Lecture 10

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Spin 1/2

$$\hat{S} = (S_x, S_y, S_z)$$

An internal degree of freedom, which means it has nothing to do with orientation in space.

$$[S_y, S_x] = i\hbar S_z$$

$$[S_z, S_y] = i\hbar S_x$$

$$[S_x, S_z] = i\hbar S_y$$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$S^2 \left|s, m\right\rangle = \hbar^2 s(s+1) \left|s, m\right\rangle$$

where s is the quantum number, where $s \in \{0, 1/2, 1, \ldots\}$ and $m \in \{-s, -s+1, \ldots, s\}$

$$S_\pm = S_x \pm i S_y$$

$$S_\pm \left|s, m\right\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)} \left|s, m \pm 1\right\rangle$$

When the spin is 1/2, we have two states, $\left|1/2, +1/2\right\rangle$ and $\left|1/2, -1/2\right\rangle$. This can also be written as $\left|\uparrow_z\right\rangle$ and $\left|\downarrow_z\right\rangle$ or $\left|1\right\rangle$ and $\left|0\right\rangle$.

Applying the spin operators

$$S_z \left|\uparrow_z\right\rangle = \frac{\hbar}{2} \left|\uparrow_z\right\rangle$$

$$S_z \left|\downarrow_z\right\rangle = -\frac{\hbar}{2} \left|\downarrow_z\right\rangle$$

The state are naturally orthonormal, meaning $\langle\uparrow|\downarrow\rangle = 0$ and $\langle\uparrow|\uparrow\rangle = 1$ and $\langle\downarrow|\downarrow\rangle = 1$.

Rewriting the operator in bracket notation in the z-basis

$$S_z = \frac{\hbar}{2}\Big(\left|\uparrow\right\rangle \left\langle\uparrow\right| - \left|\downarrow\right\rangle \left\langle\downarrow\right| \Big)$$

Lowering and raising operators

$$S_+ = \hbar \left|\uparrow\right\rangle \left\langle\downarrow\right|$$

$$S_- = \hbar \left|\downarrow\right\rangle \left\langle\uparrow\right|$$

$$S_+ \left|\downarrow\right\rangle = \hbar \left|\uparrow\right\rangle \quad , \quad S_- \left|\downarrow\right\rangle = 0$$

2D representation of the operators

$$\left|\uparrow\right\rangle \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \left|\downarrow\right\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

State vectors

$$\left|\uparrow\right\rangle \left\langle\uparrow\right| \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \left|\downarrow\right\rangle \left\langle\downarrow\right| \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Spin operators

$$S_z = \hbar/2 \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x = \hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \hbar/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The matrixes are often written as $\sigma_z, \sigma_x, \sigma_y$.

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rotating charge in a magnetic field

Classical

$$I = \frac{1}{T} = \frac{q}{2\pi r} = \frac{qv}{2\pi r}$$

A circular current is like a small magnet, where I is the current, T is the period, q is the charge, r is the radius and v is the velocity. The magnetic moment is defined as $\vec{\mu} = I\vec{A} = \frac{qvr}{2}$, where $\vec{A}$ is the area vector.

Angular momentum

$$\left|\vec{L}\right| = mvr \quad , \quad |\vec{\mu}| = \frac{2}{2m}\hat{L}$$

where m is the mass of the particle.

$$H = -\vec{\mu} \cdot \vec{B}$$

Electron spin in a magnetic field

$$\vec{\mu} = \gamma\vec{S}$$

where γ is the gyromagnetic ratio $= 2.0023\,(-e/2m)$

$$H = -\gamma\vec{S} \cdot \vec{B}$$

Spin motion in a magnetic field

Generic State:

$$|\psi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle$$

Parameterize the coefficients

$$a = \cos\frac{\alpha}{2}e^{i\phi} \quad , \quad b = \sin\frac{\alpha}{2}e^{i\varphi}$$

put the ¡-axis along $\vec{B}$.

$$H = -\gamma\vec{S} \cdot \vec{B} = -\gamma B S_z$$

The eigenstates of the Hamiltonian H is $|\uparrow\rangle$ and $|\downarrow\rangle$, with eigenvalues $-\gamma B\hbar/2$ and $\gamma B\hbar/2$ respectively. We can then rewrite the generic state as follows:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi\rangle = e^{-iHt/\hbar}(a|\uparrow\rangle + b|\downarrow\rangle)$$
$$|\psi(t)\rangle = e^{i\gamma Bt/2}a|\uparrow\rangle + e^{-i\gamma Bt/2}b|\downarrow\rangle$$

Applying the S_i operator

$$\langle\psi(t)|S_z|\psi(t)\rangle = \left(a^*e^{-i\gamma Bt/2}\langle\uparrow| + b^*e^{i\gamma Bt/2}\langle\downarrow|\right)(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)\left(e^{i\gamma Bt/2}a|\uparrow\rangle + e^{-i\gamma Bt/2}b|\downarrow\rangle\right)$$

$$\langle S_z\rangle = \hbar/2\,(a^*a - b^*b) = \frac{\hbar}{2}\cos\alpha$$

The result is time independent.

Expectation value of S_x

$$\langle S_x\rangle = \left(a^*e^{-i\gamma Bt/2}\langle\uparrow| + b^*e^{i\gamma Bt/2}\langle\downarrow|\right)\hbar/2\,(|\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow|)\left(e^{i\gamma Bt/2}a|\uparrow\rangle + e^{-i\gamma Bt/2}b|\downarrow\rangle\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\left(a^*be^{-i\gamma Bt} + ab^*e^{i\gamma Bt}\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\left(be^{-i\gamma Bt/2}|\uparrow\rangle + ae^{i\gamma Bt/2}|\downarrow\rangle\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\cos(\alpha/2)\sin(\alpha/2)e^{-i\phi}e^{i\theta}e^{-i\gamma Bt} + \underbrace{c.c}_{\text{complex conjugate}}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\sin\alpha\cos(\theta - \phi - \gamma Bt)$$

The result is time dependent and oscillates just like you would expect classically.

$$\langle S_y\rangle = -\frac{\hbar}{2}\sin\alpha\cos(\gamma Bt + \phi - \theta)$$

The expectation value of the spin precess around the field direction, with a frequency of $w_L = \gamma B$

Composite systems

Now we combine the degrees of freedom coming.

A particle with real space state $|nlm\rangle$ and spin $|\uparrow\rangle$

$$|nlm\uparrow\rangle = |nlm\rangle \otimes |\uparrow\rangle$$
$$|nlm\rangle = \psi_{nlm}(\gamma, \theta, \phi)$$
$$|s\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

The tensor product becomes a combination of all possible states:

$$|nlm, s\rangle = |nlm\rangle \otimes |s\rangle = \begin{pmatrix} a\psi_{nlm}|nlm\rangle \\ b\psi_{nlm}|nlm\rangle \end{pmatrix}$$

Two spin 1/2 particles

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad , \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

With dimensions 4×1.

Properties of Tensor Products

$$|\phi\rangle = \sum c_n |n\rangle \quad , \quad |x\rangle = \sum d_m |m\rangle$$

- The dimensions of the tensor product is the product of the dimensions of the two vectors. $a \times b \otimes c \times d = ac \times bd$
- $$|\phi\rangle \otimes |x\rangle = \sum c_n d_m |n\rangle \otimes |m\rangle = \alpha |\phi\rangle \otimes |x_1\rangle + \beta |\phi\rangle \otimes |x_2\rangle$$
- $$(\langle n'| \otimes \langle m'|) (|n\rangle \otimes |m\rangle) = \langle n'|n\rangle \cdot \langle m'|m\rangle$$