Problem Set 1

Oskar Idland

Problem 1.1 (L)

We define $|u\rangle$ and $\langle u|$.

$$|u\rangle = \alpha |u_1\rangle + \beta |u_2\rangle \quad , \quad \langle u| = |u\rangle^\dagger = \alpha^* \langle u_1| + \beta^* \langle u_2|$$

Using the distributive property of the inner product, we get

$$\langle u|w\rangle = (\alpha^* \langle u_1| + \beta^* \langle u_2|) |w\rangle = \alpha^* \langle u_1|w\rangle + \beta^* \langle u_2|w\rangle$$

Problem 1.2 (L)

Defining $|w\rangle$.

$$|w\rangle = \alpha |w_1\rangle + \beta |w_2\rangle$$

Defining the relationship between $|w\rangle$ and $\langle w|$.

$$\langle w| = |w\rangle^\dagger = (\alpha |w_1\rangle + \beta |w_2\rangle)^\dagger$$

Using the distributive property of the conjugate transpose, we arrive at

$$\langle w| = (\alpha |w_1\rangle)^\dagger + (\beta |w_2\rangle)^\dagger = \alpha^* \langle w_1| + \beta^* \langle w_2|$$

Problem 1.3 (L)

$$\langle f|g\rangle = \int_a^b f(x)^* g(x) \, \mathrm{d}x$$

$$\langle f|f\rangle = \int_a^b f(x)^2 \, \mathrm{d}x$$

All values of $f(x)^2$ are greater or equal to zero. Therefore, the integral is only zero if $f(x) = 0$ for all x.

Problem 1.4 (L)

$$\langle a_i|V\rangle = \sum_{j=1}^{N} \langle a_i| v_i |a_j\rangle$$

As all the basis vectors are orthonormal, the only non-zero term in the sum is when $i = j$. As the basis vectors are orthonormal their magnitude is one. Therefore, we get the following.

$$v_i \langle a_i|a_i\rangle = v_i$$

Problem 1.5 (L)

$$\langle x|f\rangle = \int dx' f(x') \langle x|x'\rangle = \int dx' f(x')\delta(x-x')$$

The Dirac delta picks out the only value of x' that gives a non-zero value. Therefore, we get the following.

$$\therefore \langle x|f\rangle = f(x)$$

Applying the identity $\mathbf{I}$ operator to $|f\rangle$, we get the following.

$$\mathbf{I}\,|x\rangle = \int_0^L \mathrm{d}x' \,|x'\rangle \langle x'|x\rangle = \int_0^L \mathrm{d}x' \,|x'\rangle \,\delta(x-x') = |x\rangle$$

Problem 1.6 (H)

a)

Using trig identities we know that

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\ \mathrm{d}x = \frac{1}{\pi}\left[\arctan(x/\epsilon)\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

b)

$$\delta_{\epsilon/k}(x) = k\delta_\epsilon(kx)$$

$$\frac{1}{k\pi}\frac{\epsilon}{\epsilon^2+x^2} = \frac{k}{\pi}\frac{\epsilon}{\epsilon^2+(kx)^2}$$

$$\frac{1}{k\pi}\frac{\epsilon}{\epsilon^2+x^2} = \frac{k}{k^2\pi}\frac{\epsilon}{\epsilon^2+x^2} = \underline{\underline{\frac{1}{k\pi}\frac{\epsilon}{\epsilon^2+x^2}}}$$

c)

E_1.6.pdf

Figur 1: Numerical plot of $\delta_\epsilon(x)$ for $\epsilon = 1$, $\epsilon = 0.1$ and $\epsilon = 0.01$.

d)

Assuming we have an infinite number of arguments x and values $y = \delta_1(x)$, we can use the conclusions from exercise b).

$$\delta_{0.1}(x) = 10\delta_1(10x).$$

To obtain a similar table for the new function $\delta_{0.1}$(x), we map our new table's y-value to the table of δ_1 y-value for the same x-argument times 10, and multiply that result again, by 10.

$$x_{0.1} \mapsto x_1 \quad , \quad y_{0.1} \mapsto 10y_1(10x_1)$$

Problem 1.7 (H)

As the basis are orthonormal we know $\langle b_i|b_j\rangle = 1$, if $i = j$.

$$\hat{P}_1 |A\rangle = \sum_{i=1}^{N} |b_1\rangle \langle b_1| \alpha_i |b_i\rangle = \underline{\underline{\alpha_1 |b_1\rangle}}$$

$$\hat{P}_1\hat{P}_1 |A\rangle = \hat{P}_1 \alpha_1 |b_1\rangle = \alpha_1 |b_1\rangle \underbrace{\langle b_1|b_1\rangle}_{1} = \underline{\underline{\alpha_1 |b_1\rangle}}$$

The operator $\hat{P}_1$ can be considered as a projection operator. It finds the projection of the ket $|A\rangle$ onto the basis vector $|b_1\rangle$. As the basis vectors are orthonormal, the operator $\hat{P}_1$ only picks out the component of $|A\rangle$ that is parallel to $|b_1\rangle$ together with its magnitude.

Problem 1.8 (E)

A quantum state is a complex vector (ket) in a Hilbert space. The ket vector contains all there is to know about the system. It contains the probabilities of all possible outcomes of a measurement. The difference between a quantum state and a classical state, is that the classical state does not change when measured, but the quantum state does. With a complete description of a classical state you know deterministically how it will behave in the past and future. With a quantum state it's only probabilistic what might happen in the future. To gather information of a classical state, you just need to look at the values. Its position is just its position. To gather information of a quantum state, we use operators, which acts on the ket, and gives us the probabilities of different outcomes.

Problem 1.9 (E)

$$|u\rangle \simeq \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad , \quad |v\rangle \simeq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad , \quad \alpha = a + ib$$

$$\langle u|v\rangle = \underline{\underline{u_1 v_1 + u_2 v_2}}$$

$$\langle u| \alpha |v\rangle = \langle u| (a |v\rangle + ib |v\rangle) = a \langle u|v\rangle + ib \langle u|v\rangle$$

$$\langle u| \alpha |v\rangle = \underline{\underline{a u_1 v_1 + i b u_2 v_2}}$$