Problem 11.1(L)

Consider the operator

$$A = uL^2 + \hbar v L^z$$

where u and v are real numbers and $u \gg v$. Use this operator to show that the $|nlm\rangle$-states are good states when the Coulomb-potential is perturbed by the relativistic correction term $H^1 = -\frac{(\vec{P}^2)^2}{8m^3c^2}$

Problem 11.2(L)

Consider the relativistic energy expression for a free particle

$$E_k = \sqrt{(mc^2)^2 + (\vec{P}c)^2} - mc^2$$

and expand it to order $(\vec{P}^2)^3$. To calculate the Hydrogen fine-structure we keep usually only the $(\vec{P}^2)^2$ term. What is the order of magnitude of the energy corrections induced by the $(\vec{P}^2)^3$ term?

Problem 11.3(L)

Use the Feynman-Hellman theorem to calculate the expectation values $\langle nlm|\frac{1}{r}|nlm\rangle$.

Problem 11.4(L)

Combining a spin-1/2 with an angular momentum state l gives a state with total angular momentum j:

$$|ljm_j\rangle = \alpha_j|l, m_j + \frac{1}{2}\rangle \otimes |\downarrow\rangle + \beta_j|l, m_j - \frac{1}{2}\rangle \otimes |\uparrow\rangle$$

where $|l,m\rangle$ is the state with orbital angular momentum quantum number l, z-component m, and $|\uparrow\rangle$ is the spin state of the electron with spin-component along the $+z$-axis. Show by applying the operator $\vec{J}^2 = (\vec{L}+\vec{S})^2$, and the orthonormality condition $\langle l'j'm_j'|ljm_j\rangle = \delta_{l,l'}\delta_{j,j'}\delta_{m_j,m_j'}$, that the coefficients are

$$\begin{aligned}
\alpha_{l+\frac{1}{2}} &= \sqrt{\frac{l+\frac{1}{2}-m_j}{2l+1}} \\
\beta_{l+\frac{1}{2}} &= \sqrt{\frac{l+\frac{1}{2}+m_j}{2l+1}}
\end{aligned}$$

for $j = l + 1/2$, and

$$\begin{aligned}
\alpha_{l-\frac{1}{2}} &= \sqrt{\frac{l+\frac{1}{2}+m_j}{2l+1}} \\
\beta_{l-\frac{1}{2}} &= -\sqrt{\frac{l+\frac{1}{2}-m_j}{2l+1}}
\end{aligned}$$

for $j = l - 1/2$.

Problem 11.5(H)

Consider the $|nljm_j\rangle$ eigenstates for the Hydrogen atom under weak-field Zeeman splitting. Find the energy splitting for the states with $n = 2$ in terms of the external magnetic field B and the fine structure constant α. Draw a diagram of the splitting as a function of B. When B is large, the Zeeman effect dominates over fine structure and the good quantum numbers are n,l,m_l and m_s and the eigenstates are given by $|nlm_l m_s\rangle$. Find the large-field splitting of the states with $n = 2$.

Problem 11.6(H)

Consider the electron-electron interaction as a perturbation, and compute the ground state energy of Helium correct to first order. Plug in numbers and compare your answer to that obtained in class using the variational method. Find also how the second order correction to the ground state energy depends on the parameters (you don't have to care about numerical factors).

Hint: You may use any results found in B&DD 17.1.2 for this problem.

Problem 11.7(X)

A particle with mass m moves along a line of length L. The line has its ends identified (periodic boundary conditions, a ring). Its Hamiltonian is $H = \frac{p^2}{2m}$ where $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$. Denote the energy eigenstates as $|k\rangle$. Their coordinate representation are $\psi_k(x) \equiv \langle x|k\rangle = \frac{1}{\sqrt{L}}e^{-ikx}$.

a) Find the allowed energy eigenvalues and their degree of degeneracy.

The action of the parity operator Π on $|k\rangle$ is $\Pi|k\rangle = |-k\rangle$.

b) Construct energy eigenstates that are also eigenstates of the parity operator Π.

A weak periodic potential is added to this system

$$H^1 = V_0 \cos(\frac{2\pi}{a}x)$$

where the period a matches the periodic boundary conditions, i.e. $Na = L$, where N is a positive *even* integer.

c) Use perturbation theory to compute the full energy spectrum correct up to first order in V_0.