Hand in set 5, FYS3140 spring 2024

Use the methods presented in the lectures.

Problem 1 (**Definite integrals**).
Compute the following real integrals using methods from complex analysis. You may compute the principal value of the integral whenever this is applicable.

a)
$$I = \int_{-\infty}^{\infty} \frac{x+3}{x^4+1} dx$$

b)
$$I = \int_{0}^{\infty} \frac{\cos(x)}{x^2+1} dx$$

c)
$$I = \int_{0}^{2\pi} \frac{\cos(2\theta)}{\sin(\theta)+5} d\theta$$

Hint: Reduce the integral to contour integral over the unit circle.

d)
$$I = \int_{-\infty}^{\infty} \frac{(x-1)\sin(8x-7)}{x^2-2x+5} dx$$

e)
$$I = \int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{1-x^4} dx$$

Problem 2 (**2D-electrostatics**).
Two-dimensional problems can be equivalently formulated as problems in the complex plane. In some cases, the tools of complex analysis may reveal properties of the physical system that were otherwise hidden in the real-valued two-dimensional representation. To illustrate this point, we consider a problem from electrostatics.

The electric potential field $V(x, y)$ induced by a change density ρ must satisfy the following Poison's equation in the plane as

$$-\nabla^2 V = \rho/\epsilon_0,$$

where ϵ_0 is the permittivity of vacuum and

$$\nabla^2 = \partial_x^2 + \partial_y^2$$

is the Laplace differential operator in 2D.

a) Show that if $f(z)$ is a complex analytic function with $z = x + iy$, then both the real and imaginary parts of $f = u + iv$ satisfy the Laplace's equation.

b) The electric potential of a point charge in 2D is proportional to

$$V(x, y) = \ln(r)$$

where $r = \sqrt{x^2 + y^2}$ is the distance to the point charge at the origin. Show that this corresponds to the real part of the complex potential

$$U(z) = \ln(z).$$

Draw curves in the complex plane, along which the real part of $U(z)$ is constant. Do the same for the imaginary part of $U(z)$ as well.

c) Which of these isoline curves correspond to the equipotential surfaces and which to the direction of the electric field $\vec{E}$?

d) What field is described by $\text{Im}(U(z))$? *Hint: This can be deduced by your observation in (c) and thinking about field lines and equipotentials.*

Problem 3 (Keyhole contour method for branch cuts - OPTIONAL!).

This is completely optional, i.e. not included in the exam material. This is for those of you who might be interested in learning about the keyhole contour and would like to solve this problem. Have fun with it or otherwise ignore it!

Consider the following definite integral:

$$\int_0^\infty \frac{\sqrt{x}}{x^3 + 6x^2 + 11x + 6} dx.$$

a) Draw an appropriate closed contour that you can use to compute this integral.

b) What branch are you choosing for $\sqrt{x}$ to ensure that the integrand is analytic on your contour?

c) Compute the integral.