Oblig 3

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Problem 3.1 (L)

Problem 3.2 (L)

As a continuous observable have an infinite amount of possible measurement values, yet can't be in two states at once, we need an infinite-dimensional Hilbert space. All the state vectors need to be linearly independent, and an infinite amount of linearly independent vectors must be in a infinite-dimensional Hilbert space.

Problem 3.3 (L)

The probability of measuring the the value g:

$$P(g) = |\langle g|\psi\rangle|^2 = \frac{1}{2}(-ia + ib)^2 = \frac{1}{2}\left(a^2 - b^2\right)$$

If we measure the observable to be g, there is a 100% of the state being $|g\rangle$. If g was not obtained then there is a 0% chance of the state being $|g\rangle$.

Problem 3.4 (L)

As this is a degenerate state of value 2, there are two possible eigenstates which can result in measuring +1. The probability of measuring +1 is the sum of the probabilities of measuring +1 for each of the eigenstates.

$$\hat{P}_1 = |a_1|^2 + |a_2|^2$$

Problem 3.5 (L)

TODO:

Problem 3.6 (H)

a)

It is easy to see that $\hat{H} = \hat{H}^\dagger$. Now we have to check if it has real eigenvalues.

$$\hat{H} = \begin{pmatrix} 1 & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Now we employ the eigenvalue equation:

$$\det\left(\hat{H} - \lambda\hat{I}\right) = 0$$

$$\det\left(\begin{pmatrix} 1-\lambda & \frac{i}{2} & 0 \\ -\frac{i}{2} & 1-\lambda & 0 \\ 0 & 0 & \frac{1}{2}-\lambda \end{pmatrix}\right)$$

$$\underbrace{(1-\lambda)(1-\lambda)\left(\frac{1}{2}-\lambda\right)}_{\text{term 1}}-\underbrace{\frac{i}{2}\left(-\frac{i}{2}\left(\frac{1}{2}-\lambda\right)\right)}_{\text{term 2}}+0=0$$

Beginning with the first term:

$$(1-2\lambda+\lambda^2)\left(\frac{1}{2}-\lambda\right)=\frac{1}{2}-\lambda+\frac{\lambda^2}{2}-\lambda+2\lambda^2-\lambda^3$$

$$\underline{\frac{1}{2}-2\lambda+\frac{5}{2}\lambda^2-\lambda^3}$$

Now the second term:

$$-\frac{1}{4}\left(\frac{1}{2}-\lambda\right)=\underline{-\frac{1}{8}+\frac{\lambda}{4}}$$

$$\frac{1}{2}-2\lambda+\frac{5}{2}\lambda^2-\lambda^3-\frac{1}{8}+\frac{\lambda}{4}=0$$

$$-\lambda^3+\frac{5}{2}\lambda^2-\frac{7}{4}\lambda-\frac{3}{8}=0$$

$$\underline{\underline{\lambda=\frac{3}{2}\quad,\quad\lambda=\frac{1}{2}\quad,\quad\lambda=-\frac{1}{2}}}$$

All the eigenvalues are real, and the matrix is hermitian.

b)

$$\hat{H}\left|1\right\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&\frac{i}{2}&0\\-\frac{i}{2}&1&0\\0&0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}i\\1\\0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}i+\frac{i}{2}+0\\-\frac{i}{2}i+1+0\\0+0+0\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{3}{2}i\\\frac{3}{2}\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\frac{3}{2}\begin{pmatrix}i\\1\\0\end{pmatrix}$$

$\left|1\right\rangle$ is an eigenvector with eigenvalue $\frac{3}{2}$.

$$\hat{H}\left|2\right\rangle=\begin{pmatrix}1&\frac{i}{2}&0\\-\frac{i}{2}&1&0\\0&0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\left|2\right\rangle$ is an eigenvector with eigenvalue $\frac{1}{2}$.

$$\hat{H}\left|3\right\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix}1&\frac{i}{2}&0\\-\frac{i}{2}&1&0\\0&0&\frac{1}{2}\end{pmatrix}\begin{pmatrix}-i\\1\\-1\end{pmatrix}$$

$$\frac{1}{\sqrt{3}}\begin{pmatrix}-i+\frac{i}{2}+0\\ -\frac{i}{2}i+1+0\\ 0+0-\frac{1}{2}\end{pmatrix} = \frac{1}{\sqrt{3}}\frac{1}{2}\begin{pmatrix}-i\\ 1\\ -1\end{pmatrix}$$

$|3\rangle$ is an eigenvector with eigenvalue $\frac{1}{2}$.

c)

The operator $\hat{H}$ can br written using the eigenstates as a basis $\hat{H}_E$

$$H_{11} = \langle 1|\,\hat{H}\,|1\rangle = \frac{3}{2}\,\langle 1|1\rangle = \frac{3}{2}$$

As the matrix eigenstates must be orthogonal, we only need to calculate the diagonal elements, where $i = j$.

$$H_{22} = \langle 2|\,\hat{H}\,|2\rangle = \frac{1}{2}\,\langle 2|2\rangle = \frac{1}{2}$$

$$H_{33} = \langle 3|\,\hat{H}\,|3\rangle = \frac{1}{2}\,\langle 3|3\rangle = \frac{1}{2}$$

$$\hat{H}_E = \begin{pmatrix}\frac{3}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{pmatrix}$$

This is of course an diagonal matrix, and the eigenvalues are the diagonal elements.

d)

TODO: Are they not already orthogonal?

Problem 3.7 (H)

a)

$$\hat{H} = -g \sum_{i=0}^{L-2} \left(|i\rangle \langle i+1| + \langle i+1|i\rangle \right) - V\,|0\rangle\langle 0|$$

$$\hat{H} = -g\left(|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2| \right) - V\,|0\rangle\langle 0|$$

Lets expand one term at a time.

$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad |2\rangle\langle 1| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|2\rangle\langle 3| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad |3\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{H} = \begin{pmatrix} -V & -g & 0 & 0 \\ -g & 0 & -g & 0 \\ 0 & -g & 0 & -g \\ 0 & 0 & -g & 0 \end{pmatrix}$$

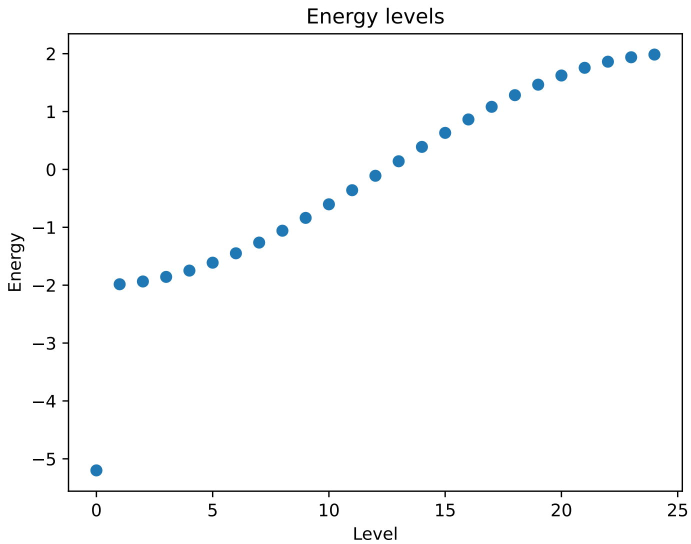
b)

The position operator $\hat{X}$ in matrix form:

$$\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

c)

The lowest energy level was approximately $E_0 = -5.2$.



Figur 1: Plot of the energy levels

d)

The energy eigenkets contains the information about the probability of finding the particle in a given state. Each eigenket correspond to each position, and each index correspond to each energy level. The probability of finding the particle in position 0 in the ground state, is the absolute value squared of the first element, in the first eigenket. The probability of finding the particle in position 1 in the ground state, is the absolute value squared of the first element, in the second eigenket. This turns out to be 96% and 39% respectively.

e)

We set our state vector $|\Psi\rangle$ to be in position 0. The we evolve it over time with the time evolution operator $\hat{U}$, defined as:

$$\hat{U} = e^{-i\hat{H}t/\hbar}$$

This gives us and expression for the stationary state $|\psi_0\rangle$ as follows:

$$|\psi_0\rangle = \sum_n^L c_n(t) |E_n\rangle$$

Where c_n is the coordinates of the state vector in the energy eigenbasis and $|E_n\rangle$ is the energy eigenkets. We calculate the coordinates c_n like this:

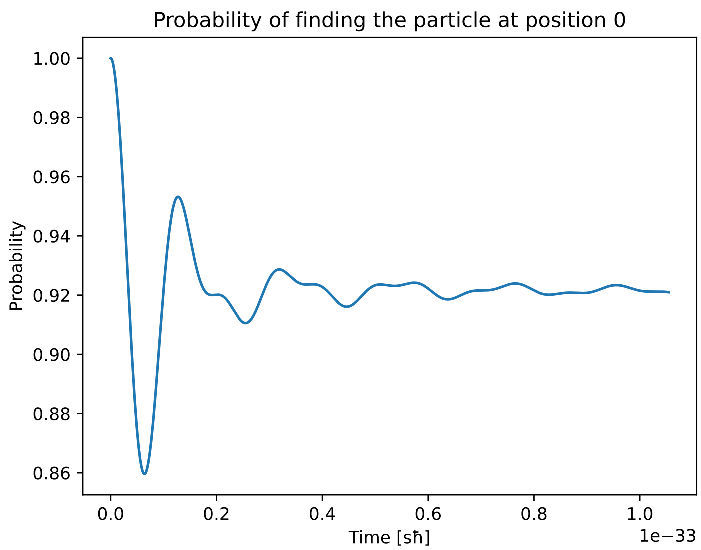
$$c_n = \langle E_n|\Psi\rangle$$

Now we can calculate the time evolution of the state vector:

$$|\Psi(t)\rangle = \sum_n^L |c_n(t)|^2 U |E_n\rangle$$

We can the plot the probability of finding the particle at position 0 as a function of time, by taking the absolute value squared of the state vector.

As we already know it is at position 0, we can use $\begin{pmatrix} 1_1 \\ 0_2 \\ \vdots \\ 0_L \end{pmatrix}$ instead of $|\Psi(0)\rangle$.



Figur 2: Probability of finding the particle at position 0 as a function of time