Oblig 5

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Problem 5.6 (H)

a)

The possible values we can measure, are the eigenvalues of the $\hat{L}_z$ operator. These are given by $\hbar m$, $m \in [-l, l] \bigcup \mathbb{Z}$ We can calculate the probability of measuring a given eigenvalue $\hbar m$ by calculating the projection of the state vector onto the eigenstate of $\hbar m$. As the state vector is given by a basis of the angular momentum eigenkets, their orthonormality dictates that $l = 1$ and $m \in [0, 1]$:

$$P(0) = |\langle 1, 0|\psi\rangle|^2 = |a_0|^2$$

$$P(\hbar) = |\langle 1, 1|\psi\rangle|^2 = |a_1|^2$$

b)

As latitude coordinates begin from the equator, but spherical coordinates begin from the north pole, we must integrate from 0 to $\pi/3$ instead of to $\pi/6$. We replace the eigenkets with their corresponding spherical harmonics.

$$Y_\ell^m(\theta, \phi) \sim e^{im\phi}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \quad , \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$$

When using spherical coordinates we must add the Jacobian determinant to the integral.

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/3} (r^2 \sin\theta) \left| -a_1 \sqrt{\frac{3}{8\pi}} \sin\theta + a_0 \sqrt{\frac{3}{4\pi}} \cos\theta \right|^2 \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}r$$

$$2\pi \frac{3}{8\pi} \int_0^{\pi/3} \sin\theta \left(|a_1|^2 \sin^2\theta + 2\,|a_0|^2 \cos^2\theta \right) \, \mathrm{d}\theta$$

$$\frac{3}{4} \left(\frac{14a_0^2 + 5a_1^2}{24} \right) = \underline{\underline{\frac{7}{16}\,|a_0|^2 + \frac{5}{32}\,|a_1|^2}}$$

c)

The lowering and raising operator working on a state. In our case, $l = 1$ which means we can rewrite the following:

$$L_+ |l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\, |l, m+1\rangle$$

$$L_- |l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\, |l, m-1\rangle$$

as:

$$L_+ |l, m\rangle = \hbar\sqrt{2 - m(m+1)}\, |l, m+1\rangle$$

$$L_- |l, m\rangle = \hbar\sqrt{2 - m(m-1)}\, |l, m-1\rangle$$

Lets first look at a general state in the x-basis:

$$|1, m\rangle_x = \alpha\, |1, -1\rangle_z + \beta\, |1, 0\rangle_z + \gamma\, |1, 1\rangle_z$$

We can apply the $\hat{L}_x$ operator to this state:

$$\hat{L}_x \left|1, m\right\rangle_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)\left|1, m\right\rangle_x$$

For simplicity, we calculate how the lowering and raising operator works on all our kets.

$$\hat{L}_+ \left|1, -1\right\rangle = \hbar\sqrt{2 - (-1)(-1+1)}\left|1, 0\right\rangle = \hbar\sqrt{2}\left|1, 0\right\rangle$$
$$\hat{L}_- \left|1, -1\right\rangle = \hbar\sqrt{2 - (-1)(-1-1)}\left|1, -2\right\rangle = 0$$
$$\hat{L}_+ \left|1, 0\right\rangle = \hbar\sqrt{2 - (0)(0+1)}\left|1, 1\right\rangle = \hbar\sqrt{2}\left|1, 1\right\rangle$$
$$\hat{L}_- \left|1, 0\right\rangle = \hbar\sqrt{2 - (0)(0-1)}\left|1, -1\right\rangle = \hbar\sqrt{2}\left|1, -1\right\rangle$$
$$\hat{L}_+ \left|1, 1\right\rangle = \hbar\sqrt{2 - (1)(1+1)}\left|1, 2\right\rangle = 0$$
$$\hat{L}_- \left|1, 1\right\rangle = \hbar\sqrt{2 - (1)(1-1)}\left|1, 0\right\rangle = \hbar\sqrt{2}\left|1, 0\right\rangle$$

We can now calculate the action of the $\hat{L}_x$ operator on our general state:

$$\hat{L}_x \left|1, m\right\rangle_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)\left|1, m\right\rangle_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)\left(\alpha\left|1, -1\right\rangle_z + \beta\left|1, 0\right\rangle_z + \gamma\left|1, 1\right\rangle_z\right)$$

$$\hat{L}_x \left|1, m\right\rangle_x = \frac{1}{2}\left(\hbar\sqrt{2}\alpha\left|1, 0\right\rangle_z + \hbar\sqrt{2}\beta\left|1, 1\right\rangle + \hbar\sqrt{2}\beta\left|1, -1\right\rangle_z + \hbar\sqrt{2}\gamma\left|1, 0\right\rangle_z\right)$$

$$\underline{\hat{L}_x \left|1, m\right\rangle_x = \frac{\hbar\sqrt{2}}{2}\left((\alpha + \gamma)\left|1, 0\right\rangle_z + \beta\left(\left|1, 1\right\rangle_z + \left|1, -1\right\rangle_z\right)\right)}$$

We also know that the operator must satisfy the following:

$$\hat{L}_x \left|1, m\right\rangle_x = \hbar m\left|1, m\right\rangle_x = \hbar m\left(\alpha\left|1, -1\right\rangle_z + \beta\left|1, 0\right\rangle_z + \gamma\left|1, 1\right\rangle_z\right)$$

We then plug in known values of m and its respective eigenket. We begin with $m = -1$.

$$\hat{L}_x \left|1, -1\right\rangle_x = \frac{\hbar\sqrt{2}}{2}\left((\alpha + \gamma)\left|1, 0\right\rangle_z + \beta\left(\left|1, 1\right\rangle_z + \left|1, -1\right\rangle_z\right)\right) = \hbar(-1)\left(\alpha\left|1, -1\right\rangle_z + \beta\left|1, 0\right\rangle_z + \gamma\left|1, 1\right\rangle_z\right)$$

This gives:

$$\gamma = -\frac{\sqrt{2}}{2}\beta$$

$$\alpha = -\frac{\sqrt{2}}{2}\beta = \gamma$$

$$\beta = -\frac{\sqrt{2}}{2}(\alpha + \gamma) = -\sqrt{2}\alpha$$

Setting $\alpha = 1$ and normalizing we get we get:

$$\underline{\underline{\left|1, -1\right\rangle_x = \frac{1}{2}\left|1, -1\right\rangle_z - \frac{1}{\sqrt{2}}\left|1, 0\right\rangle_z + \frac{1}{2}\left|1, 1\right\rangle_z}}$$

We now look at $m = 0$:

$$\hat{L}_x \left|1, 0\right\rangle = \frac{\hbar\sqrt{2}}{2}\left((\alpha + \gamma)\left|1, -1\right\rangle_z + \beta\left(\left|1, 0\right\rangle_z + \left|1, 1\right\rangle_z\right)\right) = 0$$

This gives:

$$\gamma = -\alpha$$

$$\beta = 0$$

$$\alpha = -\gamma$$

Setting $\alpha = 1$ and normalizing we get we get:

$$\underline{\underline{\left|1,0\right\rangle_x = \frac{1}{\sqrt{2}}\left|1,1\right\rangle_z - \frac{1}{\sqrt{2}}\left|1,-1\right\rangle_z}}$$

We now look at $m = 1$

$$\hat{L}_x\left|1,1\right\rangle = \frac{\hbar\sqrt{2}}{2}\left(\left(\alpha+\gamma\right)\left|1,0\right\rangle_z + \beta\left(\left|1,1\right\rangle_z + \left|1,-1\right\rangle_z\right)\right) = \hbar\left(\alpha\left|1,-1\right\rangle_z + \beta\left|1,0\right\rangle_z + \gamma\left|1,1\right\rangle_z\right)$$

We see this gives the same relationships, but with the opposite sign.

$$\gamma = \frac{\sqrt{2}}{2}\beta$$

$$\alpha = \frac{\sqrt{2}}{2}\beta = \gamma$$

$$\beta = \frac{\sqrt{2}}{2}\left(\alpha+\gamma\right) = \sqrt{2}\alpha$$

Setting $\alpha = 1$ and normalizing we get we get:

$$\underline{\underline{\left|1,1\right\rangle_x = \frac{1}{2}\left|1,-1\right\rangle_z + \frac{1}{\sqrt{2}}\left|1,0\right\rangle_z + \frac{1}{2}\left|1,1\right\rangle_z}}$$

Next we calculate the probability by taking the projection of ψ onto the eigenkets:

$$P(\left|1,-1\right\rangle_x) = \left|\left\langle 1,-1\right|_x\left|\psi\right\rangle_z\right|^2$$

$$P(\left|1,-1\right\rangle_x) = \left|\left(\frac{1}{2}\left\langle 1,-1\right| - \frac{1}{\sqrt{2}}\left\langle 1,0\right| + \frac{1}{2}\left\langle 1,1\right|\right)\left(a_1\left|1,1\right\rangle + a_0\left|1,0\right\rangle\right)\right|^2$$

$$\underline{\underline{P(\left|1,-1\right\rangle_x) = \left|\frac{1}{2}a_1 - \frac{1}{\sqrt{2}}a_0\right|^2}}$$

$$P(\left|1,0\right\rangle_x) = \left|\left(\frac{1}{\sqrt{2}}\left|1,1\right\rangle - \frac{1}{\sqrt{2}}\left|1,-1\right\rangle\right)\left(a_1\left|1,1\right\rangle + a_0\left|1,0\right\rangle\right)\right|^2$$

$$\underline{\underline{P(\left|1,0\right\rangle_x) = \frac{1}{2}\left|a_1\right|^2}}$$

$$P(\left|1,1\right\rangle_x) = \left|\left(\frac{1}{2}\left|1,-1\right\rangle + \frac{1}{\sqrt{2}}\left|1,0\right\rangle + \frac{1}{2}\left|1,1\right\rangle\right)\left(a_1\left|1,1\right\rangle + a_0\left|1,0\right\rangle\right)\right|^2$$

$$\underline{\underline{P(\left|1,1\right\rangle) = \left|\frac{1}{2}a_1 + \frac{1}{\sqrt{2}}a_0\right|^2}}$$

Problem 5.7 (H)

a)

The potential energy of a Tritium atom is the same as that for the Hydrogen.

$$E_n^H = -\frac{1}{n^2}\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{E_1}{n^2}$$

After decay, the charge inside the atom is doubled from e to $2e$. This means the potential energy is quadrupled.

$$E_n^{He} = -\frac{1}{n^2}\frac{m}{2\hbar^2}\left(\frac{2e}{4\pi\epsilon_0}\right)^2 = 4\frac{E_1}{n^2}$$

b)

We know the wave function of an electron in the ground state of a hydrogen atom is given by:

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$$

where r is the radial distance, and a_0 is the Bohr radius. This radius is determined by the potential energy. We need to compute a new value for this radius a_0', as the potential energy has changed. We know the Bohr radius is given by:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

The only change is that the charge e^2 is replaced by $2e^2$. This gives:

$$a_0' = \frac{4\pi\epsilon_0\hbar^2}{m2e^2} = \frac{a_0}{2}$$

We get a new wave function:

$$\psi_{100}' = \frac{1}{\sqrt{\pi a_0'^3}}e^{-r/a_0'} = \frac{1}{\sqrt{\pi\left(\frac{a_0}{2}\right)^3}}e^{-r/\left(\frac{a_0}{2}\right)} = \frac{2\sqrt{2}}{\sqrt{\pi a_0^3}}e^{-2r/a_0}$$

Now we can calculate the probability of finding the electron in the ground state of the Helium atom:

$$P(\text{Ground state}) = \left|\int \psi'^*\psi' \, \mathrm{d}r^3\right|^2$$

$$P(\text{Ground state}) = 16\pi^2\frac{8}{\pi^2 a_0^6}\left|\int_0^\infty r^2 e^{-3r/a} \, \mathrm{d}r\right|^2$$

Using integration by parts we get

$$P(\text{Ground state}) = \frac{16\cdot 8}{a^6}\left|\frac{2a^3}{-27}\right|^2 \approx \underline{\underline{0.702}}$$

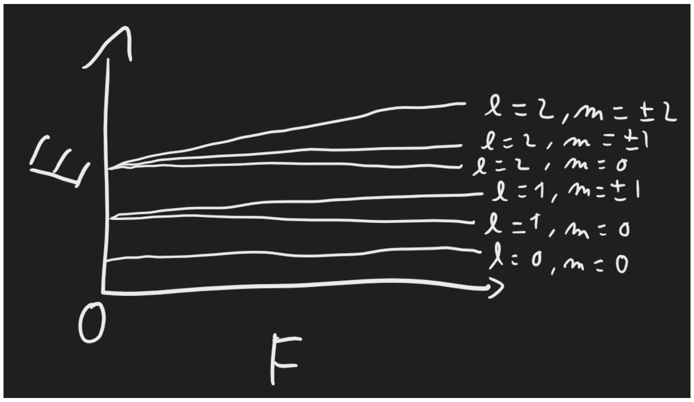
5.8 (X)

a)

The particle is in a spherically symmetrical potential, which means the potential energy is independent of the angle. This is the case as our electron is in the ground state.

b)

As we are in the ground state, the energy is not affected by the quantum number m. As the energy correction is proportional to m^2, it's only the absolute value of m that matters. We know $L_z = \hbar m$ which makes $H' = gFm^2$. This creates the following plot of the field F and the energy E



Figur 1: Plot of the field and energy as a function of m

c)

The Schrödinger equation is given by:

$$\left(-\frac{\hbar^2}{2m}\nabla + V\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

Separating the radial out of the function and rewriting the laplacian in spherical coordinates we get:

$$\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi)$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}L^2$$

We can now rewrite the Schrödinger equation with this and the eigenvalues of L as :

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}l(l+1)\right)R(r) + V(r)R(r) = ER(r)$$

Redefining $R = u/r$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{u}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} - u\right) = \frac{1}{r}\frac{\partial^2 u}{\partial r^2}$$

Plugging this back in we get:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial^2 u}{\partial r^2} - \frac{l(l+1)}{r^2}\frac{u}{r}\right) + V(r)\frac{u}{r} = E\frac{u}{r}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u(r)}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2}u(r) + V(r)u(r) = Eu(r)$$

This is the form of the1 1D Schrödinger equation with a one-dimensional potential (only r matters).

$$V_{\text{1d}} = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$$

The radial wavefunction then becomes:

$$u(r) = rR(r)$$