UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Midterm exam: FYS3110 – Quantum mechanics
Due date and time: Oct. 13 at 14:00 (on Inspera)
Remember to put your candidate number on your answer sheets (not your name).

Problem 1

A charged particle is constrained to move on the surface of a sphere. The sphere is placed in a weak magnetic field along the z-axis such that the Hamiltonian can be approximated as

$$\hat{H} = \frac{\alpha}{\hbar}\vec{L}^2 + \beta\hat{L}_z,$$

where $\hat{L}_i$ is the i'th ($i \in \{x, y, z\}$) cartesian component of the angular momentum operator and $\vec{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. α and β are real positive constants expressed in units of inverse time. In this problem use the notation $|l, m\rangle$ to denote a common eigenstate of $\vec{L}^2$ and $\hat{L}_z$ in the ususal way.

1a) (6 points) Write down the energy eigenvalues of $\hat{H}$. Make an energy-level diagram where you plot the lowest energy eigenvalues (plot ten levels) in units of $\hbar\alpha$ as a function of the ratio β/α.

1b) (6 points) Compute $\langle\psi_0|\hat{L}_z|\psi_0\rangle$, where $|\psi_0\rangle$ is the ground state of $\hat{H}$. Plot $\langle\psi_0|\hat{L}_z|\psi_0\rangle$ as a function of β/α.

Consider the state

$$|\psi\rangle = \frac{1}{2}\Big(|l = 1, m = -1\rangle + i\sqrt{2}|l = 1, m = 0\rangle - |l = 1, m = 1\rangle\Big)$$

at time $t = 0$ and let $|\psi(t)\rangle$ be the time-evolved state which coincides with $|\psi\rangle$ at $t = 0$ ($|\psi(t = 0)\rangle = |\psi\rangle$).

1c) (6 points) Write down the expression for $|\psi(t)\rangle$ and compute $\langle\psi(t)|\hat{L}_z|\psi(t)\rangle$.

1d) (6 points) A measurement of the angular momentum component along the z-axis is made on the particle in state $|\psi(t)\rangle$. Find the possible measurement results and their probabilities. Verify that your results give the expectation value calculated in problem c).

1e) (6 points) Compute $\langle\psi(t)|\hat{L}_x|\psi(t)\rangle$ and $\langle\psi(t)|\hat{L}_x^2|\psi(t)\rangle$.

1f) (6 points) A measurement of the angular momentum x-component is made on the particle in state $|\psi(t)\rangle$. Find expressions for the (possibly time-dependent) probabilities of the different measurement outcomes. Check that the sum of probabilities is unity.

1g) (6 points) The position (θ, ϕ) of the particle in the state $|\psi(t)\rangle$ is measured. Find the most probable value of the polar angle θ. (θ is the angle between the particle position vector and the z-axis).

1h) (6 points) Calculate the commutators $[\hat{L}_z, \hat{P}_x]$ and $[\hat{L}_z, \hat{P}_y]$ where $\hat{P}_i$ is the i'th component ($i \in \{x, y, z\}$) of the momentum operator. Use the results to show that $\langle l'm'|\hat{P}_x|lm\rangle = 0$ for $m' \neq m \pm 1$. Does the same result hold for $\langle l'm'|\hat{P}_y|lm\rangle$? What about $\langle l'm'|\hat{P}_z|lm\rangle$?

1i) (6 points) An extra term $\hat{H}_e = \frac{\gamma}{\hbar} i \left(\hat{L}_z \hat{L}_y - \hat{L}_y \hat{L}_z \right)$ is added to the Hamiltonian $\hat{H}$ (γ is a real positive number with units of inverse time). Find an expression for the energy eigenvalues of $\hat{H} + \hat{H}_e$. Give reasons for your answer.