Oblig 4

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Problem 1

a)

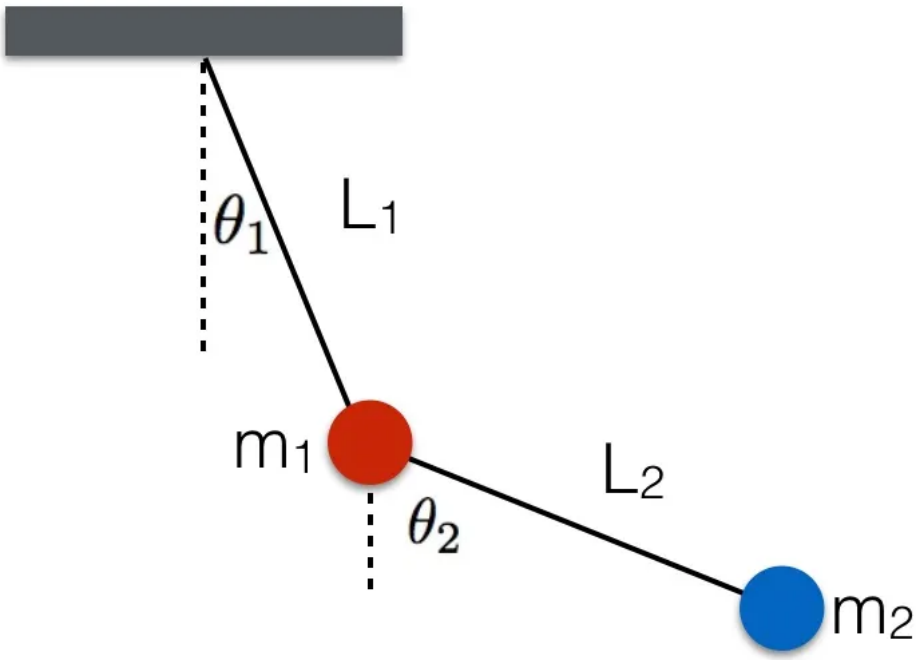

Figure 1: Scheme of the double pendulum. We have two bodies with masses m_1 and m_2, and lengths l_1 and l_2 respectively. The angles are θ_1 and θ_2.

We have 2 bodies in 2 dimensions with 2 constraints (l_1 and l_2). This gives:

$$d = 2 \cdot 2 - 2 = 2$$

We get the following coordinates:

$$\underline{\underline{q_1 = \theta_1 \quad , \quad q_2 = \theta_2}}$$

b)

We use the lengths and angles to find the x and y coordinates of the two bodies. We get:

$$x_1 = l_1 \sin\theta_1 \quad , \quad y_1 = -l_1 \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad , \quad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1 \quad , \quad \dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1$$

$$\dot{x}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \quad , \quad \dot{y}_2 = l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2$$

Then we just insert these into the formula for kinetic and potential energy:

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$\begin{aligned} T =& \frac{1}{2}m_1\left(\left(l_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1\right)^2\right) + \\ & \frac{1}{2}m_2\left(\left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2\right)\end{aligned}$$

Using this trig identity:

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$$

We get the following:

$$T = \underline{\underline{\frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)}}$$

$$V = m_1gy_1 + m_2gy_2$$

$$V = m_1g(-l_1\cos\theta_1) + m_2g(-l_1\cos\theta_1 - l_2\cos\theta_2)$$

$$V = \underline{\underline{-(m_1 + m_2)gl_1\cos\theta_1 - m_2gl_2\cos\theta_2}}$$

Putting everything together we get the Lagrangian L:

$$L = T - V$$

$$\begin{aligned} L =& \frac{1}{2}m_1\left(\left(l_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1\right)^2\right) + \\ & \frac{1}{2}m_2\left(\left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2\right) \\ & + (m_1 + m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2\end{aligned}$$

c)

Beginning with the canonical momentum p:

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = \frac{\partial T}{\partial\dot{\theta}_1} = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = \frac{\partial T}{\partial\dot{\theta}_2} = m_2l_2^2\dot{\theta}_2 + m_2l_1l_2\dot{\theta}_1\cos(\theta_1 - \theta_2)$$

Now we can find the equations of motion:

$$\frac{\partial p_{\theta_1}}{\partial t} - \frac{\partial L}{\partial\theta_1} = 0$$

$$
\begin{aligned}
\frac{\partial p_{\theta_1}}{\partial t} =& (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) \\
&- m_2 l_1 l_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)
\end{aligned}
$$

$$
\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g l_1 \sin\theta_1
$$

$$
\begin{aligned}
\frac{\partial p_{\theta_2}}{\partial t} =& m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) \\
&- m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)
\end{aligned}
$$

$$
\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2
$$

Now we can define the angles θ_1 and θ_2:

$$
\theta_1 = \underline{\underline{(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g \sin\theta_1 = 0}}
$$

$$
\theta_2 = \underline{\underline{l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1 \cos(\theta_1-\theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + g\sin\theta_2 = 0}}
$$

d)

$$
\begin{aligned}
(m_1+m_2)l_1\ddot{\theta}_1 =& -m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) \\
&- m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1
\end{aligned}
$$

$$
\ddot{\theta} = \underline{\underline{\frac{-m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1}{(m_1+m_2)l_1}}}
$$

$$
l_2\ddot{\theta}_2 = l_1\dot{\theta}_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 - l_1\ddot{\theta}_1\cos(\theta_1-\theta_2)
$$

$$
\ddot{\theta}_2 = \underline{\underline{\frac{l_1\dot{\theta}_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 - l_1\ddot{\theta}_1\cos(\theta_1-\theta_2)}{l_2}}}
$$

e)

The total energy is the naturally conserved as there is no external forces to steal energy from the system.

Problem 2

a)

$$\partial t = \frac{\partial s}{v} \quad , \quad v = \sqrt{-2gy} \quad , \quad \partial s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

$$y' = \frac{\partial y}{\partial x}$$

Simplifying the expression for ∂s:

$$\partial s = \sqrt{1 + y'^2}\mathrm{d}x$$

Finally we get the expression for ∂t:

$$\partial t = \frac{\sqrt{1 + y'^2}}{\sqrt{-2gy}}\mathrm{d}x$$

$$\int_0^T \partial t = \int_{x_A}^{x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{-2gy}} \mathrm{d}x$$

$$T(y(x)) = \int_{x_A}^{x_B} \frac{\sqrt{1 + y'^2}}{\sqrt{-2gy}} \mathrm{d}x = \int_{v_A}^{v_B} L(y, y') \mathrm{d}x$$

b)

$$H = py' - L = \frac{\partial L}{\partial y'}y' - \sqrt{\frac{1 + y'^2}{-2gy}}$$

$$\underline{\underline{H = -\frac{1}{\sqrt{-2gy}(1 + y'^2)}}}$$

The Hamiltonian is time independent, which means that it is a constant of motion.

$$-\frac{1}{\sqrt{-2gy}(1 + y'^2)} = C$$

$$1 = C^2(1 + y'^2)(-2gy)$$

$$\frac{1}{-2gC^2} = y(1 + y'^2)$$

We define k as the following:

$$k = \sqrt{\frac{1}{2gC^2}}$$

Giving:

$$y(1 + y'^2) = -k^2$$

c)

$$y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial \theta} = \frac{\partial \theta}{\partial x}$$

$$\frac{\partial x}{\partial \theta} = \frac{1}{2}k^2(1 - \cos\theta) = -y$$

$$y' = \frac{\partial y}{\partial \theta}\frac{1}{y}$$

$$(1 + y'^2)y = \left(1 + \frac{1}{y^2}\left(\frac{\partial y}{\partial \theta}^2\right)\right)y = \left(y^2 + \left(\frac{\partial y}{\partial \theta}\right)^2\right)\frac{1}{y} = -k^2$$

$$\frac{\partial y}{\partial \theta} = -\frac{k^2}{2}\sin\theta$$

$$\left(y^2 + \left(\frac{\partial y}{\partial \theta}\right)^2\right)\frac{1}{y} = \left(\left(\frac{1}{2}k^2(\cos\theta - 1)\right)^2 + \left(\frac{1}{2}k^2\sin\theta\right)^2\right)\frac{1}{y}$$

$$\left(y^2\left(\frac{\partial y}{\partial \theta}\right)^2\right)\frac{1}{y} = \frac{1}{4}k^4\left((\cos\theta - 1)^2 + \sin^2\theta\right)\frac{1}{y}$$

$$\left(y^2\left(\frac{\partial y}{\partial \theta}\right)^2\right)\frac{1}{y} = \frac{1}{4}k^4\left(2 - 2\cos\theta\right)\frac{1}{y}$$

$$\left(y^2\left(\frac{\partial y}{\partial \theta}\right)^2\right)\frac{1}{y} = -k^2 y\frac{1}{y} = -k^2$$

$$x_A = 0 \to \frac{1}{2}k^2(\theta_A - \sin\theta_A) = 0 \to \underline{\theta_A = 0}$$

$$y_A = 0 \to \frac{1}{2}k^2(\cos\theta_A - 1) = 0 \to \underline{\theta_A = 2n\pi} \quad , \quad n \in \mathbb{Z}$$

$$x_B = \frac{1}{2}k^2(\theta_B - \sin\theta_B)$$

$$y_B = \frac{1}{2}k^2(\cos\theta_B - 1)$$