Problem 10.1(L)

Consider a particle with mass m in a one dimensional potential such that the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \alpha x^4$$

Threat the quartic term as a perturbation and compute the ground state energy correct to first order in α.

Problem 10.2(L)

A spin-1/2 degree of freedom is influenced by a magnetic field that has a large z-component and a small x-component such that the Hamiltonian is

$$H = -\frac{B}{\hbar}S^z - \frac{g}{\hbar}S^x$$

Treat the x-component of the field (g) as a perturbation and
a) Compute the first order corrections to the unperturbed energy eigenvalues. Identify the dimensionless quantity that characterizes the perturbation expansion.

b) Compute the second order correction to the unperturbed energy eigenvalues.

c) Compute the first order correction to the unperturbed energy eigenstates.

Problem 10.3(H)

Use perturbation theory to estimate the first order correction to the ground state energy of Hydrogen due to the finite size of the proton. To do this, assume that the proton is a uniformly charged sphere of radius $b = 1 \cdot 10^{-15}$m. The electric potential is thus

$$V(r) = \begin{cases} \frac{e}{4\pi\epsilon_0 b}\left(\frac{3}{2} - \frac{r^2}{2b^2}\right) & r \leq b \\ \frac{e}{4\pi\epsilon_0 r} & r > b \end{cases}$$

As this problem only asks for an estimate you may expand your expressions to lowest non-vanishing order in the dimensionless quantity b/a_0 where a_0 is the Bohr radius. How does your estimate change if the proton has all its charge on its surface?

Problem 10.4(H)

This is a problem illustrating both first-order non-degenerate and degenerate perturbation theory. Consider the two-dimensional harmonic oscillator with an extra bilinear term gxy

where g is a real constant.

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + gxy$$

For $g = 0$ the exact energy eigenstates are tensor products of one-dimensional harmonic oscillator states: $|n_x, n_y\rangle = |n_x\rangle \otimes |n_y\rangle$, where $n_x, n_y \in \{0, 1, \ldots\}$. Their energies are $E_{n_x,n_y} = \hbar\omega(n_x + n_y + 1)$.

a) For $g = 0$ write down the energy eigenstates and their corresponding energies for the two lowest energy levels. What are their degeneracies?

b) Use first-order non-degenerate perturbation theory to compute how the ground state energy changes from the $g = 0$ value in a) when g is finite.

c) Use first-order degenerate perturbation theory to find how the first excited energy level splits up when g is finite.

Consider the reflection operator that interchanges $x \leftrightarrow y$

$$R|n_1\rangle \otimes |n_2\rangle = |n_2\rangle \otimes |n_1\rangle$$

d) For $g = 0$ find the eigenstates of R that are also eigenstates of H with energy $2\hbar\omega$, i.e. they belong to the first excited energy level.

e) Use non-degenerate first-order perturbation theory with the “good” states found in d) to compute how the first energy level splits up when g is finite. Compare your answer to what you got in c).

Problem 10.5(X)

Consider a quantum system described by the following (dimensionless) Hamiltonian on a three-dimensional Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, where $g \ll 1$ is a small perturbation parameter.

$$H = -|1\rangle\langle 1| + |3\rangle\langle 3| + g\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) + 2g\left(|1\rangle\langle 3| + |3\rangle\langle 1|\right)$$

a) Find the unperturbed ($g = 0$) eigenstates and eigenenergies.

b) Use perturbation theory to compute the ground *state* correct up to first order in the perturbation parameter g.

c) Use the result from b) as a variational wavefunction and use it to compute an upper limit on the ground state energy. Show that the result agrees with the ground state energy of the system computed up to second order in perturbation theory.