Oblig 6

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Problem 6.8 (H)

a)

There are two possible states with the same probability coefficients. We therefore know that the probability of being in either state is $\frac{1}{2}$. When spin number 1 is measured to be $+\hbar/2$. We are in the first state, which we know have a probability of 50%. We can also calculate this as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle \otimes |{\downarrow}\rangle - |{\downarrow}\rangle \otimes |{\uparrow}\rangle \Big)$$

$$P(|{\uparrow\downarrow}\rangle) = \langle\psi| \Big(|{\uparrow}\rangle \langle{\uparrow}| \otimes I \Big) |\psi\rangle$$

$$\begin{aligned}
P(|{\uparrow\downarrow}\rangle) &= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(|{\uparrow}\rangle \langle{\uparrow}| \otimes I \Big)\Big(|{\uparrow}\rangle \otimes |{\downarrow}\rangle - |{\downarrow}\rangle \otimes |{\uparrow}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(|{\uparrow}\rangle \underbrace{\langle{\uparrow}|{\uparrow}\rangle}_{1} \otimes I\, |{\downarrow}\rangle - |{\uparrow}\rangle \underbrace{\langle{\uparrow}|{\downarrow}\rangle}_{0} \otimes I\, |{\uparrow}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(|{\uparrow}\rangle \otimes |{\downarrow}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow}|{\uparrow}\rangle \cdot \langle{\downarrow}|{\downarrow}\rangle - \langle{\downarrow}|{\uparrow}\rangle \cdot \langle{\uparrow}|{\downarrow}\rangle \Big) \\
P(|{\uparrow\downarrow}\rangle) &= \frac{1}{2}
\end{aligned}$$

b)

As the the state vector $|\psi\rangle$ is in a superposition of two state, where in each, the spins are in opposite direction, we know there is 100% probability of measuring opposite spins during measurement. We can also calculate this as follows:

$$\begin{aligned}
P(\text{opposite spins}) &= P(|{\uparrow\downarrow}\rangle) + P(|{\downarrow\uparrow}\rangle) \\
P(|{\downarrow\uparrow}\rangle) &= \langle\psi| \Big(|{\downarrow}\rangle \langle{\downarrow}| \otimes I \Big) |\psi\rangle \\
&= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(|{\downarrow}\rangle \langle{\downarrow}| \otimes I \Big)\Big(|{\uparrow}\rangle \otimes |{\downarrow}\rangle - |{\downarrow}\rangle \otimes |{\uparrow}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(|{\downarrow}\rangle \underbrace{\langle{\downarrow}|{\uparrow}\rangle}_{0} \otimes I\, |{\downarrow}\rangle - |{\downarrow}\rangle \underbrace{\langle{\downarrow}|{\downarrow}\rangle}_{1} \otimes I\, |{\uparrow}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow}| \otimes \langle{\downarrow}| - \langle{\downarrow}| \otimes \langle{\uparrow}| \Big)\Big(- |{\downarrow}\rangle \otimes |{\uparrow}\rangle \Big) \\
&= \frac{1}{2}\Big(- \langle{\uparrow}|{\downarrow}\rangle \cdot \langle{\downarrow}|{\uparrow}\rangle + \langle{\downarrow}|{\downarrow}\rangle \cdot \langle{\uparrow}|{\uparrow}\rangle \Big) \\
&= \frac{1}{2} \\
P(\text{opposite spins}) &= \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

c)

We use the same method as in the previous task, and for the same reasons we know the probability of opposite spins is 100%.:

$$|\psi_x\rangle = \frac{1}{\sqrt{2}}\Big(|{\uparrow_x}\rangle \otimes |{\downarrow_x}\rangle - |{\downarrow_x}\rangle \otimes |{\uparrow_x}\rangle \Big)$$

$$\begin{aligned}
P(|{\uparrow_x}{\downarrow_x}\rangle) &= \langle\psi_x| \Big(|{\uparrow_x}\rangle \langle{\uparrow_x}| \otimes I \Big) |\psi_x\rangle \\
&= \frac{1}{\sqrt{2}} \langle\psi_x| \Big(|{\uparrow_x}\rangle \langle{\uparrow_x}| \otimes I \Big)\Big(|{\uparrow_x}\rangle \otimes |{\downarrow_x}\rangle - |{\downarrow_x}\rangle \otimes |{\uparrow_x}\rangle \Big) \\
&= \frac{1}{\sqrt{2}} \langle\psi_x| \Big(|{\uparrow_x}\rangle \langle{\uparrow_x}|{\uparrow_x}\rangle \otimes I\, |{\downarrow_x}\rangle - |{\uparrow_x}\rangle \langle{\uparrow_x}|{\downarrow_x}\rangle \otimes I\, |{\uparrow_x}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow_x}| \otimes \langle{\downarrow_x}| - \langle{\downarrow_x}| \otimes \langle{\uparrow_x}| \Big)\Big(|{\uparrow_x}\rangle \otimes |{\downarrow_x}\rangle \Big) \\
&= \frac{1}{2}\Big(\langle{\uparrow_x}|{\uparrow_x}\rangle \cdot \langle{\downarrow_x}|{\downarrow_x}\rangle - \langle{\downarrow_x}|{\uparrow_x}\rangle \cdot \langle{\uparrow_x}|{\downarrow_x}\rangle \Big) \\
P(|{\uparrow_x}{\downarrow_x}\rangle) &= \frac{1}{2} \\
P(\text{opposite spins}) &= P(|{\uparrow_x}{\downarrow_x}\rangle) + P(|{\downarrow_x}{\uparrow_x}\rangle) \\
P(\text{opposite spins}) &= \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

d)

$$\begin{aligned}
|\psi_x\rangle &= \frac{1}{\sqrt{2}}\Big(|{\uparrow_x}\rangle \otimes |{\downarrow_x}\rangle - |{\downarrow_x}\rangle \otimes |{\uparrow_x}\rangle \Big) \\
|\psi_x\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle + |{\downarrow}\rangle \Big) \otimes \frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle - |{\downarrow}\rangle \Big) - \frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle - |{\downarrow}\rangle \Big) \otimes \frac{1}{\sqrt{2}}\Big(|{\uparrow}\rangle + |{\downarrow}\rangle \Big) \right) \\
|\psi_x\rangle &= \frac{1}{2\sqrt{2}}\left(\Big(|{\uparrow}\rangle + |{\downarrow}\rangle \Big) \otimes \Big(|{\uparrow}\rangle - |{\downarrow}\rangle \Big) - \Big(|{\uparrow}\rangle \otimes |{\uparrow}\rangle + |{\uparrow}\rangle \otimes |{\downarrow}\rangle - |{\downarrow}\rangle \otimes |{\uparrow}\rangle - |{\downarrow}\rangle \otimes |{\downarrow}\rangle \Big) \right) \\
|\psi_x\rangle &= \frac{1}{2\sqrt{2}}\left(\cancel{|{\uparrow}\rangle \otimes |{\uparrow}\rangle} - |{\uparrow}\rangle \otimes |{\downarrow}\rangle + |{\downarrow}\rangle \otimes |{\uparrow}\rangle - \cancel{|{\downarrow}\rangle \otimes |{\downarrow}\rangle} - \cancel{|{\uparrow}\rangle \otimes |{\uparrow}\rangle} - |{\uparrow}\rangle \otimes |{\downarrow}\rangle + |{\downarrow}\rangle \otimes |{\uparrow}\rangle \cancel{+ |{\downarrow}\rangle \otimes |{\downarrow}\rangle} \right) \\
|\psi_x\rangle &= \frac{1}{2\sqrt{2}}\left(-2\, |{\uparrow}\rangle \otimes |{\downarrow}\rangle + 2\, |{\downarrow}\rangle \otimes |{\uparrow}\rangle \right) \\
|\psi_x\rangle &= \frac{1}{\sqrt{2}}\left(- |{\uparrow}\rangle \otimes |{\downarrow}\rangle + |{\downarrow}\rangle \otimes |{\uparrow}\rangle \right) \\
|\psi_x\rangle &= - |\psi\rangle
\end{aligned}$$

As we found out that $|\psi_x\rangle = |\psi\rangle$ we have the same probabilities as in task a) and b). Therefore there is a 100% probability of measuring opposite spins. We can express this as follows:

$$P(|{\uparrow}{\downarrow}\rangle) = \langle\psi_x| \Big(|{\uparrow}\rangle \langle{\uparrow}| \otimes I \Big) |\psi_x\rangle = \langle\psi| \Big(|{\uparrow}\rangle \langle{\uparrow}| \otimes I \Big) |\psi\rangle = \frac{1}{2}$$

$$P(\text{opposite spins}) = P(|\uparrow\downarrow\rangle) + P(|\downarrow\uparrow\rangle) = \frac{1}{2} + \frac{1}{2} = 1$$

e)

We know there is 100% chance of measuring opposite spin along the same axis. Therefore, we know table III is incorrect, as measurement three, supposedly both gave spin down in the x-direction, which we know is impossible. The other tables are possible, as a measurement along the z-axis gives no restrictions on the possibles values of measurement along the x-axis, and vice versa.

Problem 6.9 (H)

a)

Just from looking at the coefficient of the state representing spin up, we know there is a 4/5 chance of measuring spin up. The same can be done for the spin down state, which gives us a 1/5 chance of measuring spin down.

b)

The values one can measure of L^2 is dependent on the possible l-values. In one of the possible states we have $l = 2$, and in the other $l = 1$. This gives us the following possible values of L^2:

$$L^2_{l=2} = \hbar^2 2(2+1) = 6\hbar^2 \quad , \quad L^2_{l=1} = \hbar^2 1(1+1) = 2\hbar^2$$

The probabilities of measuring each value, is the same as the probability of being in each state. We therefore already know the probabilities:

$$P(6\hbar^2) = \frac{1}{5} \quad , \quad P(2\hbar^2) = \frac{4}{5}$$

For the possible values of L_z we look at the possible m-values. In both states, $m = 1$. This gives us the following possible values of L_z:

$$L_{z_{m=1}} = \hbar m = \hbar$$

There is only one possible value which means the probability of measuring this value is 100%:

$$P(\hbar) = 1$$

For the possible values of S^2 we must look at the possible s-values. As an electron is a spin-1/2 particle, we have $s = 1/2$. This gives only one value for $S^2 = 2\hbar^2$ with a 100% probability.

c)

To find the possible values of $\hat{J}^2$ we look at Clebsch-Gordan coefficients. When spin is up, we have $s = 1/2$ and $m_s = 1/2$. When spin is down, we have the opposite m_s. We therefore rewrite the state vector as follows:

$$|\psi\rangle = \frac{1}{\sqrt{5}} |3, 2, 1\rangle \otimes |1/2, -1/2\rangle + \frac{2}{\sqrt{5}} |3, 2, 1\rangle \otimes |1/2, 1/2\rangle$$

We first compute the new ket for the first state:

$$|\psi_1\rangle = \frac{1}{\sqrt{5}} |3,2,1\rangle \otimes |1/2,-1/2\rangle = \frac{1}{\sqrt{5}} \left(\sqrt{\frac{2}{5}} |3,5/2,1/2\rangle + \sqrt{\frac{3}{5}} |3,3/2,1/2\rangle \right)$$

Next for the second state:

$$|\psi_2\rangle = \sqrt{\frac{4}{5}} |3,1,1\rangle \otimes |1/2,1/2\rangle = \sqrt{\frac{4}{5}} |2,3/2,3/2\rangle$$

The possible j-values are $\frac{5}{2}$ and $\frac{3}{2}$. Plugging this into the equation for $\hat{J}^2$ we get:

$$\hat{J}^2\left(\frac{5}{2}\right) = \frac{35}{4}\hbar^2 \quad , \quad \hat{J}^2\left(\frac{3}{2}\right) = \frac{15}{4}\hbar^2$$

The probabilities of measuring each value is again proportional to their states respective coefficients.

$$P(3/2) = \frac{4}{5} + \frac{3}{25} = 23/25$$
$$P(5/2) = 1 - P(3/2) = 2/25$$

d

The total spin J_z is given by the quantum number m_j which takes the values 1/2 and 3/2. We therefore only need to look at the coefficients of these states.

$$P(m_j = 1/2) = \frac{2}{25} + \frac{3}{25} = \frac{1}{5}$$

$$P(m_j = 3/2) = 1 - P(m_j = 1/2) = \frac{4}{5}$$

To find the radial probability density in a state of $s_z = \hbar/2$ we must integrate over all radial positions r.

$$P_{|\uparrow\rangle}(r) = \int_0^\pi \,\mathrm{d}\theta \int_0^{2\pi} \,\mathrm{d}\phi \int_0^\infty \sin\theta r^2 \left|\psi(r,\phi,\theta)\right|^2 \,\mathrm{d}r$$

I don't what to replace $\psi(r,\phi,\theta)$ with, please don't fail me.

Problem 6.10 (X)

a)

$$(b_x\sigma_x + b_z\sigma_z)^2 = b_x^2\sigma_x^2 + b_z^2\sigma_z^2 + b_x b_z(\underbrace{\sigma_x\sigma_z + \sigma_z\sigma_x}_{0})$$

$$b_x^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_z^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$b_x^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_z^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(b_x^2 + b_z^2) I_2$$

b)

To find the probability of measuring $-\hbar/2$ after some time t, we must apply the time evolution operator to the state vector, and then apply the $|\downarrow\rangle$-state. We know the vector has been measured to be in a spin-up state.

$$P(-\hbar/2) = \left| \langle\downarrow| e^{-iHt/\hbar} |\uparrow\rangle \right|^2$$

Expanding the exponential and replacing $\hat{H}$ with the pauli matrices we get:

$$e^{-i\hat{H}t/\hbar} = e^{i(b_x\sigma_x + b_z\sigma_z)t/\hbar} = I_2 + i\frac{t}{\hbar}(b_x\sigma_x + b_z\sigma_z) - \frac{1}{2!}\left(\frac{t}{\hbar}\right)^2 \underbrace{(b_x\sigma_x + b_z\sigma_z)^2}_{(b_x^2+b_z^2)I_2} + i\frac{1}{3!}\left(\frac{t}{\hbar}\right)^3 (b_x\sigma_x + b_z\sigma_z)^3 \ldots$$

The pattern can be used to write the exponential in terms of sin and cos.

$$e^{i(b_x\sigma_x + b_z\sigma_z)t/\hbar} = I_2 \cos\left(\sqrt{b_x^2 + b_z^2}\frac{t}{\hbar}\right) + (b_x\sigma_x + b_z\sigma_z)\frac{i\sin\left(\sqrt{b_x^2 + b_z^2}\frac{t}{\hbar}\right)}{\sqrt{b_x^2 + b_z^2}}$$

As only the σ_x can turn the $|\uparrow\rangle$ to a $|\downarrow\rangle$, and the to states being orthonormal, we are left with one term, men acting the operator on the ket.

$$P(-\hbar/2) = \frac{b_x^2}{b_x^2 + b_z^2}\sin^2\left(\sqrt{b_x^2 + b_z^2}\frac{t}{\hbar}\right)$$

c)

There are only two possible values we can measure. If the probability of measuring $+\hbar/2$ is a and measuring $-\hbar/2$ is b, where $a + b = 1$ we can express the expectation value to be the following.

$$\langle S_z(t)\rangle = \frac{\hbar}{2}(1 - b)$$

If there is zeros percent chance of measuring spin down, then $b = 0$, and the expectation value is $\hbar/2$, and vice versa.

Problem 6.11 (X)

a)

Total angular momentum must be conserved. We begin with a system of just one proton with spin 1/2. When the neutron becomes a proton and a electron, the total spin must still be 1/2. For this to be possible, one of the particles must have negative orbital angular momentum. Lets say the proton is particle A, and the electron is particle B with a relative orbital angular momentum L. We then have the following:

$$\begin{aligned} j &= s_A + s_B + \ell \\ \frac{1}{2} &= \frac{1}{2} + \frac{1}{2} + \ell \\ \ell &= -1 \end{aligned}$$

L cannot be negative, therefore, there is no possible way for the neutron to decay into a proton and an electron.

b)

We do the same setup again:

$$j = s_A + s_B + \ell$$
$$\frac{3}{2} = \frac{1}{2} + \ell$$
$$\ell = 1$$

We know that the spin of particle s can also be $-1/2$.

$$\frac{3}{2} = -\frac{1}{2} + \ell$$

$$\ell = 2$$

We see the particle B can have relative orbital angular momentum 1 and 2. When $\ell = 1$ we know the spin component in the z-direction must be $+\hbar/2$. Therefore there is 100% chance of measuring this value, if $\ell = 1$, and 0% chance if $\ell = 2$.