Problem 7.1(L)

Define the permutation operator of two particles $\hat{P}$ as an operator that interchanges their labels. That is for a state describing two particles with orbital parts ψ and spin-1/2 parts χ the action of $\hat{P}$ is

$$\hat{P}|\psi_a\rangle \otimes |\psi_b\rangle \otimes |\chi_a\rangle \otimes |\chi_b\rangle = |\psi_b\rangle \otimes |\psi_a\rangle \otimes |\chi_b\rangle \otimes |\chi_a\rangle$$

where we have ordered the tensor product spaces as (orbital part particle 1 $\otimes$ orbital part particle 2 $\otimes$ spin particle 1 $\otimes$ spin particle 2), thus $\hat{P}$ is simply exchanging the particle labels $1 \leftrightarrow 2$.

a) Show that $\hat{P}$ is hermitian, and that its eigenvalues are ± 1.

Introduce the position and the usual spin-1/2 S^z basis $\langle \vec{r}_1| \otimes \langle \vec{r}_2| \otimes \langle m_1| \otimes \langle m_2|$ where the $|m\rangle \in \{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ and $|\vec{r}'\rangle \simeq \delta^3(\vec{r} - \vec{r}')$ are the single-particle spin and position eigenstates respectively.

b) Write down the representation of the operator $\hat{P}$ in this basis. i.e. What is the explicit expression for $\langle \vec{r}_1| \otimes \langle \vec{r}_2| \otimes \langle m_1| \otimes \langle m_2|\ \hat{P}\ |\vec{r}'_1\rangle \otimes |\vec{r}'_2\rangle \otimes |m'_1\rangle \otimes |m'_2\rangle$

Problem 7.2(L)

In this problem you will treat a spin-1/2 particle with a spatial state $|\psi\rangle$ and spin state $|\chi\rangle$. Denote this state

$$|\Psi\rangle = |\psi\rangle \otimes |\chi\rangle$$

In the position and spin-z eigenbasis the basis vectors are denoted $|x\rangle \otimes |\pm\rangle$ where x takes a value on the interval $[0, L]$ while $|\pm\rangle$ corresponds to the two states $|+\rangle \equiv |\uparrow\rangle$ and $|-\rangle \equiv |\downarrow\rangle$. In this basis the wavefunction is a two-component complex function of x where

$$\Psi_\pm(x) = \langle x|\psi\rangle\langle \pm|\chi\rangle$$

a) Assume that the particle is in a one dimensional harmonic oscillator potential. Write down explicit expressions for the two wavefunction components if you assume that the spin of the particle is in a state that with certainty gives the measurement result $-\hbar/2$ if the spin in the x-direction is measured.

b) Assume that you have two such spin-1/2 particles. Write down a possible two-particle state in ket notation. Write down the corresponding wavefunction. (how many arguments, components does it have?)

Problem 7.3(L)

What is the definition of fermions? What is the definition of bosons? Does it make sense to speak about a fermionic/bosonic state?

Problem 7.4(L)

Write down a state that can represent two spin-1/2 fermions in a one dimensional harmonic oscillator potential.

Problem 7.5(L)

Consider two indistinguishable particles with spin-s. Find the number of different spin states that are symmetric (antisymmetric) with respect to the interchange of the spin variables. Neglect the orbital part of the state.

Problem 7.6(H)

In this problem you will consider the changes to the hydrogen 3d-states (principal quantum number $n = 3$ and orbital angular momentum $l = 2$) caused by a particular kind of spin-orbit interaction $\hat{H}_{so}$ so that the Hamiltonian is

$$\hat{H} = \hat{H}_0 + \hat{H}_{so}, \qquad \hat{H}_{so} = \frac{\lambda}{\hbar^2}\vec{L}\cdot\vec{S}.$$

Here $\hat{H}_0 = \vec{p}^2/2m - e^2/4\pi\epsilon_0 r$, $\vec{S}$ is the spin operator of the electron and $\vec{L}$ is the orbital angular momentum operator. λ is a positive constant.

a) Find the $n = 3$, $l = 2$ energy levels and their degeneracies when the spin-orbit interaction is present.

b) The system is in the lowest energy state with $n = 3$, $l = 2$ and has a total angular momentum component along the z-axis of $+\hbar/2$, find the probability to measure the spin-z component of the electron to be $+\hbar/2$.

Now add a term $\hat{H}_b = -\frac{b}{\hbar}J_z$ to the Hamiltonian. b is a small positive constant ($b \ll \lambda$) and J_z is the z-component of the total angular momentum operator.

c) What are the changes to the $n = 3$, $l = 2$ energy levels caused by $\hat{H}_b$.

Finally consider a state of *two* electrons, each having $n = 3$ and $l = 2$. Neglect the interaction between the electrons.

d) Write down an explicit expression (in terms of one-electron states) for the lowest energy state of these two electrons. Give also its energy.

Problem 7.7(H)

Two identical non-interacting spin-1 bosons, are located in a one dimensional harmonic oscillator potential with characteristic frequency ω. Find the three lowest energy levels of this two-particle system and give their degeneracies. (Hint: Use the Clebsch-Gordan table for combining two spin-1's into symmetric and antisymmetric states).

Problem 7.8(X)

Two spin-1/2 particles interact with each other. One of the particles is subject also to a local magnetic field. The Hamiltonian is

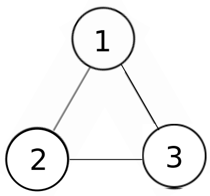
$$\hat{H} = \frac{J}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 - \frac{h_1}{\hbar}S_1^z$$

where $\vec{S}_i$ is the spin-1/2 operator for spin i. J and h_1 are positive constants. J describes the interaction of the particles, while h_1 is proportional to the local magnetic field. Only consider the spin degrees of freedom in this problem.

The particles are in the state $|\psi(0)\rangle$ at time $t = 0$. Then time goes on and the particles evolve according to $\hat{H}$ without any other external influences. At time t the following quantitites is calculated: $G_1(t) = \langle\psi(t)|S_1^z|\psi(t)\rangle$, $G_2(t) = \langle\psi(t)|S_2^z|\psi(t)\rangle$, $G_3(t) = \langle\psi(t)|\left(S_1^z + S_2^z\right)|\psi(t)\rangle$ and $G_4(t) = \langle\psi(t)|\left(S_1^z - S_2^z\right)|\psi(t)\rangle$. Which of these quantities are <u>not</u> dependent on t? Give reasons for your answers.

Problem 7.9(X)

A spinless particle is located on one of the atoms of a three-atom molecule, see figure. The particle can jump from one atom to another. Do not consider other degrees of freedom than the position of the particle in this problem.



The Hamiltonian is

$$H = -g\left(|2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3| + |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|\right)$$

where $|i\rangle$ is the state where the particle is located on atom number i. The states $|i\rangle$, $i \in \{1, 2, 3\}$ form an orthonormal set. g is a positive real number with dimension energy.

a) Write down a matrix representation for H. State explicitly the basis you are using.

The operator R is

$$R = |2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3|$$

b) Is R Hermitian? Unitary? Does R describe a symmetry-transformation? Give reasons for your answers.

c) Show that $R^3 = I$, where I is the identity operator, and use this to find the eigenvalues of R. Then find the corresponding eigenstates of R. What are the eigenstates and eigenvalues of H?

d) The particle is located on atom 2 at time $t = 0$. Compute the probability of finding the particle on the same atom 2 at a later time t.