FYS3120 Classical Mechanics and Electrodynamics

Problem set 4

February 11, 2024

Problem 1 Double trouble

A mass m_1 is attached to the ceiling by a stiff rod of length l_1 as a pendulum. Connected to the mass m_1 is another pendulum rod of length l_2, which in turn is connected to a second mass m_2. The rods are to be considered massless. There is no friction in the system. The motion is restricted to a 2D-plane.

a) Sketch the system, find the number of degrees of freedom and choose generalised coordinates.

b) Find the Lagrangian for the double pendulum.

c) Derive the equations of motion for the system.

d) Rewrite the equations of motion as a set of four coupled first order differential equations. Solve the equations numerically using appropriate initial conditions plotting the generalised coordinates and their velocities. Compare the results of two sets of initial conditions with large initial angles that differ only very slightly in one of the initial conditions. Make the comparison over a long time period. Can you explain what is going on? *Hint:* To make sure the algebra is correct, it might be wise to check the result using a symbolic algebra package. Please see the note on computer algebra systems `sympy.pdf` found with this problem set. You may also have use for some of the differential equation solving hints we gave in Problem Set 2.

e) Is the total energy in the system conserved? Discuss, using both numerical and analytical arguments.

Problem 2 The brachistochrone problem

This is a classical problem in analytical mechanics. It was discussed by Galileo Galilei, who suggested a solution (but not the correct one), and studied the problem experimentally. In 1696 the problem was formulated as a challenge to the mathematicians at the time by Johann Bernoulli. He wrote in the journal *Acta Eruditorum*:[1]

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of

[1] Johann Bernoulli, *Problema novum ad cujus solutionem Mathematici invitantur,* (A new problem to whose solution mathematicians are invited), Acta Eruditorum **18** (1696) 269.

> the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The problem he formulated was the following:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Five solutions were obtained from scientist and mathematicians we are all acquainted with, Newton, Jacob Bernoulli (the older brother of Johann), Leibniz and de L'Hôpital, in addition to Johann himself. Johann Bernoulli gave a formulation of the problem where he could use an anology to Snell's law of refraction in optics to solve the problem.

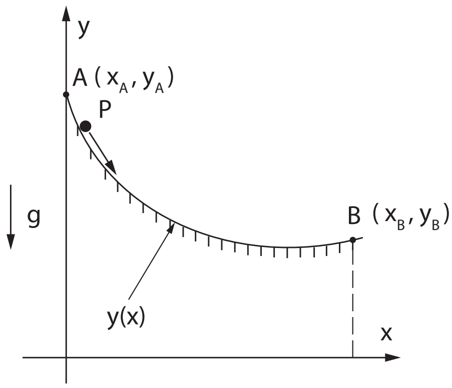

Figure 1: Illustration of the brachistochrone problem.

Let us now rephrase the problem with a few more words: Assume a small body, named P in Fig. 1, moves in a vertical plane under the influence of gravity only. It leaves a point A with zero velocity and follows (without friction) a given path in the plane which passes through a second point B, as shown in the figure. Assume the path between the two points A and B can be changed, while the points themselves stay fixed. For which path between the two points does the body spend the least time on the transit from point A to point B?

The challenge for you is the following: Find the solution to the brachistochrone problem by using the correspondence between the variational problem (finding the "path of shortest time") and the Lagrangian of mechanics, in the way discussed in the lectures.

The body P is to be treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal and y as the vertical coordinate. The boundary conditions, which fix the positions of point A and B, are specified as, $y(x_A) = y_A\,,\ y(x_B) = y_B$. To simplify the equations assume in the following that the initial coordinates are $x_A = y_A = 0$.

a) Show that the time T spent by the body on the way between A and B can be expressed as an integral of the form

$$T[y(x)] = \int_{x_A}^{x_B} L(y, y')dx, \tag{1}$$

with $y' = \frac{dy}{dx}$, and with

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}}. \tag{2}$$

Hint: Make use of energy conservation in the form $\frac{1}{2}mv^2 + mgy = 0$.

b) In this case it is not so easy to find the equations of motion directly from Lagrange's equation, however, we can use other properties from analytical mechanics. With $L(y, y')$ interpreted as a Lagrangian (x then plays the role of t in the usual formulation) the canonically conjugate momentum is $p = \frac{\partial L}{\partial y'}$ and the Hamiltonian is $H = py' - L$. Explain why H is a constant of motion and use this fact to show that $y(x)$ satisfies a differential equation of the form

$$(1 + y'^2)y = -k^2, \tag{3}$$

with k as a constant.

c) Equation (3) has a solution which can be written on *parametric form* as

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin\theta), \tag{4}$$

$$y(\theta) = \frac{1}{2}k^2(\cos\theta - 1), \tag{5}$$

where θ has been introduced as a curve parameter. Show that (4) and (5) constitute a solution of the differential equation (3) by changing the variable from x to θ in (3), and by using the above expression for $y(\theta)$. In what way are the boundary conditions taken care of by this solution?