

Oblig 3

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Problem 3.1 (L)

Problem 3.2 (L)

As a continuous observable have an infinite amount of possible measurement values, yet can't be in two states at once, we need an infinite-dimensional Hilbert space. All the state vectors need to be linearly independent, and an infinite amount of linearly independent vectors must be in a infinite-dimensional Hilbert space.

Problem 3.3 (L)

The probability of measuring the the value g :

$$P(g) = |\langle g|\psi\rangle|^2 = \frac{1}{2}(-ia + ib)^2 = \frac{1}{2}(a^2 - b^2)$$

If we measure the observable to be g , there is a 100% of the state being $|g\rangle$. If g was not obtained then there is a 0% chance of the state being $|g\rangle$.

Problem 3.4 (L)

As this is a degenerate state of value 2, there are two possible eigenstates which can result in measuring +1. The probability of measuring +1 is the sum of the probabilities of measuring +1 for each of the eigenstates.

$$\hat{P}_1 = |a_1|^2 + |a_2|^2$$

Problem 3.5 (L)

TODO:

Problem 3.6 (H)

a)

It is easy to see that $\hat{H} = \hat{H}^\dagger$. Now we have to check if it has real eigenvalues.

$$\hat{H} = \begin{pmatrix} 1 & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Now we employ the eigenvalue equation:

$$\det(\hat{H} - \lambda \hat{I}) = 0$$
$$\det \left(\begin{pmatrix} 1-\lambda & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1-\lambda & 0 \\ 0 & 0 & \frac{1}{2}-\lambda \end{pmatrix} \right)$$

$$\underbrace{(1-\lambda)(1-\lambda)\left(\frac{1}{2}-\lambda\right)}_{\text{term 1}} - \underbrace{\frac{i}{2}\left(-\frac{i}{2}\left(\frac{1}{2}-\lambda\right)\right)}_{\text{term 2}} + 0 = 0$$

Beginning with the first term:

$$(1-2\lambda+\lambda^2)\left(\frac{1}{2}-\lambda\right) = \frac{1}{2}-\lambda+\frac{\lambda^2}{2}-\lambda+2\lambda^2-\lambda^3$$

$$\underline{\frac{1}{2}-2\lambda+\frac{5}{2}\lambda^2-\lambda^3}$$

Now the second term:

$$-\frac{1}{4}\left(\frac{1}{2}-\lambda\right) = -\frac{1}{8}+\frac{\lambda}{4}$$

$$\frac{1}{2}-2\lambda+\frac{5}{2}\lambda^2-\lambda^3-\frac{1}{8}+\frac{\lambda}{4} = 0$$

$$-\lambda^3+\frac{5}{2}\lambda^2-\frac{7}{4}\lambda-\frac{3}{8} = 0$$

$$\underline{\underline{\lambda = \frac{3}{2} \quad , \quad \lambda = \frac{1}{2} \quad , \quad \lambda = -\frac{1}{2}}}$$

All the eigenvalues are real, and the matrix is hermitian.

b)

$$\hat{H}|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i + \frac{i}{2} + 0 \\ -\frac{i}{2}i + 1 + 0 \\ 0 + 0 + 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3}{2}i \\ \frac{3}{2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \frac{3}{2} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$|1\rangle$ is an eigenvector with eigenvalue $\frac{3}{2}$.

$$\hat{H}|2\rangle = \begin{pmatrix} 1 & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$|2\rangle$ is an eigenvector with eigenvalue $\frac{1}{2}$.

$$\hat{H}|3\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} -i + \frac{i}{2} + 0 \\ -\frac{i}{2} + 1 + 0 \\ 0 + 0 - \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{3}} \frac{1}{2} \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix}$$

$|3\rangle$ is an eigenvector with eigenvalue $\frac{1}{2}$.

c)

The operator \hat{H} can be written using the eigenstates as a basis \hat{H}_E

$$H_{11} = \langle 1 | \hat{H} | 1 \rangle = \frac{3}{2} \langle 1 | 1 \rangle = \frac{3}{2}$$

As the matrix eigenstates must be orthogonal, we only need to calculate the diagonal elements, where $i = j$.

$$H_{22} = \langle 2 | \hat{H} | 2 \rangle = \frac{1}{2} \langle 2 | 2 \rangle = \frac{1}{2}$$

$$H_{33} = \langle 3 | \hat{H} | 3 \rangle = \frac{1}{2} \langle 3 | 3 \rangle = \frac{1}{2}$$

$$\hat{H}_E = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

This is of course an diagonal matrix, and the eigenvalues are the diagonal elements.

d)

TODO: Are they not already orthogonal?

Problem 3.7 (H)

a)

$$\hat{H} = -g \sum_{i=0}^{L-2} (|i\rangle \langle i+1| + \langle i+1|i\rangle) - V |0\rangle \langle 0|$$

$$\hat{H} = -g (|0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 2| + |2\rangle \langle 1| + |2\rangle \langle 3| + |3\rangle \langle 2|) - V |0\rangle \langle 0|$$

Lets expand one term at a time.

$$|0\rangle \langle 1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |1\rangle \langle 0| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|1\rangle \langle 2| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |2\rangle \langle 1| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|2\rangle \langle 3| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad |3\rangle \langle 2| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|0\rangle \langle 0| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{H} = \begin{pmatrix} gV & -g & 0 & 0 \\ -g & 0 & -g & 0 \\ 0 & -g & 0 & -g \\ 0 & 0 & -g & 0 \end{pmatrix}$$

b)

The position operator \hat{X} in matrix form:

$$\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$