Problem Set 2

Problem 2.1 (L)

$$|\psi\rangle = \sum_{i=1}^{2} \psi_i |a_i\rangle = \sum_{i=1}^{2} \psi_i' |a_i'\rangle$$

We can express ψ_i in the following manner:

$$\psi_i = \sum_{i=1}^{2} \langle a_i| \psi_i' |a_i'\rangle = \underline{\underline{\sum_{i=1}^{2} \psi_i' \langle a_i|a_i'\rangle}}$$

We can then express ψ_i' in terms of ψ_i:

$$\psi_i' = \underline{\underline{\sum_{i=1}^{2} \psi_i \langle a_i'|a_i\rangle}}$$

Problem 2.2 (L)

We begin by creating a matrix T to convert from the basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ into the new basis of $B' = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$. By using the transformation matrix T given by:

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We also need the inverse matrix T^{-1} to transform from our basis B into the other basis B'.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The last two columns of the matrix forms the inverse matrix T^{-1}.

$$T^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

This gives us the operator $\hat{O}'$.

$$\hat{O}' = T^{-1}\hat{O}T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\hat{O} = \hat{O}^T}}$$

Problem 2.3 (L)

$$\langle w| \hat{O}^\dagger |w\rangle = \langle w| \hat{O} |w\rangle^*$$

We define $|w\rangle = |u\rangle + |v\rangle$.

$$(\langle u| + \langle v|) \hat{O}^\dagger (|u\rangle + |v\rangle) = (\langle u| + \langle v|) \hat{O} (|u\rangle + |v\rangle)^*$$

Problem 2.4 (L)

When an operator acts on its eigenstate, it returns the eigenvalue times the eigenstate.

$$\hat{K} |\lambda\rangle = \lambda |\lambda\rangle$$

The eigenvalues are real and the operator Hermitian, so the following must be true if we conjugate both sides:

$$\langle u| \hat{K} |\lambda\rangle = \lambda \langle u|\lambda\rangle$$
$$\left(\langle u| \hat{K} |\lambda\rangle\right)^\dagger = (\lambda \langle u|\lambda\rangle)^\dagger$$
$$\langle\lambda| \hat{K}^\dagger |u\rangle = \lambda^* \langle\lambda|u\rangle$$
$$\therefore \langle\lambda| \hat{K} |u\rangle = \lambda \langle\lambda|u\rangle$$

Problem 2.5 (L)

$$\hat{O} = |1\rangle \langle 1| - |2\rangle \langle 2|$$

Problem 2.6 (X)

An Hermitian operator is defined as an operator which is equal to its Hermitian conjugate, meaning $\hat{L} = \hat{L}^\dagger$. Hermitian operators have real eigenvalues. This is a good attribute for an operator which represent an observable, as the we only measure real values during experiments.

Proof of Real Eigenvalues:

$$\mathbf{L} |\lambda\rangle = \lambda |\lambda\rangle$$
$$\langle\lambda| \mathbf{L}^\dagger = \langle\lambda| \lambda^*$$

Multiplying the first equation with $\langle\lambda|$ and the second with $|\lambda\rangle$

$$\langle\lambda| \mathbf{L} |\lambda\rangle = \lambda \langle\lambda|\lambda\rangle$$
$$\langle\lambda| \mathbf{L}^\dagger |\lambda\rangle = \lambda * \langle\lambda|\lambda\rangle$$

As $\mathbf{L} = \mathbf{L}^\dagger$, it follows that $\lambda = \lambda^*$ and it must therefore be real

Problem 2.7 (X)

$$\hat{O} = \hat{A}\hat{B}$$
$$\hat{A}\hat{B} = A_{ij}B_{i'j'} = AB$$

We begin with a simple case of both $\hat{A}$ and $\hat{B}$ being 2×2 matrices.

$$\hat{O} = \hat{A}\hat{B} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Lets check if the following equation holds.

$$\hat{O}^\dagger \stackrel{?}{=} \hat{B}^\dagger \hat{A}^\dagger$$

Sol. 1

We first examine the left side

$$\hat{O}^\dagger = \begin{bmatrix} A_{11}^*B_{11}^* + A_{12}^*B_{21}^* & A_{21}^*B_{11}^* + A_{22}^*B_{21}^* \\ A_{11}^*B_{12}^* + A_{12}^*B_{22}^* & A_{21}^*B_{12}^* + A_{22}^*B_{22}^* \end{bmatrix}$$

Now the right side

$$\hat{A}^\dagger\hat{B}^\dagger = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^\dagger \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^\dagger = \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} B_{11}^* & B_{21}^* \\ B_{12}^* & B_{22}^* \end{bmatrix}$$

$$\begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} B_{11}^* & B_{21}^* \\ B_{12}^* & B_{22}^* \end{bmatrix} = \begin{bmatrix} A_{11}^*B_{11}^* + A_{12}^*B_{21}^* & A_{21}^*B_{11}^* + A_{22}^*B_{21}^* \\ A_{11}^*B_{12}^* + A_{12}^*B_{22}^* & A_{21}^*B_{12}^* + A_{22}^*B_{22}^* \end{bmatrix}$$

We can clearly see that

$$\hat{O}^\dagger = \hat{A}^\dagger\hat{B}^\dagger$$

And how this generalizes to any $n \times n$ matrix.

Sol. 2

Each element in $\hat{O}$ can be written as

$$O_{ij} = \sum_k^N A_{ik}B_{kj}$$

We can then take the Hermitian conjugate as we know how it operates on numbers.

$$O_{ij}^\dagger = \sum_k^N (A_{ik}B_{kj})^\dagger$$

$$O_{ji}^* = \sum_k^N B_{jk}^*A_{ki}^*$$

Therefore we know that $\hat{O}^\dagger = \hat{A}^\dagger\hat{B}^\dagger$.

Problem 2.8 (X)

To find the Hermitian conjugate of an operator we must find an operator $K^\dagger$ which satisfies the following:

$$\langle u| K |v\rangle = \langle v| K^\dagger |u\rangle$$

Where u and v can be represented as functions $u(x)$ and $v(X)$.

$$\langle u| K |v\rangle = \int_{-\infty}^{\infty} u^*(x) x \frac{\mathrm{d}}{\mathrm{d}x} v(x)\ \mathrm{d}x$$

We use integration by parts.

Definition:

$$u = u^*(x)x \quad , \quad v' = \frac{\mathrm{d}}{\mathrm{d}x}v(x)$$

$$u' = x\frac{\mathrm{d}}{\mathrm{d}x}u^*(x) + u^*(x) \quad , \quad v = v(x)$$

$$\underbrace{u^*(x)\ x\ v(x)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(u^*(x)x)v(x)\ \mathrm{d}x$$

$$-\int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}u^*(x)x + u^*(x)\right) v(x)\ \mathrm{d}x$$

$$-\int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x}u^*(x)\,(x+1)\right) v(x)\ \mathrm{d}x$$

$$-\int_{-\infty}^{\infty} \left(x\frac{\mathrm{d}}{\mathrm{d}x}u^*(x) + \frac{\mathrm{d}}{\mathrm{d}x}u^*(x)\right) v(x)\ \mathrm{d}x$$

$$\underline{\underline{K^\dagger = -x\frac{\mathrm{d}}{\mathrm{d}x} - 1}}$$

Problem 2.9 (E)

a)

By definition we know $\langle\psi| = |\psi\rangle^\dagger$.

$$\therefore \langle\psi| = c^*\left(\sqrt{3}\,\langle 0| - i\,\langle 1|\right)$$

To normalize this we must find c such that $\langle\psi|\psi\rangle = 1$.

$$\langle\psi|\psi\rangle = c^2\left(3\,\langle 0|0\rangle - \langle 1|1\rangle\right) = 1$$

$$2c^2 = 1 \quad , \quad c = \frac{1}{\sqrt{2}}$$

The normalized $|\psi\rangle$ becomes

$$\underline{\underline{|\psi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{3}\,|0\rangle + i\,|1\rangle\right)}}$$

b)

$$\underline{\underline{|\psi\rangle \simeq \begin{pmatrix} \sqrt{\frac{3}{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}}}$$

$$\hat{A}\,|0\rangle = -i\,|1\rangle$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix}$$

This gives two equations.

$$A_{11} + 0 \cdot A_{12} = 0 \rightarrow A_{11} = 0$$

$$A_{21} + 0 \cdot A_{22} = -i \rightarrow A_{21} = -i$$

$$\underline{\hat{A} \simeq \begin{pmatrix} 0 & A_{12} \\ -i & A_{22} \end{pmatrix}}$$

$$\hat{A}\left|1\right\rangle = i\left|0\right\rangle$$

$$\begin{pmatrix} 0 & A_{12} \\ -i & A_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

Again, two new equations.

$$A_{12} = i$$

$$A_{22} = 0$$

$$\underline{\underline{\hat{A} \simeq \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}}$$

c)

$$\left\langle\psi\right|\hat{A}\left|\psi\right\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3} & -i \end{pmatrix}\begin{pmatrix} -1 \\ -i\sqrt{3} \end{pmatrix} = \underline{\underline{-2\sqrt{3}}}$$

$$\left\langle\psi\right|\hat{A}\left|\psi\right\rangle = \left(\sqrt{3}\left\langle 0\right| - i\left\langle 1\right|\right)\hat{A}\left(\sqrt{3}\left|0\right\rangle + i\left|1\right\rangle\right)$$

$$\left(\sqrt{3}\left\langle 0\right| - i\left\langle 1\right|\right)\left(-i\sqrt{3}\left|1\right\rangle - \left|0\right\rangle\right)$$

$$-\sqrt{3}\left\langle 0|0\right\rangle + ii\sqrt{3}\left\langle 1|1\right\rangle$$

$$\underline{\underline{-2\sqrt{3}}}$$

Problem 2.10 (E)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a)

$$U^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

b)

To be Hermitian, U must be equal to its Hermitian conjugate.

$$U = U^\dagger$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

Both a and d must be real, and $b = c^*$.

c)

We assume that U has an eigenvector $|u\rangle$.

$$U\,|u\rangle = \lambda\,|u\rangle \quad , \quad |u\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} au_1 + bu_2 \\ cu_1 + du_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

This gives us two equations

$$au_1 + bu_2 = \lambda u_1$$

$$cu_1 + du_2 = \lambda u_2$$

$$\lambda = \frac{au_1 + bu_2}{u_1} = \frac{cu_1 + du_2}{u_2}$$

$$(au_1 + bu_2)u_2 = (cu_1 + du_2)u_1$$

$$au_1u_2 + bu_2^2 - cu_1^2 - du_1u_2 = 0$$

If this is to be true, and we take into account the demands on the elements of U from b), we get the following:

$$(au_1u_2 + bu_2^2 - cu_1^2 - du_1u_2) = (au_1u_2 + bu_2^2 - cu_1^2 - du_1u_2)^* = 0$$

$$(au_1u_2 + bu_2^2 - cu_1^2 - du_1u_2) - (au_1^*u_2^* + b^*u_2^2 - c^*u_1^2 - du_1^*u_2^*) = 0$$

Problem 2.11 (H)

Both $|\psi\rangle$ and $|\phi\rangle$ must both have real eigenvalues.