Assignment 1

The goal is to solve the 1D convection-diffusion scalar transport equation using the finite volume method. The equation (differential form) reads

$$\rho u \frac{d\phi}{dx} = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) \quad (0 \leq x \leq L)$$

where u, Γ, L are constant parameters. In this assignment $L = 1$ and the boundary conditions are

$$\phi(x=0) = 0 \quad , \quad \phi(x=L) = 1.$$

The problem has the following exact solution

$$\phi^{exact}(x) = \frac{e^{Pe\frac{x}{L}} - 1}{e^{Pe} - 1} \quad , Pe = \frac{\rho u L}{\Gamma}.$$

Task description

1.1. Discretize the integral form of the equation on a uniform 1D grid with N cells. For the diffusion term, use the linear interpolation scheme. For the advection term, once use the linear and once the upwind interpolation scheme (we call the resulting solutions linear and upwind solutions hereafter). Always use second order approximation for the derivatives at the boundaries. Show the discretized equations for a boundary cell and a normal cell in your report.

1.2. Write a program in your language of choice to compute the solution for ϕ based on the matrix of coefficients and constants obtained from task 1.1 [1]. Plot the computed ϕ as a function of x for the following cases, each for both linear and upwind solutions. Each plot should show the solution within $0 \leq x \leq 1$ and also include the analytical solution.

Case 1: $N = 100$; $Pe = 100$

Case 2: $N = 20$; $Pe = 100$

Case 3: $N = 20$; $Pe = -20$

1.3. Keep $Pe = 20$ and vary the number of cells from 10 to at least 1000. Plot the global error

$$\epsilon = \frac{\sum_{i=1}^{N} |\phi_i^{exact} - \phi_i|}{N} \quad (i : \text{cell index})$$

as a function of Δx in a double-logarithmic scale. Do that for both linear and upwind solutions and estimate the order of discretization error for each solution method.

[1] For solving the linear algebraic equation system ($A\phi = Q$), you can use the existing functions in your program of choice (e.g. matlab).