Lecture 2

Oskar Idland

Wave Function

$$|\psi\rangle = \int \,\mathrm{d}x\psi(x)\,|x\rangle$$

The wave function must not be mistaken for the state vector. The wave function $\psi(x)$ calculates the coefficients for all possible positions x in the x basis. The definition of the wave function is $|\psi\rangle = \langle x|\psi\rangle$

Operators

Definition: Takes a ket and returns a ket. It must be linear and Hermitian. As a consequence of being Hermitian the eigenvalues are real.

$$\hat{L}\,|u\rangle = |v\rangle$$

$$\hat{L}\,(|u\rangle + |v\rangle) = \hat{L}\,|u\rangle + \hat{L}\,|v\rangle$$
$$\hat{L} = \hat{L}^\dagger$$

Continuous Operators

$$|\psi\rangle = \int \langle x|\psi\rangle\,|x\rangle\;\mathrm{d}x = |x\rangle\,\langle x|\psi\rangle\;\mathrm{d}x = \underbrace{\int (|x\rangle\,\langle x|\;\mathrm{d}x)}_{\text{Identity operator }\hat{I}}\,|\psi\rangle$$

Discrete Operators

$$|\psi\rangle = \sum_i \psi_i\,|i\rangle = \sum_i \langle i|\psi\rangle\,|i\rangle = \underbrace{\sum_i |i\rangle\,\langle i|}_{\hat{I}}\,|\psi\rangle$$

$$\langle j|\psi\rangle = \sum_i \psi_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = \psi_j$$

Representation of Operators

The ket $|u\rangle$ and operator $\hat{L}$ can be represented in the following way:

$$|u\rangle \simeq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\hat{L} \simeq \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

$$|u\rangle \simeq u(x)$$

$$\hat{L} \simeq c_0(x) + c_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + c_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \dots$$

Discrete basis

$$\langle m|\,\hat{K}\,|n\rangle = K_{nm} = \text{m,n'th matrix element of } \hat{K}$$

$$|v\rangle = \sum_m V_m\,|m\rangle$$

$$\hat{K}\,|v\rangle = \langle n|\,\hat{K}\,|v\rangle = \langle n|\,\hat{K}\sum_m V_m\,|m\rangle = \sum_m V_m\,\langle n|\,\hat{K}\,|m\rangle = \sum_m V_m K_{nm}$$

Composite Operators

$$\hat{K}\hat{L}\,|v\rangle = \hat{K}(\hat{L}\,|v\rangle) = (\hat{K}\hat{L})\,|v\rangle$$
$$\langle n|\,\hat{K}\hat{L}\,|m\rangle$$

Inserting the identity operator:

$$\langle n|\,\hat{K}\underbrace{\sum_r |r\rangle\,\langle r|}_{\hat{I}}\,\hat{L}\,|m\rangle$$

$$\sum_r \langle n|\,\hat{K}\,|r\rangle\,\langle r|\,\hat{L}\,|m\rangle = \underbrace{\sum_r K_{nr}L_{rm}}_{\text{Matrix multiplication}}$$

Change of Basis

Give to different bases $\{|n\rangle\}$ and $\{|n'\rangle\}$.

$$|\psi\rangle = \sum_n \underbrace{\psi_n\,|n\rangle}_{\langle n|\psi\rangle} \quad , \quad |\psi\rangle = \sum_{n'} \underbrace{\psi_{n'}\,|n'\rangle}_{\langle n'|\psi\rangle}$$

$$\langle m'|\psi\rangle = \sum_n \psi_n\,\underbrace{\langle m'|n\rangle}_{S_{m'n}}$$

We define $S_{m'n}$ to be $\langle m'|n\rangle$ which is a matrix relation between ψ_n to $\psi_{n'}$.

$$\psi_{m'} = \sum_n S_{m'n}\psi_n$$

S is unitary meaning $S^\dagger = S^{-1}$

Proof of unitarity: Set $|\psi\rangle = |n'\rangle$

$$\underbrace{\langle m'|n'\rangle}_{\delta_{m'n'}} = \sum_n S_{m'n'} \quad \underbrace{\langle n|n'\rangle}_{\langle n'|n\rangle^* = S^*_{n'm}}$$

Remember $S_{m'n} = \langle m'|n\rangle$.

$$\langle n|n'\rangle = \langle n'|n\rangle^* = S^*_{n'm} = S^{T*}_{m'n} = S^\dagger_{m'n}$$
$$SS^\dagger = 1 \rightarrow S^\dagger = S^{-1}$$

Operators in different bases

$$K_{m'n'} = \langle m'|\hat{K}|n'\rangle = \langle m'|\hat{I}\hat{K}\hat{I}|n'\rangle$$

$$\sum_{mn} \underbrace{\langle n'|m\rangle}_{S_{m'm}} \langle m|\hat{K}|n\rangle \underbrace{\langle n|n'\rangle}_{S^{\dagger}_{nn'}}$$

$$\sum_{mn} S_{m'm} K_{mn} S^{\dagger}_{nn'} = S^{\dagger} K S = K'$$

Now we have the operator $\hat{K}$ in a new basis defined as $\hat{K}'$

Hermitian Conjugate of an Operator

Definition by the inner product:

$$\Big(|u\rangle, \hat{K}|v\rangle\Big) = \Big(\hat{K}^{\dagger}|u\rangle, |v\rangle\Big) \text{ for all } |u\rangle, |v\rangle$$

Is not as simple as "just transposing and taking the complex conjugate". A function could be Hermitian.

Definition in Dirac-Notaion:

$$\langle u|\hat{K}|v\rangle = \langle v|\hat{K}^{\dagger}|u\rangle^{*} \text{ for all } |u\rangle, |v\rangle$$

It is enough to define:

$$\langle u|\hat{K}|v\rangle = \langle v|\hat{K}^{\dagger}|u\rangle \text{ problem 2.3(L)}$$

Exercise: Find $\hat{K}^{\dagger}$

$$\hat{K} = \alpha |a\rangle \langle b|$$

$$\langle u|\hat{K}^{\dagger}|v\rangle = \langle v|\hat{K}|u\rangle^{*} = \langle v|a\rangle^{*} \langle b|u\rangle^{*}$$

$$\langle u|(|b\rangle\langle a|)|v\rangle \to \hat{K}^{\dagger} = |b\rangle\langle a|\alpha^{*}$$

Check the following correspondence

$$\hat{K}|v\rangle \leftrightarrow \langle v|\hat{K}^{\dagger}$$

We set $|w\rangle = \hat{K}|v\rangle$ and act on it with an arbitrary bra $\langle n|$.

$$\langle u|w\rangle = \langle u|\hat{K}|v\rangle = \langle v|\hat{K}^{\dagger}|u\rangle^{*} \to \langle w|u\rangle = \langle v|\hat{K}^{\dagger}|u\rangle$$

This holds for any $|w\rangle \to \langle w| = \langle v|\hat{K}^{\dagger}$

Exercise: Find the Hermitian conjugate of the following operator

$$\hat{K} \simeq \frac{\mathrm{d}}{\mathrm{d}x}$$

$$|u\rangle = u(x)$$

$$|v\rangle = v(x)$$

$$\langle u|\,\hat{K}^\dagger\,|v\rangle = \langle v|\,\hat{K}\,|u\rangle^* = \int \left(v^*(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right)^*\ \mathrm{d}x = \int v(x)\frac{\mathrm{d}}{\mathrm{d}x}u^*(x)\ \mathrm{d}x$$

$$\underbrace{v(x)u^*(x)}_{0}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}v(x)u^*(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} u^*(x)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)v(x)\ \mathrm{d}x$$