Computational Fluid Dynamics

Finite Volume Method – part 1

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Finite Volume Method

- Goal is to solve the incompressible Navier Stokes and scalar transport (energy, …) equations numerically

Step 2

$$\nabla . \vec{\mathrm{v}} = 0$$

$$\frac{\partial \rho \vec{\mathrm{v}}}{\partial t} + \nabla . (\rho \vec{\mathrm{v}} \vec{\mathrm{v}}) = -\nabla p + \nabla . (\mu \nabla \vec{\mathrm{v}}) + f_B$$

Step 1

$$\frac{\partial \rho \phi}{\partial t} + \boxed{\nabla . (\rho \phi \vec{\mathrm{v}}) = \nabla . (\Gamma \nabla \phi) + q_\phi}$$

Step 1.1 (no temporal term)

Step 1.2 (with temporal term)

Finite Volume Method

- We start with the steady scalar transport equation $\nabla.(\rho\phi\vec{\mathrm{v}}) = \nabla.(\Gamma\nabla\phi) + q_\phi$

- Finite Volume Method uses the integral form of the governing equations

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\,\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S + \int_V q_\phi\,\mathrm{d}V$$

- The equation is solved for the unknown ϕ. All other quantities are treated as known (in reality, velocity itself comes from a solution of the momentum equation).

$$\int_S \rho\boxed{\phi}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\,\nabla\boxed{\phi}\cdot\mathbf{n}\,\mathrm{d}S + \int_V q_\phi\,\mathrm{d}V$$

Unknonw: ϕ

Known parameters: ρ , $\mathbf{v}$, Γ , q_ϕ

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Finite Volume Method

$$\boxed{\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S} = \boxed{\int_S \Gamma\,\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S} + \boxed{\int_V q_\phi\,\mathrm{d}V}$$

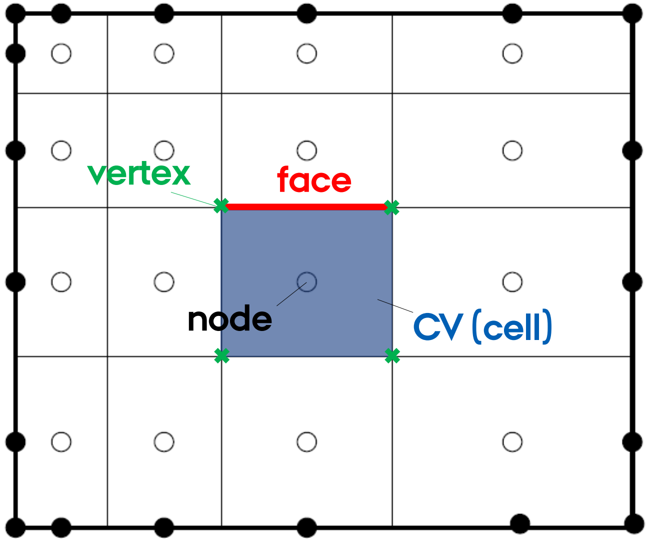
Surface integrals

Volume integral

Unknonw: ϕ

Known parameters: ρ , $\mathbf{v}$, Γ , q_ϕ

- Basic concept of FVM: split the domain into a finite number of Control Volumes (cells); satisfy the governing equation for each one of them.

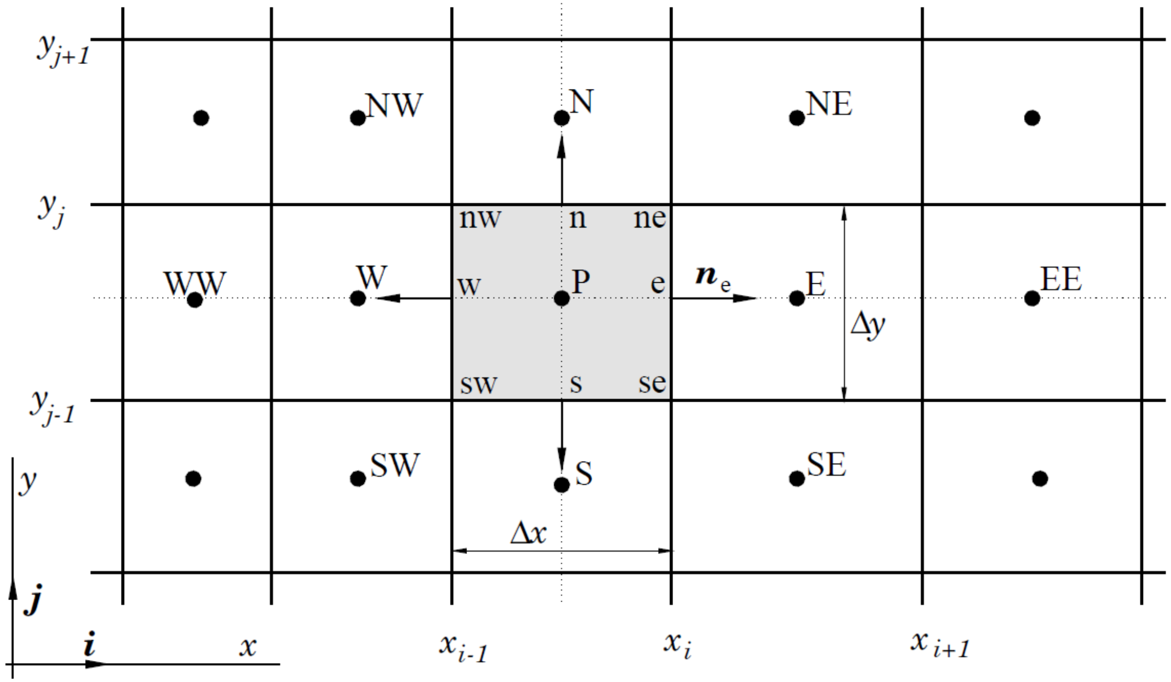


- Nodes are the locations where the value of the variable ϕ is evaluated.
- If there are N cells, there will be N equations and N unknowns (one ϕ per node). This system of equations yields the solution
- It is most common to have the nodes at the geometric center of the cell (cell-centered approach); however, other approaches are also used.

Finite Volume Method

- For simplicity, compas notationis can be used : the node of interest is labeled P, and the neighbors N, W, E, … Lower case letters are used for the faces and vertexes.

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\,\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S + \int_V q_\phi\,\mathrm{d}V$$



Finite Volume Method

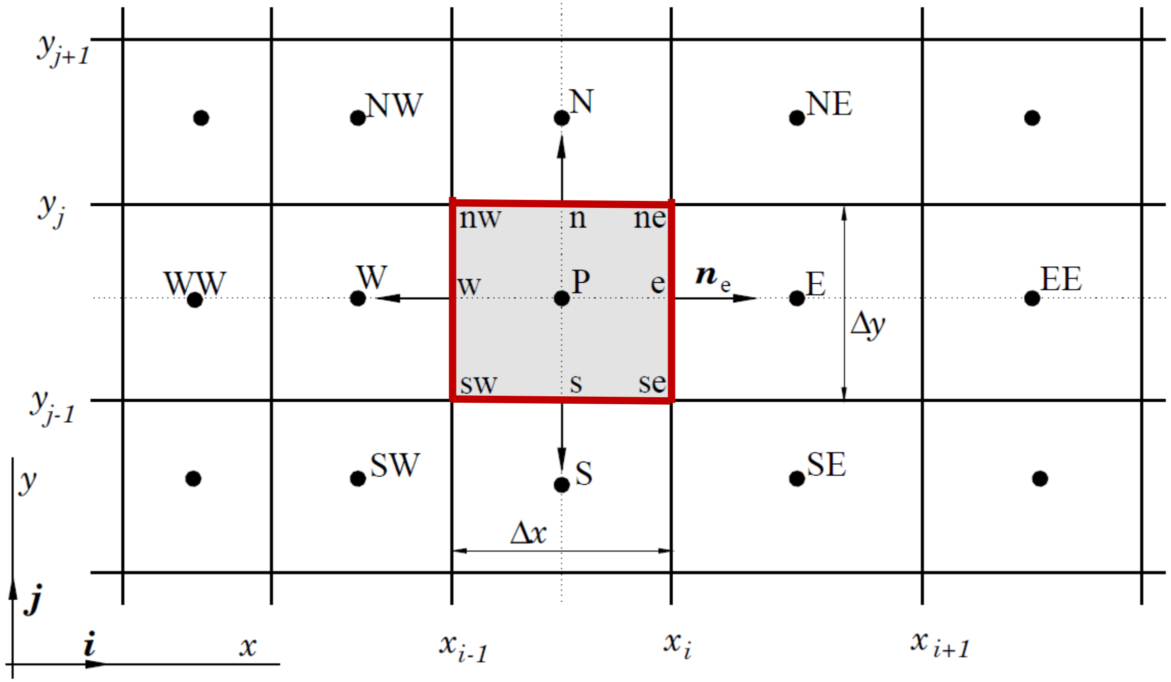
- For simplicity, compas notationis can be used: the node of interest is labeled P, and the neighbors N, W, E, … Lower case letters are used for the faces and vertexes.

- First we focus on surface integrals

$$\int_S \underbrace{\boxed{\rho\phi\mathbf{v}\cdot\mathbf{n}}}_{f^{\mathrm{c}}}\,\mathrm{d}S = \int_S \underbrace{\boxed{\Gamma\,\nabla\phi\cdot\mathbf{n}}}_{f^{\mathrm{d}}}\,\mathrm{d}S + \int_V q_\phi\,\mathrm{d}V$$

- Each surface integral is sum of surface integrals on all 4 (2D) or 6 (3D) boundaries.

$$\int_S f\ \mathrm{d}S = \sum_k \int_{S_k} f\ \mathrm{d}S$$



Approximation of surface integrals

- Calculating surface integrals is done in two steps:

Interpolation:

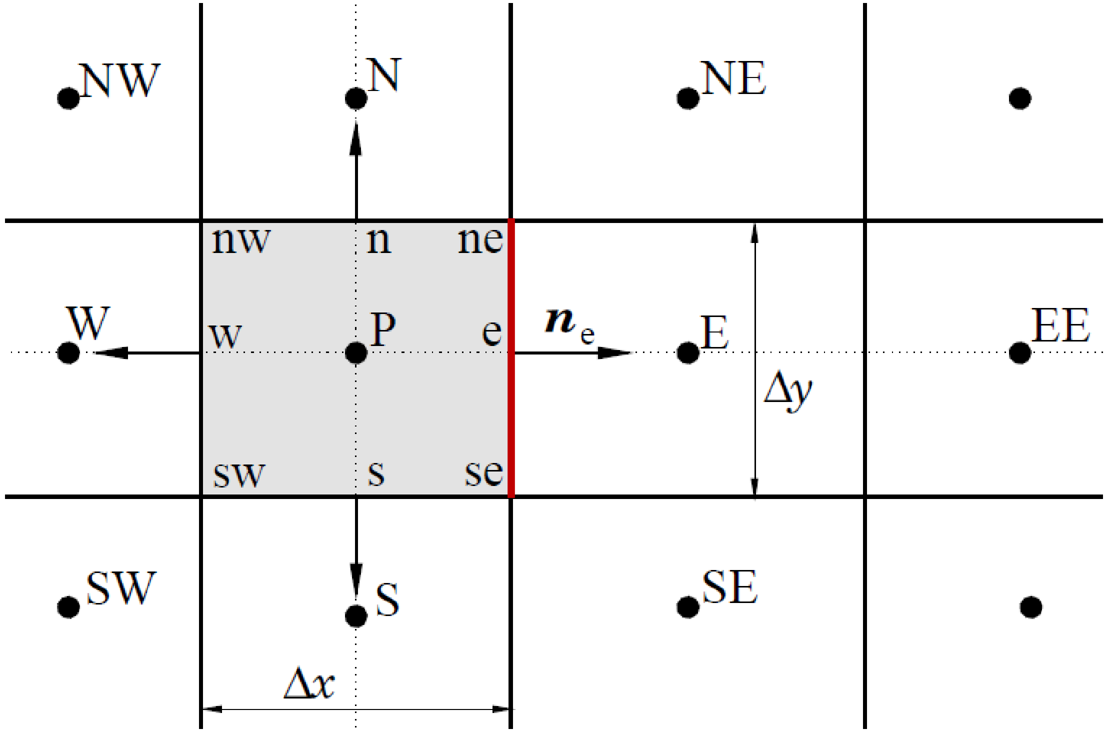
☞ Relate face values (f_e, f_{ne}, f_{se}) to nodal values $(\phi_P, \phi_E, \ldots)$

Integration:

Find face integrals $\left(\int_{e} f \; dS\right)$ from face values (f_e, f_{ne}, f_{se})

$$\int_{S_e} \overbrace{\rho \boxed{\phi} \mathbf{v} \cdot \mathbf{n}}^{f^{\mathrm{c}}} \, \mathrm{d}S \qquad\qquad \int_{S_e} \overbrace{\Gamma \boxed{\nabla \phi \cdot \mathbf{n}}}^{f^{\mathrm{d}}} \, \mathrm{d}S$$

$$\phi_e = ? \qquad\qquad \left(\frac{\partial \phi}{\partial x}\right)_e = ?$$



Interpolation schemes

$\phi_e = ?$ $\left(\frac{\partial \phi}{\partial x}\right)_e = ?$

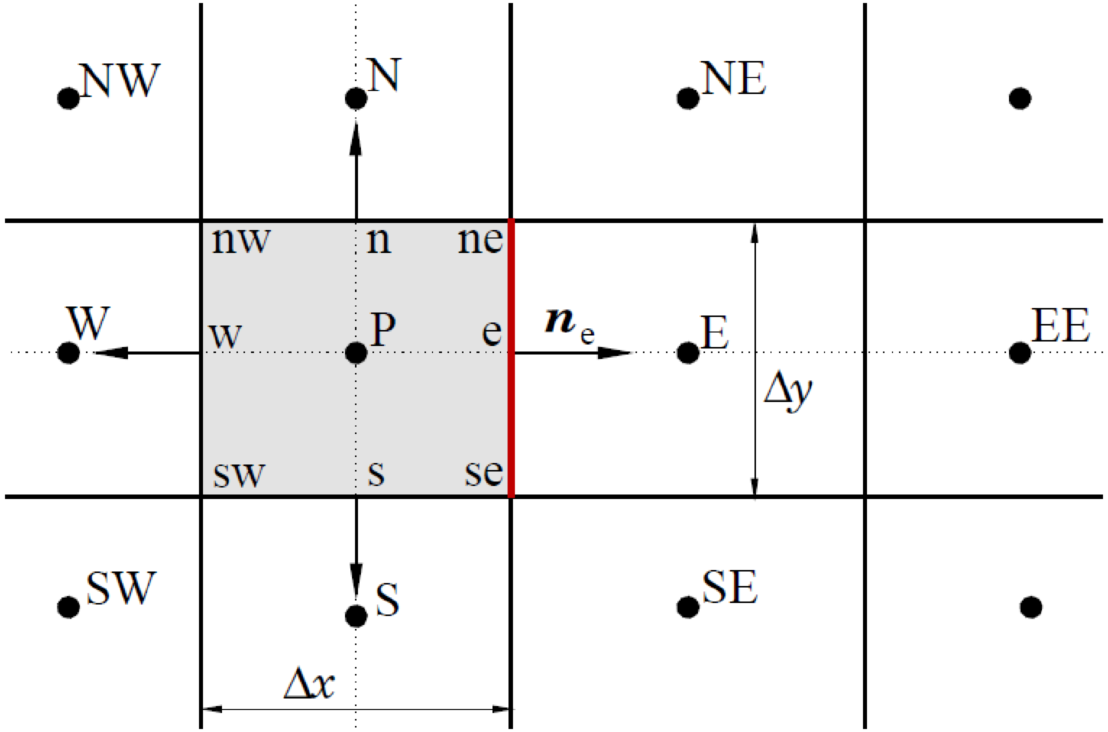
Linear Interpolation scheme

- ϕ is assumed to vary linearly between the two neighboring nodes

$$\phi_e = \phi_E \lambda_e + \phi_P (1 - \lambda_e) \qquad \lambda_e = \frac{x_e - x_P}{x_E - x_P}$$

$$\phi_e = \frac{\phi_E + \phi_P}{2}$$ (if grid is uniform)

$$\left(\frac{\partial \phi}{\partial x}\right)_e \approx \frac{\phi_E - \phi_P}{x_E - x_P}$$



- Linear scheme is effectively second order accurate

Interpolation schemes

$\phi_e = ?$ $\left(\frac{\partial \phi}{\partial x}\right)_e = ?$

Upwind Interpolation scheme

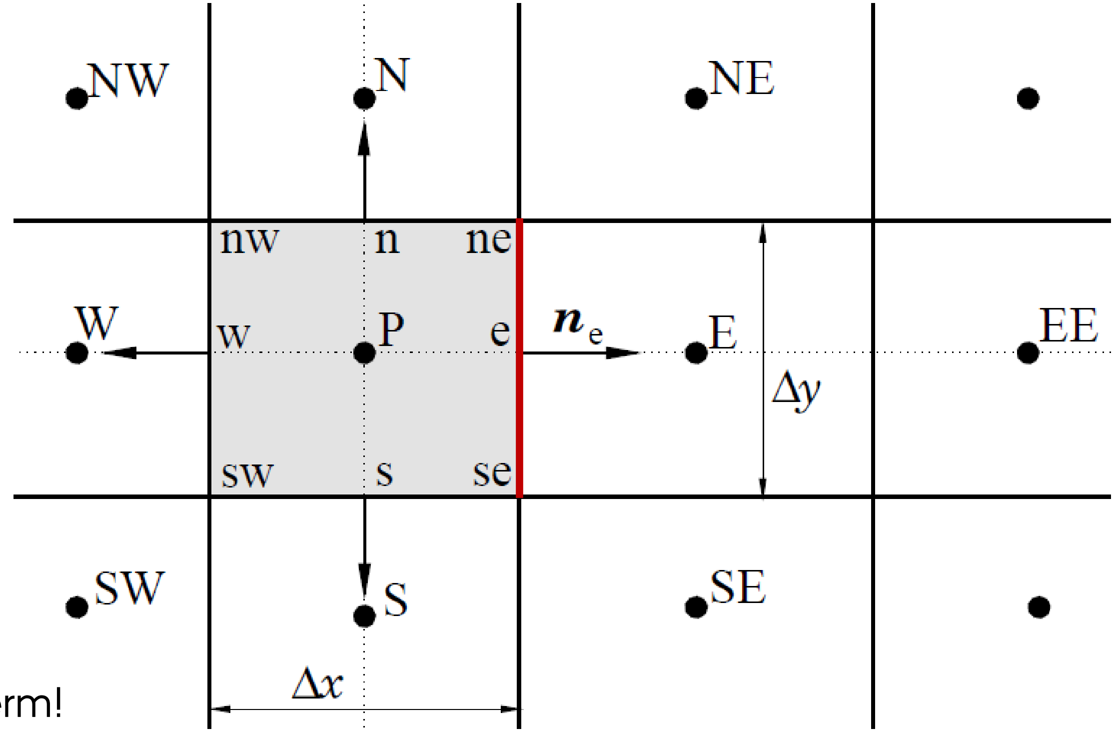
- ϕ is assumed equal to the upstream neighboring node

$$\phi_e = \begin{cases} \phi_P & \text{if } (\mathbf{v} \cdot \mathbf{n})_e > 0 \\ \phi_E & \text{if } (\mathbf{v} \cdot \mathbf{n})_e < 0 \end{cases}$$

(if the grid is orthogonal, e.g. Cartesian: $(\mathbf{v}\,.\,\mathbf{n})_e = u_e$)

$$\left(\frac{\partial \phi}{\partial x}\right)_e = 0$$

Upwind is not typically used for the diffusion term!



- Upwind scheme is first order accurate

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Approximation of surface integrals

- Calculating surface integrals is done in two steps:

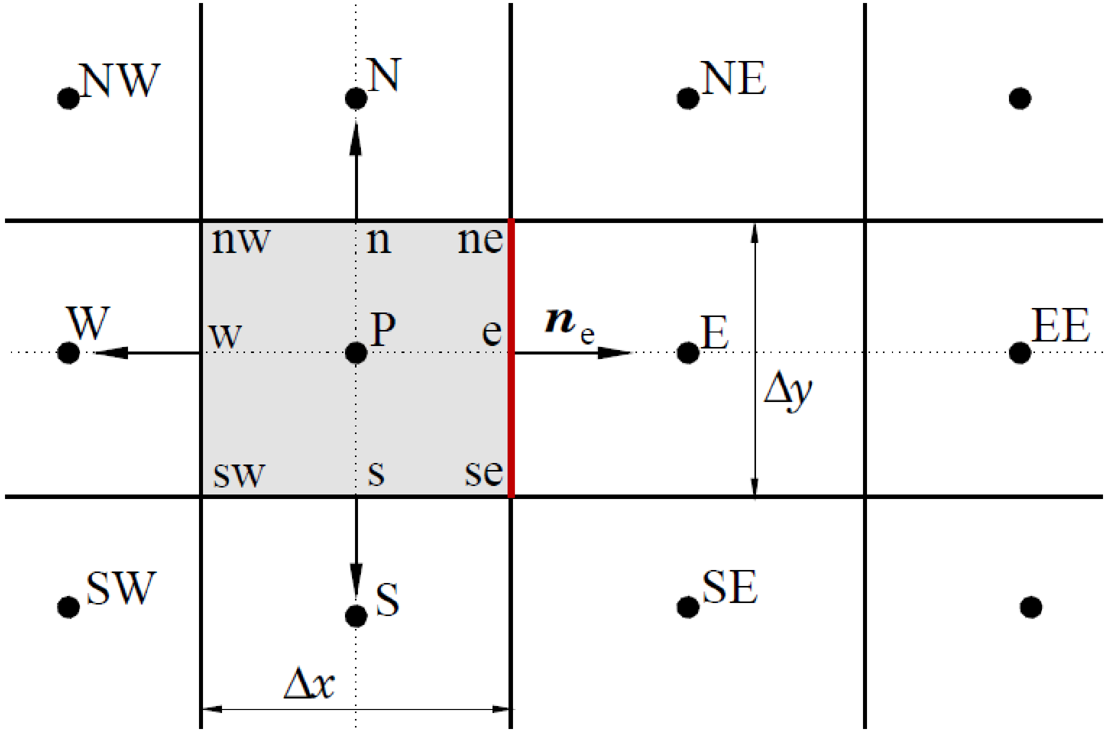
Interpolation:

☞ Relate face values (f_e, f_{ne}, f_{se}) to nodal values $(\phi_P, \phi_E, \ldots)$

Integration:

Find face integrals $(\int_e f\, dS)$ from face values (f_e, f_{ne}, f_{se})

$$\int_{S_e} \overbrace{\boxed{\rho\phi\mathbf{v}\cdot\mathbf{n}}}^{f^{\mathrm{c}}}\,\mathrm{d}S \qquad \int_{S_e} \overbrace{\boxed{\Gamma\,\nabla\phi\cdot\mathbf{n}}}^{f^{\mathrm{d}}}\,\mathrm{d}S$$



Approximation of surface integrals

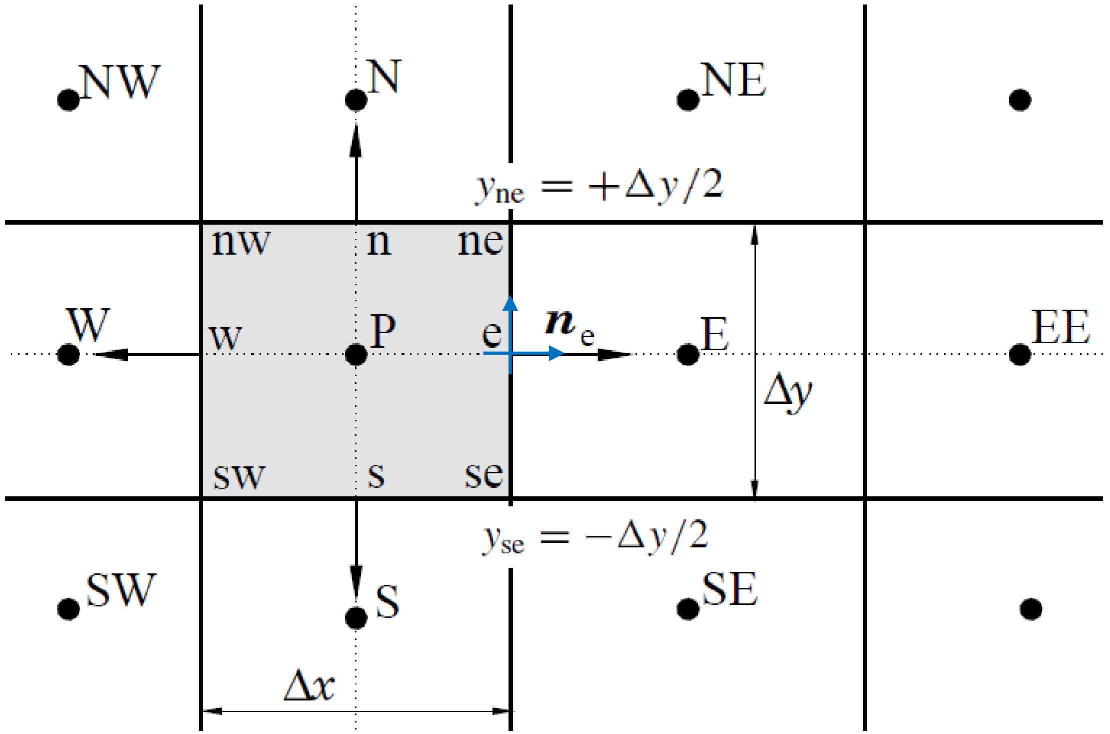
$$\int_S \underbrace{\boxed{\rho\phi\mathbf{v}\cdot\mathbf{n}}}_{} \mathrm{d}S \quad f^c \qquad \int_S \boxed{\Gamma\nabla\phi\cdot\mathbf{n}} \mathrm{d}S \quad f^d$$

Mid-point rule

- The value of ϕ at the face center is taken as the average over the face.

$$\int_{S_e} f \;\mathrm{d}S = \overline{f}_e S_e \approx f_e S_e$$

- Mid-point rule is second order accurate (only if the value is known at the 'center' of the face).



Approximation of surface integrals

$$\int_S \underbrace{\rho\phi\mathbf{v}\cdot\mathbf{n}}_{f^c}\,\mathrm{d}S \qquad \int_S \underbrace{\Gamma\,\nabla\phi\cdot\mathbf{n}}_{f^d}\,\mathrm{d}S$$

Trapezoid rule

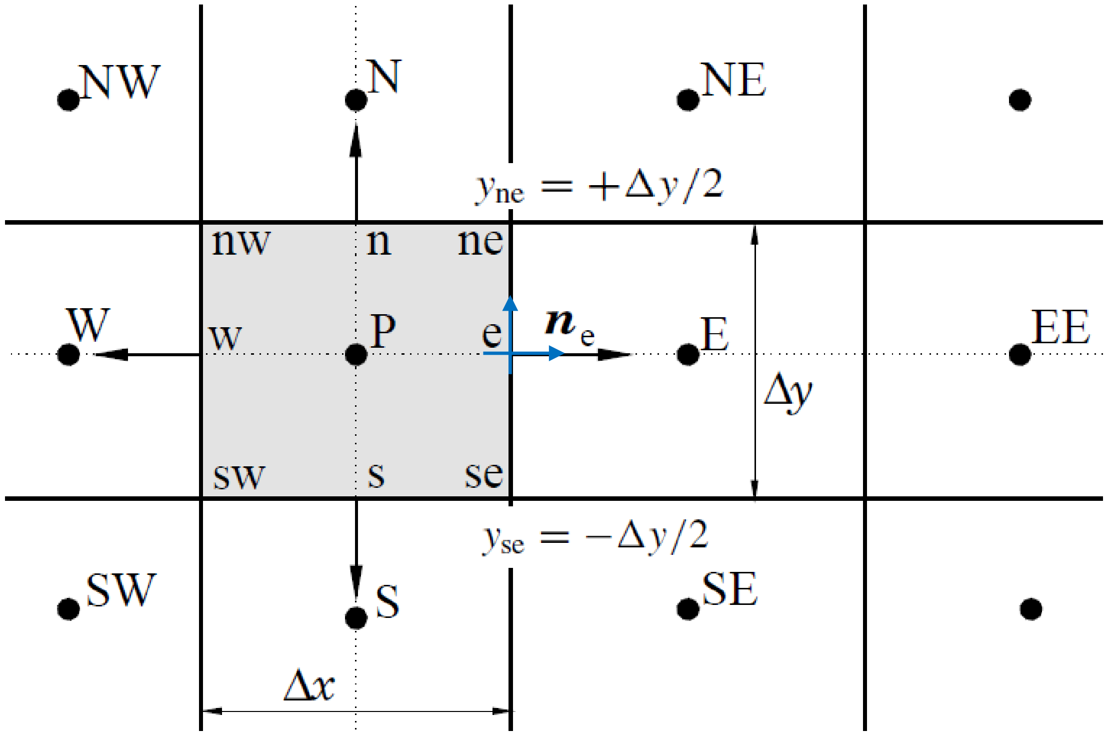
$$\int_{S_e} f\ \mathrm{d}S \approx \frac{S_e}{2}\left(f_{ne} + f_{se}\right)$$

2nd order (always)

Simson's rule

$$\int_{S_e} f\ \mathrm{d}S \approx \frac{S_e}{6}\left(f_{ne} + 4\,f_e + f_{se}\right)$$

4th order



Extension to 3D:

- Mid-point rule is smiler to 2D (simplest 2nd order approximation possible.)
- Extended Simson's rule requires 9 values: 1 face center, 4 edge centers, and 4 vertices (4th order):

$$\int_{S_P} f dS = \frac{\Delta x \Delta y}{36}\left(16 f_P + 4 f_s + 4 f_n + 4 f_e + 4 f_w + f_{se} + f_{sw} + f_{ne} + f_{nw}\right)$$

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Approximation of volume integrals

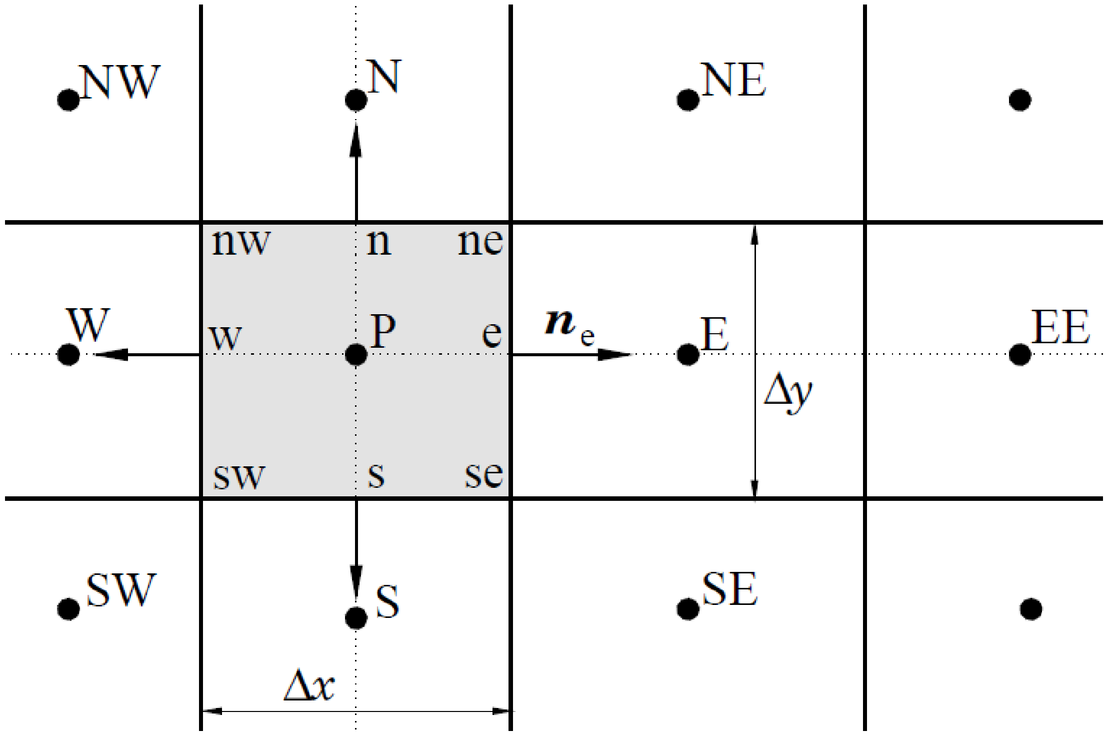
$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\,\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S + \int_V q_\phi\,\mathrm{d}V$$

<u>Cell-centre rule</u>

- The value of integrant at the cell center is taken as the average over the cell.

$$\int_V q\,\mathrm{d}V = \overline{q}\,\Delta V \approx q_\mathrm{P}\,\Delta V$$

- Cell-center rule is second order accurate (only if the node is at the 'geometric center' of the cell).
- Higher order rules can be derived (f.ex. Extended Simosn's rule for 2D cells).



How to solve a problem?

- Take the example of a convection diffusion transport equation on a uniform Cartesian grid:

$$\int_S \rho\phi \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\,\nabla\phi\cdot\mathbf{n}\,\mathrm{d}S$$

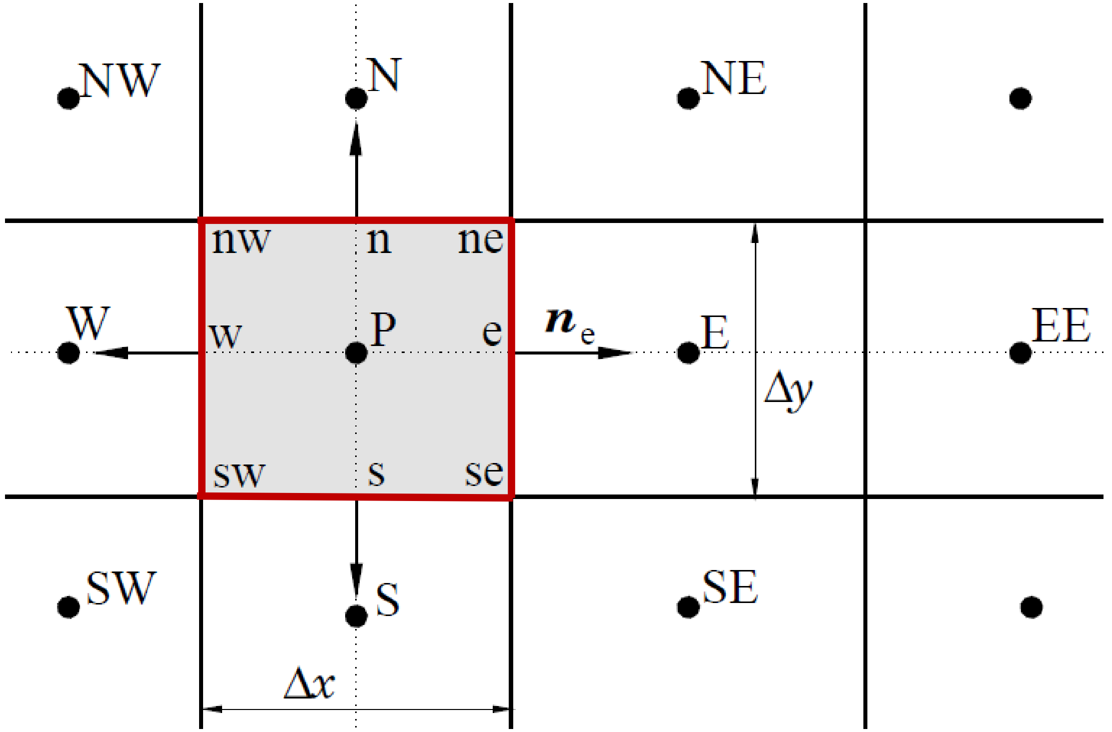
- Applying mid-point rule:

$$(\rho u)_e\phi_e\Delta y - (\rho u)_w\phi_w\Delta y + (\rho v)_n\phi_n\Delta x - (\rho v)_s\phi_s\Delta x$$

$$= \left(\Gamma\frac{\partial\phi}{\partial x}\right)_e\Delta y - \left(\Gamma\frac{\partial\phi}{\partial x}\right)_w\Delta y + \left(\Gamma\frac{\partial\phi}{\partial y}\right)_n\Delta x - \left(\Gamma\frac{\partial\phi}{\partial y}\right)_s\Delta x$$

- For simplicity we continue with the 1D problem:

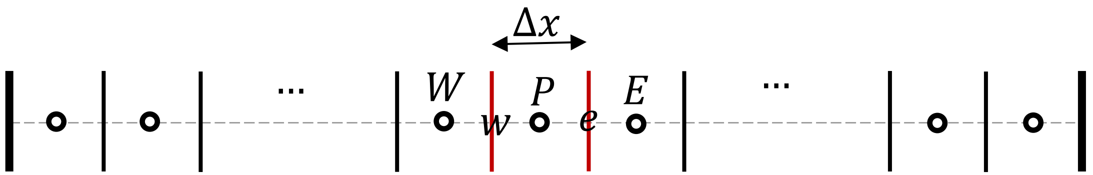
$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \left(\Gamma\frac{\partial\phi}{\partial x}\right)_e - \left(\Gamma\frac{\partial\phi}{\partial x}\right)_w$$



uniform grid
$(\Delta x = \text{const.};\ \Delta y = \text{const.})$

How to solve a problem?

- Applying the intepolation rules



uniform grid
(Δx = const.)

<u>Use linear for both advection and diffusion terms:</u>

$$(\rho u)_e \frac{\phi_P + \phi_E}{2} - (\rho u)_w \frac{\phi_P + \phi_W}{2} = \Gamma_e \frac{\phi_E - \phi_P}{\Delta x} - \Gamma_w \frac{\phi_P - \phi_W}{\Delta x}$$

- Rearranging the equations

$$\underbrace{\left[\overset{A_P^c}{\frac{(\rho u)_e - (\rho u)_w}{2}} + \overset{A_P^d}{\frac{\Gamma_e + \Gamma_w}{\Delta x}}\right]}_{A_P} \phi_P + \underbrace{\left[\overset{A_E^c}{\frac{(\rho u)_e}{2}} - \overset{A_E^d}{\frac{\Gamma_e}{\Delta x}}\right]}_{A_E} \phi_E + \underbrace{\left[\overset{A_W^c}{-\frac{(\rho u)_w}{2}} - \overset{A_W^d}{\frac{\Gamma_w}{\Delta x}}\right]}_{A_W} \phi_W = 0$$

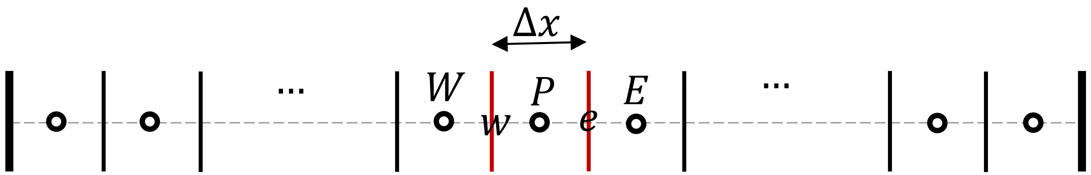
How to solve a problem?

- Applying the intepolation rules

  Use upwind for the advection term:

$$(\rho u)_e \phi_e = \max[(\rho u)_e\,,0]\,\phi_P + \min[(\rho u)_e\,,0]\phi_E$$

$$(\rho u)_w \phi_w = \max[(\rho u)_w\,,0]\phi_W + \min[(\rho u)_w\,,0]\phi_P$$



uniform grid
$(\Delta x = \text{const.})$

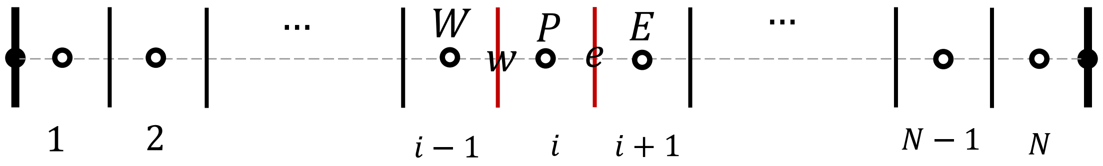
- Rearranging the equations

$$\underbrace{\left[\overset{A_P^c}{\max[(\rho u)_e\,,0] - \min[(\rho u)_w\,,0]} + \overset{A_P^d}{\frac{\Gamma_e + \Gamma_w}{\Delta x}}\right]}_{A_P}\phi_P + \underbrace{\left[\overset{A_E^c}{\min[(\rho u)_e\,,0]} - \overset{A_E^d}{\frac{\Gamma_e}{\Delta x}}\right]}_{A_E}\phi_E + \underbrace{\left[\overset{A_W^c}{-\max[(\rho u)_w\,,0]} - \overset{A_W^d}{\frac{\Gamma_w}{\Delta x}}\right]}_{A_W}\phi_W = 0$$

How to solve a problem

- The discretized equation can be written as

$$A_W\,\phi_W \;+\; A_P\,\phi_P \;+\; A_E\,\phi_E = 0$$



- If ϕ_P corresponds to the i-th cell : $\boxed{0\phi_1 + 0\phi_1 + \cdots + A_W^i\phi_{i-1} \;+\; A_P^i\phi_i \;+\; A_E^i\phi_{i+1} + \cdots + 0\phi_N = 0}$

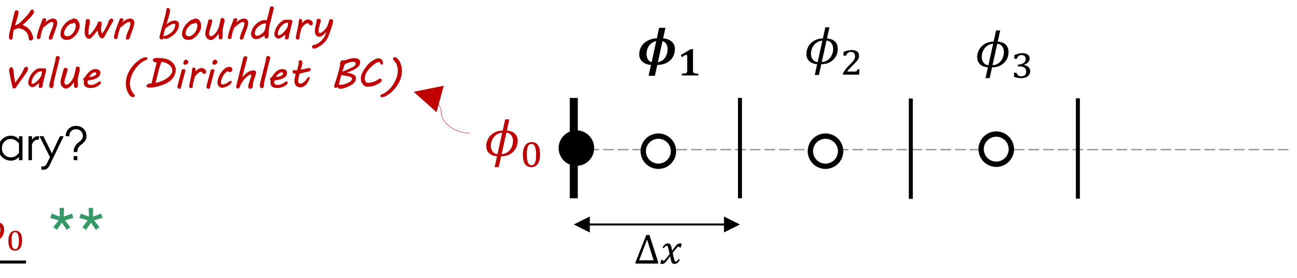
- The N equations for N cells can be written as a matrix equality:

$$\underset{A}{\begin{bmatrix} & & & & & & & & \\ 0 & \cdots & 0 & A_W^i & A_P^i & A_E^i & 0 & \ldots & 0 \\ & & & & & & & & \end{bmatrix}} \underset{\boldsymbol{\phi}}{\begin{bmatrix}\phi_1\\ \vdots \\ \phi_{i-1}\\ \phi_i \\ \phi_{i+1}\\ \vdots\\ \phi_N\end{bmatrix}} = \underset{\mathbf{Q}}{\begin{bmatrix}\vdots\\ 0\\ \vdots\end{bmatrix}} \qquad \boldsymbol{A\phi} = \mathbf{Q}$$

- $\boldsymbol{\phi} = \boldsymbol{A}^{-1}\mathbf{Q}$ yields the solution.

How to solve a problem

- How about the cells next to the boundary?



$$\underset{\substack{\uparrow\\ \text{(linear scheme)}}}{\frac{\phi_1+\phi_2}{2}} \qquad \overset{*}{\phi_0} \qquad \underset{\substack{\uparrow\\ \text{(linear scheme)}}}{\frac{\phi_2-\phi_1}{\Delta x}} \qquad \underset{\substack{\uparrow\\ \text{(linear scheme)}}}{\frac{\phi_1-\phi_0}{\Delta x/2}}\ {}^{**}$$

$$(\rho u)_e\phi_e - (\rho u)_w\phi_w = \Gamma\left(\frac{\partial \phi}{\partial x}\right)_e - \Gamma\left(\frac{\partial \phi}{\partial x}\right)_w$$

$$\Rightarrow \underbrace{\left[\frac{(\rho u)_e}{2} + \frac{\Gamma_e + 2\Gamma_w}{\Delta x}\right]}_{A_P^1}\phi_1 + \underbrace{\left[\frac{(\rho u)_e}{2} - \frac{\Gamma_e}{\Delta x}\right]}_{A_E^1}\phi_2 = \underbrace{\left[+(\rho u)_w + \frac{2\Gamma_w}{\Delta x}\right]}_{Q_1}\phi_0$$

$$\begin{bmatrix} A_P^1 & A_E^1 & 0 & \cdots & \\ \blacksquare & \blacksquare & \blacksquare & & \\ & \ddots & \ddots & \ddots & \\ & & \blacksquare & \blacksquare & \blacksquare \\ & & & \blacksquare & \blacksquare \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \\ \vdots \\ \phi_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ \\ \\ \\ \vdots \\ \\ \end{bmatrix}$$

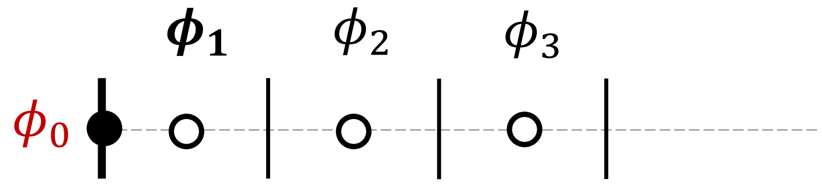
- Boundary-adjacent cells often require special treatment corresponding to the boundary condition.

* No interpolation is needed here as the boundary value is known.

** One can use any order of approximation for the derivative. Here a first order formula is used. How to make it 2nd order?! (see lecture 1, how to discretize a derivative)

How to solve a problem

Known boundary value (Dirichlet BC)



- Alternative <u>upwind</u> discretization for the boundary cell:

$$\overbrace{(\rho u)_e \phi_e}^{\substack{\max[(\rho u)_e\,,0]\,\phi_P + \min[(\rho u)_e\,,0]\phi_E \\ \text{(upwind scheme)}}} - \underbrace{(\rho u)_w \phi_w}_{\substack{\text{(upwind scheme)} \\ \max[(\rho u)_w\,,0]\,\phi_0 + \min[(\rho u)_w\,,0]\phi_P}} = \Gamma\left(\frac{\partial \phi}{\partial x}\right)_e - \Gamma\left(\frac{\partial \phi}{\partial x}\right)_w \quad \Rightarrow \quad \ldots$$

Assignment 1

Solving generic transport equation in 1D

$$\frac{\partial}{\partial x}(\rho u \phi) - \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) = 0 \quad , \qquad \rho, u\,, \Gamma \text{ are constants}$$

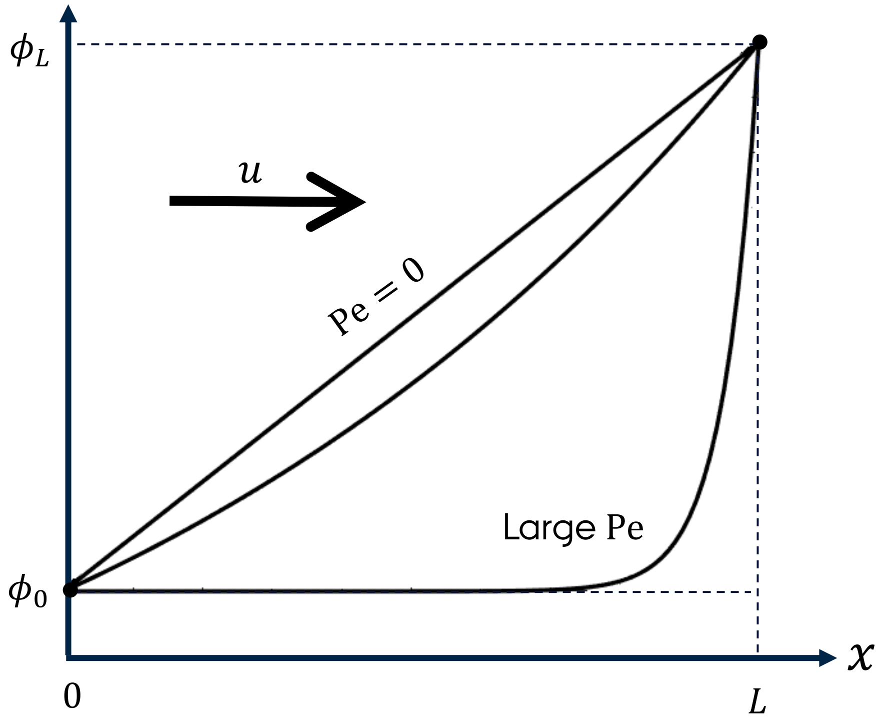
$$\phi(\mathrm{x}=0) = \phi_0$$
$$\phi(\mathrm{x}=\mathrm{L}) = \phi_L$$

- Discretize the equation, form the matrix of coefficients and constants.
- Find $\boldsymbol{\phi} = A^{-1}\boldsymbol{Q}$ and report the results according to the task description.
- You can simply use the existing functions to find $\boldsymbol{\phi} = A^{-1}\boldsymbol{Q}$
- Since the exact soution is known, error (ϵ) can be calculated. The error is dominated by discretization error, so you can estimate the order of apprixmation.

$$\epsilon = \frac{\sum_i |\phi_i^{\text{exact}} - \phi_i|}{N}$$

Analytical solution

$$\phi^{\text{exact}}(\mathrm{x}) = \phi_0 + \frac{e^{\mathrm{Pe}\frac{x}{L}} - 1}{e^{\mathrm{Pe}} - 1}(\phi_L - \phi_0) \quad , \mathrm{Pe} = \frac{\rho u L}{\Gamma}$$



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