Lecture Notes 3

Oskar Idland

Operators

Hermitian Conjugate

Definition:

$$\langle v| \hat{K}^\dagger |u\rangle = \langle u| \hat{K} |v\rangle^*$$

Discrete basis

$$\underbrace{\langle n| \hat{K}^\dagger |m\rangle}_{K^\dagger_{nm}} = \underbrace{\langle m| \hat{K} |n\rangle^*}_{K^*_{mn}}$$

$$K^\dagger = K^{*T} = K \to K^\dagger_{nm} = K^*_{mn} = K_{nm} \to K_{nn} \in \mathbb{R}$$

$$\text{when } n \neq m : K_{nm} = K^*_{mn} = K^\dagger_{nm}$$

Spectrum of an Operator

Definition: The spectrum of an operator $\hat{K}$ is the set of all eigenvalues of $\hat{K}$. Two or more linearly independent eigenvectors $|\lambda_i\rangle$ have the same eigenvalue λ, the spectrum is said to be degenerate. We can always choose the eigenvectors to be orthonormal. If there are g states $|\lambda_i\rangle$ with eigenvalue λ, then the level degeneracy is g.

Hermitian Operators

Properties

- Eigenvalues are real
- Different eigenvalues correspond to orthogonal eigenvectors
- Eigenvectors with the same eigenvalues can be chosen to be orthogonal
- The eigenkets from a complete set of basis vectors for a finite dimensional Hilbert space.

Proof of Eigenvectors Creating a Liner Compination which is also an Eigenvector

$$\hat{K}\alpha |\lambda_1\rangle = \lambda_1 \alpha |\lambda_1\rangle \quad \hat{K}\beta |\lambda_2\rangle = \lambda_2 \beta |\lambda_2\rangle$$

$$\hat{K} \left(\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle\right) = \lambda(\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle)$$

Spectral Representation of Operators

Definition: The spectral representation of an operator $\hat{K}$ in its basis of its eigenkets.

$$\langle \lambda_i| \hat{K} |\lambda_j\rangle = \langle \lambda_i|\lambda_j\rangle \lambda_j = \delta_{ij}\lambda_j$$

This shows that the matrix elements of a Hermitian operator in its eigenket basis are on the diagonal.

$$\hat{K} \simeq \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

We guess that (this is the spectral representation)

$$\hat{K} = \sum_r \lambda_r \left|\lambda_r\right\rangle \left\langle\lambda_r\right|$$

$$\left\langle\lambda_i\right| \hat{K} \left|\lambda_j\right\rangle = \left\langle\lambda_i\right| \sum_r \left|\lambda_r\right\rangle \left\langle\lambda_r|\lambda_j\right\rangle = \sum_r \lambda_r \left\langle\lambda_i|\lambda_j\right\rangle \left\langle\lambda_r|\lambda_j\right\rangle$$

r must be equal to both i and j for the sum to be non-zero.

$$\lambda_j \delta_{ij}$$

Physical Meaning

Eigenvalues: Measurement value λ

Eigenket: State on which a measurement of the quantity represented by $\hat{K}$, gives the value λ with certainty.