Oblig 7

Oskar Idland

Problem 1

a)

We know that
$$\vec{J} = \vec{L} + \vec{S} \rightarrow \vec{J}^2 = (\vec{L} + \vec{S})^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S}$$
We therefore know that:
$$\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$
The eigenvalues will then have the same values We begin with S^2
$$S^2 = \hbar^2 s(s+1) = \hbar^2 \frac{3}{4}$$
Next we have L^2:
$$L^2 = \hbar^2 l(l+1) = 6\hbar^2$$
We have j-values in integer steps in the range:
$$|l - s| \leq j \leq l + s \rightarrow \frac{3}{2} \leq j \leq \frac{5}{2}$$
Therefore we know j can be either $j_1 = 5/2$ or $j_2 = 3/2$ Next we have J^2:
$$J_1^2 = \hbar^2 j_1(j_1 + 1) = \hbar^2 \frac{35}{4}$$
$$J_2^2 = \hbar^2 j_2(j_2 + 1) = \hbar^2 \frac{15}{4}$$
Then we compute the eigenvalues of the dot product for j_1:
$$\mathrm{eig}\left(\mathbf{L} \cdot \mathbf{S}\right) = \frac{1}{2}\left(\hbar^2 \frac{35}{4} - 6\hbar^2 - \frac{3}{4}\hbar^2\right) = \hbar^2$$
And for j_2:
$$\mathrm{eig}\left(\mathbf{L} \cdot \mathbf{S}\right) = \frac{1}{2}\left(\hbar^2 \frac{15}{4} - 6\hbar^2 - \frac{3}{4}\hbar^2\right) = -\hbar^2 \frac{3}{2}$$
Therefore we know:
$$E_{so_{j=5/2}} = \lambda \quad , \quad E_{so_{j=3/2}} = -\frac{3}{2}\lambda$$
The full Hamiltonian is then dependent on the j-value:
$$E_j = E_3 + E_{so_j}$$
Adding in the energy eigenvalues we get:
$$E_j = \frac{E_1}{3^2} + E_{so_j}$$
Where
There are two possible values for the m_s, and five possible values for m_l. If $j = 3/2$ then there are $2j + 1 = 4$ possible values for m_j meaning a total degeneracy of 40. If $j = 5/2$ there are $2j + 1 = 6$ possible values giving a total degeneracy of 60.

b)

We know $n = 3$ and $l = 2$. From our previous calculations we know that the lowest energy has $j = 3/2$ and $m_j = +1/2$. With $l = 2$. As we want to go from describing our state in terms of the total angular momentum to the orbital and spin angular momentum we use the Clebsch-Gordan coefficients for angular momentum 2 and spin 1/2. We therefore have:

$$|3/2,\ +1/2\rangle = \sqrt{\frac{3}{5}}\,|1,\ -1/2\rangle - \sqrt{\frac{2}{5}}\,|0,\ +1/2\rangle$$

The probability of measuring $m_s = \hbar/2$ is therefore 2/5.

c)

The $\hat{J}_z$ operator has eigenvalues of $\hbar m_j$, which in turn gives $\hat{H}_b$ eigenvalues of $-bm_j$ We therefore get further splitting in terms of energy, where we get $2j + 1$ values of m_j. In our case that gives us 4 values for $j = 3/2$ and 6 values for $j = 5/2$.

d)

We already know the lowest energy belongs to $j = 3/2$. As the newly added term has a negative sign, we know the lowest possible energy comes from the only two positive values $m_j = 1/2$ and $m_j = 3/2$

7.7 (H)

To have a fully symmetric system as the bosons are identical, we must have both symmetric or antisymmetric spatial and spin part. We find the states which satisfy this using the Clebsch-Gordan coefficients. We look at the cases where the total spin is either 0 or 2. Starting at the upper left of the table we get:

$J = 2$

$$|j = 2, m = 2\rangle = |1, 1\rangle$$

$$|j = 2, m = 1\rangle = \frac{1}{\sqrt{2}}\,|0, 1\rangle + \frac{1}{\sqrt{2}}\,|1, 0\rangle$$

$$|j = 2, m = 0\rangle = \frac{1}{\sqrt{6}}\,|1, -1\rangle + \sqrt{\frac{2}{3}}\,|0, 0\rangle + \frac{1}{\sqrt{6}}\,|-1, 1\rangle$$

$$|j = 2, m = -1\rangle = \frac{1}{\sqrt{2}}\,|0, -1\rangle + \frac{1}{\sqrt{2}}\,|-1, 0\rangle$$

$$|j = 2, m = -2\rangle = |-1, -1\rangle$$

$J = 1$

$$|1, 1\rangle = \frac{1}{\sqrt{2}}\,|0, 1\rangle - \frac{1}{\sqrt{2}}\,|1, 0\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}\,|1, -1\rangle - \frac{1}{\sqrt{2}}\,|-1, 1\rangle$$

$$|1, -1\rangle = \frac{1}{\sqrt{2}}\,|0, -1\rangle - \frac{1}{\sqrt{2}}\,|-1, 0\rangle$$

$J = 0$

$$|j=0, m=0\rangle = \frac{1}{\sqrt{3}}|1,1\rangle - \frac{1}{\sqrt{3}}|0,0\rangle + \frac{1}{\sqrt{3}}|-1,1\rangle$$

7.8 (X)

The way to check if an operator is time-independent or not, is to see if it commutes with the time-independent Hamiltonian.

$$\left[\hat{H}, S_1^z\right]$$

Expanding the coefficients and taking out some factors we get the following:

$$\frac{J}{\hbar}i\Big(-S_1^y S_2^x + S_1^x S_2^y\Big) \neq 0$$

$$\left[\hat{H}, S_1^z\right] = \frac{J}{\hbar}i\Big(S_1^y S_2^x - S_1^x S_2^y\Big) \neq 0$$

Neither operator commutes, but as they only differ by a sign, we know that $(S_1^z + S_2^z)$ must equal zero. Naturally we also know that $(S_1^z - S_2^z)$ can't be zero and is therefore also time-dependent. In conclusion: <u>Only G_3 is time-independent</u>.

7.9 (X)

a)

We use the basis of position 1 being $(1, 0, 0)$, position 2 being $(0, 1, 0)$ and position 3 being $(0, 0, 1)$. We then have the following Hamiltonian:

$$H \simeq -g \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

b)

To be Hermtian we check the following:

$$R = R^\dagger$$

$$|2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3| \neq |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|$$

To see if it is unitary we check the following:

$$R^\dagger = R^{-1}$$

This is easy if we do the following:

$$RR^\dagger = I$$

$$\Big(|2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3|\Big)\Big(|1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|\Big)$$

$$|2\rangle\langle 2| + |3\rangle\langle 3| + |1\rangle\langle 1| = I$$

Therefore we know that R is unitary. To check for a symmetric transformation we check if it commutes with the Hamiltonian. Looking at its definition we see that it is $-g(R+R^\dagger)$.

$$[H,R] = -g\left[R+R^\dagger, R\right]$$

As R obviously commutes with itself we get:

$$-g\Big(R^\dagger R - RR^\dagger\Big) = 0$$

Therefore we know that R is a symmetric transformation.

c)

$$RR = \Big(\left|2\right\rangle\left\langle 1\right| + \left|3\right\rangle\left\langle 2\right| + \left|1\right\rangle\left\langle 3\right|\Big)\Big(\left|2\right\rangle\left\langle 1\right| + \left|3\right\rangle\left\langle 2\right| + \left|1\right\rangle\left\langle 3\right|\Big) = \left|2\right\rangle\left\langle 3\right| + \left|3\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 2\right|$$

$$R^2R = \Big(\left|2\right\rangle\left\langle 3\right| + \left|3\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 2\right|\Big)\Big(\left|2\right\rangle\left\langle 1\right| + \left|3\right\rangle\left\langle 2\right| + \left|1\right\rangle\left\langle 3\right|\Big)$$

$$\left|2\right\rangle\left\langle 2\right| + \left|3\right\rangle\left\langle 3\right| + \left|1\right\rangle\left\langle 1\right| = I$$

If $R^3 = I$, and R has an eigenvalue λ_R and I has eigenvectors $\lambda_I = 1$, then we know that $\lambda_R^3 = 1$. Which means $\lambda_R = \sqrt[3]{1} = 1$. We know R is not Hermitian which opens the door for complex values. We know that $e^{i2\pi n} = 1$. Therefore, we know that R has eigenvalues $\lambda_R = e^{i2\pi n/3}$ where $n = 0, 1, 2$.

d)

Using the time evolution operator we get:

$$e^{iHt/\hbar} = e^{-ig(R+R^\dagger)t/\hbar}$$

$$\left|\psi(t)\right\rangle = e^{-ig(R+R^\dagger)t/\hbar}\left|\psi(0)\right\rangle$$

Can' really do more as I don't have a proper state.