite iff for all vectors x we have $\mathbf{x}^T K \mathbf{x} \geq 0$.

Proof:

$$K \mathbf{x} = \lambda \mathbf{x}$$

where λ is the eigenvalue and \mathbf{x} is the eigenvector.

Suppose \mathbf{x} is an eigenvector of K and replacing $K \mathbf{v}$ with $\lambda \mathbf{v}$ (from the definition of an eigenvector and eigenvalue above):

$$\mathbf{x}^T K \mathbf{x} = \mathbf{x}^T \mathbf{x} \lambda$$

$$\mathbf{x}^T K \mathbf{x} = |\mathbf{x}|^2 \lambda$$

Therefore, as $|\mathbf{x}|^2 \geq 0$, for the equation $\mathbf{x}^T K \mathbf{x} \geq 0$, the eigenvalue must be: $\lambda \geq 0$. Since it is a semi-definite matrix, where the eigenvalues ≥ 0 , this holds true.

3.2 - Kernel Properties

3.2.1 - Prove Property $k(\mathbf{x}, \mathbf{y}) = \alpha$ is a kernel for $\alpha > 0$

$$\phi(\mathbf{x}) = \sqrt{\alpha}$$

$$k(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$$

$$k(\mathbf{x}, \mathbf{y}) = \sqrt{\alpha}\sqrt{\alpha}$$

$$k(\mathbf{x}, \mathbf{y}) = \alpha$$

3.2.2 - Prove Property $k(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) \cdot f(\mathbf{y})$ is a kernel for $f : \mathbb{R}^d \rightarrow \mathbb{R}$

$$\phi(\mathbf{x}) = f(\mathbf{x})$$

$$k(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$$

$$k(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) \cdot f(\mathbf{y})$$

3.2.3 - Prove Property If $k_1(\mathbf{x}, \mathbf{y})$ and $k_2(\mathbf{x}, \mathbf{y})$ are kernels then $k(\mathbf{x}, \mathbf{y}) = a \cdot k_1(\mathbf{x}, \mathbf{y}) + b \cdot k_2(\mathbf{x}, \mathbf{y})$ for $a, b > 0$ is a kernel

Let \mathbf{K}_1 and \mathbf{K}_2 be gram matrices

Therefore:

$$\mathbf{x}^T \mathbf{K}_1 \mathbf{x} \geq 0$$

and

$$\mathbf{x}^T \mathbf{K}_2 \mathbf{x} \geq 0$$

Let:

$$\mathbf{K}_3(\mathbf{x}, \mathbf{y}) = a\mathbf{K}_1(\mathbf{x}, \mathbf{y}) + b\mathbf{K}_2(\mathbf{x}, \mathbf{y})$$

$$\mathbf{K}_3 = a\mathbf{K}_1 + b\mathbf{K}_2$$

$$\mathbf{x}^T \mathbf{K}_3 \mathbf{x} = \mathbf{x}^T (a\mathbf{K}_1 + b\mathbf{K}_2) \mathbf{x}$$

$$\mathbf{x}^T \mathbf{K}_3 \mathbf{x} = a\mathbf{x}^T \mathbf{K}_1 \mathbf{x} + b\mathbf{x}^T \mathbf{K}_2 \mathbf{x}$$

and using 3.2.1's proof, $\mathbf{K}_3(\mathbf{x}, \mathbf{y})$ is a kernel, we get:

$$a\mathbf{x}^T \mathbf{K}_1 \mathbf{x} + b\mathbf{x}^T \mathbf{K}_2 \mathbf{x} \geq 0 \text{ for } a, b > 0$$

3.2.4 - Prove Property If $k_1(\mathbf{x}, \mathbf{y})$ is a kernel then $k(\mathbf{x}, \mathbf{y}) = \frac{k_1(\mathbf{x}, \mathbf{y})}{\sqrt{k_1(\mathbf{x}, \mathbf{x})}\sqrt{k_1(\mathbf{y}, \mathbf{y})}}$ is a kernel

Let:

$$k_1(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$$

$$\langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle = \phi(\mathbf{x}) \cdot \phi(\mathbf{y})$$

In equation $k_1(\mathbf{x}, \mathbf{y})$, replacing all the instances of $k_1(\mathbf{x}, \mathbf{y})$ with $\phi(\mathbf{x}) \cdot \phi(\mathbf{y})$:

$$\begin{aligned} k(\mathbf{x}, \mathbf{y}) &= \frac{k_1(\mathbf{x}, \mathbf{y})}{\sqrt{k_1(\mathbf{x}, \mathbf{x})}\sqrt{k_1(\mathbf{y}, \mathbf{y})}} \\ k(\mathbf{x}, \mathbf{y}) &= \frac{\phi(\mathbf{x}) \cdot \phi(\mathbf{y})}{\sqrt{\phi(\mathbf{x}) \cdot \phi(\mathbf{x})}\sqrt{\phi(\mathbf{y}) \cdot \phi(\mathbf{y})}} \\ k(\mathbf{x}, \mathbf{y}) &= \frac{\phi(\mathbf{x}) \cdot \phi(\mathbf{y})}{|\phi(\mathbf{x})||\phi(\mathbf{y})|} \\ k(\mathbf{x}, \mathbf{y}) &= \frac{k_1(\mathbf{x}, \mathbf{y})}{|\phi(\mathbf{x})||\phi(\mathbf{y})|} \end{aligned}$$

Let \mathbf{K}_1 for matrix $k_1(\mathbf{x}, \mathbf{y})$ be a gram matrix. Therefore:

$$\mathbf{x}^T \mathbf{K}_1 \mathbf{x} \geq 0$$

Multiplying each entry of k_1 will result in a positive value. Therefore, $k(\mathbf{x}, \mathbf{y})$ is a kernel.