# FYS3120 Classical Mechanics and Electrodynamics

# Collection of formulae
# 2024

## 1 Analytical Mechanics

**The Lagrangian**

$$L = L(q, \dot{q}, t) \ , \tag{1}$$

is a function of the *generalised coordinates* $q = \{q_i \, ; \, i = 1, 2, ..., d\}$ of the physical system, and their time derivatives $\dot{q} = \{\dot{q}_i \, ; \, i = 1, 2, ..., d\}$. The Lagrangian may also have an *explicit* dependence on time $t$.

**Lagrange's equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \ , \quad i = 1, 2, .., d. \tag{2}$$

There is one equation for each generalised coordinate.

**Generalised momentum**

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ , \quad i = 1, 2, .., d. \tag{3}$$

is also referred to as *canonical* or *conjugate* momentum. There is one generalised momentum $p_i$ conjugate to each generalised coordinate $q_i$.

**The Hamiltonian**

$$H(p, q) = \sum_{i=1}^{d} \dot{q}_i p_i - L, \tag{4}$$

is usually considered as a function of the generalised coordinates $q_i$ and momenta $p_i$.

**Hamilton's equations**

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \ , \quad i = 1, 2, .., d. \tag{5}$$

$$\tag{6}$$

**Standard expressions for the Lagrangian and Hamiltonian**

$$\begin{aligned} L &= K - V, \\ H &= K + V, \end{aligned} \tag{7}$$

with kinetic energy $K$ and potential energy $V$. There are cases where $H$ is *not* the total energy.

**Lagrangian and Hamiltonian for a charged particle in an electromagnetic field**

$$\begin{aligned} L = L(\mathbf{r}, \mathbf{v}) &= \frac{1}{2} m v^2 - q\phi + q\mathbf{v} \cdot \mathbf{A}, \\ H = H(\mathbf{r}, \mathbf{p}) &= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi. \end{aligned} \tag{8}$$

Here $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials. $q$ is the charge of the particle. These equations are non-relativistic.

## 2 Relativity

**Space-time coordinates as a four-vector**

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{r}). \tag{9}$$

**General Lorentz transformation**

$$x^\mu \to x'^\mu = L^\mu_{\ \nu} x^\nu. \tag{10}$$

**Special Lorentz transformation with velocity $v$ in the $x$ direction**

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1), \\ x'^1 &= \gamma(x^1 - \beta x^0), \\ x'^2 &= x^2, \\ x'^3 &= x^3, \end{aligned}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

**Condition satisfied by Lorentz transformation matrices**

$$g_{\mu\nu} L^\mu_{\ \rho} L^\nu_{\ \sigma} = g_{\rho\sigma}. \tag{11}$$

**Invariant line element**

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = g_{\mu\nu} \, \Delta x^\mu \Delta x^\nu = \Delta x_\mu \, \Delta x^\mu. \tag{12}$$

**Metric tensor**

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{13}$$

is the metric tensor for flat Minkowski space.

**Raising and lowering of indices**

$$x_\mu = g_{\mu\nu}\, x^\nu, \quad x^\mu = (ct, \mathbf{r}), \quad x_\mu = (ct, -\mathbf{r})$$
$$x^\mu = g^{\mu\nu} x_\nu \ , \quad g_{\mu\rho}\, g^{\rho\nu} = \delta_\mu^{\ \nu} \tag{14}$$

**Proper time – time dilatation**

$$d\tau = \frac{1}{c}\sqrt{ds^2} = \frac{1}{\gamma} dt, \tag{15}$$

Here $d\tau$ is the proper time interval, meaning the time measured in the (instantaneous) rest frame of a moving body by a co-moving clock, $ds^2$ is the invariant line element of an infinitesimal section of the object's world line, and $dt$ is the coordinate time interval, meaning the time interval measured in arbitrarily chosen inertial reference frame of velocity $v$ with respect to the body.

**Length contraction**

$$L = \frac{1}{\gamma} L_0 \tag{16}$$

Lengths of a body measured in the direction of its motion. $L_0$ is the length measured in the rest frame of the body, $L$ is the length measured (at simultaneity) in an arbitrarily chosen inertial frame.

**Four-velocity**

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma\,(c, \mathbf{v}) \ , \qquad U^\mu U_\mu = c^2 \tag{17}$$

**Four-acceleration**

$$A^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} \ , \quad A^\mu U_\mu = 0 \tag{18}$$

**Proper acceleration**

The proper acceleration $\mathbf{a_0}$ is the acceleration measured in the instantaneous rest frame,

$$A^\mu A_\mu = -\mathbf{a_0}^2. \tag{19}$$

**Relativistic energy and momentum**

$$E = \gamma m c^2$$
$$\mathbf{p} = \gamma m \mathbf{v} \tag{20}$$

**Four-momentum**

$$p^\mu = m\, U^\mu = m\gamma(c, \mathbf{v}) = (\frac{E}{c}, \mathbf{p}), \tag{21}$$

where $m$ is the mass of the moving body.

**Four-derivative**

$$\partial_\nu \equiv \frac{\partial}{\partial x^\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \tag{22}$$

and $\partial^\mu = g^{\mu\nu}\partial_\nu$.

## 3 Electrodynamics

**Maxwell's equations**

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} &= \mu_0\mathbf{j} \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E}+\frac{\partial}{\partial t}\mathbf{B} &= 0 \end{aligned} \tag{23}$$

**Electromagnetic field tensor**

$$\begin{aligned} F^{k0} &= \frac{1}{c}E_k\ , \quad F^{ij}=-\epsilon_{ijk}B_k \\ \tilde{F}^{k0} &= B_k\ , \quad \tilde{F}^{ij}=\frac{1}{c}\epsilon_{ijk}E_k \end{aligned} \tag{24}$$

**Maxwell's equations in covariant form**

$$\partial_\mu F^{\mu\nu}=\mu_0 j^\nu, \quad \partial_\mu\tilde{F}^{\mu\nu}=0, \quad \text{where} \quad \tilde{F}^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{25}$$

**Four-current density**

$$j^\mu=(c\rho,\mathbf{j}) \tag{26}$$

**Charge conservation**

$$\partial_\mu j^\mu=0\ , \quad \frac{\partial}{\partial t}\rho+\nabla\cdot\mathbf{j}=0 \tag{27}$$

**Electromagnetic potentials**

$$\mathbf{E}=-\nabla\phi-\frac{\partial}{\partial t}\mathbf{A}\ , \quad \mathbf{B}=\nabla\times\mathbf{A} \tag{28}$$

**Four-potential**

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\ , \quad A^\mu=(\frac{\phi}{c},\mathbf{A}) \tag{29}$$

**Lorentz force**
Force from the electromagnetic field on a point particle with charge $q$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{30}$$

**Potentials from charge and current distributions**
in Lorentz gauge, $\partial_\mu A^\mu=0$:

$$\begin{aligned} \phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}dV' \\ \mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}dV' \end{aligned} \tag{31}$$

**Retarded time**

$$t' = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'| \tag{32}$$

**Electric dipole moment**

$$\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})dV \tag{33}$$

**Electric dipole potential (dipole at the origin)**

$$\phi = \frac{\mathbf{n} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2}\ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{34}$$

**Force and torque (about the origin)**

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}\ , \quad \mathbf{M} = \mathbf{p} \times \mathbf{E} \tag{35}$$

**Magnetic dipole moment**

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{j}(\mathbf{r})\ dV \tag{36}$$

**Magnetic dipole potential (dipole at the origin)**

$$\mathbf{A} = \frac{\mu_0\ \mathbf{m} \times \mathbf{n}}{4\pi r^2}\ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{37}$$

**Force and torque (about the origin)**

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})\ (\text{current loop})\ , \quad \mathbf{M} = \mathbf{m} \times \mathbf{B} \tag{38}$$

**Lorentz transformation of the electromagnetic field**

$$F'^{\mu\nu} = L^{\mu}_{\ \rho} L^{\nu}_{\ \sigma} F^{\rho\sigma} \tag{39}$$

**Lorentz invariants for electromagnetic fields**

$$\mathbf{E}^2 - c^2\mathbf{B}^2 = -\frac{c^2}{2} F_{\mu\nu}F^{\mu\nu}, \quad \mathbf{E} \cdot \mathbf{B} = \frac{c}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} \tag{40}$$

**Lorentz transformations for electric and magnetic fields**

$$\begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}\ , \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}) \\ \mathbf{B}'_{||} &= \mathbf{B}_{||}\ , \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}/c^2) \end{aligned} \tag{41}$$

The fields are decomposed in a parallel component ($||$), in the direction of transformation velocity $\mathbf{v}$, and a perpendicular component ($\perp$), orthogonal to $\mathbf{v}$.

**Electromagnetic field energy density**

$$u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \frac{\epsilon_0}{2}(E^2 + c^2B^2) \tag{42}$$

**Electromagnetic energy current density (Poynting's vector)**

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{43}$$

**Monochromatic plane waves, plane polarized**

$$\begin{aligned}
&\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \ , \quad \mathbf{E}_0 = E_0 \mathbf{e}_1 \\
&\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \, ; \quad \mathbf{B}_0 = B_0 \mathbf{e}_2 \\
&\mathbf{E}_0 \cdot \mathbf{k} = \mathbf{B}_0 \cdot \mathbf{k} = 0 \ , \quad \mathbf{B}_0 = \frac{1}{c}\mathbf{n} \times \mathbf{E}_0 \ , \quad \mathbf{n} = \frac{\mathbf{k}}{k}
\end{aligned} \tag{44}$$

**Monochromatic plane waves, circular polarized**

$$\begin{aligned}
&\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left(\mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]\right) \ , \quad \mathbf{E}_0 = E_0 \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2) \\
&\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}\left(\mathbf{B}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]\right) \ , \quad \mathbf{B}_0 = B_0 \frac{1}{\sqrt{2}}(\mathbf{e}_2 \mp i\mathbf{e}_1)
\end{aligned} \tag{45}$$

**Polarization vectors**

$$\mathbf{e}_1 \cdot \mathbf{k} = \mathbf{e}_2 \cdot \mathbf{k} = 0 \ , \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \ , \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \tag{46}$$

**Four-wave vector**

$$k^\mu = (\frac{\omega}{c}, \mathbf{k}) \ , \quad \omega = ck \tag{47}$$

**Radiation fields, in the wave zone** ($r >> r', \lambda$)

$$\begin{aligned}
&\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \frac{d}{dt}\int \mathbf{j}(\mathbf{r}', t') dV' \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \\
&\mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}
\end{aligned} \tag{48}$$

**Electric dipole radiation**

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \ddot{\mathbf{p}}(t - r/c) \ , \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n} \tag{49}$$

**Radiation from accelerated, charged particle**

$$\begin{aligned}
&\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi c r}[\mathbf{a} \times \mathbf{n}]_{ret} \, , \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}_{ret} \\
&\mathbf{n} = \mathbf{R}/R \, , \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)
\end{aligned} \tag{50}$$

where $\mathbf{r}(t)$ is the vector showing the particle's path.

**Radiated power, Larmor's formula**

$$P = \frac{\mu_0 q^2}{6\pi c}\mathbf{a}^2 \tag{51}$$

## 4 Classical relativistic field theory

**Lagrange's equation for a relativistic field** $\phi$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{52}$$