Lecture Notes

Oskar Idland

Bras, Kets and Dirac-Delta

Oskar Idland

1 Inner Product

There are many ways to write the inner product

- I.
$$(|u\rangle, |v\rangle) = (|v\rangle, |u\rangle)^*$$
- II. Second linearity makes first linearity impossible.
$$(|u\rangle, \alpha |v_1\rangle + \beta |v_2\rangle) = \alpha (|u\rangle, |v_1\rangle) + \beta (|u\rangle, |v_1\rangle)$$
- III.
$$(\alpha |v_1\rangle + \beta |v_2\rangle, |u\rangle) = \alpha^* ||v_1\rangle, |u\rangle\rangle + \beta^* (|v_2\rangle, |u\rangle)$$
- IV.
$$\underbrace{(|v\rangle, |v\rangle)}_{\mathbb{R}} \geq 0$$

1.1 Dirac Notation

We denote the inner product like this in Dirac-notation.

$$(|u\rangle, |v\rangle) = \langle u|v\rangle \in \mathbb{C}$$

- I
$$\langle u|v\rangle = (\langle v|u\rangle)^*$$
- II
$$|v'\rangle = \alpha |v_1\rangle + \beta |v_2\rangle$$
- III
$$\langle v|v\rangle \geq 0$$

1.2 Representation of Bras

The bra can be written as an operator operating on a ket, producing a number

$$\langle A| \simeq \int \mathrm{d}x A^*(x)$$

$$\langle A|B\rangle = \int \mathrm{d}x A^*(x) B(x)$$

$$\langle B|A\rangle = \int \mathrm{d}x B^*(x) A(x)$$

2 Sets of Kets

this is a ket $|u\rangle$

3 Discrete and Continuous Basis

3.1 Discrete

$$|f\rangle = \sum_{i=1}^{\infty} f_i \, |i\rangle$$

For orthonormal basis

$$\langle i|j\rangle = \delta_{ij} \rightarrow f_j = \langle j|f\rangle$$

3.2 Continuous

$$|f\rangle = \int_0^L f(x') \, |x'\rangle \; \mathrm{d}x$$

$$\langle x|f\rangle = f(x)$$

$$\langle x|f\rangle = \int_0^L \langle x| \, f(x) \, |x'\rangle \; \mathrm{d}x = \int_0^L f(x) \, \underbrace{\langle x|x'\rangle}_{\delta} \; \mathrm{d}x = f(x)$$

In a short interval $[-\epsilon, \epsilon]$ the function f becomes approximately constant. Using the definition of the Dirac-delta we get the following

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} f(x)\delta(x - x') \; \mathrm{d}x = f(x) \; \underbrace{\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \delta(x - x') \; \mathrm{d}x}_{1} = f(x)$$

Lecture 2

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Wave Function

$$|\psi\rangle = \int \, \mathrm{d}x \psi(x) \, |x\rangle$$

The wave function must not be mistaken for the state vector. The wave function $\psi(x)$ calculates the coefficients for all possible positions x in the x basis. The definition of the wave function is $|\psi\rangle = \langle x|\psi\rangle$

Operators

Definition: Takes a ket and returns a ket. It must be linear and Hermitian. As a consequence of being Hermitian the eigenvalues are real.

$$\hat{L} \, |u\rangle = |v\rangle$$

$$\hat{L} \, (|u\rangle + |v\rangle) = \hat{L} \, |u\rangle + \hat{L} \, |v\rangle$$

$$\hat{L} = \hat{L}^\dagger$$

Continuous Operators

$$|\psi\rangle = \int \langle x|\psi\rangle \, |x\rangle \; \mathrm{d}x = |x\rangle \, \langle x|\psi\rangle \; \mathrm{d}x = \underbrace{\int (|x\rangle \, \langle x| \; \mathrm{d}x)}_{\text{Identity operator } \hat{I}} \, |\psi\rangle$$

Discrete Operators

$$|\psi\rangle = \sum_i \psi_i \, |i\rangle = \sum_i \langle i|\psi\rangle \, |i\rangle = \underbrace{\sum_i |i\rangle \, \langle i|}_{\hat{I}} \, |\psi\rangle$$

$$\langle j|\psi\rangle = \sum_i \psi_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = \psi_j$$

Representation of Operators

The ket $|u\rangle$ and operator $\hat{L}$ can be represented in the following way:

$$|u\rangle \simeq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\hat{L} \simeq \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

$$|u\rangle \simeq u(x)$$

$$\hat{L} \simeq c_0(x) + c_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + c_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots$$

Discrete basis

$$\langle m|\hat{K}|n\rangle = K_{nm} = \text{m,n'th matrix element of } \hat{K}$$

$$|v\rangle = \sum_m V_m |m\rangle$$

$$\hat{K}|v\rangle = \langle n|\hat{K}|v\rangle = \langle n|\hat{K}\sum_m V_m|m\rangle = \sum_m V_m \langle n|\hat{K}|m\rangle = \sum_m V_m K_{nm}$$

Composite Operators

$$\hat{K}\hat{L}|v\rangle = \hat{K}(\hat{L}|v\rangle) = (\hat{K}\hat{L})|v\rangle$$

$$\langle n|\hat{K}\hat{L}|m\rangle$$

Inserting the identity operator:

$$\langle n|\hat{K}\underbrace{\sum_r |r\rangle\langle r|}_{\hat{I}}\hat{L}|m\rangle$$

$$\sum_r \langle n|\hat{K}|r\rangle\langle r|\hat{L}|m\rangle = \underbrace{\sum_r K_{nr}L_{rm}}_{\text{Matrix multiplication}}$$

Change of Basis

Give to different bases $\{|n\rangle\}$ and $\{|n'\rangle\}$.

$$|\psi\rangle = \sum_n \underbrace{\psi_n |n\rangle}_{\langle n|\psi\rangle} \quad , \quad |\psi\rangle = \sum_{n'} \underbrace{\psi_{n'}|n'\rangle}_{\langle n'|\psi\rangle}$$

$$\langle m'|\psi\rangle = \sum_n \psi_n \underbrace{\langle m'|n\rangle}_{S_{m'n}}$$

We define $S_{m'n}$ to be $\langle m'|n\rangle$ which is a matrix relation between ψ_n to $\psi_{n'}$.

$$\psi_{m'} = \sum_n S_{m'n}\psi_n$$

S is unitary meaning $S^\dagger = S^{-1}$

Proof of unitarity: Set $|\psi\rangle = |n'\rangle$

$$\underbrace{\langle m'|n'\rangle}_{\delta_{m'n'}} = \sum_n S_{m'n'} \underbrace{\langle n|n'\rangle}_{\langle n'|n\rangle^* = S^*_{n'm}}$$

Remember $S_{m'n} = \langle m'|n\rangle$.

$$\langle n|n'\rangle = \langle n'|n\rangle^* = S^*_{n'm} = S^{T*}_{m'n} = S^\dagger_{m'n}$$

$$SS^\dagger = 1 \rightarrow S^\dagger = S^{-1}$$

Operators in different bases

$$K_{m'n'} = \langle m'| \hat{K} |n'\rangle = \langle m'| \hat{I}\hat{K}\hat{I} |n'\rangle$$

$$\sum_{mn} \underbrace{\langle n'|m\rangle}_{S_{m'm}} \langle m| \hat{K} |n\rangle \underbrace{\langle n|n'\rangle}_{S^{\dagger}_{nn'}}$$

$$\sum_{mn} S_{m'm} K_{mn} S^{\dagger}_{nn'} = S^{\dagger} K S = K'$$

Now we have the operator $\hat{K}$ in a new basis defined as $\hat{K}'$

Hermitian Conjugate of an Operator

Definition by the inner product:

$$\Big(|u\rangle , \hat{K} |v\rangle\Big) = \Big(\hat{K}^{\dagger} |u\rangle , |v\rangle\Big) \text{ for all } |u\rangle , |v\rangle$$

Is not as simple as "just transposing and taking the complex conjugate". A function could be Hermitian.

Definition in Dirac-Notaion:

$$\langle u| \hat{K} |v\rangle = \langle v| \hat{K}^{\dagger} |u\rangle^{*} \text{for all } |u\rangle , |v\rangle$$

It is enough to define:

$$\langle u| \hat{K} |v\rangle = \langle v| \hat{K}^{\dagger} |u\rangle \text{ problem 2.3(L)}$$

Exercise: Find $\hat{K}^{\dagger}$

$$\hat{K} = \alpha |a\rangle \langle b|$$

$$\langle u| \hat{K}^{\dagger} |v\rangle = \langle v| \hat{K} |u\rangle^{*} = \langle v|a\rangle^{*} \langle b|u\rangle^{*}$$

$$\langle u| (|b\rangle \langle a|) |v\rangle \to \hat{K}^{\dagger} = |b\rangle \langle a| \alpha^{*}$$

Check the following correspondence

$$\hat{K} |v\rangle \leftrightarrow \langle v| \hat{K}^{\dagger}$$

We set $|w\rangle = \hat{K} |v\rangle$ and act on it with an arbitrary bra $\langle n|$.

$$\langle u|w\rangle = \langle u| \hat{K} |v\rangle = \langle v| \hat{K}^{\dagger} |u\rangle^{*} \to \langle w|u\rangle = \langle v| \hat{K}^{\dagger} |u\rangle$$

This holds for any $|w\rangle \to \langle w| = \langle v| \hat{K}^{\dagger}$

Exercise: Find the Hermitian conjugate of the following operator

$$\hat{K} \simeq \frac{\mathrm{d}}{\mathrm{d}x}$$

$$|u\rangle = u(x)$$

$$|v\rangle = v(x)$$

$$\langle u|\,\hat{K}^{\dagger}\,|v\rangle = \langle v|\,\hat{K}\,|u\rangle^{*} = \int \left(v^{*}(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right)^{*}\,\mathrm{d}x = \int v(x)\frac{\mathrm{d}}{\mathrm{d}x}u^{*}(x)\,\mathrm{d}x$$

$$\underbrace{v(x)u^{*}(x)}_{0}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}v(x)u^{*}(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}u^{*}(x)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)v(x)\,\mathrm{d}x$$

Lecture Notes 3

Oskar Idland

Operators

Hermitian Conjugate

Definition:

$$\langle v| \hat{K}^\dagger |u\rangle = \langle u| \hat{K} |v\rangle^*$$

Discrete basis

$$\underbrace{\langle n| \hat{K}^\dagger |m\rangle}_{K^\dagger_{nm}} = \underbrace{\langle m| \hat{K} |n\rangle^*}_{K^*_{mn}}$$

$$K^\dagger = K^{*T} = K \to K^\dagger_{nm} = K^*_{mn} = K_{nm} \to K_{nn} \in \mathbb{R}$$

$$\text{when } n \neq m : K_{nm} = K^*_{mn} = K^\dagger_{nm}$$

Spectrum of an Operator

Definition: The spectrum of an operator $\hat{K}$ is the set of all eigenvalues of $\hat{K}$. Two or more linearly independent eigenvectors $|\lambda_i\rangle$ have the same eigenvalue λ, the spectrum is said to be degenerate. We can always choose the eigenvectors to be orthonormal. If there are g states $|\lambda_i\rangle$ with eigenvalue λ, then the level degeneracy is g.

Hermitian Operators

Properties

- Eigenvalues are real
- Different eigenvalues correspond to orthogonal eigenvectors
- Eigenvectors with the same eigenvalues can be chosen to be orthogonal
- The eigenkets from a complete set of basis vectors for a finite dimensional Hilbert space.

Proof of Eigenvectors Creating a Liner Compination which is also an Eigenvector

$$\hat{K}\alpha |\lambda_1\rangle = \lambda_1 \alpha |\lambda_1\rangle \quad \hat{K}\beta |\lambda_2\rangle = \lambda_2 \beta |\lambda_2\rangle$$

$$\hat{K} (\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle) = \lambda(\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle)$$

Spectral Representation of Operators

Definition: The spectral representation of an operator $\hat{K}$ in its basis of its eigenkets.

$$\langle \lambda_i| \hat{K} |\lambda_j\rangle = \langle \lambda_i|\lambda_j\rangle \lambda_j = \delta_{ij}\lambda_j$$

This shows that the matrix elements of a Hermitian operator in its eigenket basis are on the diagonal.

$$\hat{K} \simeq \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

We guess that (this is the spectral representation)

$$\hat{K} = \sum_r \lambda_r \left|\lambda_r\right\rangle \left\langle\lambda_r\right|$$

$$\left\langle\lambda_i\right| \hat{K} \left|\lambda_j\right\rangle = \left\langle\lambda_i\right| \sum_r \left|\lambda_r\right\rangle \left\langle\lambda_r|\lambda_j\right\rangle = \sum_r \lambda_r \left\langle\lambda_i|\lambda_j\right\rangle \left\langle\lambda_r|\lambda_j\right\rangle$$

r must be equal to both i and j for the sum to be non-zero.

$$\lambda_j \delta_{ij}$$

Physical Meaning

Eigenvalues: Measurement value λ

Eigenket: State on which a measurement of the quantity represented by $\hat{K}$, gives the value λ with certainty.