FYS3120 Classical Mechanics and Electrodynamics

Problem set 6

February 26, 2024

Problem 1

a) Below we have four equations that involve tensors of different ranks. Clearly the consistency rules for covariant equations are not satisfied in all places. Show where there are errors in each equation, and show how the equations can be modified to bring them to correct covariant form (there will be multiple alternative solutions but we prefer the simple ones).

$$C^\mu = T^\mu{}_\nu\, A^\mu\,, \quad D_\nu = T^\mu{}_\nu\, A_\mu\,, \quad E_{\mu\nu\rho} = T_{\mu\nu}\, S^\nu{}_\rho\,, \quad G = S_{\mu\nu}\, T^\nu{}_\alpha\, A^\alpha . \tag{1}$$

b) Assume A^μ and B^μ to be four-vectors and $T^{\mu\nu}$ to be a rank-2 tensor. Show that by making products of these and by lowering and contracting indices, one can form several new four-vectors and scalars.

c) Calculate the following four-derivatives: $\partial_\mu x^\nu$, $\partial_\mu x_\nu$, and $\partial_\mu x^\mu$, where we remind you that the differential operator ∂_μ is defined by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{2}$$

d) We have defined the following four tensor fields as functions of the space-time coordinates $x = (x^0, x^1, x^2, x^3)$,

$$f(x) = x_\mu x^\mu\,, \quad g^\mu(x) = x^\mu\,, \quad b^{\mu\nu}(x) = x^\mu x^\nu\,, \quad h^\mu(x) = \frac{x^\mu}{x_\nu x^\nu}. \tag{3}$$

Calculate the following derivatives,

$$\partial_\mu f(x)\,, \quad \partial_\mu g^\mu(x)\,, \quad \partial_\mu b^{\mu\nu}(x)\,, \quad \partial_\mu h^\mu(x). \tag{4}$$

Hint: If you are uncertain about results for tensors, a convenient way to check these is to specify the index values explicitly, *e.g.*, in the first case, by choosing $\mu = 1$, which gives $\partial_\mu = \frac{\partial}{\partial x}$, and writing $f(x) = (ct)^2 - (x^2 + y^2 + z^2)$.

e) Show that the gauge transformations of the scalar and vector potentials, $\phi(x) = \phi(\mathbf{r}, t)$ and $\mathbf{A}(x) = \mathbf{A}(\mathbf{r}, t)$, can be written in terms of the four-potential $A^\mu(x) = \left(\frac{1}{c}\phi(x), \mathbf{A}(x)\right)$ as

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) - \partial^\mu \chi(x), \tag{5}$$

where $\chi(x)$ is some well-behaved function of four-vector spacetime x. What sort of tensor is $\partial^\mu \chi(x)$?

f) Show that the electromagnetic field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

where A^μ is the four-potential, is invariant under gauge transformations.

g) Explain why $F^{\mu\nu}F_{\mu\nu}$ is invariant under Lorentz transformations.

h) Find an explicit expression for the matrix for $F^{\mu\nu}$ in terms of the components of the electric and magnetic fields.

i) Find $F^{\mu\nu}F_{\mu\nu}$ in terms of electric and magnetic fields.