Hand in set 1, FYS3140 spring 2024

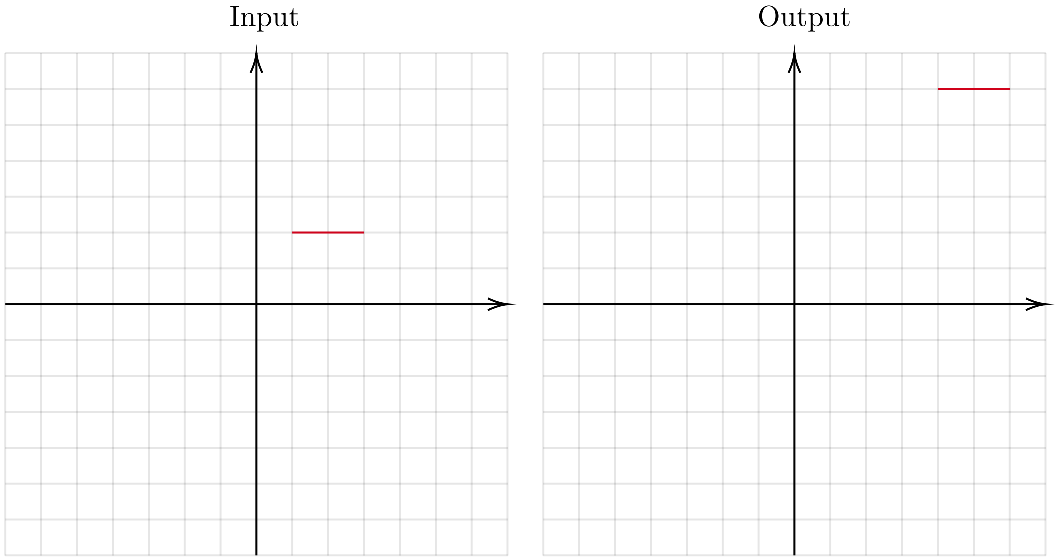
Problem 1 (Geometry and transformations).
In this problem we are going to present you with a curve (or region) in the complex plane along with a complex function. Your task will be to draw the input curve (or region) and the output obtained by the given function. For example if the curve is given as the following set

$$\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \in [1,3], \ \mathrm{Im}(z) = 2\},$$

and the function is

$$f(z) = z - (3 + 4i),$$

we want you to draw the following:



Do this for the following curves (regions) and functions:

a) Curve: $\{z \in \mathbb{C} \ : \ |z| = 1\}$. Function: $f(z) = 3(z + 2 - i)$.

b) Region: $\{z \in \mathbb{C} \ : \ |z - 2| \leq 1\}$. Function: $f(z) = 2(z - 3 + 2i)$.

c) Region: $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$. Function: $f(z) = 1/z$.

d) Curve: $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) = 0\}$. Function: $f(z) = e^z$.

Problem 2 (Power series).
Determine the disk of convergence for the following complex power series and draw the region of convergence in the complex plane.

a)

$$\sum_{n=0}^{\infty} \frac{(z + 4 - 2i)^{2n}}{3^n}$$

b)

$$\sum_{n=1}^{\infty} \frac{(z-3+i)^n}{n^2+2n}$$

c)

$$\sum_{n=2}^{\infty} \frac{\ln(n)\, z^n}{n}$$

Problem 3 (Elementary functions).

a) Find a formula for $\cos^3(x)$ that only uses first powers of $\cos(x)$ and $\sin(x)$. (*Hint: Euler's identity*).

b) Show that

$$\tan^{-1}(z) = -\frac{i}{2} \ln\left(\frac{i-z}{i+z}\right).$$

c) Compute the integral

$$\int_0^{\pi/2} \cos^2(x)\, \sin(x)\, dx$$

using the exponential forms of $\cos(x)$ and $\sin(x)$.