Lecture 11

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2 spin - 1/2 system

$$|t\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \right)$$

Probability to measure spin 1 to be $\hbar/2$ along z-axis We define spin 1 to be when the first spin is up and the second is down. Spin 2 is when the first spin is down and the second is up.

$$P_t = \langle t| \Big(|\uparrow\rangle \langle\uparrow| \otimes I \Big) |t\rangle = \frac{1}{2} \Big(\langle\uparrow| \otimes \langle\downarrow| + \langle\downarrow| \otimes \langle\uparrow| \Big) \underbrace{\Big(|\uparrow\rangle \langle\uparrow| \otimes I \Big) \Big(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \Big)}_{|\uparrow\rangle \otimes |\downarrow\rangle}$$

$$P_t = \frac{1}{2} \Big(\langle\uparrow|\uparrow\rangle \langle\downarrow|\downarrow\rangle + \langle\downarrow|\uparrow\rangle \langle\uparrow|\downarrow\rangle \Big) = \frac{1}{2}$$

Total Angular Momentum

The total angular momentum is the sum of orbital and spin angular momentum

$$\vec{J} = \vec{L} + \vec{S}$$

This is actually represented on the form:

$$\vec{J} = \vec{L} \otimes I + I \otimes \vec{S}$$

Commutation

$$\Big[\hat{L}, \hat{S} \Big] = \Big[\hat{L} \otimes I, \hat{S} \otimes I \Big]$$

$$(L \otimes I)(I \otimes S) - (I \otimes S)(L \otimes I)$$

$$L \otimes S - L \otimes S = 0$$

Total spin of two spin 1/2 particles

Total spin of two particles spinning up

$$\vec{S}^{\text{tot}} = \vec{S} \otimes I + I \otimes \vec{S}$$

$$\vec{S}^{\text{tot}} = S_z \otimes I + I \otimes S_z$$

$$S_z^{\text{tot}} |\uparrow\rangle \otimes |\uparrow\rangle = \Big(S_z \otimes I + I \otimes S_z \Big) |\uparrow\rangle \otimes |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \otimes |\uparrow\rangle + \frac{\hbar}{2} |\uparrow\rangle \otimes |\uparrow\rangle = \hbar |\uparrow\rangle \otimes |\uparrow\rangle$$

Total spin of one particle spinning up and one spinning down

$$S_z^{\text{tot}} |\uparrow\rangle \otimes |\downarrow\rangle = \Big(S_z \otimes I + I \otimes S_z \Big) |\uparrow\rangle \otimes |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \otimes |\downarrow\rangle + \frac{\hbar}{2} |\uparrow\rangle \otimes |\downarrow\rangle = 0$$

Naturally we se that the total spin is just the sum of the spin of each particle. Its easy then to see that

$$S_z^{\text{tot}} |\downarrow\rangle \otimes |\uparrow\rangle = 0 \quad , \quad S_z^{\text{tot}} |\downarrow\rangle \otimes |\downarrow\rangle = -\hbar |\downarrow\rangle \otimes |\downarrow\rangle$$

Eigenstate of total spin $S^{\textbf{tot}}$ squared

$$\vec{S}^{\text{tot}^2} = \vec{S}^{\text{tot}} \cdot \vec{S}^{\text{tot}}$$

$$\vec{S}^{\text{tot}^2} = \left(\vec{S} \otimes I + I \otimes \vec{S}\right) \cdot \left(\vec{S} \otimes I + I \otimes \vec{S}\right)$$

$$\vec{S}^{\text{tot}^2} = \vec{S}^2 \otimes I + I \otimes \vec{S}^2 + 2\left(S_x \otimes S_x + S_y \otimes S_y + S_z \otimes S_z\right)$$

We must define S_i in terms of the raising and lowering operators.

$$S_x \otimes S_x = \frac{1}{4}\left(S_+ + S_-\right) \otimes \left(S_+ + S_-\right)$$

$$S_y \otimes S_y = \frac{1}{4}\left(S_+ S_-\right) \otimes \left(S_+ + S_-\right)$$

$$S_x \otimes S_x + S_y \otimes S_y = \frac{1}{2}\left(S_+ \otimes S_- + S_- \otimes S_+\right)$$

We can now derive the total:

$$S^{\text{tot}^2} = S^2 \otimes I + I \otimes S^2 + S_+ \otimes S_- + S_- \otimes S_+ + 2S_z \otimes S_z$$

Eigenstate of total spin $S^{\textbf{tot}}$ squared for two particles spinning up

$$S^{\text{tot}^2} \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle = \left(S^2 \otimes I + I \otimes S^2 + S_+ \otimes S_- + S_- \otimes S_+ + 2S_z \otimes S_z\right) \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle$$

$$S^{\text{tot}^2} \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle = \hbar^2 \frac{3}{4} \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle + \hbar^2 \frac{3}{4} \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle + \hbar^2 \left|\text{NULL}\right\rangle \otimes \left|\downarrow\right\rangle + \left|\text{NULL}\right\rangle + 2\left(\frac{\hbar}{2}\right)^2 \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle$$

$$S^{\text{tot}^2} \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle = \hbar^2 \left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right) \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle = 2\hbar^2 \left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle$$

Eigenstate of total spin $S^{\textbf{tot}}$ of two particles spinning down

$$S^{\text{tot}^2} \left|\downarrow\right\rangle \otimes \left|\downarrow\right\rangle = 2\hbar^2 \left|\downarrow\right\rangle \otimes \left|\downarrow\right\rangle$$

Eigenstate of total spin $S^{\textbf{tot}}$ of one particle spinning up and one spinning down

$$S^{\text{tot}^2} \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle = \left(S^2 \otimes I + I \otimes S^2 + S_+ \otimes S_- + S_{\otimes} S_+ + 2S_z \otimes S_z\right) \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle$$

$$\hbar^2 \frac{3}{4} \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle + h^2 \frac{3}{4} \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle + 0 + \hbar^2 \left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle + 2\frac{\hbar}{2} \cdot \left(\frac{\hbar}{2}\right) \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle$$

$$\hbar^2 \left(\left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle + \left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle\right)$$

Eigenstate of total spin $S^{\textbf{tot}}$ of one particle down up and one spinning up

$$\vec{S}^{\text{tot}^2} \left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle = \hbar^2 \left(\left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle + \left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle \right)$$

Conclusion: $\left|\uparrow\right\rangle \otimes \left|\downarrow\right\rangle$ and $\left|\downarrow\right\rangle \otimes \left|\uparrow\right\rangle$ are not eigenstates but $\left|\uparrow\right\rangle \otimes \left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle \otimes \left|\downarrow\right\rangle$ are.

Creating Linear Combinations

$$S^{\text{tot}^2}\Big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\Big) = 0$$

This is an eigenstate. Also known as the singlet state

$$S^{\text{tot}^2}\Big(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\Big) = 2\hbar^2\Big(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\Big)$$

This is one of the three parts of the triplet state. The other being $|\uparrow\rangle \otimes |\uparrow\rangle$ and $|\downarrow\rangle \otimes |\downarrow\rangle$

General Combinations of ang.mom. states

$$|sm\rangle = \sum_{m_1 m_2} C^{S_1 S_2 S}_{m_1 m_2} |s_1 m_1\rangle \otimes |s_2 m_2\rangle$$

where C is the Clebsch-Gordan coefficient.

$$|s_1 m_1\rangle \otimes |s_2 m_2\rangle = \sum_{m_1 m_2} C^{S_1 S_2 S}_{m_1 m_2} |sm\rangle$$

a spin - 3 / 2 with a spin-1 to get a total spin 3/2, m = 3/2

$$|3/2, +|3/2\rangle\rangle$$

Reading from the table we see the column of 3/2 and +3/2, having values 3/5 square root of 3/5 for m1 = +3/2 and s1 = 0. We get the negative of the square root of 2/5 for m1 = +1/2 and s1 = 1.

$$\left|\underbrace{3/2}_{s}, +\underbrace{|3/2\rangle}_{m}\right\rangle = \sqrt{\frac{3}{5}}\left|\underbrace{3/2}_{s_1}, +\left|\underbrace{3/2}_{m_1}\right\rangle\right\rangle \otimes \left|\underbrace{1}_{s_2}, \underbrace{0}_{m_2}\right\rangle - \sqrt{\frac{2}{5}}\left|\underbrace{3/2}_{s_1}, +\left|\underbrace{1/2}_{m_1}\right\rangle\right\rangle \otimes \left|\underbrace{1}_{s_2}, \underbrace{+1}_{m_2}\right\rangle$$