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Mathematical Methods in Physics

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Introduction

This is a compendium of lecture notes used in the mathematical methods in physics (FYS3140) course at the University of Oslo.

It is intended to bachelor and master level students in physics. Students are encouraged to use the compendium to follow the classroom lectures, AND Boas' textbook as an alternative presentation of the topics and an excellent source of problems to solve.

The course is structured around five main topics: I) Complex analysis, II) Variational calculus, III) ODE's, IV) Integral transformations, and V) PDE's. Basic theorems and mathematical techniques are introduced for each topic and then applied in the context of different physics problems.

The lecture notes were assembled in this compendium format thanks to Hiba Guessous who took this course in 2022. Many thanks to the teaching assistants Vidar Skogvoll, Jonas Eidesen and Larissa Bravina whose valuable and constant inputs throughout the course have greatly improved these notes. This compendium is a work in process and the feedback from students are extremely important to improve the presentation.

Part I

Complex Analysis

Lecture 1: Complex numbers

1.1 Complex numbers

Square root of negative numbers: Complex numbers are powerful generalizations of real numbers. Whilst real numbers are lined up on the real axis, complex numbers can extend into the plane, hence they are defined by a pair of number (x, y). To represent the additional axis of the plane, we start off by defining the square root of negative one

$$i = \sqrt{-1}$$

as the *imaginary number*. The imaginary number i has a conjugate number, namely $-i$, because their square is the same negative number:

$$i^2 = (-i)^2 = -1$$

A COMPLEX NUMBER is denoted as $z = x + iy$, where $x = Re(z)$ is called the real part and $y = Im(z)$ is the imaginary part. Both x and y must be real numbers !

COMPLEX NUMBERS are useful to do algebraic manipulations beyond real algebra. They also have a natural geometric representation as points in the plane and are a powerful alternative tool to vector analysis and real calculus. In the following chapters, we will introduce the complex analysis toolbox.

Example 1.1 (Example of complex numbers in algebra:). The quadratic equation $z^2 + 4 = 0$ has complex solutions which are complex conjugates $z_1 = \sqrt{-4} = 2i, \quad z_2 = -\sqrt{-4} = -2i$.

Example 1.2 (Example of complex number in geometry:). $z = 2 + 3i$ is a complex number and we can think of it as a pair of two real numbers $(2, 3)$ where 2 is the real part and 3 is the imaginary part. This also suggests a useful geometric representation of complex numbers.

EACH COMPLEX NUMBER $z = x + iy$ has a complex conjugate denoted $\bar{z} = x - iy$. The complex conjugate has the same real part and an imaginary part with opposite sign, because of the duality between i and $-i$. The real and imaginary parts follow from these transformations:

$$\boxed{x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}}$$

Check that $z \cdot \bar{z} = x^2 + y^2$. This representation $z = x + i$ is also called the canonical form or the Cartesian representation.

Example 1.3. Find the solutions of $z^2 - 2z + 3 = 0$.

Solution:

$$z = \frac{2 \pm \sqrt{4 - 12}}{2} = 1 \pm i\sqrt{2}$$

Example 1.4. Write $z = \frac{1+i}{1-i}$ in the canonical form $z = x + iy$.

Solution:

We multiply and divide by $1 + i$, so that the denominator becomes real :

$$z = \frac{(1 + i)^2}{1^2 + 1^2} = \frac{1 + 2i + i^2}{2} = i$$

Example 1.5. Write $z = \frac{3i-7}{i+4}$ in the canonical form $z = x + iy$

Solution:

We multiply and divide by $4 - i$, so that the denominator becomes real:

$$z = \frac{(-7 + 3i)(4 - i)}{4^2 + 1^2} = \frac{-28 + 7i + 12i - 3i^2}{17} = \frac{25}{17} + i\frac{19}{17}$$

1.2 Complex plane and complex representations

1.2.1 Cartesian form:

We are used to real numbers represented on a line. Complex numbers, on the other hand, are sets of two real numbers and lie as points in the (x, y)-plane also called the complex plane or the Argand diagram. The x-axis is called the *real axis* and the y-axis is called the *imaginary axis*. It corresponds to the rectangular representation of a complex number as: $z = x + iy$.

1.2.2 *Polar form:*

As coordinates, x and y follow basic coordinate transformations to the polar coordinates :

$$x = r\cos\theta, \qquad y = r\sin\theta$$

where the radius is determined by the absolute value or magnitude of z: $r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} \equiv |z| \equiv mod\ z$

The angle θ (measured in radians) determines the **argument** of z up to an arbitrary 2π rotation. This is because any (2π) rotation returns to the same point in the plane. Thus, the argument of z is intrinsically multi-valued, and we can see this clearly when we represent a (unique) complex number z in a polar form

$$z = r\cos(\theta + 2\pi n) + ir\sin(\theta + 2\pi n), \qquad n \in \mathbb{N}$$

To eliminate this ambiguity, it is useful to define the argument of z as the *principal value* of θ which is in the basic interval $[-\pi, \pi)$ (this is the convention used in this course unless otherwise stated). The complex plane is partitioned into 4 quadrants depending on the value of θ.

Then, the polar form of z is simply

$$z = r\cos(\theta) + ir\sin(\theta), \qquad \theta \in [-\pi, \pi)$$

What is θ? The complex plane is divided into four domains (quadrants) labeled counterclockwise. Depending on which quadrant the complex number is located, the principal value of the phase $\theta \in [-\pi, \pi)$ is determined as follows:

$$\theta = \begin{cases} \arctan(y/x), & x > 0, y > 0 \text{ Quadrant I} \\ \pi + \arctan(y/x), & x < 0, y > 0 \text{ Quadrant II} \\ -\pi + \arctan(y/x), & x < 0, y < 0 \text{ Quadrant III} \\ \arctan(y/x), & x > 0, y < 0 \text{ Quadrant IV} \end{cases} \tag{1.1}$$

1.2.3 *Euler formula:*

This is a useful formula that links the trigonometric or polar form of z with the complex exponential form:

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta}} \tag{1.2}$$

Check that the complex exponential has unit magnitude $|e^{i\theta}| = 1$.

Example 1.6. Write $z = -\sqrt{3} + i$ in polar form and in terms of the complex exponential. Represent it in the complex plane.

Solution:

The complex number is in Quadrant II. Thus, its argument is $\theta = \pi + \arctan(-\frac{1}{\sqrt{3}}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The absolute value is $r = \sqrt{3+1} = 2$. Hence, z in the polar form reads as

$$z = 2[\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})] = 2e^{i5\pi/6}$$

Example 1.7. In complex analysis, we often represent curves and domains in the complex plane by equations and inequalities. For example, what is the curve made by the points in the complex plane satisfying the equation $|z - 2 + i| = 2$? What is the domain of points satisfying $|z - 2 + i| \leq 2$

Solution:

This is the circle with the centre at point $z_0 = 2 - i$ and of radius 2. The domain satisfying this inequality is the disk enclosed the circle.

Example 1.8. Express $e^{i\pi} + e^{-i\pi}$ in the $x + iy$ form.

Solution:

$$\cos(\pi) + i\sin(\pi) + \cos(-\pi) + i\sin(-\pi) = -2$$

Example 1.9. : Express $\frac{(i-\sqrt{3})^3}{1-i}$ in the $x + iy$ form.

Solution:

$$\frac{(i-\sqrt{3})^3}{1-i} = \frac{1+i}{2}(i^3 + (-\sqrt{3})^3 + 3i^2(-\sqrt{3}) + 3i(-\sqrt{3})^2)$$

$$= \frac{1+i}{2}(-i - 3\sqrt{3} + 3\sqrt{3} + 9i) = 4i(1+i) = -4 + 4i$$

The modulus of z is $r = \sqrt{4^2 + 4^4} = 4\sqrt{2}$ and the argument is $\theta = \pi + \arctan(4/(-4)) = 3\pi/4$ (in QII). Thus,

$$\frac{(i-\sqrt{3})^3}{1-i} = 4\sqrt{2}\left[\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})\right] = 4\sqrt{2}e^{3i\pi/4}$$

1.3 *Application to classical mechanics*

1.3.1 *Particle motion in the plane*

The trajectory of a particle moving in the (x, y) plane can be readily expressed as a time-dependent complex number $z(t) = x(t) + iy(t)$. This is a path in the complex plane. We can compute the magnitude of velocity and acceleration in terms of the time derivatives of $z(t)$.

Example 1.10. : Let us consider a trajectory given by

$$z(t) = 1 + 3e^{2it} = 1 + 3\cos(2t) + 3i\sin(2t)$$

It means that $x(t) = 1 + 3\cos(2t)$ and $y(t) = 3\sin(2t)$.

The velocity in the complex form is determined by the time derivative of z:

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

Therefore, the speed follows as the magnitude of dz/dt: $|\frac{dz}{dt}| = |6ie^{2it}| = 6$ which, in this example, is independent of time. This means that the particle is moving at a constant speed. The distance of $z(t)$ away from 1 is $|z-1| = |3e^{2it}| = 3$ and also a constant. So the particle moves uniformly in a circle centred at point $(1;0)$. Notice that the magnitude of acceleration is nonzero, $|\frac{d^2z}{d^2t}| = 12$ and corresponds to a non-zero centripetal acceleration.

Lecture 2: Functions of one complex variable

2.1 *Complex power series*

A COMPLEX SERIES is defined as the sum of complex numbers

$$s_N = x_N + iy_N = \sum_{n=0}^{N} (a_n + ib_n),$$

where a_n and b_n are real numbers for every term in the series, i.e. $n = 0, 1, 2 \cdots$.

For finite N, the sum s_N is also a complex number. The question is if s_N converges to a a finite values $s = x + iy$ in the limit of $N \to \infty$, i.e.

$$s = \lim_{N \to \infty} s_N$$

To answer this question, we can use the convergence criteria of the real series formed by the real and imaginary parts. Here we are going to use the absolute convergence criterion. That is, the complex series is (**absolutely**) convergent if the coefficients of successive terms follow this inequality:

$$\boxed{\rho = \lim_{n \to \infty} \left| \frac{a_{n+1} + ib_{n+1}}{a_n + ib_n} \right| < 1} \tag{2.1}$$

Example 2.1. Test if $\sum_n \frac{(1+3i)^n}{n!}$ is convergent.

Solution:

$$\left|\frac{(1+3i)^{n+1}n!}{(n+1)!(1+3i)^n}\right| = \left|\frac{(1+3i)}{(n+1)}\right| = \frac{|1+3i|}{n+1} = \frac{\sqrt{10}}{n+1} \to_{n\to\infty} 0$$

Thus, the series is convergent.

A **complex power series** is defined as a series of powers of z,

$$\boxed{f_N(z) = \sum_{n=0}^{N} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots,}$$

where a_n are complex numbers. Notice that the power series is a complex function of z. The domain of z for which $f_N(z)$ converges to a sum is determined by the convergence criterion:

$$\boxed{\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}z}{a_n}\right| < 1}. \tag{2.2}$$

This inequality determines the radius of disk of convergence, i.e. $f_N(z)$ is a convergent series for all z inside the disk of convergence.

POWER SERIES are very useful in real analysis as many functions can be conveniently Taylor expanded as power series. Power series are perhaps even more useful in complex analysis as they map out whole regions in the complex plane where functions are well-behaved.

Example 2.2. Find the radius of convergence of this power series

$$f_N(z) = \sum_{n=0}^{N} \frac{(iz)^n}{n^2}$$

Solution:

The coefficient of z^n is i^n/n^2 . Let us look at the convergence criterion:

$$\lim_{n\to\infty} \left|\frac{izn^2}{(n+1)^2}\right| = |iz| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = |iz| < 1$$

Thus, the convergence condition is $|iz| = |z| < 1$. This corresponds to the unit disk centered at the origin.

Example 2.3. Find the radius of convergence of the following power series :

$$f_N(z) = \sum_{n=0}^{N} \frac{z^n}{n!}$$

Solution:

Let us look at the convergence criterion:

$$\lim_{n\to\infty} |\frac{z}{(n+1)}| = |z| \lim_{n\to\infty} \frac{1}{(n+1)} \to 0$$

This power series is convergent for any value of z. Based on the analogy with real power series, we may recognise this as the power series expansion of the complex exponential.

Example 2.4. Prove Euler formula :

$$\cos\theta + i\sin\theta = e^{i\theta}$$

Solution:

Let us use the Taylor expansion of the trigonometric real functions:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

Combing them in the complex form and re-arranging the terms:

$$\begin{aligned}
\cos\theta + i\sin\theta &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots) \\
&= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} \cdots \\
&= e^{i\theta}
\end{aligned}$$

Example 2.5. Find the disk of convergence for :

$$\sum_{n=0}^{\infty} \left(\frac{z-2+i}{2}\right)^n$$

Solution:

The radius of convergence is

$$\rho = \lim_{n\to\infty} \left| \frac{(z-2+i)^{n+1}/2^{n+1}}{(z-2+i)^n/2^n} \right| = \frac{|z-2+i|}{2} < 1$$

This power series is convergent when the complex number z satisfies the inequality $|z-2+i| < 2$. Re-writing it in the rectangular form, we see that it corresponds to $|(x-2)+i(y+1)| < 2$, which is the disk centered at the point $(2,-1)$ and of radius 2.

2.2 Elementary functions of one complex variable

A COMPLEX FUNCTION $f(z)$ is a map that takes one complex number $z = x + iy$ and returns another complex number, i.e. $f : \mathbb{C} \to \mathbb{C}$. The rectangular form of $f(z)$ can be written as

$$\boxed{f(z) = u(x,y) + iv(x,y)}$$

where $u(x,y) = Re(f(z))$ is the real part and $v(x,y) = Im(f(z))$ is the imaginary part. This is also called the *normal form*. Just like with complex numbers, the functions u and v need to also be real-valued functions!

Definition 2.1. $f(z)$ IS SINGLE-VALUED when $f(z)$ takes a unique value for each z. In polar form, this implies restricting the argument of z to the basic interval, i.e. $\theta \in [-\pi, \pi)$.

Next, we are going to introduce important elementary functions and their normal form.

2.2.1 Exponential function

THE EXPONENTIAL holds a special place in complex analysis through the powerful Euler's formula connecting the exponential with the trigonometric functions

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Using the rectangular form $z = x + iy$, we can also write the exponential of z as

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y), \tag{2.3}$$

where e^x is the familiar, non-negative defined exponential that blows up for positive values of x or decreases to zero for negative x. However, the complex exponential has a more subtle behavior in the plane due to the trigonometric dependence on y which changes sign.

Another common complex exponential is e^{iz} which can be expressed using Euler's formula as

$$e^{iz} = \cos(z) + i\sin(z)$$

and relates to the complex trigonometric functions as discussed further below.

Some useful identities:

$$\boxed{e^z e^{\bar{z}} = e^{2x}} \qquad \boxed{e^z e^{-\bar{z}} = e^{2iy}}$$

Example 2.6. Write $e^{-i\pi/4+\ln 3}$ in the rectangular form.

Solution:
$e^{-i\pi/4+\ln 3} = 3(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = \frac{3}{\sqrt{2}} - i\frac{3}{\sqrt{2}}$.

2.2.2 *Power and root functions*

THE POWER FUNCTION takes a complex number to a power n:

$$f(z) = z^n, \quad n \in \mathbb{Z}$$

De Moivre formula: This is a useful identity that follows from the Euler formula and provides a trigonometric (polar) form of the power function :

$$\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

Proof: Use the Euler's formula: $z^n = (re^{i\theta})^n = r^n e^{in\theta}$.

Similarly, the root function takes the n-th root of a complex number:

$$f(z) = z^{1/n}, \quad n \in \mathbb{Z}, \quad n \neq 0$$

De Moivre formula: It provides a trigonometric form also for the root functions

$$\boxed{z^{1/n} = r^{1/n}[\cos(\theta/n) + i\sin(\theta/n)]}$$

2.2.3 *Trigonometric functions*

THESE ARE ELEMENTARY trigonometric functions that are now applied to a complex number. We have useful representations of $\sin z$ and $\cos z$ in terms of the complex exponential function:

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}} \tag{2.4}$$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \tag{2.5}$$

Proof: We use the series representation of the exponential and trigonometric functions

$$\begin{aligned}
\frac{e^{iz} - e^{-iz}}{2i} &= \frac{1}{2i} \sum_n \frac{(iz)^n (1 - (-1)^n)}{n!} \\
&= \frac{1}{2i} (2iz - 2i\frac{z^3}{3!} + 2i\frac{z^5}{5!} \cdots) \\
&= \sin z
\end{aligned}$$

Similar rules of trigonometry apply to the trigonometric functions of complex variables. A useful identity is:

$$\boxed{\sin^2 z + \cos^2 z = 1}$$

Proof:

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} + e^{-2iz} - 2}{-4}$$

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + e^{-2iz} + 2}{4}$$

By adding them up, we get that

$$\sin^2 z + \cos^2 z = 1$$

2.2.4 *Hyperbolic functions*

THESE ARE GENERALIZATIONS of the hyperbolic functions, like sinh and cosh to complex variables. Like with the trigonometric functions, we have useful representations of the hyperbolic functions in terms of the complex exponentials:

$$\boxed{\sinh z = \frac{e^z - e^{-z}}{2}} \tag{2.6}$$

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2}} \tag{2.7}$$

Useful identity 1:

$$\boxed{\cosh^2 z - \sinh^2 z = 1} \tag{2.8}$$

Proof:

$$\sinh^2 z = \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + e^{-2z} - 2}{4}$$
$$\cosh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 = \frac{e^{2z} + e^{-2z} + 2}{4}$$

By subtracting them, we have that

$$\cosh^2 z - \sinh^2 z = 1$$

Useful identity 2:

$$\boxed{\sinh iz = i \sin z} \tag{2.9}$$

Proof: We have that

$$\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$$

2.2.5 *Logarithmic function*

The natural representation of the logarithm is using the polar form z so in this case we can clearly see that the function is in general multi-valued through its dependence on the argument $arg(z)$:

$$\boxed{\ln z = \ln r + i(\theta + 2\pi n).}$$

However, we may define the single-valued logarithm in terms of the principal value of θ as :

$$\boxed{\ln z = \ln r + i\theta, \qquad \theta \in (-\pi, \pi].}$$

Example 2.7. Write $\ln(1 + i\sqrt{3})$ in the rectangular form.

Solution:

The modulus is $r = |z| = 2$ and argument $\theta = \arctan(\sqrt{3}) = \pi/3$. Hence: $\ln z = \ln 2 + i\frac{\pi}{3}$.

2.2.6 *Complex roots and powers*

The exponent of root and power functions can be a complex number such as,

$$\boxed{z^w = e^{w \ln z}}$$

where both z and w are complex number.

Example 2.8. Write $(1+i)^{1-i}$ in a polar form.

Solution:

$$\begin{aligned}
(1+i)^{1-i} &= e^{(1-i)\ln(1+i)} = e^{(1-i)[\ln\sqrt{2}+i(\pi/4+2n\pi)]} \\
&= e^{\ln\sqrt{2}+\pi/4+2n\pi+i(-\ln\sqrt{2}+\pi/4+2n\pi)} \\
&= \sqrt{2}e^{\pi/4+2n\pi}e^{i(-\ln\sqrt{2}+\pi/4)} \\
&= \sqrt{2}e^{\pi/4+2n\pi}[\cos(\pi/4-\ln\sqrt{2})+i\sin(\pi/4-\ln\sqrt{2})]
\end{aligned}$$

Lecture 3: Analytic functions

Now that we are more familiar with complex numbers and complex functions, we are ready to learn about complex differentiation and basic properties of analytic functions.

3.1 *Complex differentiation:*

Definition 3.1 (Derivative of a complex function:). THE concept of differentiation is extended to complex variables such that formally, we define a derivative of $f(z)$ as a limit of the ratio between increments of $f(z)$ and increments in z

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3.1}$$

However, there is a subtlety here when we realise that increments can point in infinitely many directions in the complex plane. There are infinitely many ways in which we can infinitesimally vary z! How is this freedom going to affect changes in $f(z)$ determines whether or not $f(z)$ has a well-defined derivative at that point z.

A COMPLEX FUNCTION $f(z)$ has a derivative at a point z_0 if $f'(z_0)$ has a unique value no matter how we approach z_0 from all directions in the complex plane, i.e. independent on how Δz approaches zero. In other words $f(z)$ is differentiable at z_0 if $f'(z_0)$ is unique (independent on the path of vanishing Δz).

Complex analysis follows also similar rules of differentiation as in real analysis, namely:

Sum rule: $\frac{d}{dz}(f+g) = f'(z) + g'(z)$

Product rule: $\frac{d}{dz}(f \cdot g) = f'(z)g(z) + f(z)g'(z)$

Chain rule: $\frac{d}{dz} f(g(z)) = f'(g)g'(z)$

Definition 3.2 (Analytic function:). A COMPLEX FUNCTION $f(z)$ IS ANALYTIC in a region of the complex plane if it is differentiable at every point in that region. The property of being analytic requires that the function is differentiable at every point in a given domain. This is a fundamental property with important implications in complex analysis. One such implication, is that the real and imaginary parts of $f(z)$ are related with each other through the Cauchy-Riemann conditions, as discussed next. Later on, we will find out that analytical functions has unique derivatives of any order, i.e. it is differentiable at any order.

Example 3.1. : Show that e^z is differentiable at every point in the complex plane, hence it is analytic.

Solution:

The derivative at an arbitrary point is unique:

$$\begin{aligned} \frac{d}{dz} e^z &= \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z} \\ &= \lim_{\Delta z \to 0} e^z \frac{e^{\Delta x}(\cos \Delta y + i \sin \Delta y) - 1}{\Delta x + i \Delta y}. \end{aligned}$$

Since Δx and Δy are infinitesimal real-valued increments, we can Taylor expand to first order the trigonometric functions and the real exponential,

$$\begin{aligned} \frac{d}{dz} e^z &= e^z \lim_{\Delta z \to 0} \frac{(1+\Delta x)(1+i\Delta y) - 1}{\Delta x + i \Delta y} \\ &= e^z \lim_{\Delta z \to 0} \frac{\Delta x + i \Delta y}{\Delta x + i \Delta y} \\ &= e^z \end{aligned}$$

This calculation is independent of the value of z, hence the complex exponential is analytic in the complex plane.

Example 3.2. Show that $\bar{z}$ is not differentiable.

Solution:

$$\begin{aligned}
\frac{d}{dz}\bar{z} &= \lim_{\Delta z \to 0} \frac{(\bar{z} + \Delta \bar{z}) - \bar{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + \Delta y}
\end{aligned}$$

For any z, if we approach it long the x-axis ($\Delta y = 0$), then the derivative is 1. However, if we approach it along the y-axis ($\Delta x = 0$), then the same derivative takes the value -1. The derivative takes different values depending on which direction we approach z. Hence, $\bar{z}$ is not differentiable.

Example 3.3. Show that $|z|^2$ is not analytical.

Solution:

$$\begin{aligned}
\frac{d}{dz}|z|^2 &= \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\bar{z} + \Delta \bar{z}) - |z|^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z\Delta \bar{z} + \bar{z}\Delta z}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{2x\Delta x + 2y\Delta y}{\Delta x + i\Delta y}
\end{aligned}$$

The numerator is real, while the denominator is a complex number. This is a problem! If we approach an arbitrary point z along the x-axis (i.e. $\Delta y = 0$), then the derivative on this path equals $2x$. On the other hand, if we choose the path along the y-axis (i.e. $\Delta x = 0$), then the derivative is instead equal to $-2iy$. This tells us that the value of $f'(z)$ is path-dependent. The function is not differentiable for any point in the complex plane, hence, it is not analytical.

Definition 3.3 (Holomorphic function). A COMPLEX FUNCTION $f(z)$ is analytical at a point $z = z_0$ if we can define a region around z_0 where $f(z)$ has a unique derivative at every point inside it.

Definition 3.4 (Regular point:). IT IS A POINT z_0 at which $f(z_0)$ is analytic. In other words, there exists a neighborhood around z_0 where $f(z)$ is differentiable.

Definition 3.5 (Singular point:). A POINT OF z_0 IS SINGULAR when $f(z)$ is not analytic at z_0, i.e. there is no neighborhood of z_0 where $f(z)$ is differentiable.

Definition 3.6 (Isolated singularity:). A POINT z_0 IS AN ISOLATED SINGULARITY when $f(z)$ is analytic everywhere else except inside a small disk centered at z_0.

3.2 *Cauchy-Riemann conditions*

Theorem 3.4. IF $f(z) = u(x,y) + iv(x,y)$ IS ANALYTIC *in a region of the complex plane, then $u(x,y)$ and $v(x,y)$ must satisfy the* ***Cauchy-Riemann conditions*** *in that region,*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{3.2}$$

Proof: We use the rule of differentiation of $f(z)$ as an implicit function of x and y through $z = x + iy$:

$$\partial_x f = \frac{\partial f(z)}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = f'(z), \qquad \partial_y f = \frac{\partial f(z)}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = if'(z)$$

$f(z)$ is analytic implies that $f'(z)$ is uniquely defined, hence the two expressions above must be equal:

$$\partial_x f = -i\partial_y f$$

Evaluating the partial derivatives of $f(z)$ in terms of the partial derivatives of $u(x,y)$ and $v(x,y)$, we find equivalent expressions of $f'(z)$:

$$\boxed{f'(z) = \partial_x u + i\partial_x v = -i\partial_y u + \partial_y v} \tag{3.3}$$

Taking the real and imaginary parts, we get the Cauchy-Riemann relations.

Theorem 3.5. IF $u(x,y)$ AND $v(x,y)$ *and their partial derivatives with respect to x and y are differentiable and satisfy the Cauchy-Riemann (C-R) conditions in a region of the complex plane, then $f(z)$ is analytic inside that region (not necessarily on the boundary).*

Proof: We want to show that $f'(z)$ is uniquely defined (path-independent) for each point in the region where $u(x,y)$ and $v(x,y)$ and their derivatives are continuous and C-R conditions are satisfied.

$$\frac{df}{dz} = \frac{du + idv}{dx + idy}.$$

Since $u(x,y)$ and $v(x,y)$ are differentiable, we have that

$$\begin{aligned} du &= \partial_x u dx + \partial_y u dy \\ dv &= \partial_x v dx + \partial_y v dy. \end{aligned}$$

Inserting this back into the differentiation formula and rearranging terms, we get

$$\frac{df}{dz} = \frac{(\partial_x u + i\partial_x v)dx + (\partial_y u + i\partial_y v)dy}{dx + idy}$$

By using the C-R relations, it follows that

$$\begin{aligned} \frac{df}{dz} &= \frac{(\partial_x u + i\partial_x v)dx + (-\partial_x v + i\partial_x u)dy}{dx + idy} \\ &= \frac{(\partial_x u + i\partial_x v)dx + i(\partial_x u + i\partial_x v)dy}{dx + idy} \\ &= \partial_x u + i\partial_x v \end{aligned}$$

which is well-defined since u and v are differentiable functions.

Example 3.6. Check that $f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$ satisfies the C-R conditions everywhere outside the origin $z_0 = 0$.

Solution:

First, we rewrite $f(z) = \frac{\bar{z}}{|z|^2}$ in the canonical form to determine u and v functions, namely:

$$u(x,y) = \frac{x}{x^2 + y^2}, \qquad v(x,y) = -\frac{y}{x^2 + y^2}$$

Them, we compute their partial derivatives:

$$\partial_x u(x,y) = \frac{-x^2 + y^2}{(x^2 + y^2)^2}, \qquad \partial_y u(x,y) = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\partial_y v(x,y) = \frac{-x^2 + y^2}{(x^2 + y^2)^2}, \qquad \partial_x v(x,y) = \frac{2xy}{(x^2 + y^2)^2}$$

We immediately notice that the C-R conditions are satisfied, i.e. $\partial_x u(x,y) = \partial_y v(x,y)$ and $\partial_y u(x,y) = -\partial_x v(x,y)$.

Example 3.7. Similarly you can show that the simple fraction $f(z) = \frac{1}{z-z_0}$ satisfies the C-R conditions everywhere except at the isolated singular point z_0. Hence, it is analytic outside its isolated singularity. Simple fractions are a convenient way to presented isolated singularities.

Example 3.8. Let $f(z) = u + iv$ be an analytic function, and let $\vec{F} = v\vec{i} + u\vec{j}$ be a vector field with the components determined by the real and imaginary parts of $f(z)$. Show that div $\vec{F} = 0$ and curl $\vec{F} = 0$ correspond to C-R conditions.

Solution:

The C-R relations follow straightforwardly from the definitions of divergence and curl of a 2D vector field:

$$\text{div } \vec{F} = 0 \to \partial_x v + \partial_y u = 0$$
$$\text{curl } \vec{F} = 0 \to \partial_x u - \partial_y v = 0.$$

Definition 3.7 (Harmonic function). A REAL FUNCTION $\phi(x, y)$ that satisfies the Laplace equation in two dimensions, i.e.

$$\nabla^2\phi = \partial_x^2\phi + \partial_y^2\phi = 0,$$

is called a **harmonic function**.

Theorem 3.9. *(Part 1) If $f(z) = u + iv$ is analytic in a region, then u and v are* ***conjugate harmonic functions****, i.e. they satisfy the Laplace's equation in that region.*

(Part 2) Any function u (or v) satisfying the Laplace's equation in a simply-connected region is the real (or imaginary part) of an analytic function.

Proof of Part I: We use that C-R conditions satisfied by u and v and differentiate them once more.

$$\partial_x^2 u \equiv \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}, \qquad \partial_y^2 u \equiv \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

Hence, $\partial_x^2 u + \partial_y^2 u = 0$. Similarly,

$$\partial_x^2 v \equiv \frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y}, \qquad \partial_y^2 v \equiv \frac{\partial}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$$

Hence $\partial_x^2 v + \partial_y^2 v = 0$.

Example 3.10. Show that the corresponding u and v of the analytic function $f(z) = \frac{1}{z}$ are conjugate harmonic functions.

Solution:

We have shown in Example 3.7 that $f(z)$ is an analytic function. Here we show that

$$u(x,y) = \frac{x}{x^2+y^2}, \qquad v(x,y) = -\frac{y}{x^2+y^2}$$

are harmonic functions. Let's differentiate $u(x,y)$:

$$\begin{aligned} \partial_x^2 u(x,y) &= \partial_x\left(\frac{-x^2+y^2}{(x^2+y^2)^2}\right) = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}, \\ \partial_y^2 u(x,y) &= -\partial_y\left(\frac{2xy}{(x^2+y^2)^2}\right) = -\frac{2x(x^2-3y^2)}{(x^2+y^2)^3} \end{aligned}$$

Hence $\nabla^2 u(x,y) = 0$. Similarly, you can show that $\nabla^2 v(x,y) = 0$.

Cauchy-Riemann conditions in polar form: For some $f(z)$, it may be difficult to separate $u(x,y)$ and $v(x,y)$ using the rectangular form of z. In such cases, we resort on the polar form $z = re^{i\theta}$, where $-\pi < \theta \leq \pi$ as natural coordinates of u and v

$$\boxed{f(z) = u(r,\theta) + iv(r,\theta)}$$

The corresponding Cauchy-Riemann conditions in polar form read as:

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \tag{3.4}$$

Proof: Let us take the implicit derivative with respect to r and θ:

$$\partial_r f(z) = f'(z)\frac{dz}{dr} = \frac{z}{r}f'(z), \quad \partial_\theta f(z) = f'(z)\frac{dz}{d\theta} = izf'(z)$$

Since $f(z)$ is analytic, the two expressions for $f'(z)$ must be identical:

$$\frac{r}{z}(\partial_r u + i\partial_r v) = -i\frac{1}{z}(\partial_\theta u + i\partial_\theta v)$$

Separating out the real and imaginary parts, we obtain the C-R relations in polar form.

Example 3.11. Show that $f(z) = z^n$ is an analytic function.

Solution:

It is easier to identify u and v using the polar form $z = re^{i\theta}$ and the de Moivre formula for the power function:

$$f(z) = r^n[\cos(n\theta) + i\sin(n\theta)] \to u(r,\theta) = r^n\cos(n\theta), \quad v(r,\theta) = r^n\sin(n\theta)$$

By taking their partial derivatives, we obtain

$$\partial_r u(r,\theta) = nr^{n-1}\cos(n\theta) = \frac{1}{r}\partial_\theta v(r,\theta)$$

$$\partial_r v(r,\theta) = nr^{n-1}\sin(n\theta) = -\frac{1}{r}\partial_\theta u(r,\theta)$$

We deduce that this power function is analytic everywhere in the complex plane.

Lecture 4: Complex integration

In this lecture, we will learn how to solve complex line integrals using line parameterization and, in particular, how to solve contour integrals using Cauchy's theorem and Cauchy's integral formula.

4.1 *Line integrals:*

Unlike real definite integrals $\int_a^b dx f(x)$, it is not sufficient to specify the end points of a complex integral. Obviously, that is because there are infinitely many paths to connect two points in the complex plane.

Let Γ be a simple curve[1]. The integral of $f(z)$ on the path Γ is defined as

$$\boxed{I = \int_{\Gamma} f(z)dz} \tag{4.1}$$

[1] A simple curve is one which does not cross itself.

We solve this integral by specifying an appropriate parameterization of Γ, which allows us to write the complex variable as a function of the parameter that defines the curve. This is analogous to how we solve line integrals in vector analysis. Let's illustrate this by simple examples.

Example 4.1. (Exercise 4 in 14.3 Boas) Compute

$$I = \int_{\Gamma} \frac{1}{1-z^2} dz,$$

where Γ is the positive imaginary axis (y-axis).

Solution:

We parameterized Γ as $z(y) = iy$, where $y > 0$. We have that $dz = idy$. Hence, the integral becomes

$$\begin{aligned} I &= \int_0^\infty \frac{idy}{1-(iy)^2} = i\int_0^\infty \frac{dy}{1+y^2} \\ &= i\arctan(y)|_0^\infty \\ &= \frac{i\pi}{2} \end{aligned}$$

Example 4.2. (Exercise 1 in 14.3 Boas) Compute

$$I = \int_i^{i+1} zdz,$$

along a straight line parallel to the x-axis.

Solution:

In this case, the curve Γ is a horizontal segment between i and $i+1$, which means that $y = 1$ and $x \in [0,1]$. We use the parametrization $z(x) = x+i$, with $x \in [0,1]$. Thus, $dz = dx$ and the integral becomes

$$\begin{aligned} I &= \int_0^1 (x+i)dx = \frac{1}{2}(x+i)^2|_0^1 = \frac{1}{2}[(1+i)^2 - i^2] \\ &= \frac{1+2i}{2} \end{aligned}$$

4.2 *Contour integral and Cauchy's theorem:*

Definition 4.1. CONTOUR INTEGRAL of $f(z)$ is the line integral on a closed simple curve called contour C. By default, the counter is traversed **counter-clockwise**.

$$\boxed{I = \oint_C f(z)dz} \tag{4.2}$$

A canonical example of a contour is the circle centered at a point z_0 and of radius r and it is represented as $|z - z_0| = r$. But, the contour could take various shapes and, in fact, even be, by virtue of Cauchy's theorem, completely arbitrary.

Theorem 4.3. ***Cauchy's theorem****: Let C be a simple closed curve with a continuously turning tangent except possibly at a finite number of points (that is the curve is smooth except for a finite number of corners). If $f(z)$ is* ***analytic*** *on and inside C, then the contour integral on C vanishes*

$$\boxed{\oint_C f(z)dz = 0} \tag{4.3}$$

Proof: This contour integral can be written in terms of two real contour integrals as follows:

$$\oint_C (u+iv)(dx+idy) = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

USING Green's theorem from vector analysis

$$\oint_C \vec{F}\cdot d\vec{l} = \iint_D \mathrm{curl}(\vec{F})dxdy,$$

we can express the above real contour integrals as area integrals

$$\oint_C (udx - vdy) = \iint_D (-\partial_x v - \partial_y u)dxdy \tag{4.4}$$

$$\oint_C (vdx + udy) = \iint_D (\partial_x u - \partial_y v)dxdy \tag{4.5}$$

where D is the domain enclosed by C.

Since $f(z)$ is analytic, C-R conditions

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v \tag{4.6}$$

imply that the integrands of the area integrals vanish and thus the integrals also vanish. Since both the real and the imaginary part of the contour integral are zero, it follows that

$$\oint_C dz f(z) = 0.$$

The Cauchy's theorem is, in many ways, the bedrock theorem on which complex analysis is build. It has many ramifications and many subsequent theorems can be deduced from it. One important implication is that it allows us to perform integrals of complex functions with singularities inside the integration domain. The simplest case is that of a simple pole (isolated singularity) at z_0 (*see* Fg. (4.1)). We represent this by the simple fraction $\frac{1}{z-z_0}$. The Cauchy's integral formula allows us to quickly evaluate contour integrals of functions that have a simple pole at z_0 and can be cast into this form as

$$\oint_C \frac{f(z)}{z-z_0}dz$$

where $f(z)$ is analytic inside the integration domain. It also provides us a natural integral representation of complex functions.

4.3 Cauchy's integral formula

Theorem 4.4. *If $f(z)$ is* ***analytic*** *on and inside a simple contour C, then the value of $f(z)$ at a point $z = z_0$ inside the domain bounded by C is given by the following contour integral along C traversed counterclockwise:*

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint\limits_C \frac{f(z)}{z - z_0} dz, \qquad z_0 \text{ inside } C} \tag{4.7}$$

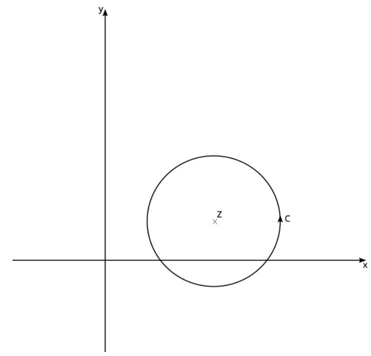


Figure 4.1: Contour enclosing z_0.

See figure 4.1

Since z_0 is an arbitrary *point inside C, it can be taken as a complex variable such that the function $f(z)$ can be represented as a contour integral for an arbitrary z:*

$$\boxed{f(z) = \frac{1}{2\pi i} \oint\limits_C \frac{f(w)}{w - z} dw, \qquad z \text{ inside } C} \tag{4.8}$$

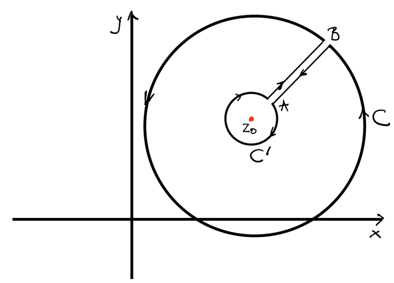


Figure 4.2: Contour isolating z_0.

Remember that the path along C is taken in the counterclockwise direction.

Proof: Let us define the integrand as

$$\phi(z) = \frac{f(z)}{z - z_0}.$$

$\phi(z)$ is analytic except in a small neighborhood around z_0. In other words, $\phi(z)$ has an isolated singularity at z_0. We can define a small circle C' that encloses z_0 such that z_0 can be cut out from the integration domain as illustrated in Fig. 4.2. Thus, we can rewrite the big contour C_0 as $C \bigcup C' \bigcup AB \bigcup BA$. The line integrals on the segments AB and BA cancel out, and the only non-zero contribution comes from the contour integral around C' circle in the limit where the radius goes to zero.

By Cauchy's theorem, using that $\phi(z)$ is analytic outside C',

$$\oint\limits_{C_0} \phi(z) dz = 0$$

implies that

$$\oint\limits_C \phi(z) dz + \oint\limits_{C' \text{ clockwise}} \phi(z) dz = 0.$$

We used that the line integrals along the two incisions cancel out because they have identical integrands and the segments are traversed in opposite directions: $\int_{AB} \phi dz = -\int_{BA} \phi dz$. Equivalently, going counterclockwise on both contours :

$$\oint_C \phi(z)dz = \oint_{C'} \phi(z)dz$$

Let us represent the inner contour C' by a small circle centered at z_0, $z = z_0 + \rho e^{i\theta}$ such that an infinitesimal line element on the circle is given as $dz = \epsilon i e^{i\theta} d\theta$. Now we can perform this contour integral using the contour parameterization as

$$\begin{aligned} \oint_C \phi(z)dz &= \oint_{C'} \phi(z)dz \\ &= \int_0^{2\pi} \phi(z_0 + \epsilon e^{i\theta})\rho i e^{i\theta} d\theta \\ &= \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta})}{\epsilon e^{i\theta}} \epsilon i e^{i\theta} d\theta \\ &= i \int_0^{2\pi} f(z_0 + \epsilon e^{i\theta}) d\theta \\ &\underset{\epsilon \to 0}{=} i f(z_0) \int_0^{2\pi} d\theta \\ &= 2\pi i f(z_0). \end{aligned}$$

Example 4.5. Consider

$$I = \oint_C \frac{1}{z^2 - 1} dz,$$

where C is a counterclockwise directed contour including $z = -1$ but not $z = 1$ as illustrated in Fig. 4.3

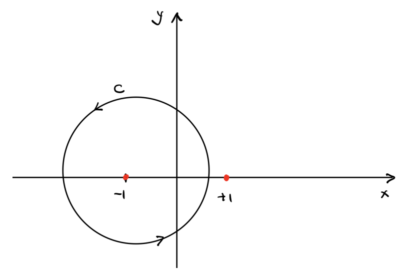


Figure 4.3: Contour used in Exercise 4.5.

Solution:

This is our first example where we use partial fraction decomposition to represent the integrand into simple fractions as

$$\frac{1}{z^2 - 1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

This is something that we will use again and again in different contexts! Then, the contour integral becomes:

$$I = \oint_C \frac{dz}{2(z-1)} - \oint_C \frac{dz}{2(z+1)},$$

The first integral vanish since its integrand is analytic everywhere inside C. For the second integral, since $z = -1$ is inside C, we apply the Cauchy's integral formula. Therefore,

$$I = -\frac{1}{2}(2\pi i) = -\pi i.$$

Example 4.6. (Ex 17 in 14.3 Boas, *see figure 4.4*) Compute the contour integral :

$$I = \oint_C \frac{\sin z}{2z - \pi} dz = \frac{1}{2} \oint_C \frac{\sin z}{z - \pi/2} dz$$

where C is:
(a) $|z| = 1$ oriented counterclockwise.
(b) $|z| = 2$ oriented counterclockwise.

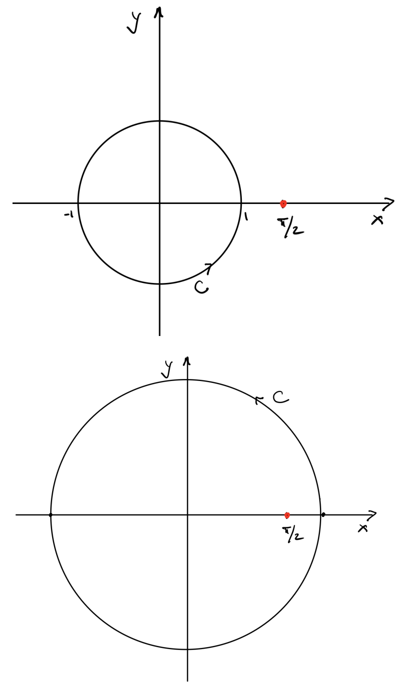


Figure 4.4: (top) Contour $|z| = 1$. (bottom) Contour $|z| = 2$ used in Exercise 4.6.

Solution:
Let us first cast the integral in the form for which we can readily apply the Cauchy's integral formula

$$I = \frac{1}{2} \oint_C \frac{\sin z}{z - \pi/2} dz$$

from which we see that $f(z) = \sin z$. Now, we need to find if $z_0 = \pi/2$ is inside the integration domain or not.

a) $z_0 = \pi/2$ is outside the unit disk centered at 0. For the contour integral over this unit circle vanishes by Cauchy's theorem:

$$I = 0$$

b) In this case, $z_0 = \pi/2$ is inside the disk of radius 2 enclosed by C given by $|z| = 2$. Thus, by the Cauchy's integral formula

$$I = \frac{1}{2} 2\pi i f(\pi/2) = i\pi \sin(\pi/2) = i\pi.$$

Example 4.7. (Ex 17 in 14.3 Boas, *see figure 4.5*) Compute the contour integral :

$$I = \oint_{C'} \frac{z^2 e^z dz}{2z + i\pi} dz$$

where C' is the unit circle centered at origin: $|z| = 1$ and is traversed clockwise.

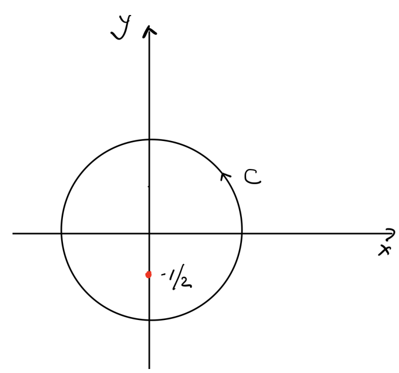


Figure 4.5: Contour $|z| = 1$ for Exercise 1.7

Solution:
First, we rewrite the integral as

$$I = -\frac{1}{2} \oint_C \frac{z^2 e^z}{z + i/2} dz, \qquad C : |z| = 1 \text{ counterclockwise}$$

From the integrand, we see that $f(z) = z^2 e^z$ is analytic and $z_0 = -i/2$ is inside C. Hence, by Cauchy's line integral, the integral is evaluated to

$$I = -\frac{1}{2} 2\pi i f(-i/2) = \pi i \frac{1}{4} e^{-i/2}.$$

Example 4.8. (*see figure 4.6*) Compute the contour integral :

$$I = \oint_C \frac{\cos z}{z^2 - 4} dz$$

where C is the rectangular contour with corners $i, -i, 3-i, 3+i$ traversed in the counter-clockwise direction.

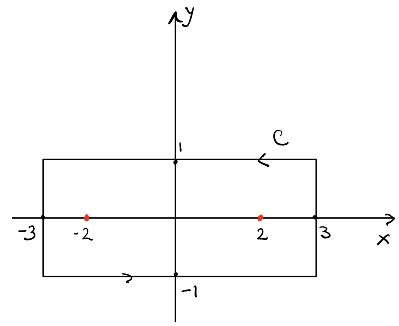


Figure 4.6: Rectangular contour from Exercise 1.8

Solution:

The integrand has isolated singularities at ± 2, but only $z_0 = 2$ is inside the contour. Thus, we can use the Cauchy's integral formula with $f(z) = \frac{\cos(z)}{z+2}$ which is analytic inside C which encloses $z_0 = 2$.

$$I = \oint_\Gamma \frac{f(z)}{z-2} dz$$

This immediately gives that

$$I = 2\pi i f(2) = 2\pi i \frac{\cos(2)}{4} = i\pi \cos(2)$$

Lecture 5: Generalized Cauchy's integral formula

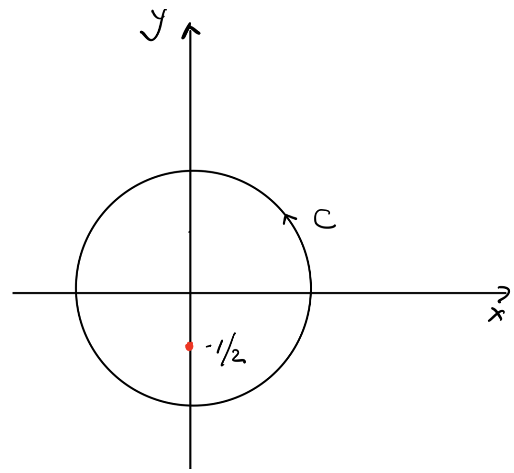


Figure 5.1: Contour $|z| = 1$ for Exercise 5.1

5.1 More examples using the Cauchy's integral formula:

Example 5.1. (Ex 17 in 14.3 Boas, *see figure 5.1*) Compute the contour integral

$$I = \oint_{C'} \frac{z^2 e^z dz}{2z + i} dz$$

where C' is the unit circle centered at origin: $|z| = 1$ and is traversed clockwise.

Solution:

First we write the integral with the contour in the counterclockwise direction and with the denominator in the form $z - z_0$

$$I = -\frac{1}{2} \oint_C \frac{z^2 e^z dz}{z + i/2} dz, \qquad C : |z| = 1 \text{ counterclockwise}$$

The function $f(z) = z^2 e^z$ is analytic in the whole complex plane, hence also inside C. Also, $z_0 = -i/2$ is inside C. Hence, by Cauchy's line integral :

$$I = -\frac{1}{2} 2\pi i f(-i/2) = \pi i \frac{1}{4} e^{-i/2} = \frac{\pi}{4}(\sin(1/2) + i\cos(1/2))$$

Example 5.2. (*see figure 5.2*)

Compute the contour integral

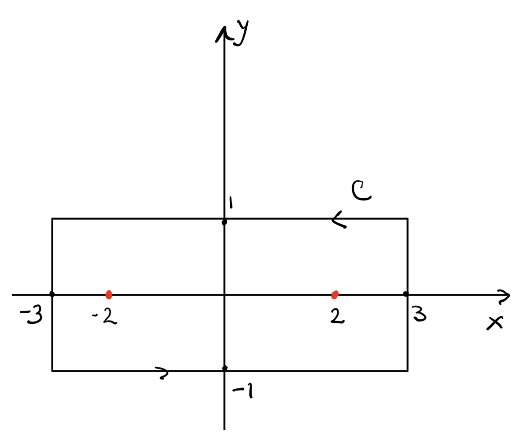


Figure 5.2: Rectangular contour used in Exercise 5.2

$$I = \oint_\Gamma \frac{\cos z}{z^2 - 4} dz$$

where Γ is the rectangular contour with corners $-3-i, -3+i, 3+i, 3-i$ traversed in the counter-clockwise direction.

Solution:

By partial fraction decomposition, we can decompose the integral into the difference between two contour integrals :

$$I = \frac{1}{4}\oint_\Gamma \frac{\cos z}{z-2} dz - \frac{1}{4}\oint_\Gamma \frac{\cos z}{z+2} dz$$

The integrands have each an isolated singularity at $z_0 = \pm 2$ inside the rectangle and the function $f(z) = \cos(z)/4$ is analytic. Thus, by the Cauchy's integral formula, we have that

$$I = 2\pi i \frac{\cos(2)}{4} - 2\pi i \frac{\cos(-2)}{4} = 0.$$

5.2 Generalized Cauchy's integral theorem

Differential form of Cauchy's integral formula: If $f(z)$ is analytic on and inside a simple closed curve C, and z_0 is a point inside C, then the value of the n'th derivative $f^{(n)} = d^n f/dz^n$ at $z = z_0$ has an integral form given by

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \qquad z_0 \text{ inside } C} \tag{5.1}$$

The function $f(z)$ that is analytic inside C is differentiable to any order.

Proof: This can be shown by differentiating recursively Cauchy's integral formula:

$$
\begin{aligned}
f'(z_0) &= \frac{1}{2\pi i}\oint\limits_C \frac{f(z)}{(z-z_0)^2}dz \\
f^{(2)}(z_0) &= \frac{2!}{2\pi i}\oint\limits_C \frac{f(z)}{(z-z_0)^3}dz \\
&\cdots \\
f^{(n)}(z_0) &= \frac{n!}{2\pi i}\oint\limits_C \frac{f(z)}{(z-z_0)^{n+1}}dz
\end{aligned}
$$

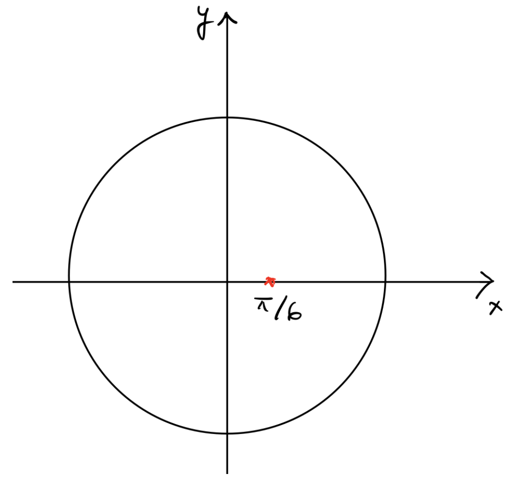


Figure 5.3: Contour $|z| = 1$ for Exercise 5.3

Example 5.3. (see figure 5.3)

Evaluate

$$I = \oint\limits_C dz \frac{2\sin(2z)}{(6z-\pi)^3},$$

where C is the circle $|z| = 3$ (counter-clockwise oriented).

Solution:

We notice that $z_0 = \pi/6$ is inside the circle at 0 and of radius 3. Furthermore, $f(z) = \frac{2\sin(2z)}{6^3}$ is analytic inside C. Hence, the conditions of the Cauchy's Integral formula apply, and we get for $n = 2$:

$$
\begin{aligned}
I &= \oint\limits_C dz \frac{2\sin(2z)}{(6z-\pi)^3} = \frac{2\pi i}{n!} f^{(2)}(z_0) \\
&= -\pi i \frac{2^3 \sin(\pi/3)}{6^3} = i\frac{\pi}{3^3}\frac{\sqrt{3}}{2}
\end{aligned}
$$

5.3 Cauchy's inequality

THE CAUCHY'S INEQUALITY is useful in a number of problems and proofs. We will use it to prove the Taylor and Laurent expansions of complex functions.

Triangle inequality: Let C be a closed curve bounding a disk of radius R and centered at z_0. If $f(z)$ is bounded **on** C, i.e. $|f(z)| \leq M$ for all z on C, then :

$$\boxed{\left|\oint\limits_C f(z)dz\right| \leq 2\pi RM} \qquad (5.2)$$

Proof:

$$\left| \oint_C f(z)dz \right| \leq \oint_{|z-z_0|=R} |f(z)||dz| \leq M \oint_{|z-z_0|=R} |dz|$$

USING the parameterization $z = z_0 + Re^{i\theta}$, with $0 \leq \theta < 2\pi$ and $|dz| = |iRe^{i\theta}d\theta| = Rd\theta$, we have that

$$\oint_{|z-z_0|=R} |dz| = R\int_0^{2\pi} d\theta = 2\pi R.$$

Thus,

$$\left| \oint_C f(z)dz \right| \quad \leq \quad 2\pi RM.$$

Example 5.4. Find an upper bound for the absolute value of

$$I = \oint_C \frac{e^z}{z^2-1}dz,$$

where the contour is the circle of radius 2 centered at the origin and traversed in the counterclockwise direction.

Solution:

The contour C corresponds to $|z| = 2$ with the circumference of length $2\pi R = 4\pi$. Next, we need to determine the upper bound M of $|f(z)|$.

$$|e^z| = |e^x e^{iy}| = e^x \leq e^2$$

We use the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

adapted to our case as

$$|(z^2-1)+1| \leq |z^2-1| + 1 \to |z^2-1| \geq |z^2| - 1$$

Combining this with the upper bound for the magnitude of z, we have that

$$|z^2-1| \geq |z|^2 - 1 = R^2 - 1 \geq 3$$

Hence,

$$|f(z)| = \frac{|e^z|}{|z^2-1|} \leq \frac{e^2}{3}$$

and integral is upper bounded in magnitude by

$$|I| \leq \frac{4\pi e^2}{3}$$

(Evaluate the integral using Cauchy's integral formula and check that it's indeed smaller than this upper bound)

<u>Generalized Cauchy's inequality:</u> Let $f(z)$ be analytic on and inside a circle C of radius R centered at z_0. If $f(z)$ is bounded, i.e. $|f(z)| \leq M$ for all z on C, then its derivatives are upper-bounded in the center of the disk by

$$\boxed{|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}}. \tag{5.3}$$

This is a useful inequality in proving important theorems. We will use it later in the proof of the Taylor expansion of analytic functions.

Proof: For a compact region centered at z_0 and of radius R where $f(z)$ is analytic, it also attains a maximum value. In other words, $f(z)$ is bounded in this region, i.e. there exits $M > 0$ such that $|f(z)| \leq M$. Let us start from the generalized Cauchy's integral formula :

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \oint\limits_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \\ &\leq \frac{n!}{2\pi} \oint\limits_{|z-z_0|=R} \frac{|f(z)|}{|z-z_0|^{n+1}} |dz| \\ &\leq \frac{n!}{2\pi} \frac{M}{R^n} \oint\limits_{|z-z_0|=R} |dz| \end{aligned}$$

USING the parameterization $z = z_0 + Re^{i\theta}$, with $0 \leq \theta < 2\pi$ and $|dz| = |iRe^{i\theta} d\theta| = Rd\theta$, we have that $\oint\limits_{|z-z_0|=R} |dz| = 2\pi R$.

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$$

We may notice that that this upper bound may not apply for derivatives that are evaluated at other points inside the disk than its center z_0.

Example 5.5. Let us consider the function $f(z) = z^2 e^z$ from previous examples, and evaluate the upper found of $f^{(2)}(0)$ at the center of the circle $|z| = 2$.

Solution:

First, let us determine the upper bound M of $f(z)$ on the circle $|z| = 2$:

$$|f(z)| = |z|^2 |e^z| = |z|^2 e^x \leq 2^2 e^2$$

Thus, $M = 2^2e^2$ and $R = 2$. According to the generalized Cauchy's inequality:

$$|f^{(2)}(0)| \leq 2e^2$$

Theorem 5.6 (Liouville's theorem). *If $f(z)$ is analytic and bounded in the entire complex plane, then $f(z)$ is constant.*

Proof: We want to prove that $f(z)$ is constant hence $f'(z) = 0$. Since $f(z)$ is analytic and bounded by M everywhere i.e. $|f(z)| < M$ for every $z \in \mathbb{C}$. This implies that $f(z)$ is analytic in a disk around an *arbitrary* point $z_0 \in \mathbb{C}$ ($|z - z_0| = R$). By Cauchy's inequality, there exists an upper bound M_R for $f(z)$ in this disk of radius R, such that the derivative is bounded by :

$$\boxed{|f'(z_0)| \leq \frac{M_R}{R}}$$

SINCE the disk is of arbitrary size R and center z_0, the inequality is also satisfied in the limit of $R \to \infty$ and for any z_0. But, since the absolute value of positive defined, it means that

$$\boxed{0 \leq |f'(z_0)| \leq 0 \to f'(z)_0 = 0.}$$

5.4 Applications: 2D Stationary flows (Optional material)

COMPLEX ANALYSIS IS A POWERFUL TOOL for solving problems in two dimensional problems in electrostatics, electromagnetism, hydrodynamics, elasticity, etc. Here, we present how 2D hydrodynamics can be formulated in terms of complex fields and potentials.

SUPPOSE we have a fluid flowing with a velocity $\vec{F} = [u(x,y), v(x,y)]$ in a domain in the (x,y) plane. We can represent the velocity vector field by a scalar complex field :

$$f(z) = u(x,y) + iv(x,y)$$

Definition 5.1 (Incompressible flow). A FLUID IS SAID TO BE INCOMPRESSIBLE, when the fluid density is constant throughout the whole domain. It follows then from the fluid mass conservation law

$$\partial_t \rho + \nabla \cdot (F\rho) = 0$$

that the velocity field $\vec{F}$ must be divergence free $\nabla \cdot F = 0$:

$$\nabla \cdot F = \partial_x u + \partial_y v = 0$$

Definition 5.2 (Irrotational flow). A FLOW IS IRROTATIONAL if the velocity field is curl free :

$$curl(\vec{F}) = \partial_x v - \partial_y u = 0$$

THE CURL measures the infinitesimal rotation of the vector field. This means that the flow has no infinitesimal rotational, so no vortices. Physically, it means that the flow is not spinning around any infinitesimal vortices.

WE QUICKLY RECOGNIZE that an incompressible and irrotational flow corresponds to the Cauchy-Riemann relations for the u and v. Thus, the complex velocity field $f(z) = u + iv$ is analytic everywhere in the complex plane. This also implies that regions in the complex plane where the flow may be compressible or rotational corresponds to where $f(z)$ ceases to be analytic. These regions may contain singular points which physically correspond to sinks or sources (of fluid mass) or vortices (rotating flows).

Hydrodynamic potential theory: 2D hydrodynamics has a neat complex representation in terms of a complex potential field. Similar complex representation can be also formulated for 2D electrostatics/electrodynamics fields. When flows are incompressible and irrotational, then the hydrodynamic potential is an analytic function as shown below.

AN INCOMPRESSIBLE FLOW corresponds to a potential flow, i.e. the flow velocity is determined as the gradient of a potential field $\phi(x, y)$

$$\vec{F} = \nabla\phi$$

and since $\nabla \cdot \vec{F} = 0$, this means that the potential field is a harmonic function $\nabla^2\phi = 0$.

AN IRROTATIONAL FLOW has $curl(\vec{F}) = 0$ and this is satisfied by a flow velocity represented in terms of a stream function ψ,

$$\vec{F} = curl(\psi).$$

THE STREAM FUNCTION is also a harmonic function $curl(curl(\psi)) = \nabla^2\psi = 0$. When the flow is both incompressible and irrotational, this means that we can define an analytic complex potential as :

$$\Phi = \phi + i\psi$$

such that the velocity field can be reconstructed either as the gradient of the potential real field :

$$u = \partial_x \phi, \qquad v = \partial_y \phi$$

or as the curl of the stream function :

$$u = \partial_y \psi, \qquad v = -\partial_x \psi$$

THE POTENTIAL FIELD ϕ and the stream function ψ are connected with each other through the Cauchy-Riemann conditions. This implies that the gradient of ϕ is orthogonal to the gradient of ψ. The derivatives of the complex potential is :

$$\Phi' = \partial_x \Phi = \partial_x \phi + i\partial_x \psi = \partial_x \phi - i\partial_y \phi$$

to be continued ....

Lecture 6: Taylor expansion

After getting familiar with some of the basic rules of differentiation and integration of complex functions, we now introduce how to represent complex functions as complex power series expansions. This is extremely useful as it represents a complex function in a form which can reveal the function's isolated singularities in a given integration domain. In this lecture, we focus on the Taylor expansion of analytic functions and apply it to few examples.

6.1 Taylor expansion

Theorem 6.1. *If $f(z)$ is analytic at z_0 then it can be represented as a complex power (Taylor) series as*

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n}, \tag{6.1}$$

where complex coefficients are uniquely defined by the derivatives of $f(z)$ evaluated at z_0,

$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint \frac{f(w)dw}{(w - z_0)^{n+1}}} \tag{6.2}$$

The complex function $f(z)$ has a ***unique*** *Taylor series expansion around a regular point z_0. The disk of convergence is the disk centered at z_0 and extending to the nearest singularity.*

Proof: First thing to show is that the Taylor series is convergent for any z in the region centered at z_0 where $f(z)$ is analytic.

For a compact region centered at z_0 and of radius R and where $f(z)$ is analytic, $f(z)$ is bounded, i.e. there exits an $M > 0$ such that

$$|f(z)| \leq M$$

Using the Cauchy's inequality: $|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \rightarrow |a_n| \leq \frac{M}{R^n}$.

HENCE,

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n \leq M\frac{r^n}{R^n}, \qquad r = |z-z_0|$$

WE RECOGNIZE the real power series $\sum_n (r/R)^n$ as being the geometric series which converges to $(1-\frac{r}{R})^{-1}$ for $r < R$. Since $r = |z-z_0|$ is the radius of a small region inside the analytic domain of size R, it follows that $r < R$. Thus, each term in the Taylor series is bounded by the corresponding term of the geometric series, which implies that the Taylor series is also absolutely convergent (textitsee figure 6.1)

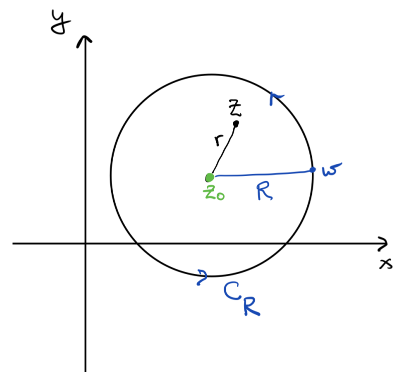


Figure 6.1: Disk of convergence for Taylor series

NEXT THING IS TO SHOW THAT the coefficients a_n of the power series are determined by the derivatives of $f(z)$. For this, we will use the connection to the geometric series. We start from the Cauchy's integral formula :

$$f(z) = \frac{1}{2\pi i}\oint_{C_R} \frac{f(w)}{w-z}dw$$

where the contour is the circle centered at z_0 and of radius R.

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C_R} \frac{f(w)}{(w-z_0)-(z-z_0)}dw \\
&= \frac{1}{2\pi i}\oint_{C_R} \frac{f(w)}{w-z_0}\left(1-\frac{z-z_0}{w-z_0}\right)^{-1} dw \\
&= \frac{1}{2\pi i}\oint_{C_R} \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw, \qquad \frac{|z-z_0|}{|w-z_0|} = \frac{r}{R} < 1 \\
&= \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\oint_{C_R} \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)(z_0)}}{n!}(z-z_0)^n
\end{aligned}$$

6.1.1 *Definitions: Zeros*

SUPPOSE $f(z)$ IS ANALYTIC on a disk $|z - z_0| < R$ such that it has a unique Taylor expansion around z_0.

A zero: If $f(z_0) = 0$, then z_0 is a *zero* of $f(z)$ when $f(z_0) = 0$ and not all the a_n are zero.

A zero of order n: z_0 is called *zero of order* n of $f(z)$ when $a_m = 0$ for $m < n$. Then, the Taylor expansion of $f(z)$ can be expressed as

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + +a_{n+2}(z - z_0)^{n+2} + \cdots \\ f(z) &= (z - z_0)^n \sum_{k=n}^{\infty} a_k(z - z_0)^{k-n}. \end{aligned}$$

ZEROS ARE ISOLATED. If $f(z)$ has more than one zero, we can always isolate them by enclosing each zero by a small disk that does not contain any other zeros.

Example 6.2. Find the Taylor series expansion of $f(z) = e^{2z}$ around a point z_0 in the complex plane.

Solution:
Since the exponential is analytic everywhere in the complex plane, it has a well-defined Taylor expansion around any point z_0. To determine the Taylor coefficients, we compute :

$$f^{(n)}(z) = \frac{d^n}{dz^n} e^{2z} = 2^n e^{2z}$$

HENCE,

$$e^{2z} = e^{2z_0} \sum_n \frac{2^n}{n!}(z - z_0)^n$$

Example 6.3. Find the Taylor expansion of $f(z) = \frac{\sin(z)}{z}$ at $z_0 = 0$.

Solution:
Apparently, there seems to be a problem at $z = 0$ because of the singular behavior of $1/z$. However, on a closer inspection, we may recognise that this is removed by the first order term in the Taylor expansion of $\sin(z)$ which is analytic at $z_0 = 0$. Thus, $\sin(z)/z$ is analytic and has a Taylor expansion at $z_0 = 0$. To check this let us first look at the he Taylor expansion of $\sin(z)$ at $z_0 = 0$. It's coefficients are:

$$\begin{aligned}
a_0 &= \sin(z)|_{z=0} = 0 && (6.3)\\
a_1 &= \cos(z)|_{z=0} = 1 && (6.4)\\
a_2 &= \frac{1}{2!}(-1)\sin(z)|_{z=0} = 0 && (6.5)\\
a_3 &= \frac{1}{3!}(-1)\cos(z)|_{z=0} = \frac{1}{3!}(-1) && (6.6)\\
&\cdots && (6.7)\\
a_{2k+1} &= \frac{1}{(2k+1)!}(-1)^k && (6.8)
\end{aligned}$$

Thus, its Taylor series is given as

$$\sin(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1}$$

from which it follows that

$$f(z) = \frac{\sin(z)}{z} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k}$$

Furthermore, we observe that the disk of convergence spans the whole complex plane.

Example 6.4. (*see figure 6.2*) Find the Taylor expansion of the principal value of the logarithm $f(z) = \ln(z)$ around the point $z_0 = 1$ where it is analytic.

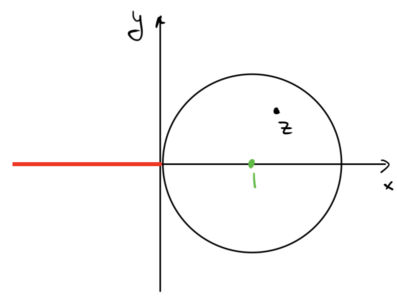


Figure 6.2: Disk of convergence for Taylor expansion of $\ln z$ around $z = 1$.

Solution:

RECALL that

$$\ln(z) = ln(re^{i\theta}) = \ln(r) + i\theta, -\pi < \theta \leq \pi.$$

WE NOTICE THAT $\ln(z)$ is not analytic at the origin because of the singularity in $\ln(r)$. This singular point is however not isolated, and this is because of the sudden jump in the phase along the real non-positive axis. The origin is called the **branch point** and the line of phase discontinuity is the **branch cut**. The branch point is uniquely defined, however the branch cut is not because the argument is multi-valued and we have the freedom to choose its principal value.

NOW, we are looking at a point on the positive real axis. Hence, the disk of convergence, being the circular disk touching the nearest singularity, will be $|z-1| < 1$. Using that $\frac{d\ln(z)}{dz} = \frac{1}{z}$:

$$f^{(n)}(z) = \frac{d^n}{dz^n}\ln(z) = (-1)^{n-1}(n-1)!\frac{1}{z^n}$$

HENCE,

$$\begin{aligned}
\ln(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n \\
&= \ln(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n
\end{aligned}$$

LET US SEE what happens if we differentiate once the series expansion of the logarithm. We get :

$$\begin{aligned}
\frac{d}{dz}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n &= \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}(-1)^{n-1}(1-z)^{n-1} = \sum_{n=0}^{\infty}(1-z)^n \\
&= \frac{1}{1-(z-1)} = \frac{1}{z}
\end{aligned}$$

valid for $|1-z| < 1 \to |z-1| < 1$. The series for the derivative of $\ln(z)$ has the same disk of convergence as the Taylor series for $\ln(z)$ itself.

An analytic function has derivatives that are also analytic ! If a function $f(z)$ is analytic at z_0, the Taylor series for $f'(z)$ around z_0 can be obtained by term-wise differentiation of the Taylor series for f around z_0. Both these Taylor expansions will have the same disk of convergence.

Lecture 7: Laurent expansion

A FUNCTION $f(z)$ may be represented by different series expansions at a given point z_0 depending on their convergence domains. The general representation of $f(z)$ as a power series with both positive and negative powers is called the Laurent expansion. As we will see in this lecture, the Laurent expansion at a point z_0 is determined by its convergence domain. In other words, the function $f(z)$ may have different Laurent expansions at z_0 for different domains occupied by z. This contrasts with the Taylor expansion at z_0 which is uniquely determined and corresponds to one convergence domain.

IN PROVING THE LAURENT'S THEOREM and in examples of Laurent expansions, we will use the Taylor expansion of the simple fraction:

$$\boxed{\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ for } |z| < 1.} \tag{7.1}$$

7.1 Laurent expansion

Theorem 7.1 (Laurent's theorem). *Let C_1 and C_2 be two circles of radius R_1 and $R_2 > R_1$, respectively and centered at some point z_0. When, $f(z)$ is analytic in an annulus $R_1 < |z - z_0| < R_2$, it can be expanded in a Laurent series:*

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}} \tag{7.2}$$

with the coefficients uniquely determined by $f(z)$ and the analytic domain where the function is expanded :

$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz} \tag{7.3}$$

*where C is an **arbitrary** closed contour inside the annulus, and encircling z_0 (does not have to be the same contour for both integrals).*

Regular part: $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges for $|z-z_0| < R_2$. This series is called the **analytic or regular part** of the Laurent series.*

Singular part: $\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ *is convergent for $|z-z_0| > R_1$. This series is called the **principal or singular part** of the Laurent series.*

BOTH *the regular part and the singular part of the Laurent series are convergent inside the annulus $R_1 < r < R_2$, where $f(z)$ is analytic.*

ALTERNATIVELY, *we can write the Laurent expansion in this compact form*

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n} \tag{7.4}$$

where

$$\boxed{c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz.} \tag{7.5}$$

Proof (see figure 7.1) :

The point z_0, around which $f(z)$ is expanded, could be anything, i.e. either regular or singular. However, $f(z)$ needs to be analytic in the domain where it is expanded to ensure that the Laurent series is convergent. This convergence domain is determined by the annulus $R_1 < |z-z_0| < R_2$. We encircle the point z by a contour joining C_2 and C_1 as illustrated in Fig. 7.1. Contour C_1 has counterclockwise (positive) orientation, while the contour C_2 has clockwise (negative) orientation and the paths $(\pm\Gamma)$ bridging these two contours have opposite orientation. Since $f(z)$ is analytic inside this closed path, we can use Cauchy's integral formula to represent $f(z)$ as:

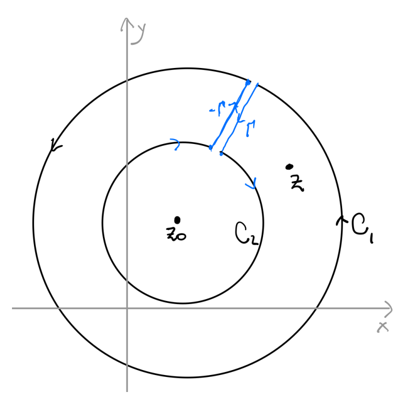


Figure 7.1: Contour circumventing z_0 and enclosing the region where we know $f(z)$ is analytic.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint\limits_{C} \frac{f(w)}{w-z} dw \\
&= \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(w)}{w-z} dw
\end{aligned}
$$

FOR THE FIRST INTEGRAL, we have that z is inside C_1 hence $|z - z_0| < |w - z_0| \to \left|\frac{z-z_0}{w-z_0}\right| < 1$. This implies that the integral over this contour becomes

$$
\begin{aligned}
\frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{(w-z_0)-(z-z_0)} dw \qquad (7.6) \\
&= \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{w-z_0} \frac{1}{1-\frac{z-z_0}{w-z_0}} dw \qquad (7.7)
\end{aligned}
$$

NOTICE that since $|z - z_0| < |w - z_0|$, the simple fraction can be expanded as a geometric series for every w on C_1, thus

$$
\begin{aligned}
\frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n dw \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(w) dw}{(w-z_0)^{n+1}} \right) (z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n. \qquad (7.8)
\end{aligned}
$$

WHAT WE HAVE SHOWN is that the first integral reduces to the regular series which is convergent inside C_1. Notice that the coefficients a_n can be related to the derivatives of $f(z)$ through the Cauchy's integral formula only when $f(z)$ is analytic at z_0!

NOW, for the second contour integral, z is outside the contour C_2 hence $|z - z_0| > |w - z_0| \to \left|\frac{w-z_0}{z-z_0}\right| < 1$. Thus, the contour integral can be rewritten as:

$$
\begin{aligned}
\frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(w)}{(w-z_0)-(z-z_0)} dw \\
&= -\frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(w)}{z-z_0} \frac{1}{1-\frac{w-z_0}{z-z_0}} dw. \qquad (7.9)
\end{aligned}
$$

We use that $|z-z_0| > |w-z_0|$ on this contour to expand the simple fraction as a geometric series as:

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}dw &= -\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n dw \\
&= -\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{(w-z_0)^{-n}}\right)\frac{1}{(z-z_0)^{n+1}} \\
&= -\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)dw}{(w-z_0)^{-n+1}}\right)\frac{1}{(z-z_0)^{n}} \\
&= -\sum_{n=1}^{\infty} b_n\frac{1}{(z-z_0)^n}. \qquad (7.10)
\end{aligned}$$

This second integral reduces to the singular part of the Laurent series which is convergent for z outside C_1 but inside C_2.

The domain where both series are convergent simultaneously corresponds to $R_2 < |z-z_0| < R_1$, and determines the convergence domain for the Laurent expansion at z_0. In summary,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C}\frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}dw \\
&= \sum_{n=0}^{\infty}a_n(z-z_0)^n + \sum_{n=1}^{\infty}b_n\frac{1}{(z-z_0)^n}
\end{aligned}$$

7.2 *Definitions: Poles and Residue*

Definition 7.1 (Regular point). If all the $b_n = 0$ for all n's, $f(z)$ is analytic at z_0, also called a *regular point*. Then, the domain where $f(z)$ is expanded can also include z_0.

Definition 7.2 (Simple Pole). If $b_1 \neq 0$ and $b_k = 0$, $k > 1$, then z_0 is called a *simple pole*

$$\frac{b_1}{z-z_0} + \sum_n a_n(z-z_0)^n$$

Definition 7.3 (Pole of order n:). If $b_n \neq 0$ for some n, but $b_k = 0$ for all $k > n$, then $f(z)$ is said to have a *pole of order n* at z_0.

$$\frac{b_1}{z-z_0} + \cdots \frac{b_n}{(z-z_0)^n} + \sum_n a_n(z-z_0)^n$$

Definition 7.4 (Residue:). When $f(z)$ is expanded at z_0 in a domain near z_0, then the coefficient b_1 of the Laurent expansion is called the *residue* of $f(z)$ at z_0: $Res(f, z_0) = b_1$. When z_0 is a regular point, the function is analytic at z_0 and is Taylor expanded in a small region centered at z_0. Thus, the residue at a regular point is zero. However, when z_0 is an isolated singularity, the Laurent expansion in a disk punctuated at z_0, will pick up singular terms and the residue can be non-zero. Next chapter focuses more on the residue of a function and various methods to find it. For now, we will exemplify how to determine the residue from the Laurent expansion near z_0.

7.3 *Examples:*

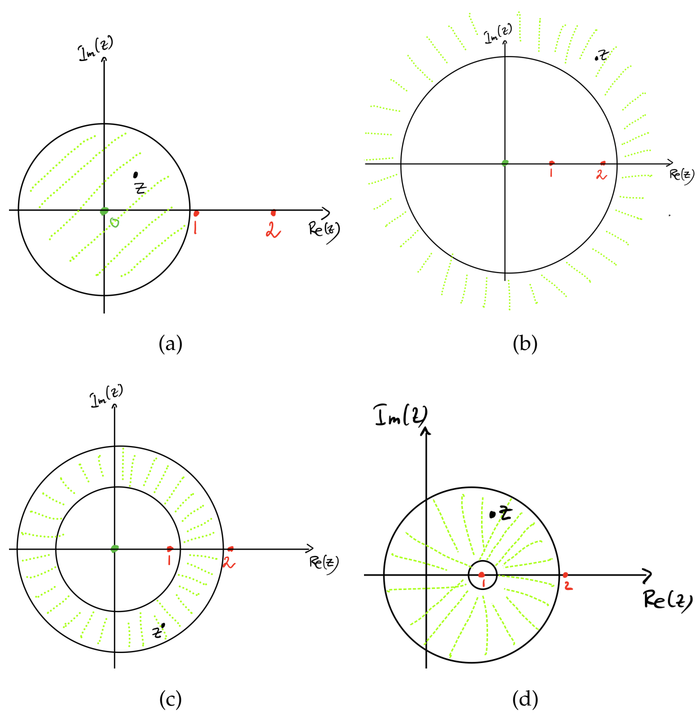


Figure 7.2: Expansion domains used Examples 7.2 (a), 7.3 (b), 7.4 (c), and 7.5 (d).

Example 7.2. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

at $z_0 = 0$ for $|z| < 1$. What is its residue at $z_0 = 0$?

Solution:

Method I: Determine the c_n coefficients WE SEEK to find the Laurent expansion around z_0 in this form:

$$f(z) = \frac{1}{(z-1)(z-2)} = \sum_{n=-\infty}^{\infty} c_n z^n \tag{7.11}$$

where

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz. \tag{7.12}$$

for a contour that is inside the disk $|z| < 1$. We notice that $f(z)$ is analytic inside the contour C.

Because $f(z)$ is analytic at $z_0 = 0$, we can relate c_n with derivatives of $f(z)$ for $n \geq 0$:

$$c_n = \frac{f^{(n)}(0)}{n!}, \qquad n \geq 0. \tag{7.13}$$

For $n < 0$, the integrand $f(z)z^{-n-1}$ is analytic and, by the Cauchy's theorem, it follows that the contour integral vanishes, hence

$$c_n = 0, \qquad n \leq -1. \tag{7.14}$$

Thus, the singular part of the Laurent series vanishes and we are left with the Taylor series expansion at the regular point $z_0 = 0$:

$$f(z) = \frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{7.15}$$

which is convergent for $|z| < 1$. Now we need to determine the derivatives:

$$\begin{aligned} \frac{d^n}{dz^n}\left(\frac{1}{(z-2)(z-1)}\right) &= \frac{d^n}{dz^n}\frac{1}{(z-2)} - \frac{d^n}{dz^n}\frac{1}{(z-1)} \\ &= \frac{(-1)^n n!}{(z-2)^{n+1}} - \frac{(-1)^n n!}{(z-1)^{n+1}} \end{aligned}$$

Hence,

$$f^{(n)}(0) = (-1)^{n-n-1} n! 2^{-n-1} - (-1)^{n-n-1} n! = n!(1-2^{-n-1})$$

such that the series expansion becomes

$$f(z) = \frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} (1-2^{-n-1}) z^n \tag{7.16}$$

Method II: Partial fraction decomposition WE START by using partial fraction decomposition to identify the isolated singularities:

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1},$$

which correspond to $z = 1$ and $z = 2$ (see Fig.7.2.). The disk with $|z| < 1$ does not contain these singular points, i.e. $f(z)$ is analytic inside it. We fulfill the conditions for a Taylor expansion, hence we are expecting that the Laurent series will contain only the regular part.

We expand the simple fractions at $z_0 = 0$ for their corresponding disks of convergence centered at $z_0 = 0$. For $(z-2)^{-1}$ this corresponds to $|z| < 2$:

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-z/2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \tag{7.17}$$

The disk of convergence $|z| < 2$ includes the unit disk.

SIMILARLY, for $(z-1)^{-1}$ we use the convergence disk $|z| < 1$:

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty}(z)^n\,. \tag{7.18}$$

Hence $f(z)$ is expanded as

$$\frac{1}{(z-2)(z-1)} = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n,$$

which coincides with the Taylor series. The residue at $z_0 = 0$ equals $b_1 = 0$, thus $Res(f, 0) = 0$.

This is the more trivial case when z_0 is a regular point *inside* the analytic domain for the expansion. This is the case when we obtain a Taylor expansion. Now, we will consider the same function and same $z_0 = 0$, but take different analytic regions for the expansion which do not include z_0. Then we will see that the Laurent expansion is different for different domains. However, for each domain there is the Laurent expansion is unique!

Example 7.3. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

around the origin $z_0 = 0$ for $|z| > 2$.

Solution:

In this case, the domain of expansion is detached from $z_0 = 0$ and corresponds to the outside of the disk $|z| > 2$ (see Fig.7.2). The geometric series used previously are not convergent in this domain. So, we need to adapt the simple fractions. This is done in the following way:

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{(1-1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}, \quad |z| > 1 \tag{7.19}$$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{(1-2/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}, \quad |z| > 2 \qquad (7.20)$$

Collecting the two series, we find that

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty}\frac{2^n - 1}{z^{n+1}}, |z| > 2$$

which contains only the singular part of Laurent expansion.

Example 7.4. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

around the origin $z_0 = 0$ for $1 < |z| < 2$.

Solution:

In this case, z is inside the annulus determined by the two concentric circles with center at the origin (see Fig.7.2). Eqs. 7.19 and 7.18 are the convergent geometric series for this domain, thus the Laurent expansion of $f(z)$ can be written as

$$\frac{1}{(z-1)(z-2)} = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}, 1 < |z| < 2$$

which contains both the regular and the singular parts of Laurent expansion.

Example 7.5. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

around the $z_0 = 1$ for $0 < |z-1| < 1$ (disk punctuated at $z_0 = 1$ and of radius smaller than 1).

Solution:

$1/(z-2)$ is analytic inside this domain including $z_0 = 1$, thus we can Taylor expand it at $z_0 = 1$. Hence, the Laurent expansion of $f(z)$ at $z_0 = 1$ is

$$\frac{1}{(z-1)(z-2)} = -\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{z-1}, \quad 0 < |z-1| < 1$$

The singular part contains a single term so $z_0 = 1$ is a simple pole and the residue $Res(f, 1) = -1$.

Example 7.6. Find the residue of $f(z) = \frac{e^z}{z-1}$ at $z_0 = 1$.

Solution:
We use that the exponential e^z is analytic to Taylor expand it at $z_0 = 1$ as

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n e^z}{dz^n} \right)_{z=1} (z-1)^n = \sum_{n=0}^{\infty} \frac{e^1}{n!} (z-1)^n$$

The function is analytic for $|z-1| > 0$ (disk punctuated at $z_0 = 1$ and of arbitrary radius) and can be expanded as

$$\begin{aligned} f(z) &= \frac{e}{z-1} \left(1 + (z-1) + \frac{1}{2!}(z-1)^2 + \frac{1}{3!}(z-1)^3 + \cdots \right) \\ &= \frac{e}{z-1} + e \left(1 + \frac{1}{2!}(z-1) + \frac{1}{3!}(z-1)^2 + \cdots \right) \\ &= \frac{e}{z-1} + e \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(z-1)^n \end{aligned}$$

The singular part has only the first term $b_1 = e$, hence $\text{Res}(1) = e$ and $z_0 = 1$ is a simple pole.

Example 7.7. Find the residue at $z_0 = 0$ of $f(z) = \frac{1}{z(z+1)}$.

Solution:
We use the partial fraction decomposition

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$$

$\frac{1}{z+1}$ is analytic at $z_0 = 0$ and for $|z| < 1$, we can Taylor expand it as the geometric series

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n$$

HENCE, for $0 < |z| < 1$, the function is expanded as

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \sum_{n=0}^{\infty} (-1)^n z^n$$

The singular part has only the first term $b_1 = 1$, hence $\text{Res}(0) = 1$.

Essential singularity: If $b_n \neq 0$ for any n, then z_0 is an *essential singularity*.

Behavior of a differentiable function near an essential singularity:
Suppose that $f(z)$ is differentiable for $|z| > R$. Then we can define

$$g(w) = f(1/z) \qquad |w| < 1/R.$$

Since $g'(w) = -z^{-2} f'(1/z)$, it follows that $f(1/z)$ is differentiable for $|z| < 1/R$. By definition, the behavior of $f(z)$ at infinity is the same the behavior $f(1/z)$ at the origin.

Example 7.8. Find the Laurent expansion of

$$f(1/z) = e^{1/z}$$

at $z_0 = 0$ for $|z| > 0$ (complex plane punctured at origin). Find its residue at $z_0 = 0$.

Solution:

We use the fact the Laurent expansion is unique for a given analytic domain. We first consider the simple exponential function $f(w) = w^z$ which is an analytic function everywhere in the complex plane. Its Taylor expansion at $w = 0$ is given by

$$f(w) = e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n.$$

For $|z| > 0$, the $e^{1/z}$ is analytic and has a unique expansion. Therefore, it can be obtained also by substituting $w = 1/z$ in the Taylor expansion,

$$e^{1/z} = f(1/z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

which is a singular series with all $b_n \neq 0$, thus $z_0 = 0$ is an essential singularity. Its residue at $z_0 = 0$ is $Res(e^{1/z}, 0) = 1$.

Example 7.9. Find the Laurent expansion of

$$f(1/z) = \cos(1/z)$$

at $z_0 = 0$ for $|z| > 0$ (complex plane punctured at origin).

Solution:

Again we use the Laurent expansion is unique for a given analytic domain. In this case, we consider $f(w) = \cos(w)$ which is analytic at Taylor expansion at $w = 0$ is given by

$$f(w) = \cos(w) = \sum_{n=0}^{\infty} a_n w^n,$$

where

$$a_n = \frac{1}{n!}\frac{d^n}{dw^n}\cos(w)\Big|_{w=0} \tag{7.21}$$

$$= \begin{cases} \frac{1}{(2k)!}(-1)^k, & n = 2k \\ 0, & n = 2k+1 \end{cases} \tag{7.22}$$

Thus,

$$f(w) = \cos(w) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k w^{2k},$$

For $|z| > 0$, the $\cos(1/z)$ is analytic and has a unique expansion. Therefore, it can be obtained by substituting $w = 1/z$ in the Taylor expansion,

$$f(1/z) = \cos(1/z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\frac{1}{z^{2k}}.$$

Lecture 8: Methods of finding the residue

8.1 *Finding the residue*

The residue of a $f(z)$ at an isolated singular point z_0 captures the "strength" of the singular behavior. Finding the residue of $f(z)$ is useful in evaluating complex integrals such as

$$\oint_{|z-\pi/2|=\pi/2} \tan(z) dz,$$

where isolated singularities are "baked" into the integrand function, and it is not straightforward to apply the Cauchy integral formula.

By the residue theorem, the contour integral of $f(z)$ is determined by the residues at all the isolated singularities enclosed by the contour. The residue is extracted from the Laurent expansion. In practice, however, we can often make use of a couple of short-cuts and rules to quickly find the residue without actually doing the Laurent expansion.

8.1.1 *Laurent expansion of $f(z)$:*

The Laurent expansion of $f(z)$ at z_0 and in an analytic domain surrounding

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}} \tag{8.1}$$

gives us the residue of $f(z)$ at z_0 defined as

$$\boxed{\mathbf{Res}(z_0) = b_1} \tag{8.2}$$

Example 8.1. Find the residue at $z_0 = \frac{1}{2}$ of $f(z) = \frac{\sin(\pi z)}{4z^2-1}$.

Solution:

To find the residue at $z_0 = 1/2$, we need to Laurent expand $f(z)$ in a region around z_0. The $\sin(\pi z)$ function is analytic at $z_0 = 1/2$.

Now let us decompose the fraction into simple fractions,

$$\frac{1}{4z^2-1} = \frac{1}{(2z-1)(2z+1)} = \frac{1}{2}\left(\frac{1}{2z-1} - \frac{1}{2z+1}\right)$$

$\frac{1}{2z+1}$ is analytic at $z_0 = 1/2$, hence it does not contribute to the residue. Thus, the residue of $f(z)$ at $z_0 = 1/2$ is determined by the residue of this function

$$\frac{1}{4}\frac{\sin(\pi z)}{z-1/2}$$

The $\sin(\pi z)$ has the Taylor expansion at $z_0 = 1/2$ given as:

$$\begin{aligned}\sin(\pi z) &= \sin\left(\frac{\pi}{2}\right) + \pi\cos\left(\frac{\pi}{2}\right)(z-\frac{1}{2}) - \frac{\pi^2}{2!}\sin\left(\frac{\pi}{2}\right)(z-\frac{1}{2})^2\cdots \\ &= 1 - \frac{\pi^2}{2!}(z-\frac{1}{2})^2 + \frac{\pi^4}{4!}(z-\frac{1}{2})^4 - \cdots\end{aligned}$$

By dividing it with $\frac{1}{4(z-1/2)}$, we find the corresponding Laurent expansions for $|z-1/2| > 0$:

$$\frac{1}{4}\frac{1}{z-1/2} - \frac{1}{4}\frac{\pi^2}{2!}(z-\frac{1}{2}) + \frac{1}{4}\frac{\pi^4}{4!}(z-\frac{1}{2})^3 - \cdots$$

Thus, $b_1 = 1/4$ and $\text{Res}(f(z), 1/2) = 1/4$.

8.1.2 *Simple pole rule*

Suppose z_0 is a simple pole of $f(z)$ such that we write $f(z)$ as

$$f(z) = \frac{g(z)}{z-z_0}$$

with $g(z)$ being analytic at z_0 and $g(z_0) \neq 0$. The residue at z_0 can be determined by

$$\boxed{\text{Res}(z_0) = \lim_{z\to z_0}(z-z_0)f(z) = g(z_0)} \tag{8.3}$$

Example 8.2. Find the residue at $z_0 = 0$ of $f(z) = \frac{\cos(z)}{z}$.

Solution:

First we notice that $zf(z) = g(z) = \cos(z)$ which is analytic at z_0 and cos(0)= 1. Hence, the residue is

$$\text{Res}(0) = \cos(0) = 1$$

Example 8.3. Find the residue at $z_0 = i$ of $f(z) = \frac{\sin(z)}{(1-z^4)}$.

Solution:

$$\frac{\sin(z)}{(1-z^4)} = \frac{\sin(z)}{(z-i)(z+i)(1-z^2)} = \frac{g(z)}{z-i},$$

where

$$g(z) = \frac{\sin(z)}{(z+i)(1-z^2)}$$

which is analytic at $z_0 = i$ and $g(i) = \sin(i)/(4i)$. Hence $z_0 = i$ is a simple pole,

$$f(z) = \frac{g(i)}{z-i} + \text{regular part}$$

Hence, the residue

$$\text{Res}(i) = g(i) = \frac{\sin(i)}{4i} = \frac{\sinh(1)}{4}$$

Theorem 8.4 (L'Hôpital rule for a simple pole:). *When*

$$\boxed{f(z) = \frac{g(z)}{h(z)}}$$

with $g(z)$ and $h(z)$ being analytic at z_0 and

$$g(z_0) \neq 0, \qquad h(z_0) = 0, \qquad h'(z_0) \neq 0,$$

then $f(z)$ has a simple pole at z_0 with the residue determined by the L'Hôpital rule:

$$\boxed{Res(z_0) = \frac{g(z_0)}{h'(z_0)}} \tag{8.4}$$

Proof:

$$\text{Res}(z_0) = \lim_{z\to z_0} \frac{(z-z_0)g(z)}{h(z)} = g(z_0) \lim_{z\to z_0} \frac{(z-z_0)}{h(z)} = g(z_0) \lim_{z\to z_0} \frac{1}{h'(z)} = \frac{g(z_0)}{h'(z_0)}$$

Alternative way to determine that $f(z)$ has a simple pole at z_0: By comparing the smallest powers in the Taylor expansions of $g(z)$ and $h(z)$ at z_0. When $h(z)$ has the leading order term **one** order higher than the leading order term in $g(z)$, then z_0 is a simple pole.

Example 8.5. Find the residue at $z_0 = 0$ of $f(z) = \frac{\cos(2z)}{\sin(2z)}$.

Solution:

The cos and sin functions are analytic. Furthermore,

$$\cos(0) = 1, \qquad \sin(0) = 0, \qquad \frac{d\sin(2z)}{dz}|_{z=0} = 2\cos(0) = 2.$$

Hence, z_0 is a simple pole and the residue is

$$\text{Res}(0) = \frac{\cos(0)}{\cos(0)} = \frac{1}{2}$$

8.1.3 *Multiple pole rule:*

If z_0 is a pole of order n of $f(z)$, then the residue at z_0 is

$$\boxed{\text{Res}(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\big|_{z=z_0}, \qquad m \geq n} \tag{8.5}$$

Proof: Let us use the Laurent expansion for $f(z)$ at z_0 which is a pole of order n. This means that the singular part is a finite sum with n terms:

$$\frac{b_1}{z-z_0} + \cdots + \frac{b_n}{(z-z_0)^n} + \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

Then,

$$(z-z_0)^m f(z) = b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \cdots + b_n(z-z_0)^{m-n} + \sum_{k=0}^{\infty} a_k(z-z_0)^{k+m}$$

By taking the $(m-1)$ derivative of this and evaluate it at z_0, we can determine an expression for the b_1 coefficient

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\Big|_{z=z_0} = (m-1)!b_1$$

and thus determine the residue

$$\text{Res}(z_0) = b_1 = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\Big|_{z=z_0}$$

Example 8.6. Find the residue at $z_0 = \pi$ of $f(z) = \frac{z\sin z}{(z-\pi)^3}$.

Solution:

The nominator $z\sin(z)$ is analytic, hence $z_0 = \pi$ is a pole of order of at most 3. So, we can use $m = 3$ and apply the rule above

$$\begin{aligned}\text{Res}(\pi) &= \frac{1}{2!}\frac{d^2}{dz^2}[z\sin z]\big|_{z=\pi} \\ &= \frac{1}{2}\frac{d}{dz}[\sin z + z\cos z]\big|_{z=\pi} \\ &= \frac{1}{2}[2\cos z - z\sin z]\big|_{z=\pi} \\ &= -1\end{aligned}$$

Example 8.7 (Proposed by you in the class)**.** Find the residue at $z_0 = 0$ of

$$f(z) = \frac{\cos(z)}{z^4}$$

Solution:

$\cos(z)$ function is analytic at $z_0 = 0$ and has the Taylor expansion

$$\cos(z) = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 \cdots$$

Thus,

$$f(z) = \frac{\cos(z)}{z^4} = \frac{1}{z^4} - \frac{1}{2z^2} + \text{ regular terms}$$

We notice something very interesting here: the singular part has the highest order negative power given by $n = 4$ which means that we are dealing with pole of order 4, but due to the symmetry of the function, the first term vanishes and therefore $b_1 = 0$ which implies that $Res(0) = 0$ (as if this were a regular point). So, what is happening here? This a great example of an isolated singularity which does not contribute to the contour integral, because the simple fractions

$$\frac{1}{z^2}, \quad \frac{1}{z^4},$$

which, even though clearly singular at $z_0 = 0$, give no contribution to the contour integrals, i.e.

$$\oint_{|z|=R} \frac{dz}{z^2} = 0$$

and

$$\oint_{|z|=R} \frac{dz}{z^4} = 0$$

by virtue of the generalised Cauchy's line formula.

8.2 Residue theorem

THE RESIDUE THEOREM is very useful in evaluating complex integrals as well as many definite real integrals.

Theorem 8.8. *Let z_0 be an isolated singular point of $f(z)$ and C be a positive-oriented simple contour* ***enclosing*** *z_0* ***and no other singularities****. Then, the integral of $f(z)$ on C is determined by the residue of $f(z)$ at z_0:*

$$\boxed{\oint_C f(z)dz = 2\pi i Res(f, z_0),} \tag{8.6}$$

with $Res(z_0) = b_1$ is the residue of $f(z)$ at z_0 determined by the Laurent expansion of $f(z)$ at z_0 for $z \neq 0$ inside C.

Proof: For any $z \neq z_0$ inside C, $f(z)$ is analytic, hence it has a well-defined Laurent expansion. Now, the regular part of the Laurent series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic everywhere including at z_0. Hence by the Cauchy's theorem, its contour integral vanishes:

$$\sum_{n=0}^{\infty} a_n \oint_C (z - z_0)^n dz = 0.$$

This means that the contour integral of $f(z)$ is non-zero because it picks up a net contribution from the singular part of the Laurent series :

$$\oint_C f(z)dz = \sum_{n=1}^{\infty} b_n \oint_C \frac{1}{(z - z_0)^n} dz$$

It turns out that only the first term survives this integral while all the higher order terms vanish upon integration. From the Cauchy's integral formula, the contour integral of the first term reduces to the b_1 coefficient

$$b_1 \oint_C \frac{1}{(z - z_0)} dz = 2\pi i b_1,$$

and therefore it is the *residue*, as the only non-zero contribution to the contour integral. All the higher order terms, vanish by the generalized Cauchy's formula :

$$\oint_C \frac{dz}{(z - z_0)^{n+1}} = 0$$

HENCE, the contour integral of $f(z)$ is determined by its residue at z_0 as

$$\oint_C f(z)dz = 2\pi i b_1$$

Theorem 8.9 (Residue theorem II). *Let $z_0, z_1, z_2, \cdots$ be isolated singularities of $f(z)$. Then, the integral of $f(z)$ around a simple closed curve C traversed counterclockwise and surrounding all singular points is determined by:*

$$\boxed{\oint_C f(z)dz = 2\pi i \sum_k Res(z_k).} \tag{8.7}$$

Proof: We draw a small circle around each isolated singularity and connect them through cuts with the main closed curve. By Cauchy's theorem:

$$\oint_{C-C_0-C_1-C_2-\cdots} f(z)dz = 0$$

which implies that the integral over the main contour is determined by the sum of the integrals over the inner, small circles enclosing the singularities

$$\oint_C f(z)dz = \sum_k \oint_{C_k} f(z)dz.$$

Since inside each circle we have only one isolated singularity z_i, the contour integral with z_k inside is given by the b_1 of the Laurent series expansion of $f(z)$ at z_k, i.e. the residue of $f(z)$ at z_k, $\text{Res}(z_k)$. Hence,

$$\oint_C f(z)dz = 2\pi i \sum_k \text{Res}(z_k)$$

Example 8.10. Use the residue theorem to evaluate this integral

$$I = \oint_{|z|=4} \frac{z+2}{z^2+9} dz.$$

Solution:

The integrand function:

$$f(z) = \frac{z+2}{z^2+9} = \frac{z+2}{(z+3i)(z-3i)}$$

has two simple poles at $z_0 = \pm 3i$ which are enclosed by the disk of radius 4. Hence, the integral is determined by the residue of $f(z)$ at these poles.

$$\oint\limits_{|z|=4} \frac{z+2}{z^2+9} dz = 2\pi i [\mathrm{Res}(3i) + \mathrm{Res}(-3i)]$$

The residues can be quickly determined in this case by the simple pole rule:

$$\mathrm{Res}(3i) = \lim_{z \to 3i} (z-3i) f(z) = \frac{3i+2}{6i} = \frac{1}{2} - i\frac{1}{3}$$

$$\mathrm{Res}(-3i) = \lim_{z \to 3i} (z+3i) f(z) = \frac{-3i+2}{-6i} = \frac{1}{2} + i\frac{1}{3}$$

HENCE, $I = 2\pi i$

Example 8.11 (Exercise (14.31) Boas:)**.** Evaluate the integral

$$I = \oint\limits_{|z|=1} \frac{e^{3z} - 3z - 1}{z^4} dz$$

Solution:

We notice that the integrand has one pole of order 4 at $z_0 = 0$. Thus, by the residue theorem,

$$\oint\limits_{|z|=1} \frac{e^{3z} - 3z - 1}{z^4} dz = 2\pi i \mathrm{Res}(0).$$

We observe that $z_0 = 0$ is a pole of order 4. We determine the the residue at $z_0 = 0$ using the multiple pole rule for $m = 4$:

$$\begin{aligned} \mathrm{Res}(0) &= \frac{1}{3!} \frac{d^3}{dz^3} [e^{3z} - 3z - 1] \Big|_{z=0} \\ &= \frac{1}{3!} [3^3 e^{3z}] \Big|_{z=0} \\ &= \frac{9}{2} \end{aligned}$$

Hence, $I = 9\pi i$. Equivalently, we could use generalized Cauchy's integral formula for $f(z) = e^{3z} - 3z - 1$ which is analytic inside the unit disk:

$$\oint\limits_{|z|=1} \frac{e^{3z} - 3z - 1}{z^4} dz = 2\pi i \frac{1}{3!} \left(\frac{d^3}{d^3} [e^{3z} - 3z - 1] \right)_{z=0}$$

Example 8.12. Evaluate the integral

$$\oint_{|z-\pi/2|=\pi/2} \tan(z)dz,$$

Solution:

We notice that $\tan(z) = \frac{\sin z}{\cos z}$ is analytic inside the disk $|z - \pi/2| = \pi/2$ except for $z_0 = \pi/2$. Thus, the residue theorem,

$$\oint_{|z-\pi/2|=\pi/2} \tan(z)dz = 2\pi i Res\left(\tan(z), z = \frac{\pi}{2}\right)$$

To determine the order of this pole, we Taylor expand to lowers order sin and cos functions as

$$\sin(z) = 1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \cdots,$$

$$\cos(z) = -\left(z - \frac{\pi}{2}\right) + \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^3 + \cdots,$$

from which we see that it is a simple pole. Thus, we can use the L'Hopital rule to determine the residue

$$Res\left(\tan(z), z = \frac{\pi}{2}\right) = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

Lecture 9: Definite integrals

Complex analysis provides us with a powerful tool to evaluate definite integrals, which otherwise are more challenging to solve within real integral calculus. In this lecture, we will go through several techniques of evaluating different types of definite integrals by mapping them to complex contour integrals, which are evaluated by the residue theorem. The different types of integrals are distinguished by the properties of the integrand function.

9.1 *Definite integral with trigonometric integrand*

For an integral of this form

$$I = \int_0^{2\pi} f(\theta)d\theta$$

it is useful to use the mapping to the unit circle $z = e^{i\theta}$ in the complex plane. Namely,

$$z = e^{i\theta} \to dz = ie^{i\theta}d\theta \to d\theta = \frac{dz}{iz},$$

such that the real integral maps into a complex contour integral over the unit circle $|z| = 1$,

$$\boxed{\int_0^{2\pi} f(\theta)d\theta = \oint_{|z|=1} \frac{f(z)}{iz}dz} \tag{9.1}$$

which can be evaluated by the residue theorem for $f(z)/(iz)$ is analytic inside the unit disk except for a finite number of poles inside it, none on the circle.

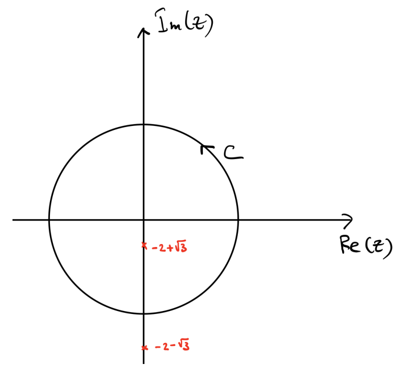


Figure 9.1: Illustration for Example 9.1

Example 9.1. Evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{2+\sin(\theta)}.$$

Solution:

We use the transformation to the complex exponential:

$$z = e^{i\theta} \to d\theta = \frac{dz}{iz}.$$

The complex representation of $\sin(\theta)$ follows from the Euler's formula:

$$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

Hence, the integral becomes:

$$\begin{aligned} I &= \oint\limits_{|z|=1} \frac{dz}{iz}\frac{2i}{4i+(z-1/z)} \\ &= \oint\limits_{|z|=1} \frac{dz}{iz}\frac{2iz}{z^2+4iz-1} \\ &= \oint\limits_{|z|=1} dz\frac{2}{(z+i(2+\sqrt{3}))(z+i(2-\sqrt{3}))}. \end{aligned}$$

We recognise that the integrand has two simple poles at $z_0 = \pm i(2-\sqrt{3})$, but only one of them is inside the unit circle, as shown in Fig. 9.1. Thus, by the residue theorem, we can evaluate this integral terms of the residue at $z_0 = -i(2-\sqrt{3})$

$$I = 2\pi i \text{Res}\left(z_0 = -i(2-\sqrt{3})\right),$$

where

$$\text{Res}\left(-i(2-\sqrt{3})\right) = \lim_{z\to z_0}\frac{2}{(z+i(2+\sqrt{3}))} = \frac{2}{2i\sqrt{3}}$$

thus the definite integral is equal to

$$I = \frac{2\pi}{\sqrt{3}}$$

9.2 Definite integral with rational integrand

Let us consider an integral of this generic form

$$I = \int_{-\infty}^{\infty} f(x)dx,$$

where the integrand function extended to the complex plane satisfied this two conditions:

1. $f(z)$ is analytic in the complex plane except for a finite number of poles, none on the real axis
2. $f(z)$ is upper-bounded on the semi-circle Γ_ρ in one of the half-planes as

$$|f(z)| \leq \frac{A}{\rho^2}, \text{ for } |z| = \rho \text{ and } A \text{ a constant}$$

Then this integral can be equal to the contour integral over one the half-planes

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \oint_C f(z)dz = 2\pi i \sum_k \text{Res}\,(f(z), z_k)} \tag{9.2}$$

where z_k are the poles of $f(z)$ inside the contour. In most examples this applies for a rational integrand

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, such that $deg(P) \leq deg(Q) + 2$ and $Q(x)$ has no zeros on the real axis.

Example 9.2. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

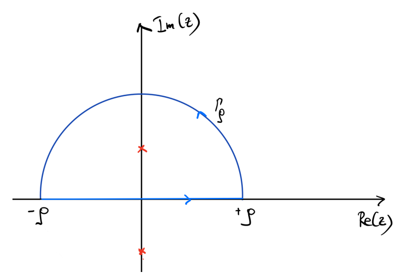


Figure 9.2: Illustration for Example 9.2

Solution:
Let us examine the contour integral in the upper half-plane:

$$I_C = \oint\limits_C \frac{dz}{1+z^2} = \oint\limits_C \frac{dz}{(z+i)(z-i)},$$

where C is the closed loop with the upper semicircle Γ_ρ as shown in Figure 9.2. The integrand has two simple poles but only $z_0 = i$ is enclosed by C. The residue of the integrand at this point is

$$\text{Res}(i) = \frac{1}{(z+i)}\bigg|_{z=0} = \frac{1}{2i}.$$

Hence, by residue theorem

$$I_C = 2\pi i \text{Res}(i) = \pi.$$

Now, we want to relate I_C to the real integral I. For this, we decompose the contour into the integral over the real axis and the integral over the semicircle Γ_ρ:

$$I_C = \pi = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_{\Gamma_\rho} \frac{dz}{1+z^2}$$

such that the integral can be expressed as

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi - \lim_{\rho\to\infty} \int_{\Gamma_\rho} \frac{dz}{1+z^2}$$

We want to show that the line integral over semicircle vanishes in the limit of $\rho \to \infty$. We use the Cauchy's inequality to show that the upper bound of the integral vanishes in the limit of an infinite radius. The integrand $f(z) = 1/(1+z^2)$ is upper bounded on Γ_ρ by:

$$|f(z)| \leq \frac{1}{|1+z^2|} \leq \frac{1}{|z|^2 - 1} = \frac{1}{\rho^2 - 1}$$

HENCE, by Cauchy's inequality,

$$\left| \int_{\Gamma_\rho} f(z)dz \right| \leq \pi\rho|f(z)| \leq \frac{\pi\rho}{\rho^2 - 1},$$

and in the limit $\rho \to \infty$, the upper bound vanishes

$$\lim_{\rho\to\infty} \left| \int_{\Gamma_\rho} f(z)dz \right| \leq \lim_{\rho\to\infty} \frac{\pi\rho}{\rho^2 - 1} = 0$$

HENCE, $I = \pi$. Now, in practice, it suffices to check that the degree of polynomial in the denominator is higher then that of the nominator. In this example, $deg(Q) = 2$ and $deg(P) = 0$.

9.3 *Jordan's lemma*

An integral of this generic form

$$I = \int_{-\infty}^{\infty} f(x)e^{ix}dx,$$

where $f(z)$ satisfies the following conditions in the complex plane:

1. $f(z)$ is analytic in the upper-half plane except for a finite number of poles, none on the real axis.

2. $f(z)$ is upped-bounded on the semicircle Γ_ρ as

$$|f(z)| \leq \frac{A}{\rho} \quad \text{for } |z| = \rho \text{ and A is a constant,}$$

can be extended to into the *upper-half plane* and equated to the contour integral

$$\boxed{\int_{-\infty}^{+\infty} f(x)e^{ix}dx = \oint_C f(z)e^{iz}dz = 2\pi i \sum_k \text{Res}\left(f(z)e^{iz}, z_k\right)} \tag{9.3}$$

where z_k are the poles of $f(z)$ in the upper-half plane.

This can be generalized to any integer power m of the complex exponential

$$\boxed{\int_{-\infty}^{+\infty} f(x)e^{imx}dx = \oint_C f(z)e^{imz}dz = 2\pi i \sum_k \text{Res}\left(f(z)e^{imz}, z_k\right)} \tag{9.4}$$

where C is in the *upper-half* plane for $m > 0$ the contour and C is in the *lower-half* plane for $m < 0$ the contour. The half-plane is selected such that the corresponding line integral over the semi-circle vanishes in the limits of infinite radius. This is related to the behavior of the complex exponential in the plane

$$|e^{imz}| = |e^{imx}||e^{-my}| = e^{-my},$$

and becomes clear by examples.

In most practical cases, $f(x)$ is on the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, such that $deg(P) \leq deg(Q) + 2$ and $Q(x)$ has no zeros on the real axis.

Example 9.3. Evaluate the following integral:

$$I = \int_{-\infty}^{+\infty} \frac{x \sin x}{1+x^2} dx$$

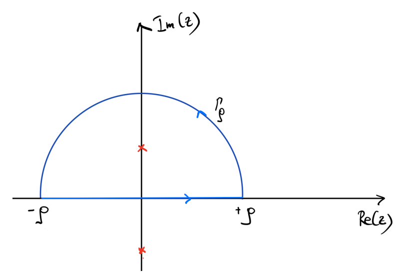


Figure 9.3: Illustration for example 9.4

Solution:

First, we rewrite this as:

$$I = \frac{1}{2i}\left[\int_{-\infty}^{+\infty} \frac{xe^{ix}}{1+x^2}dx - \int_{-\infty}^{+\infty} \frac{xe^{-ix}}{1+x^2}dx\right] = \text{Im}\left(\int_{-\infty}^{+\infty} \frac{x}{1+x^2}e^{ix}dx\right)$$

Using that

$$|e^{iz}| = e^{-y}$$

we select the upper-half plane $y > 0$, where the exponential is decaying! Let us now evaluate the corresponding contour integral in this *upper-half* plane

$$\begin{aligned} I_{C_\rho} &= \oint_{C_\rho} \frac{ze^{iz}}{(z+i)(z-i)} dz \\ &= 2\pi i \text{Res}\left(\frac{ze^{iz}}{(z+i)(z-i)}, i\right) \\ &= \frac{i\pi}{e}, \end{aligned}$$

where C_ρ contains the semicircle on the positive half-plane as shown in Fig. 9.3. The integral over semi-circle vanishes as its upper bound vanishes in the limit of infinite radius by virtue of the Cauchy's inequality:

$$\left| \int_{\Gamma_\rho} \frac{ze^{iz}}{z^2+1} dz \right| \leq \frac{\pi \rho^2 e^{-\rho}}{\rho^2 - 1} \xrightarrow[\rho \to \infty]{} 0.$$

Hence,

$$I = \text{Im}\left(\frac{i\pi}{e}\right) = \frac{\pi}{e}$$

9.4 Singular integrals

9.4.1 Cauchy's principal value

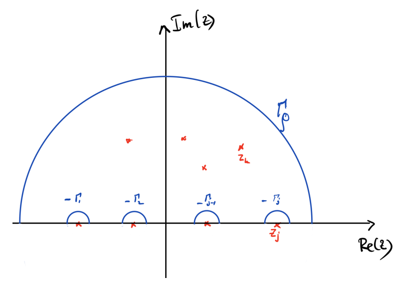


Figure 9.4: Contour circumventing the simple poles on the real axis

THE PRINCIPAL VALUE METHOD applies to improper integrals that have a singular behavior around isolated points. For an integral

$$\int_{-\infty}^{\infty} f(x) dx$$

which has a singularity at x_0, we can extract the non-singular value of this integral by *the principal value (P.V.)* integral defined as

$$P.V. \int_{-\infty}^{\infty} f(x) dx = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{x_0 - \epsilon} f(x) dx + \int_{x_0 + \epsilon}^{\infty} f(x) dx \right].$$

This principal value integral can be determined by complex integration. The idea is to introduce the contour integral in the half-plane as illustrated in Fig. 9.4 to isolate the singularities located at x_j on the real axis. The contour integral can be decomposed into

$$\oint_C f(z) dz = P.V. \int_{-\infty}^{+\infty} f(x) dx - \lim_{\epsilon \to 0} \sum_j \int_{\Gamma_{\epsilon,j}} f(z) dz + \int_{\Gamma_\rho} f(z) dz,$$

where the minus sign on the lhs accounts for the fact the small semi-circles $\Gamma_{\epsilon,j}$ (centered at the poles x_j and of radius ϵ) have a negative orientation (clockwise). Now, the line integral over the large semi-circle Γ_ρ vanishes in the limit of infinite radius since $f(z)$ decays sufficiently fast (the same argument as in the other classes of integrand functions). This works both when have an integrand of the form $f(z) = \frac{P(z)}{Q(z)}$ where $deg(Q) = deg(P) + 2$ or $f(z) = \frac{P(z)}{Q(z)}e^{imz}$, in which case we choose the half-plane for which the complex exponential decays (Jordan's lemma). Either way we have that

$$\left| \int\limits_{\Gamma_\rho} f(z)dz \right| \to 0 \text{ as } \rho \to \infty$$

Thus,

$$\boxed{P.V. \int_{-\infty}^{+\infty} f(x)dx = \oint_C f(z)dz + \lim_{\epsilon \to 0} \sum_j \int\limits_{\Gamma_{\epsilon,j}} f(z)dz.} \tag{9.5}$$

The function $f(z)$ might have additional poles z_k in the complex plane away from the real axis which will be picked up the contour integral over C. By the residue theorem, this integral reduces to

$$\oint\limits_C f(z)dz = 2\pi i \sum_k \text{Res}(f(z), z_k).$$

When the isolated singularities are simple **simple** poles x_j on the real axis, then we can also evaluate the line integrals over the small semi-circles in terms of the residues at x_j. To show this, we use the line parameterization on the small-semicircle centered at x_j given by $z = x_j + \epsilon e^{i\theta}$, such the

$$\lim_{\epsilon \to 0} \int\limits_{\Gamma_{\epsilon,j}} f(z)dz \quad = \quad \lim_{\epsilon \to 0} \int\limits_0^\pi f(x_j + \epsilon e^{i\theta}) i\epsilon e^{i\theta} d\theta$$

We are stuck here unless we make some assumptions about the structure of $f(z)$ near the poles x_j. It is here where we use that the x_j are **simple** poles to write the function on this semi-circle as

$$f(z) = \frac{g(z)}{z - x_j}$$

where $g(z)$ is analytic at x_j. Inserting this into the line integral from above, we have that

$$\begin{aligned}
\lim_{\epsilon\to 0}\int\limits_{\Gamma_{\epsilon,j}} f(z)dz &= \lim_{\epsilon\to 0}\int\limits_{0}^{\pi}\frac{g(x_j+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta \\
&= ig(x_j)\int\limits_0^{\pi} d\theta \\
&= i\pi g(x_j) = i\pi \mathrm{Res}(f(z), x_j),
\end{aligned}$$

where we used that $g(z)$ is analytic at x_j thus the limit of zero ϵ is well-defined and equal to the value $g(x_j)$ which also determines the residue of $f(z)$ at x_j (the simple pole rule). Notice that the poles on the real axis contribute with a **factor** πi instead of $2\pi i$ because they are surrounded by a semi-circle!

Finally, collecting all these residues, we can express the P.V. integral as

$$\boxed{P.V.\int_{-\infty}^{+\infty} f(x)dx = 2\pi i\sum_k \mathrm{Res}(f(z), z_k) + \pi i\sum_j \mathrm{Res}(f(z), x_j)} \tag{9.6}$$

Let us take two examples to illustrate how to apply the Cauchy principal value method in practice. One in which the integrand is on the form $f(z) = \frac{P(z)}{Q}$ and the other on the form $f(z) = \frac{P(z)}{Q}e^{imz}$.

Example 9.4. Evaluate this integral

$$I = P.V.\int_{-\infty}^{+\infty}\frac{1}{(x+1)(x^2+4)}dx$$

Solution:

We notice that the integrand as an simple pole on the real axis $z_0 = -1$. Additionally, there are two simple poles in the imaginary axis at $\pm i4$

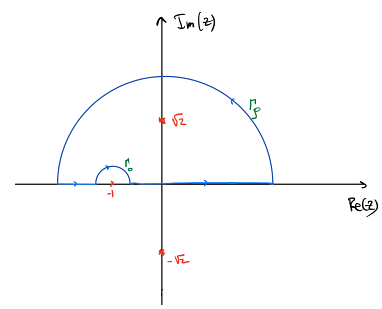


Figure 9.5: Contour circumventing the singular point at $z_0 = -1$.

$$I = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-1-\epsilon}\frac{1}{(x+1)(x^2+4)}dx + \int_{-1+\epsilon}^{+\infty}\frac{1}{(x+1)(x^2+4)}dx\right]$$

In Fig. 9.5, we illustrate the contour in the upper half-plane. The complex integral over the outer half-circle Γ_ρ vanishes in the limit of an infinite radius since the integrand is rational

$$\frac{P(x)}{Q(x)} = \frac{1}{(x+1)(x^2+4)},$$

and that $\deg(P) = 0$, while $\deg(Q) = 3$ (same argument as for integrals with rational integrands).

Thus, the principal value is determined by the contour integral in the half-plane and the semicircle of radius ϵ centered at the pole.

$$I = \oint_C \frac{1}{(z+1)(z^2+4)} dz + \lim_{\epsilon \to 0} \int_{\Gamma_{\epsilon,-1}} \frac{1}{(z+1)(z^2+4)} dz,$$

where $\Gamma_{\epsilon,-1}$ is the semi-circle of radius ϵ centered at -1. Using the residue at simple pole -1 and $2i$ in the upper-half plane, we then evaluate the principal value as:

$$I = 2\pi i \text{Res}(2i) + \pi i \text{Res}(-1) = \frac{\pi(1-2i)}{10} + i\frac{\pi}{5} = \frac{\pi}{10}$$

We can also show that the line integral over the semi-circle gives this by using the parameterization $z - 1 = \epsilon e^{i\theta}$:

$$\begin{aligned} \int_{\Gamma_{\epsilon,-1}} \frac{1}{(z+1)(z^2+4)} dz &= \int_0^\pi d\theta i\epsilon e^{i\theta} \frac{1}{\epsilon e^{i\theta}((\epsilon e^{i\theta}-1)^2+4)} \\ &=_{\epsilon \to 0} \quad i\frac{\pi}{5} = i\pi \text{Res}(-1). \end{aligned}$$

Example 9.5. Evaluate this integral

$$I = P.V. \int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{x+1} dx$$

Solution:

Let us rewrite the integral as

$$I = P.V. \int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{x+1} dx = Im \left[P.V. \int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x+1} dx \right]$$

By Jordan's lemma, the function $f(z) = \frac{e^{i\pi z}}{z+1}$ is decaying in the upper half-plane $y > 0$, thus we take the contour C_+ in this domain. By the Cauchy principal value method, the integral reduces to

$$P.V. \int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x+1} dx = \oint_{C^+} \frac{e^{i\pi z}}{z+1} dz + \lim_{\epsilon \to 0} \int_{\Gamma_{\epsilon,-1}} \frac{e^{i\pi z}}{z+1} dz.$$

The $f(z)$ is analytic in the half-plane enclosed by C_+, thus by the Cauchy's theorem, this contour integral vanishes. The integral over the semi-circle centered at the simple pole -1 reduces to the residue at -1, thus

$$\begin{aligned} P.V. \int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{x+1} dx &= i\pi Res\left(\frac{e^{i\pi z}}{z+1}, -1\right) \\ &= i\pi e^{-i\pi} = i\pi \left[\cos(\pi) - i\sin(\pi)\right] \\ &= -i\pi. \end{aligned}$$

Thus the solution to the integral is $I = -\pi$.

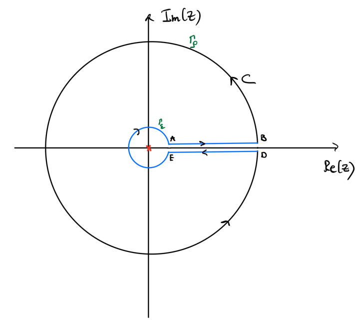


Figure 9.6: Key-hole contour

9.4.2 *Improper integral with branch cuts:*

We consider one example on how to solve improper integrals that contain branch cuts on the real axis. The technique is to integrate around a keyhole loop.

Example 9.6. Evaluate the principal value of this integral :

$$I = P.V. \int_0^\infty \frac{1}{(x+1)\sqrt{x}} dx$$

Solution:

The $f(z) = \sqrt{z}$- function is a multi-valued function, i.e. it can take multiple values at the same point. To see this, let's take $z = re^{i\theta}$, with $\theta \in [0, 2\pi]$, then $z^{1/2} = e^{\frac{1}{2}\ln z} = \sqrt{r}e^{i\theta/2}$. Then, for any θ, if we apply a 2π-rotation and evaluate the function, we see that

$$\sqrt{z} = \begin{cases} \sqrt{r}e^{i\theta/2} \\ \sqrt{r}e^{i(\theta+2\pi)/2} = -\sqrt{r}e^{i\theta/2} \end{cases}$$

This line of discontinuity is called a branch cut. There are infinitely many branch cuts and all start from the same branch point at $z = 0$. We choose which branch cut we want to work with by defining the principal value of θ. For our purpose, we place the branch cut on non-negative real axis, $x \geq 0$. This implies that $\theta \in (0, 2\pi)$. For $\theta \in (-\pi, \pi)$, the branch cut is on the non-positive real axis $x \leq 0$.

To avoid crossing the branch-cut, we construct the contour as a keyhole loop (see Fig.9.6), so that we can evaluate the contour integral through the residue theorem :

$$I_C = \oint_C \frac{1}{\sqrt{z}(z+1)} dz = 2\pi i \text{Res}(-1) = \frac{2\pi i}{\sqrt{-1}} = 2\pi$$

$$\oint_\Gamma \frac{1}{\sqrt{z}(z+1)} dz = \left[\int_{\Gamma_\rho} + \int_{\Gamma_\epsilon} + \int_{AB} + \int_{DE} \right] \frac{1}{\sqrt{z}(z+1)} dz$$

The integrals over Γ_ρ and Γ_ϵ vanish. By changing to complex variable :

$$z = re^{i\theta} \to dz = ie^{i\theta} d\theta$$

we can show that the integrand vanishes both in the limit of $r \to 0$ and $r \to \infty$:

$$\int_\Gamma \frac{1}{\sqrt{z}(z+1)} dz = i \int_\Gamma \frac{\sqrt{r}e^{i\theta/2} d\theta}{1 + re^{i\theta}} \to 0$$

We are left with the integrals above and below the branch cut.

on AB: we set $\theta = 0$, hence $z = r$:

$$\int_0^\infty \frac{1}{\sqrt{r}(r+1)} dr = I$$

On DE: we set $\theta = 2\pi$, hence $z = re^{2\pi i}$:

$$\int_{-\infty}^0 \frac{e^{-i\pi}}{\sqrt{r}(r+1)} dr = \int_0^\infty \frac{1}{\sqrt{r}(r+1)} dr = I$$

The keyhole contour is equal to the sum of these integrals, hence $2I = 2\pi$ and $I = \pi$.

Part II

Variational Calculus

Lecture 10: Calculus of variations: Euler stationarity condition

Many physics problems necessitate computations of the ground state or the equilibrium state or finding optimal geometrical shapes or equation of motion. For this, we use a powerful technique called *variational calculus* or *calculus of variations*.

In variational calculus, we generalize the notion of a stationary point to a stationary curves and surfaces. The equilibrium or ground state of a system corresponds to the minimum of some appropriate global quantity, e.g. total energy. For a function $f(x)$, we compute its extreme points (minima, maxima or inflection) by the stationarity condition $f'(x) = 0$. Variational calculus generalizes this stationarity condition to a **functional** $I[y_1, y_2, y_3 \cdots]$ which is a function of functions, namely it is applied on an entire function and outputs a number! We distinguish it from the ordinary function which takes a real number and outputs another real number, i.e. $f : \mathcal{R} \to \mathcal{R}$. By contrast the functional I is a map on the space of smooth functions to real numbers, i.e.

$$I : \mathcal{R}^\infty \to \mathcal{R}$$

The functional I is defined as a definite integral, which in the general form is

$$I[y_1, y_2, y_3 \cdots] = \int_{x_1}^{x_2} F(x, y_1(x), y_2(x), y_3(x) \cdots) dx$$

where the integrand is a function which may depend explicitly on x or implicitly through its additional variables $y_1 = y_1(x)$, $y_2 = y_2(x)$, $y_3 = y_3(x)$ that are ordinary real functions of x, and act as **independent variables** for the functional $I[y_1, y_2, y_3 \cdots]$. Because of this, they are also called the **dependable variables** of the functional I.

WE SPECIALIZE OURSELVES to a particular class of functionals that depend on a function of **one** real variable $y(x)$, and its first derivative $y' = \frac{dy}{dx}$. Then, our functional I is defined by this definite integral

$$I[y, y'] = \int_{x_1}^{x_2} F(x, y, y')dx.$$

In this lecture, we will derive the stationarity condition also known as the Euler condition for this class of functionals using calculus of variations.

10.1 Euler condition

OUR GENERAL AIM is to find the function $y(x)$ for which the functional $I[y, y']$ is stationary. In practical applications, this optimal function $y(x)$ could represent:

- Geodesic, i.e. the minimum-distance curve between points on a surface. The total distance between two points represents the functional that we want to minimize and the corresponding stationarity condition represents the equation of the geodesic.

- Path travelled in the least amount of time (Fermat's principle from optics). The total time spend to travel between two points is the functional minimised by the optimal path.

- Trajectory of stationary action (Hamilton's principle in Lagrangian mechanics). By minimising the action, we determine the equation of motion.

THE STATIONARITY CONDITION includes minima, maxima and inflections. However, in physics problem, the functional (often relating to an appropriate energy) is minimised by the stationary solution corresponding to an equilibrium state. In optimization problems though we can use the variational calculus to find other stationary solutions, not only the minima.

10.1.1 *Calculus of variations*

Let us apply the calculus of variations to this type of functional

$$I[y,y'] = \int_{x_1}^{x_2} F(x,y(x),y'(x))dx, \tag{10.1}$$

which has two dependable variables that are the curve $y = y(x)$ and its first derivative $y'(x)$.

OUR AIM is to determine the optimal curve $y_0(x)$ for which $I[y,y']$ is stationary. This means that **any** variations (infinitesimal perturbations) around the stationary curve $y_0(x)$ will leave $I[y,y']$ unchanged.

Tuning parameter method: For this let us consider an **arbitrary** curve $\eta(x)$ with the boundary conditions $\eta(x_1) = \eta(x_2) = 0$ (we will see why later) and which is superimposed to the stationary one :

$$y(x) = y_0(x) + \epsilon\eta(x),$$

where ϵ is a free tuning parameter such that when $\epsilon \to 0$, we obtain the desired stationary curve. This procedure means that we consider a variational of the curve, and often the variation is denoted as (see below the derivation using this variational notation)

$$\delta y \equiv \epsilon\eta(x).$$

Using the functional remain unchained by this *arbitrary variation*, we can write it as

$$I[y_0,y_0'] = I[y,y'] = I[y_0 + \epsilon\eta, y_0' + \epsilon\eta'] \equiv I(\epsilon),$$

which means that I as a regular function of the tuning parameter ϵ. This is nice, because now we can use the stationarity condition applied to an ordinary function, namely that its first derivative evaluated at the stationary point must vanish, i.e.

$$\frac{dI}{d\epsilon}\Big|_{\epsilon=0} = 0. \tag{10.2}$$

Inserting the integral expression from Eq.10.1 into Eq. 10.2, we obtain that

$$\begin{aligned}
\frac{dI}{d\epsilon} &= \int_{x_1}^{x_2} \frac{d}{d\epsilon} F(x,y_0(x)+\epsilon\eta(x),y_0'(x)+\epsilon\eta'(x))dx \\
&= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx
\end{aligned} \tag{10.3}$$

After an integration by parts of the second term, we arrive at

$$\frac{dI}{d\epsilon} = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx \qquad (10.4)$$

We now use the boundary conditions satisfied by the variational, i.e. $\eta(x_1) = \eta(x_2) = 0$ to remove the boundary term, and we are left with

$$\begin{aligned} \frac{dI}{d\epsilon} &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)dx \\ &= 0 \end{aligned} \qquad (10.5)$$

This integral vanishes for an **arbitrary** variation $\eta(x)$ when the expression in the square brackets vanishes. The resulting equation is the **Euler condition** satisfied by the stationary solution $y_0(x)$:

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \qquad (10.6)$$

where $F = F(x, y(x), y'(x))$.

Variational notation: An alternative calculus can be performed using the variational symbol δ to indicate that we differentiate with respect to a variational (virtual) function. The variations of the dependable variables are defined as

$$\delta y(x) \equiv \epsilon\eta, \qquad \delta y'(x) = \frac{d}{dx}\delta y \equiv \epsilon\eta'$$

The variation of the functional I defined from the derivative with respect to ϵ as

$$\delta I \equiv \frac{dI}{d\epsilon}d\epsilon$$

with the stationarity condition $\delta I|_{y=y_0} = 0$. Similarly, by the chain rule of differentiation, the variation of the integrand function $F(x, y, y')$ is

$$\delta F \equiv \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

We can repeat the previous derivation in this variational notation. Using the integral expression of I for the variational, we have that

$$\begin{aligned} \delta I &= \int_{x_1}^{x_2} \delta F dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx \\ &= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\delta y(x)dx \\ &= \int_{x_1}^{x_2} \frac{\delta I}{\delta y}\delta y(x)dx \end{aligned} \qquad (10.7)$$

where the **functional derivative** of I with respect to $y(x)$ is defined as (for this type of functionals)

$$\frac{\delta I}{\delta y} \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}$$

We see that the stationarity Euler condition corresponds to a vanishing functional derivative :

$$\boxed{\frac{\delta I}{\delta y} = 0 \longleftrightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \tag{10.8}$$

The solution of this differential equation gives us the stationary curve. The parameterization $y(x)$ is particularly useful when $\partial F/\partial y = 0$, such that the equation above reduces to the **first integral** of the Euler equation :

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \to \frac{\partial F}{\partial y'} = \text{const.}} \tag{10.9}$$

which is easier to solve for.

Example 10.1. Find the curve with the minimum distance between two points (x_1, y_1) and x_2, y_2 in the (x, y) plane. The function corresponding to the distance between two points is the integral over the infinitesimal distance computed from the Pythagorean theorem as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}dx$$

such that the total length L as a functional of the curve $y(x)$ and $y'(x)$ can be written as

$$L = \int_0^L ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}dx$$

Solution:

We notice that our integrand function $F(y') = \sqrt{1 + (y')^2}$ is independent of both x and y. Applying first integral of the Euler equation, we have that :

$$\frac{y'}{\sqrt{1 + (y')^2}} = \kappa$$

where κ is an integration constant. Equivalently, this means that :

$$y' = \frac{\kappa}{\sqrt{1 - \kappa}} = c_1$$

where c_1 is a constant. This is the equation of the straight line :

$$y(x) = c_1 x + c_2$$

with the integration constants c_1 and c_2 determined by the boundary conditions. This is a trivial problem, but the advance of this formalism is that elegantly applies to solving non-trivial problems as we will see later in other examples.

AN IMPORTANT THING TO REMEMBER is that the functional I depends on a curve and their derivatives. It is up to us how we choose to represent the curve as $y = y(x)$, $x = x(y)$, $\theta = \theta(r)$, or by a parameterization $(x(t), y(t))$. To solve for $y_0(x)$, we might exploit the freedom that we have in choosing how to represent the curve in order to reduce the Euler equation to its first integral form. This helps us find the solution without getting lost into cumbersome math. Let us discuss some alternative curve representations and derive the corresponding first integral form of the Euler equation.

10.1.2 *Curve representation $x = x(y)$*

Sometimes, the Euler equation might be easier to solve when we use $x = x(y)$ as the dependable variable and integrate over y, instead of x. For this, the functional is

$$I[x, x'] = \int_{y_1}^{y_2} F(y, x(y), x'(y))dy \tag{10.10}$$

with the corresponding Euler equation

$$\boxed{\frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0} \tag{10.11}$$

We can go from one formulation to the over through the transformation of variables

$$dx = dy/y' = x'dy$$

How do we know which is more efficient? One clue is to look at the integrand. If F is independent of x (not explicitly dependent on x), i.e. $\partial F/\partial x = 0$, then the parametrization $x = x(y)$ is handy because the Eq. 10.11 reduces to its *first integral* form

$$\boxed{\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0 \rightarrow \frac{\partial F}{\partial x'} = \text{const.}} \tag{10.12}$$

Example 10.2 (Soap film between two wire rings:). This is a classical problem to finding surfaces of revolution. It is a simpler case of the Plateau's problem of finding the surface with the minimum area for a given sets of bounding curves. The free energy of the soap is equal to twice (one for each liquid-air interface) the surface tension times the area of the soap film. Therefore, the equilibrium surface is that of minimum surface area. Due to the axial symmetry for the two concentric rings, the minimum surface is the surface of revolution about the x-axis. Therefore, for a rotation about x, the area of the surface of revolution is

$$S = 2\pi \int y ds = 2\pi \int_{x_1}^{x_2} y\sqrt{1+(y')^2}dx$$

and this is the functional that we want to minimize to find the equilibrium profile $y_0(x)$.

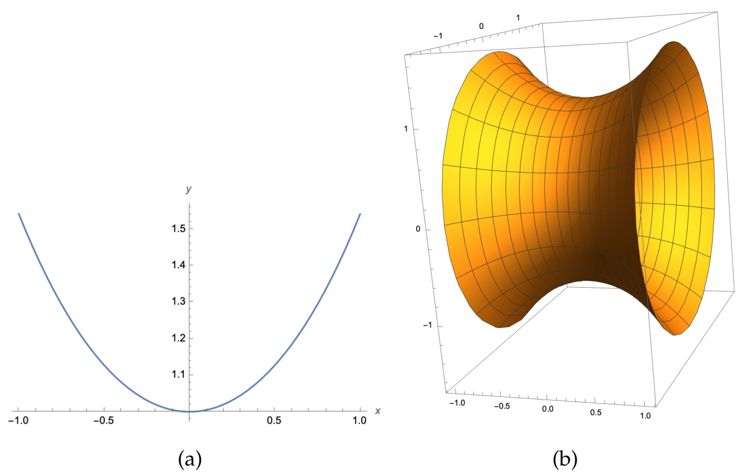


Figure 10.1: (a) The equilibrium profile $y(x)$ of the soap film. (b) Surface of revolution obtain by rotating the profile $y(x)$ about the x-axis.

Solution:

We notice that the integrand

$$F(y, y') = 2\pi y\sqrt{1+(y')^2}$$

is independent of x, i.e. $\partial F/\partial x = 0$. Hence, if we parameterize the profile as $x(y)$, we can rewrite the area as an integral over y,

$$S = 2\pi \int_{y_1}^{y_2} y\sqrt{1+(x')^2}dy$$

and apply the first integral of the Euler equation in these coordinates to determine the stationarity condition as

$$\frac{yx'}{\sqrt{1+(x')^2}} = \kappa_1$$

where κ_1 is an arbitrary constant fixed by the boundary conditions. This is a first order differential equation (ode) in $x(y)$ as rewritten equivalently:

$$\frac{dx}{dy} = \frac{\kappa_1}{\sqrt{y^2 - \kappa_1^2}}$$

We can solve this ode by the separation of variable method

$$\int dx = \kappa_1 \int dy \frac{1}{\sqrt{y^2/\kappa_1^2 - 1}}.$$

We now use the substitution $y = \kappa_1 \cosh(t)$ such that $dy = \kappa_1 \sinh(t) dt$. Then, the integral reduces to

$$\int dx = \kappa_1 \int dt \to t = \frac{x}{\kappa_1} + \kappa_2,$$

Using the $y(t)$ substitution, we find the equilibrium profile

$$y(x) = \kappa_1 \cosh\left(\frac{x}{\kappa_1} + \kappa_2\right)$$

with κ_1 and κ_2 determined by boundary conditions. This is also called the *catenary* equation.

For the particular values of the integration constants $\kappa_1 = 1$ and $\kappa_2 = 0$, the catenary curve

$$y(x) = \cosh(x)$$

and shown in Fig. 10.2 (a). In Fig. 10.2 (b), we see catenoid surface between two circles centered at $x_{1,2} = \pm 1$ and of radius $y_1 = y_2 = \cosh(1)$ as the surface of revolution of the catenary curved.

A quick primer on the area of a surface of revolution: The surface of revolution is the surface generated by a rotation of the profile curve $y = y(x)$ around the x-axis. A surface element dS is a band revolving around the x-axis and of a thickness given by the arclength at a certain point on the profile. Therefore, we can approximate this surface element as an infinitesimal cylinder of height given by the arclength of the profile at a given point $ds = \sqrt{1 + [y'(x)]^2} dx$ and a circular base of radius $y(x)$. Thus, the area element is the area of the infinitesimal cylinder

$$dS = 2\pi y(x) ds = 2\pi y(x) \sqrt{1 + [y'(x)]^2} dx$$

The total area is the superposition of all of these area elements along the surface profile, hence

$$S = \int dS = 2\pi \int_{x_1}^{x_2} y(x) \sqrt{1 + [y'(x)]^2} dx$$

Lecture 11: Cycloids and geodesics

IN THIS LECTURE, we will apply the stationarity condition to the classical brachistochrone problem and to geodesics. For each problem, we will make use of the appropriate curve parameterization to reduce the Euler equation to its first integral form.

11.1 Curve representation $x = x(y)$

Brachistochrone problem: Find the trajectory of a particle starting from a rest point at (x_1, y_1) and reaching (x_2, y_2) in the *shortest time* and under the force of gravity alone (in the y direction). This condition is carried in the greek etymology of Brachistochrone = brákhistos (shortest) + khrónos (time).

It turns out that this shortest time trajectory under gravity is actually a particular solution in the family of cycloid curves. A cycloid is a curve traced by a point on a rolling circle (See Fig. 11.1). This is the trajectory of a speck of dust stuck on the bicycle's wheel while you are biking.

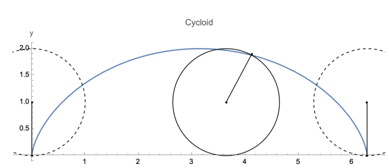


Figure 11.1: Cycloid curve

Solutions: The functional that we want to construct is the one that measures the total time between the two end points, namely

$$T = \int_0^T dt.$$

We want to use an appropriate integration variable such that $y(x)$ is the dependable variable for our functional are $T[y, y']$. How do we change from time to space variables? For this, we use that the curve results from an equation of motion of a particle falling under gravity.

Physically, the infinitesimal arclength ds around a given point on the curve relates to the speed at that point as:

$$ds = v(x, y(x))dt$$

Geometrically, the arclength can also be expressed as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}dx$$

By combing these two expressions, we find the transformation of variables

$$dt = \frac{\sqrt{1 + y'^2}}{v(x, y(x))}dx$$

In the absence dissipative forces (e.g. friction), the total energy is conserved. Let us place out coordinate system such that (x_1, y_1) is the origin and the ground zero relative to which we measure the gravitational potential (*see figure 11.2*). Thus, the initial total energy of a particle is zero (no kinetic and potential energy) and it must stay zero at any later time. Therefore, at a given time, the potential and kinetic energy must balance each other out

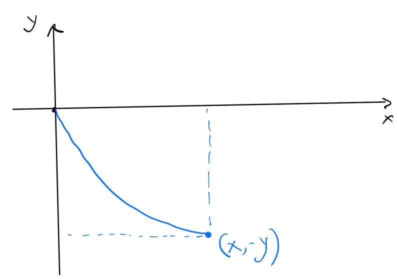


Figure 11.2: Sketch of the trajectory.

$$\frac{1}{2}mv^2 - mgy = 0 \to v = \sqrt{2gy}$$

Now we are ready to define our functional that we want to minimize:

$$T[y, y'] = \int dt = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + [y'(x)]^2}{y(x)}}dx,$$

where the integrand function

$$F(y(x), y'(x)) = \sqrt{\frac{1 + [y'(x)]^2}{y(x)}}$$

depends on both $y(x)$ and $y'(x)$ and does not depend explicitly on x. It is harder to solve Euler's equation for this curve parameterization. But, if instead we let x be the dependable variable $x = x(y)$ and integrate over y, then we can use the first integral for the Euler equation. For this, we use the change of variables

$$dy = x'dx, \qquad y' = \frac{dy}{dx} = \frac{1}{x'},$$

such that the function can be expressed equivalently as

$$T[y, x'] = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1 + [x'(y)]^2}{y(x)}}dy$$

which is independent of x. The corresponding Euler equation reduces to its first integral form

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0 \rightarrow \frac{x'}{\sqrt{y(1+x'^2)}} = \sqrt{c},$$

where c is an constant of integration.

Solving for x', we find that

$$x' = \frac{dx}{dy} = \sqrt{\frac{cy}{1-cy}}.$$

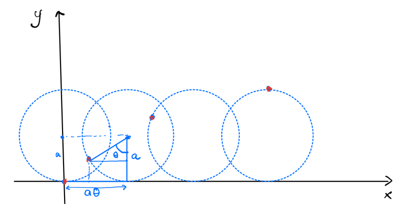


Figure 11.3: Sketch of a point on a rolling circle tracing different points on the cycloid.

The generic solution of this first order ode can be determined by the separation of variables

$$dx = \sqrt{\frac{cy}{1-cy}}dy.$$

We use the substitution $cy = \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta)$ with $cdy = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta$ such that

$$\begin{aligned} dx &= \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\frac{1}{c}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta \\ &= \frac{1}{c}\sin^2\left(\frac{\theta}{2}\right)d\theta \rightarrow \\ \int dx &= \frac{1}{2c}\int(1-\cos\theta)\,d\theta \\ x &= \frac{1}{2c}(\theta-\sin\theta)+c', \end{aligned}$$

where c' is the other integration constant. This is set to 0 by the initial condition that $x(\theta=0) = x_1 = 0$. Thus, for an arbitrary c, we obtain a family of parameterised curves in terms of θ :

$$x(\theta) = \frac{1}{2c}(\theta-\sin\theta) \tag{11.1}$$

$$y(\theta) = \frac{1}{c}\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2c}(1-\cos\theta), \tag{11.2}$$

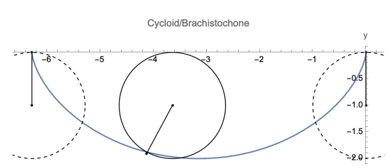


Figure 11.4: Brachistochrone as a cycloid curve under the x-axis.

which is also called the family of cycloids. The parameter $1/(2c)$ fixes the axis of the rolling circle. The name is suggestive and means that a cycloid is the curve traced by a fix point on a rolling circle. (*see figure 11.3*). The brachistochrone curve is a specific kind of cycloid where the circle is rolling under the x-axis as shown in Fig. 11.4.

11.2 *Polar representation $\theta = \theta(r)$, $r = r(\theta)$*

PROBLEMS THAT HAVE rotational symmetry about an axis are easier to handle when we use polar coordinates. For such problems, the curve is represented as $\theta = \theta(r)$, such that the functional has the angle as its main dependable variable:

$$I[\theta, \theta'] = \int_{r_1}^{r_2} F(r, \theta(r), \theta'(r))dr \tag{11.3}$$

where $\theta' = d\theta/dr$. The corresponding Euler equation for this stationary curve is

$$\boxed{\frac{\partial F}{\partial \theta} - \frac{d}{dr}\left(\frac{\partial F}{\partial \theta'}\right) = 0 \qquad \theta = \theta(r)} \tag{11.4}$$

When the integrand is independent of θ, i.e. $\partial F/\partial \theta = 0$, Eq. 11.4 reduces to its *first integral*,

$$\boxed{\frac{\partial F}{\partial \theta'} = \text{const.}} \tag{11.5}$$

SIMILARLY, if instead the curve representation $r = r(\theta)$, the Euler equation for the corresponding $I[r, r']$ is then:

$$\boxed{\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) = 0, \qquad r = r(\theta)} \tag{11.6}$$

with its first integral form

$$\boxed{\frac{\partial F}{\partial r'} = \text{const.}} \tag{11.7}$$

This polar representation is often useful when finding geodesics. We illustrate how to derive parametric equations for geodesics on a cylinder and on a cone using polar representation and the corresponding first integral form.

Geodesic on a cylinder: Find the geodesics on the surface of a circular cylinder corresponding to

$$x = R\cos(\theta), \qquad y = R\sin\theta, \qquad z = z$$

where R is the fixed radius of the circular base.

Solution: The infinitesimal arclength of curve on the surface of the cylinder is determined as

$$dl^2 = R^2 d\theta^2 + dz^2$$

thus,

$$dl = \sqrt{R^2 + z'^2} d\theta$$

where $z' = dz/d\theta$. The total distance between two points on the cylinder is then the following functional

$$L = \int dl = \int d\theta \sqrt{R + (z')^2}$$

We notice that the integrand is only a function of z', $F(z') = \sqrt{R + (z')^2}$ hence the corresponding Euler equation (Eq. 11.6) simplifies to

$$\frac{d}{d\theta} \frac{\partial F}{\partial z'} = 0$$

from which it follows that

$$\frac{z'}{\sqrt{R^2 + z'^2}} = constant$$

Since R is also a constant, this is equivalent to

$$z' = a$$

which has the solution the helix

$$z(\theta) = a\theta + b$$

where a and b are integration constants. Fig. 11.5 show a helix on the cylinder for $a = 1$ and $b = 0$.

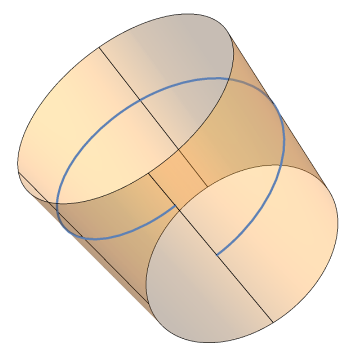


Figure 11.5: Geodesic on a cylinder

Obs: In the (x, y) plane, there is a unique geodesic that connects two points, i.e. the straight line. However, this property does not hold on curved spaces. Notice that the same point on the cylinder corresponds to any $\theta + 2\pi n$, where $n \in \mathcal{Z}$, i.e. θ plus an arbitrary number of 2π-rotations. Hence the geodesic connecting two points is also determined up to an arbitrary number of turns (revolutions)

$$z(\theta) = a(\theta + 2\pi n) + b$$

Hence, there are infinitely many geodesics connecting two points on the cylinder and they differ in the number of 2π-rotations or the direction of rotation. Out of all of them, there will be one geodesic that has the shortest distance compared to all the others. To get an intuition about this, imagine that we look at the geodesic connecting two points with no net 2π rotation. This is the straight line obtained by rolling the cylinder once and flatting it on the plane. The geodesic with a 2π-rotation corresponds to the straight line after two rollings of the cylinder in the plane (so the two points end up further apart). The more rotations, the further apart will the two points in the cylinder be located in the plane.

Geodesic on a cone: Find the geodesics on the surface of the cone

$$z = \sqrt{x^2 + y^2}.$$

Solution:

We can make use of the cone symmetry around the z axis, by using the polar coordinates in the (x, y)-plane with $r = \sqrt{x^2 + y^2}$ being the radius of the circle. Thus, an infinitesimal arclength of a curve on the surface of the cone is determined as

$$dl^2 = ds^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Using that on the surface of the cone, z and r are related to each other as $z = r$, we obtain

$$dl = \sqrt{2dr^2 + r^2 d\theta^2} = \sqrt{2 + r^2(\theta')^2}dr.$$

To find the geodesic on this cone, we want to minimize the functional given by the total length of a curve between two points on the cone:

$$L = \int dl = \int_{r_1}^{r_2} \sqrt{2 + r^2(\theta')^2}dr$$

Since the functional depends only on θ', we can use the first integral form of the stationarity condition in Eq. 11.4, which leads to

$$\frac{\partial}{\partial \theta'} \sqrt{2 + r^2(\theta')^2} = \frac{r^2\theta'}{\sqrt{2 + r^2(\theta')^2}} = \kappa$$

where κ is the integration constant. Hence, the ode satisfied by $\theta(r)$ is

$$\frac{d\theta}{dr} = \sqrt{2}\frac{\kappa}{r}\frac{1}{\sqrt{r^2 - \kappa^2}}$$

By the separation of variables, we find that

$$\int d\theta = \sqrt{2} \int dr \frac{\kappa}{r^2} \frac{1}{\sqrt{1 - \kappa^2/r^2}}$$

Using the transformation of variables $\omega = \kappa/r$, we write the integral on the left hand side as

$$\int d\theta = -\sqrt{2} \int d\omega \frac{1}{\sqrt{1 - \omega^2}}$$

For $r > \kappa$, we substitute $\omega = \cos\alpha$ such that

$$\int d\theta = \sqrt{2}\int d\alpha \frac{\sin\alpha}{\sqrt{1-\cos^2\alpha}} = \sqrt{2}\int d\alpha$$

$$\theta + \theta_0 = \sqrt{2}\arccos\left(\frac{\kappa}{r}\right)$$

or equivalently,

$$r\cos\left(\frac{\theta+\theta_0}{\sqrt{2}}\right) = \kappa, \qquad r = \kappa\sec\left(\frac{\theta+\theta_0}{\sqrt{2}}\right).$$

where θ_0 is an integration constant. Both κ and θ_0 are fixed by the boundary conditions. The parametric equations for the geodesic on the cone are given by

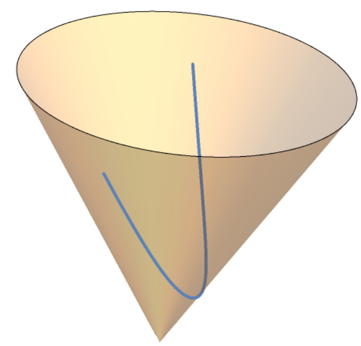


Figure 11.6: Geodesic on the cone

$$\begin{aligned} x &= \kappa\sec\left(\frac{\theta+\theta_0}{\sqrt{2}}\right)\cos\theta \\ y &= \kappa\sec\left(\frac{\theta+\theta_0}{\sqrt{2}}\right)\sin\theta \\ z &= \kappa\sec\left(\frac{\theta+\theta_0}{\sqrt{2}}\right) \end{aligned} \tag{11.8}$$

Fig. 11.6 shows a geodesic corresponding to $\theta_0 = 0$ and $\kappa = 1$.

Lecture 12: Fermat's principle and Hamilton's principle

Variational calculus is a powerful tool used in optics through Fermat's principle and in Hamiltonian mechanics through Hamilton's principle. In this lecture, we will apply the Euler condition to find the path of a ray of light (path of minimum time) and its relation to the classical Snell's law in optics. We will use the Euler condition corresponding to the stationary action to derive the Euler-Lagrange equation of motion for Hamiltonian (conserved) systems.

12.1 *Fermat's principle and Snell's law*

Fermat's principle postulates that a ray of light travels in an optical medium on the path of least time. An optical medium is characterized by its **index of refraction** or **refractive index**, n which determines the speed of light through it as

$$v = c/n,$$

where c is the speed of light in the vacuum. Thus, in any optical medium that is not a vacuum, light travels at a lower speed. As fascinating as it can get nowadays, we can slow down light and almost trap it inside a cloud of ultra cold atoms, such as Bose Einstein condensates which has a refraction index of about 1.57×10^7 corresponding to a light speed of about $v \sim 19 m/s$ thus comparable with the car speed on the highway.

12.1.1 *Snell's law*

Let us consider a light ray traveling in the (x, y) plane in a layered optical medium, i.e. the refractive index is position dependent, e.g. $n(x)$. Further more, we assume that there is a sharp jump from one constant value to another constant value at $x = 0$ (like an air-water interface). We aim to derive the Snell's law using the Euler condition corresponding to Fermat's principle of least time. The path of the light ray is a curve $y(x)$ which minimises the total time spent in that path. Hence, our functional integral is total time. The infinitesimal time increment is related to the infinitesimal arclength on the path traversed by light though the speed of light, thus

$$dt = \frac{ds}{v} = \frac{1}{c} n(x) \sqrt{1 + y'^2} dx.$$

The total time is a functional with dependable variables y and y' given by

$$T[y, y'] = \int dt = \frac{1}{c} \int n(x) \sqrt{1 + y'^2} dx$$

where $y' = dy/dx$. In fact, we have chosen the appropriate curve parameterization since the function is only dependent on y'. Thus, the corresponding Euler equation reduced to its first integral form given by

$$\frac{d}{dx} \frac{\partial}{\partial y'} \left[n(x) \sqrt{1 + y'^2} \right] = 0,$$

which implies that

$$n(x) \frac{y'}{\sqrt{1 + y'^2}} = \kappa,$$

where κ is the integration constant. This relation holds everywhere along the path, i.e. for every x. Let us consider as an example a layer of two homogeneous media, whereby the refractive index is constant in each medium and jumps abruptly at their interface,

$$n(x) = \begin{cases} n_1 & x > 0 \\ n_2 & x \leq 0 \end{cases} \tag{12.1}$$

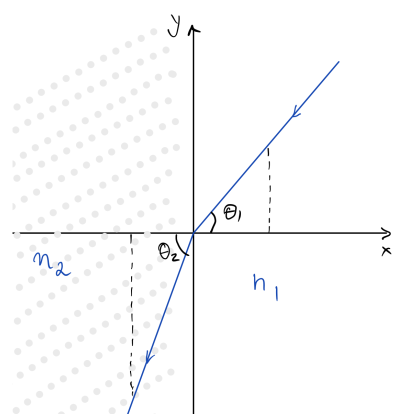


Figure 12.1: Snell's law.

We solve the stationarity condition for each homogeneous side and match the solutions at the interface, i.e. $x = 0$. For $x \neq 0$, the refractive index is a constant, hence it can be absorbed into the integration constant, and the solution is a straight line in each half-plane

$$\frac{y'}{\sqrt{1 + y'^2}} = a \to y(x) = ax + b, \qquad x \neq 0,$$

where the slope a is determined by the interfacial condition at $x = 0$. The chance in slope is related to the change in the refraction by the

Snell's law. Applying the stationarity condition on each side of $x = 0$, we have that

$$n_1 \frac{a_1}{\sqrt{1+a_1^2}} = n_2 \frac{a_2}{\sqrt{1+a_2^2}}.$$

Using that the slopes are tangents: $a_1 = \tan(\theta_1)$ and $a_2 = \tan(\theta_2)$ with $\theta_{1,2}$ being the angles between the lines meeting at $x = 0$ and x axis (see Fig. 12.1), the above interfacial condition reduces to the classical **Snell's law**:

$$\begin{aligned} n_1 \frac{\tan\theta_1}{\sqrt{1+\tan^2\theta_1}} &= n_2 \frac{\tan\theta_1}{\sqrt{1+\tan^2\theta_1}} && (12.2) \\ n_1 \sin\theta_1 &= n_2 \sin\theta_2. && (12.3) \end{aligned}$$

When θ_1 is the *incidence angle*, θ_2 is the *refractive angle*.

12.2 Hamilton's principle and Euler-Lagrange equations

12.2.1 1D trajectory of motion $x = x(t)$:

A curve could also corresponds to the trajectory of a particle moving in a one-dimensional space (or the trajectory along one coordinate), hence $x = x(t)$. Then, we introduce a functional defined as

$$I[x, \dot{x}] = \int_{t_1}^{t_2} F(t, x(t), \dot{x}(t)) dt \tag{12.4}$$

where $\dot{x} = \frac{dx}{dt}$ such that its corresponding Euler equation becomes

$$\boxed{\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0} \tag{12.5}$$

which is also known as the Euler-Lagrange equation of motion of one particle moving in the x direction. When $\partial F/\partial x = 0$, then Eq. 12.5 reduces the *first integral*

$$\boxed{\frac{\partial F}{\partial \dot{x}} = \text{const.}} \tag{12.6}$$

12.2.2 Several dependable variables:

THE VARIATIONAL CALCULUS is a very powerful method in Lagrangian/Hamiltonian mechanics. and it is cast as the *principle of least action*. The idea is to introduce a **Lagrangian** function $L = T - V$ where T is the total **kinetic energy** and V is the **potential energy**.

Both energies are expressed in terms of generalized position and velocity coordinates, $(q_i, \dot{q}_i)$ where i labels each degree of freedom (or component of motion),

$$L = \frac{1}{2}\sum_i m\dot{q}_i^2 + V(q_1, \cdots, q_n)$$

FOR A SINGLE POINT particle moving in 3D, $q_i \equiv x, y, z$ and $\dot{q}_i \equiv \dot{x}, \dot{y}, \dot{z}$. The Lagrangian of the system determines the **action** as

$$S[q_1, q_n, \cdots \dot{q}_1 \cdots \dot{q}_n] = \int_{t_i}^{t_f} L(t, \{q_i, \dot{q}_i\})dt,$$

which is a functional that has several dependable variables corresponding to many curves. Each curve represents the motion along a coordinate axis, and in the language of statistical physics, this is a degree of freedom.

FOR SIMPLICITY, let us consider the case of two independent degrees of freedom corresponding to a curve parameterized with respect to time as $q_1 = q_1(t)$ and $q_2 = q_2(t)$. For instance, this could be the motion of a point particle in the plane or of two point particles in one dimension. Lagrange showed that the classical Newton's equations of motion can be deduced from the condition of stationary action with respect to variations of the trajectory $(q_1(t), q_2(t))$ that leaves the initial and final points fixed,

$$\delta S[q_1, \dot{q}_1, q_2, \dot{q}_2] = 0.$$

This variational procedure is also called the **Hamilton's principle of stationary action**.

12.2.3 *Euler-Lagrange's equations*

BY TAKING THE VARIATIONAL of the action for two degrees of freedom, we have that

$$\begin{aligned}
\delta S[q_1, \dot{q}_1, q_2, \dot{q}_2] &= \int dt \left[\frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \frac{\partial L}{\partial \dot{q}_2}\delta \dot{q}_2\right] \\
&= \int dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\delta q_1 + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\delta q_2\right] \\
&= 0
\end{aligned}$$

USING that δq_1 and δq_2 are *independent* variations, we find that each integrand must vanish independently, HENCE, each degree of freedom satisfies its own Euler condition:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0 \tag{12.7}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0 \tag{12.8}$$

AS A SET OF EQUATIONS, these are also called the **Euler-Lagrange equations**. Let us see that indeed these equations reduce to Newton's law $F = m\ddot{x}$. The Lagrangian function for the two variables is

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = T(\dot{q}_1, \dot{q}_2) - V(q_1, q_2) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - V(q_1, q_2)$$

INSERTING this into Eq. 12.7 and 12.8, and rearranging terms, we find that

$$m\frac{d^2q_1}{dt^2} = -\frac{\partial V}{\partial q_1}$$
$$m\frac{d^2q_2}{dt^2} = -\frac{\partial V}{\partial q_2},$$

which we recognize as the Newtonian dynamics for two particles in a potential field which exerts a potential (gradient) force $\vec{F} = -\nabla V$ on each particle.

Example 12.1. Let us derive the equations of motion for a point particle of mass m and moving in the $x - y$ plane and in a radially-symmetric potential field, $V(r)$. Thus, the potential depends only on the distance from the origin of the moving particle, i.e. is independent of orientation. This also means that perhaps it is useful to work in polar coordinates. The kinetic energy depends on velocity squared and this can be determined as the time derive of an arclength $ds^2 = dr^2 + r^2 d\theta^2$ along the particle's trajectory, i.e.

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

where the first component is the radial speed and and the second one is the angular speed. The Lagrangian function at a given point on the trajectory is

$$L = \frac{1}{2}mv^2 - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

such that the action functional is

$$S[r, \theta, \dot{r}, \dot{\theta}] = \int_0^T dt \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right]$$

The corresponding Euler-Lagrange Eqs. 12.7-12.8 are

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + V'(r) = 0 \qquad (12.9)$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0, \qquad (12.10)$$

where $V'(r) = dV(r)/dr$. Equivalently, this is

$$m(\ddot{r} - r\dot{\theta}^2) = -V'(r) \qquad (12.11)$$

$$m(r^2\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0, \qquad (12.12)$$

We may recognize the second equation as being the conservation of angular momentum, $l = mr^2\dot{\theta}$, which is due to the rotational symmetry of the potential and the Hamiltonian dynamics (no dissipation).

Theorem 12.2 (Noether's theorem:). *This deep connection between symmetry and conservation laws has been exploited by Emma Noether using variational calculus in Lagrangian mechanics.*

$$\boxed{\textit{Symmetry} \Longleftrightarrow \textit{Conservation Law}} \qquad (12.13)$$

In the previous example, the rotational invariance translates into the conservation of angular momentum under Hamiltonian dynamics. Similarly, a system that has *translational symmetry* will conserve *momentum*. A system that is invariant under *time translations* conserves *energy*. The idea is that we can determine the conservation law directly from the variational of the action integral taking into account the symmetries of the Lagrangian without having to derive all the equations of motion.

We illustrate this technique for the previous example of a point particle moving under a rotationally-symmetric force. Let us look again at the corresponding action integral

$$S[r, \theta, \dot{r}, \dot{\theta}] = \int_0^T dt \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right]$$

Notice that the Lagrangian remains unchanged by a rotation with an *arbitrary constant* phase

$$\theta(t) \to \theta(t) + \epsilon\alpha$$

For a stationary action the requirement is that the Lagrangian remains unchanged by an arbitrary rotational with any time-dependent phase, including this one that is inspired by the rotational symmetry

$$\theta(t) \to \theta(t) + \epsilon(t)\alpha$$

Thus, we relate the variational of θ with the rotation rate α

$$\delta\theta(t) \equiv \epsilon(t)\alpha$$

Inserting, this into the variational of the action and applying the usual integration by parts (with vanishing boundary terms), we find that

$$\begin{aligned}
\delta S &= \int_0^T dt m r^2 \dot{\theta}\delta\dot{\theta} && (12.14)\\
&= \alpha \int_0^T dt m r^2 \dot{\theta}\dot{\epsilon} && (12.15)\\
&= -\alpha \int_0^T dt \frac{d}{dt}(mr^2\dot{\theta})\epsilon && (12.16)\\
&= 0 && (12.17)
\end{aligned}$$

The stationary condition $\delta S = 0$ for an arbitrary ϵ immediately results in the conservation law of angular momentum

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0.$$

NOTICE that in this example the Lagrangian is also not explicitly dependent on time, i.e.

$$\frac{\partial L}{\partial t} = 0$$

This means that it is invariant under time translations, $t \to t + \epsilon$, with an *arbitrary constant* time increment ϵ. It implies that the Lagrangian remains unchanged to the following transformation of the variable q:

$$q(t) \to q(t+\epsilon)$$

For an infinitesimal ϵ, we can Taylor expand to first order in ϵ such that the above transformation reduces to a global translation of the coordinate variable

$$q(t) \to q(t) + \epsilon\dot{q}$$

This leads us to consider variations that mimics the invariant transformation by making ϵ time-dependent:

$$\delta q \equiv \epsilon(t)\dot{q}$$

The variational calculus of the action with this variation is

$$\begin{aligned}
\delta S &= \int dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right] && (12.18)\\
&= \int dt \left[\frac{\partial L}{\partial q}\epsilon\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\epsilon\dot{q})\right] && (12.19)\\
&= \int dt \left[\left(\frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\right)\epsilon + \frac{\partial L}{\partial \dot{q}}\dot{\epsilon}\dot{q}\right] && (12.20)\\
& && (12.21)
\end{aligned}$$

The first term in the parenthesis represents the total time derivative,

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

using that $\frac{\partial L}{\partial t} = 0$. Thus, the action can be rewritten, after an integration by parts of the last term, as

$$\begin{aligned} \delta S &= \int dt \left[\frac{dL}{dt} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)\right]\epsilon && (12.22) \\ &= 0 && (12.23) \end{aligned}$$

Thus, time invariance leads to the conservation of the Hamiltonian

$$\frac{d}{dt}\left[\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right] = 0 \Longleftrightarrow \frac{d}{dt}H = 0 \qquad (12.24)$$

where the Hamiltonian is defined as the conjugate of the Lagrangian by the Legendre transform

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = T(\dot{q}) + V(q).$$

Part III

Ordinary differential equations

Lecture 13: First order ODEs

13.1 *Ordinary Differential equations*

General definitions:

Definition 13.1. An ordinary differential equation (ode) is a general expression containing derivatives of a function of one variable, $y = y(x)$

$$F(x, y, y', y'', y''', \cdots y^{(n)}) = 0,$$

where $y' = dy/dx$, $y'' = d^2y/dx^2, \cdots y^{(n)} = d^n y/dx^n$ are derivatives of $y(x)$.

Definition 13.2. The order of an ode is given by the highest derivative of $y(x)$.

Example 13.1. $y'' + x^2 y' + y^2 = 5$ is an ODE of order 2

Definition 13.3. Linear ode's have a general form

$$a_0 y + a_1 y' + a_2 y'' + a_3 y''' + \cdots + a_n y^{(n)} = b,$$

where $a_0, a_1 \cdots, a_n$ and b are **constants** or **functions of** x. Each term is a linear function of $y(x)$ and its derivatives.

Example 13.2. The equation of the linear harmonic oscillator

$$y'' + \omega^2 y = 0$$

has constant coefficients.

Linear ode's have **general** solutions or **family** of solutions, which have n *arbitrary* constants, n is the **order** of the ode.

Example 13.3. The general solution of

$$y'' + \omega^2 y = 0$$

is family of solutions

$$y(x) = A\sin(\omega x) + B\cos(\omega x),$$

with A and B arbitrary constants.

Boundary or initial conditions are required to find the *specific* solution fulfilling these conditions. For a unique specific solution, the number of boundary conditions must equal the number of coefficients determined by the order of the ODE. If all the conditions at imposed at $x = 0$ then they are called **initial conditions**.

THERE ARE SEVERAL GENERIC METHODS that work for certain kinds of ode's of a given order. In this lecture, we will go through these methods and apply them to 1st and 2nd order ode's.

13.2 First order ODE's

FIRST ORDER differential equations are expressions that link the variable x and the function $y(x)$ with the first derivative of $y(x)$

$$F(x, y, y') = 0.$$

The ode is a **linear** superposition of y and y' in the general form

$$y' + P(x)y = Q(x).$$

If the coefficients in front of $y(x)$ and $y'(x)$ depend on y and y', then the ODE is *nonlinear*.

We will go through the basic methods of solving 1st order ODE's, such as the integrating factor method that applies only to the linear case, and two other generic methods that can be used to solve both linear and some types of nonlinear 1st order ode's.

13.2.1 Linear 1st order ODE's

The generic form of a **linear** first order ODE is give by

$$y' + P(x)y = Q(x), \tag{13.1}$$

where $P(x)$ and $Q(x)$ are known continuous functions of x or constants. The task is to find a general solution of $y(x)$ (no specific boundary conditions) which satisfies this ODE. We introduce the anti-derivative of $P(x)$:

$$I(x) = \int_{-\infty}^{x} P(s)ds$$

and multiply both sides of Eq.13.1 with the exponential factor $e^{I(x)}$. This results in

$$\begin{aligned} y'e^{I(x)} + P(x)ye^{I(x)} &= Q(x)e^{I(x)} \\ \frac{d}{dx}\left[y(x)e^{I(x)}\right] &= Q(x)e^{I(x)} \\ y(x)e^{I(x)} &= \int_{-\infty}^{x} dzQ(z)e^{I(z)} + C, \end{aligned}$$

where C is the integration constant. Hence, the general solution can be written as

$$y(x) = e^{-I(x)} \int_{-\infty}^{x} dzQ(z)e^{I(z)} + Ce^{-I(x)}, \qquad I(x) = \int_{-\infty}^{x} P(s)ds. \tag{13.2}$$

Example 13.4. Find the general solution of this linear first order ode

$$y' + y = e^x \tag{13.3}$$

Solution:

The coefficient in front of y is 1, hence we multiply on both sides with e^x:

$$\begin{aligned} y'e^x + ye^x &= e^{2x} \\ \frac{d}{dx}[y(x)e^x] &= e^{2x} \\ y(x)e^x &= \frac{1}{2}e^{2x} + C, \end{aligned}$$

where C is the integration constant. Hence, the general solution is

$$y(x) = \frac{1}{2}e^x + Ce^{-x}. \tag{13.4}$$

13.2.2 *Method of separation of variables*

Suppose we now take a 1-st order ODE of this general form

$$g(y)y' = f(x) \tag{13.5}$$

such that all the y-dependent terms can be separated from the x-dependent terms as

$$g(y)dy = f(x)dx$$

We call this a *separable* equation, and the solution $y(x)$ is obtained by integrating each side of the differential form

$$\int g(y)dy = \int f(x)dx.$$

Example 13.5. Find the solution of

$$xy' = y$$

Solution:

We write it in the differential form and separate the variables

$$\frac{dy}{y} = \frac{dx}{x},$$

which we can solve when $y \neq 0$ and $x \neq 0$. Integrating on both sides, we get

$$\ln y = \ln x + C \rightarrow y(x) = Ax.$$

where $A = e^C$ is the integration constant. The general solution is a family of straight lines of various slopes that intersect at $y = x = 0$. Notice there is also the trivial solution $y = 0$ (for any x) which is the horizontal line in the (x, y) plane which is not part of the family of solutions obtained above, i.e. it cannot be obtained from the the general solution by specializing the arbitrary constant A. It makes sense that it is outside since the differential form that we solved was valid for $y \neq 0$ and $x \neq 0$.

13.2.3 *Method of integrating factors*

LET US now consider a first order *nonlinear* ODE of this type:

$$Q(x, y)y' + P(x, y) = 0 \tag{13.6}$$

or in the differential form

$$P(x, y)dx + Q(x, y)dy = 0. \tag{13.7}$$

If the coefficients $P(x, y)$ and $Q(x, y)$ are functions that satisfy the integrability condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

that corresponds to the condition an an *exact* differential form of a function $F(x, y)$, i.e.

$$dF = P(x, y)dx + Q(x, y)dy,$$

WHERE

$$Q = \frac{\partial F}{\partial y}, \quad P = \frac{\partial F}{\partial x}.$$

The integrability condition corresponds to

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

The function F is determined by its *integrating factors* P and Q. Then the general solution of Eq. (5) is given by solving

$$dF = 0 \to F(x,y) = const. \to y(x).$$

The general expression of $F(x,y)$ is obtained by integrating the equations for P and Q.

$$\begin{array}{c} P(x) = \frac{\partial F}{\partial x} \to F(x,y) = \int P(x)dx + f(y) \\ Q(y) = \frac{\partial F}{\partial y} \to F(x,y) = \int Q(y)dy + g(x) \end{array}$$

Matching the integration factors, we have that

$$F(x,y) = \int P(x)dx + \int Q(y)dy.$$

The general solution is obtained from

$$\int P(x)dx + \int Q(y)dy = C$$

which is the same as solving the equation directly from separation of variables. The advantage however that it is a more general technique than the separation of variables.

Particular case: separation of variables We can show that the separation of variables is a particular case of the method of integrating factors. That is when we have an ODE of this form

$$Q(y)y' + P(x) = 0 \to Q(y)dy + P(x)dx = 0 \tag{13.8}$$

which we can clearly see that it is a separable equation. But we also see that functions P and Q satisfy the integrability condition for an exact differential

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example 13.6. (*see figure 13.1*) Find the solution $y(x)$ of this nonlinear 1st order ode

$$2xy + (x^2 - y^2)y' = 0$$

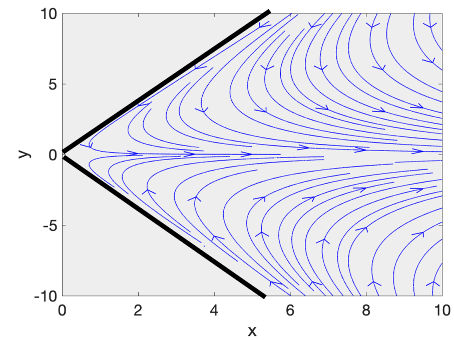


Figure 13.1: Flow streamlines

Solution:

The ode has a differential for

$$2xydx + (x^2 - y^2)dy = 0$$

and corresponds to the exact differential of a field $F(x,y)$ to be determined from the integrability condition:

$$\frac{\partial}{\partial y}(2xy) = \frac{\partial}{\partial x}(x^2 - y^2) = 2x$$

The function $F(x,y)$ is determined by the integrating factors

$$2xy = \frac{\partial F}{\partial x} \to F(x,y) = 2y\int xdx + f(y) = yx^2 + f(y)$$
$$x^2 - y^2 = \frac{\partial F}{\partial y} \to F(x,y) = \int(x^2 - y^2)dy + g(x) = yx^2 - \frac{y^3}{3} + g(x)$$

By identifying those two expressions, $F(x,y) = yx^2 - \frac{y^3}{3}$ and the general solution is when $F(x) = C$ is a constant,

$$yx^2 - \frac{y^3}{3} = C,$$

where the curves $y(x)$ correspond to the flow streamlines of an incompressible fluid flowing into a corner bounded by walls meeting at the origin at 60^0 angle as shown in Fig. 13.1. The streamline function $F(x,y)$ determines the incompressible velocity flow field

$$u_x = \frac{\partial F}{\partial y}, \qquad u_y = -\frac{\partial F}{\partial x}$$

(check that the flow is incompressible $\nabla \cdot \vec{u} = 0$).

For our case the velocity field is

$$u_x = x^2 - y^2, \qquad u_y = -2xy.$$

13.3 Linear second order ODE's

General form of a linear 2nd order ode is

$$\boxed{y'' + P(x)y' + Q(x)y = R(x)}, \tag{13.9}$$

where $P(x), Q(x)$ and $R(x)$ are continuous known functions of x or just constants.

Homogeneous ODE i.e. $R(x) = 0$***:*** The **general solution** is given as a linear superposition of two independent solutions $y_1(x)$ and $y_2(x)$

$$\boxed{y_h(x) = c_1 y_1(x) + c_2 y_2(x)} \tag{13.10}$$

Non-homogeneous ODE i.e. $R(x) \neq 0$***:*** The **general solution** is given by the general solution of the homogeneous equation from Eq. 13.10 and a *particular* solution $y_p(x)$ determined by the source term $R(x)$.

$$\boxed{y(x) = y_h(x) + y_p(x)}. \tag{13.11}$$

A *specific* solution is fixed by specializing the arbitrary constants c_1 and c_2 such that the general solution satisfies given **boundary conditions**.

13.3.1 *2nd order ODE's with constant coefficients*

Method of undetermined coefficients

Homogeneous ODE $R(x) = 0$*:* Let us consider a differential equation of this form

$$\boxed{y'' + ay' + by = 0}, \tag{13.12}$$

The task is to find the two independent solutions for this homogeneous equation. We introduce the *auxiliary* or *characteristic* equation

$$\lambda^2 + a\lambda + b = 0,$$

with the roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Thus, we can factorize the 2nd order differential into a product of 1st order differentials as

$$\left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right) y = 0,$$

which we can solve by successive integrations. We do this by reducing the 2nd order to a set of two 1st order ode's:

$$\begin{aligned} \left(\frac{d}{dx} - \lambda_1\right) u(x) &= 0 \\ \left(\frac{d}{dx} - \lambda_2\right) y(x) &= u(x) \end{aligned}$$

The first equation has the general solution

$$u(x) = c_1 e^{\lambda_1 x}$$

which implies that the integral solution of the second equation is

$$y(x) = c_2 e^{\lambda_2 x} + c_1 e^{\lambda_2 x} \int ds e^{(\lambda_1 - \lambda_2)s}.$$

We have the following two situations:

a) $\lambda_1 \neq \lambda_2$ and λ_1: Then we integrate the exponential integrand and obtain, the general solution of $y(x)$ given by

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

b) $\lambda_1 = \lambda_2 = \lambda$. In this case, $e^{(\lambda_1 - \lambda_2)s} = 1$ and its integral is just x. Thus, the general solution reduces to

$$\boxed{y(x) = (c_1 x + c_2) e^{\lambda x}}$$

Example 13.7. Find the general solution of this equation

$$y'' - 4y' + 3y = 0 \tag{13.13}$$

Solution: The characteristic equation

$$\lambda^2 - 4\lambda + 3 = 0,$$

has distinct real roots $\lambda_1 = 1$ and $\lambda_2 = 3$. Thus, the general solution is

$$y(x) = c_1 e^x + c_2 e^{3x}$$

with arbitrary coefficients determined by boundary conditions.

Example 13.8. Find the general solution of this equation

$$y'' - 2y' + y = 0 \tag{13.14}$$

Solution: The characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0,$$

has equal roots $\lambda = \lambda_{1,2}$. The ode can be written equivalently as:

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} - 1\right) y = 0$$

Let us use the substitution
$$\left(\frac{d}{dx}-1\right)y \equiv u(x)$$
so that the main ode is
$$\left(\frac{d}{dx}-1\right)u = 0.$$
The corresponding general solution is
$$u(x) = c_1 e^x.$$

We solve the remaining equation from substitution

$$\left(\frac{d}{dx}-1\right)y = c_1 e^x$$
which leads to
$$e^{-x}y' - ye^{-x} = c_1 \rightarrow \frac{d}{dx}\left(e^{-x}y\right) = c_1 \rightarrow ye^{-x} = c_1 x + c_2$$

HENCE the general solution is:
$$y(x) = (c_1 x + c_2)e^x$$

Example 13.9. Find the general solution of this equation
$$y'' - 2y' + 2y = 0 \tag{13.15}$$

Solution: The characteristic equation
$$\lambda^2 - 2\lambda + 2 = 0,$$
has the complex conjugate roots $\lambda_{1,2} = 1 \pm i$. The ode can be written equivalently as:
$$\left(\frac{d}{dx}-1+i\right)\left(\frac{d}{dx}-1-i\right)y = 0$$
Solve
$$y' - (1-i)y = 0$$
$$\frac{dy}{y} = (1-i)dx \rightarrow \ln y = (1-i)x + c \rightarrow y(x) = ce^x e^{-ix}.$$
The general solution of
$$y' - (1+i)y = 0$$
is
$$y(x) = ce^x e^{ix}.$$
Hence, the general solution is
$$y(x) = e^x(c_1 e^{-ix} + c_2 e^{ix})$$

Lecture 14: Linear second order ODEs

14.1 *2nd order ODE's with constant coefficients*

This is a class of ode's that are analytically tractable. We present how to apply the method of undetermined coefficients to find the general solution of non-homogeneous odes.

14.1.1 *Method of undetermined coefficients*

Non-homogeneous ODE $R(x) \neq 0$ A non-homogeneous, linear 2nd order ode with constant coefficients has a general form

$$\boxed{y'' + ay' + by = R(x)}$$

Recall that the homogeneous solution

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

is obtained through the characteristic equation $\lambda^2 + a\lambda + b = 0$ that has the roots $\lambda_{1,2}$ which determine the independent solutions y_1 and y_2 (as discussed in the previous lecture). In addition, we also have a particular solution $y_p(x)$ that is determined by the *forcing* term $R(x)$. Below, we discuss three classes of forcing terms for which we determine y_p. The corresponding system of 1st order ODEs are

$$\begin{aligned} \left(\frac{d}{dx} - \lambda_1\right) u(x) &= R(x) \\ \left(\frac{d}{dx} - \lambda_2\right) y(x) &= u(x) \end{aligned}$$

$R(x)$	cases	$y_p(x)$
Ae^{kx}	$k \neq \lambda_1 \neq \lambda_2$	Be^{kx}
	$k = \lambda_1$ or $k = \lambda_2$ and $\lambda_1 \neq \lambda 2$	Cxe^{kx}
	$k = \lambda_1 = \lambda_2$	Dx^2e^{kx}
Ae^{ikx}	$k \neq \lambda_1 \neq \lambda_2$	Be^{ikx}
	$k = \lambda_1$ or $k = \lambda_2$ and $\lambda_1 \neq \lambda 2$	Cxe^{ikx}
	$k = \lambda_1 = \lambda_2$	Dx^2e^{ikx}
$e^{kx}P_n(x)$	$k \neq \lambda_1 \neq \lambda_2$	$e^{kx}Q_n(x)$
	$k = \lambda_1$ or $k = \lambda_2$ and $\lambda_1 \neq \lambda 2$	$xe^{kx}Q_n(x)$
	$k = \lambda_1 = \lambda_2$	$x^2e^{kx}Q_n(x)$

Table 14.1: Table of particular solutions for different forms of $R(x)$. A is constant, $P_n(x)$ and $Q_n(x)$ are polynomials of order n.

with the integral solutions

$$\begin{aligned} u(x) &= c_1 e^{\lambda_1 x} + e^{\lambda_1 x} \int ds R(s) e^{-\lambda_1 s} \\ y(x) &= c_2 e^{\lambda_2 x} + e^{\lambda_2 x} \int ds u(s) e^{-\lambda_2 s}. \end{aligned}$$

Again depending on the value of $\lambda_{1,2}$ and the form of the sourcing term $R(x)$, we distinguish different cases. This is best explored by concrete examples.

Example 14.1. Find the general solution for this ODE:

$$y'' - 4y' + 3y = 2e^{-x}$$

Solution: From the characteristic equation

$$\lambda^2 - 4\lambda + 3 = 0$$

the roots are $\lambda_1 = 1$ and $\lambda_2 = 3$. Hence,

$$\begin{aligned} \left(\frac{d}{dx} - 1\right) u(x) &= 2e^{-x} \\ \left(\frac{d}{dx} - 3\right) y(x) &= u(x). \end{aligned}$$

The solution of the first equation is

$$u' - u = 2e^{-x} \to \frac{d}{dx}(ue^{-x}) = 2e^{-2x} \to u(x) = e^x(-e^{-2x} + c_1) = -e^{-x} + c_1 e^x.$$

Thus,

$$\begin{aligned} y' - 3y &= -e^{-x} + c_1 e^x \to \frac{d}{dx}(ye^{-3x}) = -e^{-4x} + c_1 e^{-2x} \\ y(x) &= e^{3x}\left(\frac{1}{4}e^{-4x} - \frac{1}{2}c_1 e^{-2x} + c_2\right) \end{aligned}$$

The general solution is

$$y(x) = c_1 e^x + c_2 e^{3x} + \frac{1}{4}e^{-x} = y_h + y_p, \qquad y_p = \frac{1}{4}e^{-x}.$$

In general, the particular solutions corresponding to a forcing term that is an exponential or a complex exponential are summarized in Table 14.1.1.

Example 14.2. Find the general solution for this ODE:

$$y'' - 4y' + 3y = xe^{-x}$$

Solution: We first solve,

$$\left(\frac{d}{dx} - 3\right) y \equiv u(x)$$

which leads to:

$$\begin{aligned} u' - u &= xe^{-x} \to \frac{d}{dx}(ue^{-x}) = xe^{-2x} \\ u(x) &= e^x(-\frac{1}{4}(1+2x)e^{-2x} + c_1) \\ &= -\frac{1}{4}(1+2x)e^{-x} + c_1 e^x \end{aligned}$$

Now, we solve for $y(x)$ from the substitution equation

$$\begin{aligned} y' - 3y &= -\frac{1}{4}(1+2x)e^{-x} + c_1 e^x \to \\ \frac{d}{dx}(ye^{-3x}) &= -\frac{1}{4}(1+2x)e^{-4x} + c_1 e^{-2x} \to \\ y(x) &= e^{3x}\left(\frac{1}{32}(3+4x)e^{-4x} - \frac{1}{2}c_1 e^{-2x} + c_2\right) \end{aligned}$$

Thus, the general solution is

$$y(x) = c_1 e^x + c_2 e^{3x} + \frac{1}{32}(3+4x)e^{-x} = y_h + y_p, \qquad y_p = \frac{1}{32}(3+4x)e^{-x}$$

In general, for the class of odes with $R(x) = e^{kx}P_n(x)$ where $P_n(x)$ is a polynomial of order n, the particular solutions are summarized in Table 14.1.1.

14.2 2nd order ODE with variable coefficients

Now we consider some methods of solving 2nd order ODE with variable coefficients

$$\boxed{y'' + P(x)y' + Q(x)y = R(x),} \tag{14.1}$$

using the same principle of reducing the order of the ODE.

14.2.1 Homogeneous case: $R(x) = 0$

Method of variation of constants: This methods applies when we already know one independent solution, e.g. $y_1(x)$. Then, the other independent solution can be found by the ansatz

$$y_2(x) = c(x)y_1(x)$$

where $c(x)$ is determined by inserting this solution into the ODE:

$$c''y_1 + 2c'y_1' + y''c + P(c'y_1 + cy_1') + Qcy_1 = 0.$$

This leads to a *linear* 1st order ode in $u(x) \equiv c'(x)$ which can be integrated out by e.g. the integrating factor method,

$$u'y_1(x) + [2y_1'(x) + P(x)y_1(x)]u = 0 \to u(x) \to c(x) = \int dxu(x).$$

When we keep track of the integration constants, then we actually obtain the general solution

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Example 14.3. Let us consider this equation

$$y'' + \frac{y'}{x^2} - \frac{y}{x^3} = 0, \qquad x \neq 0$$

for which you can show that $y_1(x) = x$ is a solution. We want to find the other independent solution by the variation of constant method:

$$y_2(x) = c(x)y_1(x)$$

The corresponding equation for $u(x) = c'(x)$ is

$$x^2u' + (2x+1)u = 0$$

which can be solved by the separation of variables

$$\frac{du}{u} = -\frac{2x+1}{x^2}dx \to \tag{14.2}$$

$$\ln u(x) = -2\ln x + \frac{1}{x} \tag{14.3}$$

$$u(x) = \frac{1}{x^2}e^{1/x} \tag{14.4}$$

up to the arbitrary integration constant which we disregard here. Integrating it once more, we find

$$\frac{dc}{dx} = \frac{1}{x^2}e^{1/x} \to c(x) = e^{1/x}, \tag{14.5}$$

up to the second integration constant. Thus, the other particular solution is

$$y_2(x) = c(x)y_1(x) = -xe^{1/x}$$

and the general solution solution is then

$$y(x) = c_1x - c_2xe^{1/x}.$$

14.2.2 *Non-homogeneous case: $R(x) \neq 0$*

Method of factorization: This is a generic method of finding a particular solution $y_p(x)$ when we know at least one of the independent solutions of the homogeneous equation, e.g. $y_1(x)$. We then write the particular solution as

$$y_p = v(x)y_1(x),$$

and determine $v(x)$ from the inserting this solution into the ode:

$$y_p' = y_1'v + y_1v', \qquad y_p'' = y_1''v + 2y_1'v' + y_1v''$$

Inserting this particular solution into the non-homogeneous ODE

$$v(y_1'' + Py_1' + Qy_1) + v'(2y_1' + y_1P) + v''y_1' = R,$$

we can reduces it to a *linear 1st order ode* in v'

$$v''y_1' + v'(2y_1' + y_1P) = R.$$

We can find the solution of v' up to an arbitrary integration constant, and integrate it once more to determine $v(x)$ up to another integration constant. When we also include the contributions with the arbitrary integration constants, then we actually obtain the general solution

$$y(x) = y_1(x) + y_2(x) + y_p(x).$$

Therefore, by neglecting the integration constant, we are hunting only for the particular solution.

Example 14.4. Find the particular solution of this equation

$$y'' + \frac{y'}{x^2} - \frac{y}{x^3} = \frac{5}{x^3}.$$

Let us use again $y_1(x) = x$ to construct the ansatz of the particular solution as

$$y_p = xv(x).$$

The corresponding 1st order ODE for $u(x) = v'(x)$ is then

$$xu' + \left(2 + \frac{1}{x}\right)u = 5x^3$$

or equivalently

$$u' + \left(\frac{2}{x} + \frac{1}{x^2}\right)u = \frac{5}{x^4}.$$

This can be integrated by multiplying on both sides with $e^{2\ln x - 1/x} = x^2e^{-1/x}$, such that

$$\frac{d}{dx}\left(ux^2e^{-1/x}\right) = \frac{5}{x^2}e^{-1/x}.$$

By integrating this up to the arbitrary constant, we find

$$ux^2 e^{-1/x} = 5e^{-1/x} \to u(x) = \frac{5}{x^2}.$$

Thus,

$$v'(x) = \frac{5}{x^2} \to v(x) = -\frac{5}{x}$$

which implies that one particular solution is

$$y_p = -5.$$

Variation of parameters: This is another generic method of finding a particular solution $y_p(x)$ once we have determined the homogeneous solution

$$y_h = c_1 y_1(x) + c_2 y_2(x).$$

This is based on the following ansatz (which we will show later that can be determined through the *Green's function method*)

$$\boxed{y_p(x) = f_1(x) y_1(x) + f_2(x) y_2(x),} \tag{14.6}$$

where we replaced the constants c_1, c_2 by unknown functions $f_1(x), f_2(x)$ to be determined by the ODE and the additional constraint that

$$f_1' y_1 + f_2' y_2 = 0.$$

By taking successive derivatives of $y_p(x)$

$$y_p' = f_1 y_1' + f_2 y_2' \tag{14.7}$$

$$y_p'' = f_1' y_1' + f_1 y_1'' + f_2' y_2' + f_2 y_2'', \tag{14.8}$$

and insert them into the main ODE, we find the corresponding equation satisfied by f_1 and f_2 and get to system of equations for f_1', f_2'

$$\begin{aligned} f_1' y_1 + f_2' y_2 &= 0 \\ f_1' y_1' + f_2' y_2' &= R(x) \end{aligned}$$

This has a unique solution when the determinant - the *Wronskian*- is nonzero, namely

$$W(x) \equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

The Wronskian is indeed non-zero because y_1 and y_2 are independent functions. Thus, the solutions for f_1' and f_2' are

$$\boxed{f_1'(x) = -\frac{y_2(x)}{W(x)} R(x), \quad f_2'(x) = \frac{y_1(x)}{W(x)} R(x),} \tag{14.9}$$

which, by integration, leads to the particular solution

$$\boxed{y_p(x) = -y_1(x)\int dx \frac{y_2(x)}{W(x)}R(x) + y_2(x)\int dx \frac{y_1(x)}{W(x)}R(x).} \tag{14.10}$$

Example 14.5. Find the particular solution $y_p(x)$ for this ODE

$$y'' - y = \cosh x$$

using the variation of parameters method.

Solution: The homogeneous equation reduces to

$$y'' - y = 0 \to \left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 1\right) y = 0,$$

and has the independent solutions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. We compute the Wronskian for these independent solutions at an arbitrary point x to show that it is everywhere nonzero

$$W(x) \equiv \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0.$$

The solutions for $f_1(x)$ and $f_2(x)$ follow from integrating their corresponding equations

$$\begin{aligned} f_1'(x) &= -\frac{y_2(x)}{W(x)}R(x) = \frac{e^{-x}}{2}\cosh(x) = \frac{e^{-x}}{2}\frac{e^x + e^{-x}}{2} \\ f_2'(x) &= \frac{y_1(x)}{W(x)}R(x) = -\frac{e^x}{2}\cosh(x) = -\frac{e^x}{2}\frac{e^x + e^{-x}}{2}, \end{aligned}$$

$$\begin{aligned} f_1'(x) &= \frac{1 + e^{-2x}}{4} \to f_1(x) = \frac{1}{4}(x - \frac{1}{2}e^{-2x}) \\ f_2'(x) &= -\frac{e^{2x} + 1}{4} \to f_1(x) = -\frac{1}{4}(x + \frac{1}{2}e^{2x}). \end{aligned}$$

Thus, the particular solution is

$$y_p(x) = \frac{1}{4}(x - \frac{1}{2}e^{-2x})e^x - \frac{1}{4}(x + \frac{1}{2}e^{2x})e^{-x},$$

or by rearranging terms,

$$y_p(x) = \frac{1}{4}x(e^x - e^{-x}) - \frac{1}{8}(e^x + e^{-x}).$$

Example 14.6. Solve the same ODE

$$y'' - y = \cosh x$$

using the factorization method.

Solution: Now we start with the ansatz for the particular solution $\tilde{y}_p(x) = c(x)e^x$ given in terms of one of the independent solutions (let's take $y_1(x) = e^x$) and varying the coefficient. Inserting this ansatz into our ODE, we obtain a 1st order ODE for $c'(x)$:

$$c'' + 2c' = e^{-x}\frac{e^x + e^{-x}}{2},$$

which we can solve by the integration factor method

$$\frac{d}{dx}(c'e^{2x}) = \frac{e^{2x}+1}{2} \to c' = \frac{1}{4} + \frac{1}{2}xe^{-2x} \to c(x) = \frac{1}{4}x(1-e^{-2x}) - \frac{1}{8}e^{-2x}$$

Thus, the particular solution is

$$\tilde{y}_p(x) = \frac{1}{4}x(e^x - e^{-x}) - \frac{1}{8}e^{-x}$$

What's going on? The particular solution is **not unique**. However, we must always check that $y_p - \tilde{y}_p$ is a linear combinations of the independent solutions:

$$y_p - \tilde{y}_p = -\frac{1}{8}e^x.$$

14.2.3 *Homogeneous Euler-Cauchy equation*

THE EULER–CAUCHY EQUATION is intimately related to solving the Laplace's equation in polar coordinates, and is relevant to many physics applications in quantum and classical mechanics, fluid dynamics, statistical physics, etc. The homogeneous Euler-Cauchy equation is given in the general form as

$$\boxed{x^2y'' + a_1xy' + a_0y = 0}, \qquad (14.11)$$

where a_1 and a_0 are constants. We will go through two equivalent methods that are tailored to this equation.

Method 1: We use the ansatz that the independent solutions are power functions of the form x^m and determined m by inserting this into Eq (14.11)

$$[m(m-1) + a_1m + a_0]x^m = 0 \to m^2 + (a_1 - 1)m + a_0 = 0, \qquad x \neq 0$$

If $m_1 \neq m_2$, then we have determined two independent solutions and the general solution follows by linear superposition

$$y(x) = c_1x^{m_1} + c_2x^{m_2}, \qquad m_1 \neq m_2$$

If $m_1 = m_2 = m$, then we still know one independent solution $y_1 = x^m$ and can determine the other one by the variation of constant method

$$y_2(x) = c(x)y_1(x) = c(x)x^m.$$

Example 14.7. Find the general solution of this Euler-Cauchy equation

$$x^2y'' + xy' - 9y = 0 \tag{14.12}$$

Solution: We insert an ansatz x^m and determine the quadratic form satisfied by m:

$$m^2 - 9 = 0 \to m_{1,2} = \pm 3,$$

for any $x \neq 0$. Thus, the general solution follows by a linear superposition of the independent solutions,

$$y(x) = c_1x^3 + c_2x^{-3}$$

which is valid for $x \neq 0$.

Example 14.8. Find the general solution of this Euler-Cauchy equation

$$x^2y'' - xy' + y = 0 \tag{14.13}$$

by this method.

Solution: We insert a solution of this form x^m and determined the quadratic form satisfied by m:

$$m^2 - 2m + 1 = 0 \to m_1 = m_2 = m \pm 1,$$

again for any $x \neq 0$. Let us look at the solution for $x > 0$. Thus, $y_1(x) = x$ and $y_2(x) = c(x)x$, where $c(x)$ is obtained by inserting $y_2(x)$ in Eq. (14.13):

$$xc'' + c' = 0 \to \frac{dc'}{c'} = -\frac{dx}{x} \to c' = \frac{1}{x} \to c = \ln x$$

and $y_2(x) = x \ln x$. The general solution is given as

$$y(x) = c_1x + c_2x \ln x, x > 0$$

A similar general form holds also for $x < 0$, but the coefficient might be different

$$y(x) = \tilde{c}_1x + \tilde{c}_2x \ln(-x), x < 0$$

Change of variable method: Another way to solve the Euler-Cauchy equation is to use the following change of variable:

$$\begin{aligned} x &= e^z, \qquad x > 0 \\ x &= -e^z, \qquad x < 0, \end{aligned}$$

such that the Eq. (14.11) transforms to an ODE with *constant* coefficients:

$$\frac{d^2y}{dz^2} + (a_1 - 1)\frac{dy}{dz} + a_0 y(z) = 0, \tag{14.14}$$

which can be solved through the method of characteristic equation.

$$y(x) = y(z) = c_1 y_1(z) + c_2 y_2(z) = c_1 y_1(\ln x) + c_2 y_2(\ln x)$$

HOWEVER, we need to consider separately the two branches of solutions for $x > 0$ and $x < 0$, because of the singular point at $x = 0$, as we have seen already with other method.

Example 14.9. Solve Eq. (14.12) by the change of variable method.

Solution: Using the substitution $x = e^z$ for $x > 0$, Eq. (14.12) transforms to

$$\frac{d^2y}{dz^2} - 9y(z) = 0 \to y(z) = c_1 e^{3z} + c_2 e^{-3z}$$

THUS the general solution is

$$y(x) = c_1 x^3 + c_2 x^{-3}$$

which is valid for $x \neq 0$ and the coefficients c_1, c_2 could be different for the two branches of solutions ($x < 0$ and $x > 0$).

Example 14.10. Solve the other example from Eq. (14.13) by this method.

Solution: Use the change of variable $x = e^z$ for $x > 0$, such that Eq. (14.12) transforms to

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y(z) = 0 \to \frac{du}{dz} - u(z) = 0 \to u(z) = Ae^z,$$

where

$$\frac{dy}{dz} - y(z) = u(z) = \tilde{A}e^z \to \frac{d}{dz}(ye^{-z}) = A \to y(z)e^{-z} = Az + B$$

THUS, the general solution for $x > 0$ is

$$y(x) = y(z) = Aze^z + Be^z \to y(x) = Ax \ln x + Bx,$$

and general solution for $x < 0$ has a similar form but the coefficients can be different

$$y(x) = y(z) = \tilde{A}x \ln |x| + \tilde{B}x, \qquad x < 0$$

Lecture 15: Green function method

15.1 *Green functions: Prelude*

Green functions are named after the mathematician and physicist George Green who developed this method and applied it to electricity and magnetism, in the late 19th Century. Nowadays, the Green function method is a powerful tool to solve linear response problems, widely applied in all corners of physics. Due to Feynman's diagrammatic representation of a Green function associated with the wave operator, the Green function is also referred to as the "propagator" between two states. In statistical physics, the Green function tells us about the linear response of a system at one point in space and time due to a perturbation at another point. Green functions are used to solve **linear, non-homogeneous differential equations** for given initial or boundary value conditions. They represent solutions to differential equations that have a localized force source, e.g. the electrical field induced by a point charge, or the temperature profile induced by a pointwise heat source, etc.

In this lecture, we will introduce the method of Green function to solve 2nd order ode. First, we will start by introducing the Dirac delta function to represent pointwise forcing terms.

15.2 Dirac delta function

To prepare the ground for the Green function method, let us first introduce the Dirac function $\delta(x)$ and its associated calculus. The Dirac $\delta(x)$ function is a highly-peaked distribution function defined through its integral:

$$\boxed{\int_a^b dx f(x)\delta(x-x_0) = \begin{cases} f(x_0) & \text{if } a < x_0 < b \\ 0, & \text{otherwise} \end{cases}} \tag{15.1}$$

The particular case of $f(x) = 1$ gives the normalization condition

$$\boxed{\int_{-\infty}^{\infty} dx\delta(x) = 1} \tag{15.2}$$

The $\delta(x)$, a limit of sharp pulse/spike of very short duration/extent, can be defined in terms of the Dirac delta sequences of strongly-peaked, differentiable functions $\{\phi_n(x)\}$ $(n = 1, 2, 3...)$ such that

$$\int_{-\infty}^{\infty} dx\delta(x) \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\phi_n(x-a) = 1 \tag{15.3}$$

and

$$\int_{-\infty}^{\infty} dx f(x)\delta(x-x_0) \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\phi_n(x-a)f(x) = f(a) \tag{15.4}$$

Within this definition, Eqs. (15.1) and (15.2) are understood as short-hand notation for the limit integration over the Dirac delta sequences. Notice that the sequence functions ϕ_n do not have a proper limit by themselves, i.e. only the limit of their integral is well-defined and thus they are "meaningful" only inside integrals. We can see this clearly by considering some examples of Dirac delta sequences.

As a concrete example, let us consider

$$\phi_n(x) = \begin{cases} 0 & |x| \geq \frac{1}{n} \\ \frac{n}{2}, & |x| \leq \frac{1}{n} \end{cases}, \tag{15.5}$$

We can see that ϕ_n are normalized for all n:

$$\int_{-\infty}^{\infty} dx\phi_n(x) = \frac{n}{2}\int_{-1/n}^{1/n} dx = 1$$

Also,

$$\int_{-\infty}^{\infty} dx\phi_n(x)f(x) = \frac{n}{2}\int_{-1/n}^{1/n} f(x)dx = f(y), \qquad -\frac{1}{n} \leq y \leq \frac{1}{n}$$

where we use the **mean value theorem** for integrals: that for $f(x)$ continuous in $[a, b]$, there exits a value $y \in [a, b]$ for which

$$\int_a^b dx f(x) = f(y)(b-a).$$

Thus, in the limit of $n \to \infty$, $y \to 0$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} dx \phi_n(x) f(x) = f(0)$$

Other examples of Dirac delta sequences:

$$\phi_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2} \tag{15.6}$$

$$\phi_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \tag{15.7}$$

$$\phi_n(x) = \frac{1}{n\pi} \frac{\sin^2(nx)}{x^2} \tag{15.8}$$

$$\tag{15.9}$$

Show that Eqs. 15.3 and 15.4 are satisfied by each of these delta sequences.

Heaviside step function is the step function defined as

$$H(x-a) = \begin{cases} 0 & x < a \\ 1, & x > a \end{cases},$$

and the anti-derivative of the Dirac delta function. Thus, it can be defined as the limit of the *smooth* Heaviside sequence functions

$$H(x) \equiv \lim_{n\to\infty} \psi_n(x),$$

where $\psi_n(x)$ are the anti-derivatives of the Dirac delta sequence functions:

$$\psi_n(x) = \int_{-\infty}^{x} \phi_n(x')dx' \to \frac{d\psi_n}{dx} = \phi_n(x)$$

such that the derivative of the Heaviside function is the delta function

$$H'(x) \equiv \lim_{n\to\infty} \psi_n'(x) = \lim_{n\to\infty} \phi_n(x) = \delta(x).$$

Dirac delta calculus means that we perform calculus-like operations involving delta functions, e.g. Eqs. (15.1)-(15.2), but with the understanding that this is just a shorthand notation for well-defined expressions involving limits of delta sequences. This also includes integration by parts. Since there exist delta sequences of differentiable functions, we perform integration by parts on the integral

$$\int_{-\infty}^{\infty} \phi_n'(x) f(x) dx = [\phi_n(x) f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi_n(x) f'(x) dx$$

The surface term vanishes, since each ϕ_n is a peaked function that decay fast at $\pm\infty$. Thus, taking the limit $r \to \infty$, we get that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n'(x) f(x) dx = -\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f'(x) dx$$

In a shorthand notation using the Dirac delta function, this is written as

$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx = -f'(0).$$

By a change of variables, we get the more general result

$$\int_{-\infty}^{\infty} \delta'(x-a) f(x) dx = -f'(a).$$

The generalization to higher-order derivatives of the delta function is that

$$\int_{-\infty}^{\infty} \delta^{(k)}(x-a) f(x) dx = (-1)^k f^{(k)}(a).$$

15.3 *Linear 2nd order ODEs: Green function method*

Let us consider a non-homogeneous linear 2nd order ODEs in the normal form

$$Dy(x) = R(x)$$

where D is the *2d order linear differential* operator

$$D \equiv \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)$$

and $R(x)$ is the forcing term. The Green function $G(x,z)$ is uniquely determined by the homogeneous part of the ODE, along with **specific** boundary conditions. It is **independent** of the forcing term $R(x)$. In principle, we could have any boundary conditions, but for now we consider either **homogeneous boundary conditions**:

$$y(a) = y(b) = 0. \tag{15.10}$$

or the **homogeneous initial conditions**

$$y(0) = y'(0) = 0. \tag{15.11}$$

The **specific** solution under boundary conditions is determined by a **convolution** integral of the Green function with the forcing term $R(x)$:

$$\boxed{y(x) = \int_a^b dz G(x,z) R(z), \qquad a \le x, z \le b.} \tag{15.12}$$

Similarly, the **specific** solution under initial conditions is determined by the **convolution** integral:

$$\boxed{y(x) = \int_0^\infty dz G(x,z) R(z), \qquad x,z \geq 0.} \tag{15.13}$$

The Green function is determined by the following properties:

Property 1: $G(x,z)$ is defined as a function that satisfies the ode with a Dirac delta forcing:

$$\boxed{DG(x,z) = \delta(x-z)} \tag{15.14}$$

Except for the singular point $x = z$ where ode is undetermined, this ode is otherwise homogeneous. Now, z is a variable for $G(x,z)$, so the singular point dependents on z, and we will see the implications of this later. We can derive Eq. (15.14) by applying the differential operator D on both sides of the specific solution in Eq. **??**:

$$\begin{aligned} Dy(x) &= \int_a^b [DG(x,z)]R(z) \rightarrow \\ \int_a^b \delta(x-z)R(z) \equiv R(x) &= \int_a^b [DG(x,z)]R(z) \rightarrow \\ DG(x,z) &= \delta(x-z) \end{aligned}$$

where z, x are both inside $[a,b]$. The singular point $x = z$ imposes additional restrictions on our $G(x,z)$.

Property 2:

$$\boxed{G(x,z) \text{ is continuous at } x = z} \tag{15.15}$$

Property 3:

$$\boxed{G(x,z) \text{ has a unit jump at } x = z} \tag{15.16}$$

We want to determine $G(x,z)$ as a **continuous** function everywhere in the integration domain $[a,b]$, hence also at the singular point $x = z$. Integrating on both sides of Eq. 15.14 over an infinitesimal domain around the singular point $[z-\epsilon, z+\epsilon]$:

$$\int_{z-\epsilon}^{z+\epsilon} dx DG(x,z) = \int_{z-\epsilon}^{z+\epsilon} dx \delta(x-z) = 1.$$

USING the definition of the differential operator, we find that

$$\int_{z-\epsilon}^{z+\epsilon} dx \frac{d^2}{dx^2} G(x,z) + \int_{z-\epsilon}^{z+\epsilon} dx P(x) \frac{d}{dx} G(x,z) + \int_{z-\epsilon}^{z+\epsilon} dx Q(x) G(x,z) = 1.$$

In the limit of $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \left[\frac{d}{dx} G(x,z) \right]_{z-\epsilon}^{z+\epsilon} + P(z) \lim_{\epsilon \to 0} \left[G(x,z) \right]_{z-\epsilon}^{z+\epsilon} = 1.$$

The continuity assumption implies that

$$\boxed{\lim_{\epsilon \to 0} \left[G(x,z) \right]_{z-\epsilon}^{z+\epsilon} = 0} \tag{15.17}$$

Thus, it follows that the jump in the first derivative must be equal to 1:

$$\boxed{\lim_{\epsilon \to 0} \left[\frac{d}{dx} G(x,z) \right]_{z-\epsilon}^{z+\epsilon} = 1} \tag{15.18}$$

Property 4:

$$\boxed{G(x,z) \text{ satisfies the same boundary/initial conditions as } y(x)} \tag{15.19}$$

The last property of $G(x,z)$ is determined by what kind of boundary conditions we impose on the initial ODE. We consider **homogeneous** boundary conditions given in Eq. 15.10. From Ef. **??**, it implies that for the homogeneous boundary conditions Eq. 15.10, $G(x,z)$ must also satisfy that

$$G(a,z) = G(b,z) = 0$$

For the homogeneous initial conditions from Eq. 15.11, $G(x,z)$ must satisfy that

$$G(0,z) = \frac{d}{dx} G(x,z)\Big|_{x=0} = 0$$

THESE four properties are sufficient to determined uniquely the function $G(x,z)$, and thus by the **specific** solution $y(x)$ from Eqs. 15.12 and 15.13.

Example 15.1. Find the Green function corresponding to

$$y'' + y = R(x),$$

with the homogeneous boundary conditions

$$y(0) = y(\pi/2) = 0$$

and for an arbitrary forcing term $R(x)$.

Solution: The Green function is determined by following ode

$$\frac{d^2}{dx^2}G(x,z) + \frac{d}{dx}G(x,z) = \delta(x-z)$$

with the homogeneous boundary conditions

$$G(0,z) = G(\pi/2,z) = 0.$$

The general solution of the homogeneous equation $y'' + y = 0$ follows quickly by successive integration method and is given by

$$y_h(x) = c_1 e^{ix} + c_2 e^{-ix}.$$

Thus, for $x \neq z$ the Green's function $G(x,z)$ given by this homogeneous solution. The singular point at $x = z$ introduces a jump in the coefficients c_1 and c_2 for $x < z$ and $x > z$. The two branches of solutions are

$$G(x,z) = \begin{cases} A(z)e^{ix} + B(z)e^{-ix}, & x \leq z \\ C(z)e^{ix} + D(z)e^{-ix}, & x \geq z \end{cases},$$

where $0 \leq x, z \leq \pi/2$ and the functions $A(z), B(z), C(z), D(z)$ are determined by the boundary conditions, the continuity of $G(x,z)$ at $x = z$ and the fact that the derivative $\frac{d}{dx}G(x,z)$ at $x = z$ has a jump equal to 1.

First, we notice that the boundary point $x = 0$ is located in the branch of solutions corresponding to $0 = x < z$, while $x = \pi/2$ picks up the other solution branch for $\pi/2 = x > z$.

Boundary conditions: $G(0,z) = G(\pi/2,z) = 0$ imply that:

$$G(0,z) = A(z) + B(z) = 0 \rightarrow B(z) = -A(z)$$

$$G(\pi/2,z) = C(z) - D(z) = 0 \rightarrow D(z) = C(z)$$

Thus,

$$G(x,z) = \begin{cases} A(z)(e^{ix} - e^{-ix}) = \tilde{A}(z)\sin x, & x \leq z \\ C(z)(e^{ix} + e^{-ix}) = \tilde{C}(z)\cos x, & x \geq z \end{cases}$$

Continuity at $x = z$ implies that

$$G(x \searrow z, z) = G(x \nearrow z, z)$$

$$\tilde{A}(z)\sin z = C(z)\cos z \rightarrow C(z) = \tilde{A}(z)\frac{\sin z}{\cos z}$$

Unit jump at $x = z$ implies that

$$\left(\frac{dG}{dx}\right)_{x\searrow z} - \left(\frac{dG}{dx}\right)_{x\nearrow z} = 1$$

which leads to

$$-\tilde{A}(z)\cos z - \tilde{A}(z)\frac{\sin z}{\cos z}\sin z = 1 \rightarrow \tilde{A}(z) = -\cos(z)$$

which means that $C(z) = -\sin(z)$. Thus, the Green function becomes

$$G(x,z) = \begin{cases} -\cos z \sin x, & x \leq z \\ -\sin z \cos x, & x \geq z. \end{cases}$$

The specific solution for this homogeneous boundary conditions follows by the convolution integral

$$\begin{aligned} y(x) &= \int_0^{\pi/2} dz G(x,z) R(z) \\ &= -\cos(x)\int_0^x dz \sin(z) R(z) - \sin(x)\int_x^{\pi/2} dz \cos(z) R(z) \end{aligned}$$

Lecture 16: Green function method Examples

16.1 *Green function method*

LET US recap the general steps in applying the Green function method to a generic 2nd order **linear and non-homogeneous** ODE,

$$Dy(x) = R(x), \qquad D \equiv \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x),$$

where D is the *linear differential* operator with either *homogeneous boundary conditions*:

$$y(a) = y(b) = 0$$

or *homogeneous initial conditions*

$$y(0) = y'(0) = 0.$$

The Green function $G(x, z)$ is determined by the following properties:

1. $DG(x, z) = \delta(x - z)$, thus $x = z$ is a singular point
2. $G(x, z)$ is assumed to be a *continuous* function at $x = z$.
3. $\frac{d}{dx}G(x, z)$ has a jump of size 1 at $x = z$.
4. $G(x, z)$ satisfies the same boundary conditions as $y(x)$

   $$G(a, z) = G(b, z) = 0$$

   or initial conditions

   $$G(0, z) = \frac{d}{dx}G(x, z)|_{x=0} = 0$$

Example 16.1. Find the Green function for a forced harmonic oscillator given by

$$y'' + y = R(x),$$

with the homogeneous initial conditions

$$y(0) = y'(0) = 0$$

and for an arbitrary forcing term $R(x)$.

Solution: The Green function is determined by the following ODE:

$$\frac{d^2}{dx^2}G(x,z) + G(x,z) = \delta(x-z)$$

with the homogeneous initial conditions

$$G(0,z) = \frac{d}{dx}G(x,z)|_{x=0} = 0.$$

For $x \neq z$, the Green function is the solution of the homogeneous equation $y'' + y = 0$, which has the independent solutions $e^{\pm ix}$. Hence, we can write the Green function as

$$G(x,z) = \begin{cases} A(z)e^{ix} + B(z)e^{-ix}, & x \leq z \\ C(z)e^{ix} + D(z)e^{-ix}, & x \geq z \end{cases},$$

where the coefficients $A(z), B(z), C(z), D(z)$ are functions of z and determined by the properties of $G(x,z)$ at the singular point $x = z$. The homogeneous initial conditions at $x = 0$ imply that

$$\begin{aligned} A(z) + B(z) &= 0 \\ A(z) - B(z) &= 0 \end{aligned}$$

thus $A(z) = B(z) = 0$ and

$$G(x,z) = \begin{cases} 0, & x \leq z \\ C(z)e^{ix} + D(z)e^{-ix}, & x \geq z \end{cases},$$

The continuity of $G(x,z)$ and the discontinuous derivative of magnitude 1 at $x = z$ translate to the following set of equations:

$$\begin{aligned} C(z)e^{iz} + D(z)e^{-iz} &= 0 \\ iC(z)e^{iz} - iD(z)e^{-iz} &= 1 \end{aligned}$$

with the solution: $C(z) = \frac{1}{2i}e^{-iz}$ and $D(z) = -\frac{1}{2i}e^{iz}$. Thus,

$$G(x,z) = \begin{cases} 0, & x \leq z \\ \frac{1}{2i}(e^{i(x-z} - e^{-i(x-z)}) = \sin(x-z), & x \geq z \end{cases}$$

The corresponding specific solution under *homogeneous initial conditions* is then given by the convolution integral of the above Green function with the forcing

$$\begin{aligned} y(x) &= \int_0^\infty dz G(x,z) R(z) \\ &= \int_0^x dz \sin(x-z) R(z) \end{aligned}$$

Example 16.2. Find the specific solution to the forced harmonic oscillator

$$y'' + y = \frac{1}{\sin x},$$

with the homogeneous initial conditions

$$y(0) = y'(0) = 0.$$

Solution: We have already derived the Green function as

$$G(x,z) = \begin{cases} 0, & x \le z \\ \sin(x-z), & x \ge z \end{cases}$$

The specific solution is then the convolution of $G(x,z)$ with the forcing term $\frac{1}{\sin z}$:

$$\begin{aligned} y(x) &= \int_0^\infty dz G(x,z) \frac{1}{\sin z} && (16.1) \\ &= \int_0^x dz \frac{\sin(x-z)}{\sin z} && (16.2) \\ &= \int_0^x dz \frac{\sin x \cos z - \cos x \sin z}{\sin z} && (16.3) \\ &= \sin x \int_0^x dz \frac{\cos z}{\sin z} - \cos x \int_0^x dz \frac{\sin z}{\sin z} && (16.4) \end{aligned}$$

We can of course easily solve for the integrals. However, we also notice that this integral form allows us to connect the variation of parameters method (i.e. the one with the Wronskian) with the Green function method. Let us compute the Wronskian corresponding to the independent solutions $y_1(x) = \sin x$ and $y_2(x) = \cos x$:

$$W(x) \equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

Thus, from the variation of parameters method, the specific solution can be written in this form

$$\begin{aligned} y(x) &= -y_1(x) \int_0^x dz \frac{y_2(z)}{W(z)} R(z) + y_2(x) \int_0^x dz \frac{y_1(z)}{W(z)} R(z) \\ &= y_1(x) \int_x^0 dz \frac{y_2(z)}{W(z)} R(z) + y_2(x) \int_0^x dz \frac{y_1(z)}{W(z)} R(z) \end{aligned}$$

Notice that each integral gives a function of x and a constant (the boundary conditions) which are multiplied with y_1 and y_2. Thus, we can determine $y_p(x)$ by dropping the constant limits and write the solution in terms of the indefinite integrals

$$\begin{aligned} y_p(x) &= -y_1(x)\int dx \frac{y_2(x)}{W(x)}R(x) + y_2(x)\int dx\frac{y_1(x)}{W(x)}R(x) \\ &= \sin x \int dx \frac{\cos x}{\sin x} - \cos x \int dx \frac{\sin x}{\sin x} = \sin x \ln|\sin x| - x\cos x \end{aligned}$$

Example 16.3. Find the specific solution for the forced harmonic oscillator

$$y'' + y = \frac{1}{\sin z},$$

with the homogeneous boundary conditions

$$y(0) = y(\pi/2) = 0.$$

Find the particular solution $y_p(x)$.

Solution: The Green function for this ODE was derived in the previous lecture and given as

$$G(x,z) = \begin{cases} -\cos z \sin x, & x \le z \\ -\sin z \cos x, & x \ge z \end{cases}$$

Thus, the specific solution follows as the convolution:

$$\begin{aligned} y(x) &= \int_0^{\pi/2} dz G(x,z)\frac{1}{\sin z} \\ &= -\sin x \int_x^{\pi/2} dz \frac{\cos z}{\sin z} - \cos x \int_0^x dz \sin z \frac{1}{\sin z} \end{aligned}$$

We recognize that this is a particular form of the general expression in terms of y_1 and y_2

$$y(x) = y_1(x)\int_x^b dz \frac{y_2(z)}{W(z)}R(z) + y_2(x)\int_a^x dz \frac{y_1(z)}{W(z)}R(z)$$

where a and b are the boundary limits. The particular solution is obtained by replacing the definite integrals (x is the upper limit on both) with indefinite integrals

$$\begin{aligned} y_p(x) &= -y_1(x)\int dx \frac{y_2(x)}{W(x)}R(x) + y_2(x)\int dx\frac{y_1(x)}{W(x)}R(x) \\ &= \sin x \int dx \frac{\cos x}{\sin x} - \cos x \int dx \sin x \frac{1}{\sin x} \\ &= \sin x \ln|\sin x| - x\cos x \end{aligned}$$

Notice, the particular solution is determined by the forcing term and is independent of the boundary conditions. Therefore, $y_p(x)$ is the same in these two examples.

16.1.1 *Connection to the variation of parameters method:*

Let us consider the general case again

$$Dy(x) = R(x),$$

with the homogeneous initial condition $y(0) = y'(0) = 0$ and compute the specific solution to this initial condition as well as the particular solution as part of the general solution for arbitrary boundary/initial conditions.

The Green function is written as linear superpositions of the corresponding independent solution $y_1(x)$ and $y_2(x)$ on each side of the singular point at $x = z$, thus the two branches

$$G(x,z) = \begin{cases} A(z)y_1(x) + B(z)y_2(x), & x < z \\ C(z)y_1(x) + D(z)y_2(x), & x > z \end{cases}$$

From the initial condition $G(0,z) = 0$ and $\frac{d}{dx}G(x,z)|_{x=0} = 0$:

$$\begin{aligned} Ay_1(0) + By_2(0) &= 0 \\ Ay_1'(0) + By_2'(0) &= 0 \end{aligned}$$

When the Wronskian $W(0) \neq 0$, the unique solution is the trivial solution $A(z) = B(z) = 0$.

Hence,

$$G(x,z) = \begin{cases} 0, & x < z \\ C(z)y_1(x) + D(z)y_2(x), & x < z \end{cases}$$

The other constants are determined from the continuity of $G(x,z)$ and the unit jump in its derivative at $x = z$:

$$\begin{aligned} Cy_1(z) + Dy_2(z) &= 0 \\ Cy_1'(z) + Dy_2'(z) &= 1 \end{aligned}$$

The solution is this system of equations can be written in terms of the Wronskian $W(z)$:

$$C(z) = -\frac{y_2(z)}{W(z)}, \qquad D(z) = \frac{y_1(z)}{W(z)}$$

Thus,

$$G(x,z) = \begin{cases} 0, & x < z \\ -y_1(x)\frac{y_2(z)}{W(z)} + y_2(x)\frac{y_1(z)}{W(z)}, & x > z \end{cases}$$

The specific solution then follows as the convolution integral

$$\begin{aligned} y(x) &= \int_0^\infty dz G(x,z)R(z) \\ &= -y_1(x)\int_0^x dz\frac{y_2(z)}{W(z)}R(z) + y_2(x)\int_0^x dz\frac{y_1(z)}{W(z)}R(z) \end{aligned}$$

A particular solution is obtained by replacing the definite with indefinite integrals in x

$$y_p(x) = -y_1(x)\int dx\frac{y_2(x)}{W(x)}R(x) + y_2(x)\int dx\frac{y_1(x)}{W(x)}R(x)$$

which, by default, is generic for arbitrary boundary conditions. This is exactly the same formula that we arrived at using the variation of constants method. *Note that, often, we can determine $y_p(x)$ as being the same as the specific solution with homogeneous boundary conditions because then any applied boundary conditions will be imposed on the homogeneous solution $y_h(x) = c_1y_1(x) + c_2y_2(x)$.*

Example 16.4. Find the solution for the damped, harmonic oscillator

$$y'' + 2y' + y = f(t)$$

where

$$f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & t > 0 \end{cases}$$

with the initial condition $y(0) = y'(0) = 0$.

Solution: The corresponding linear differential operator is

$$D = \frac{d^2}{dt^2} + 2\frac{d}{dt} + 1$$

The independent solutions of the homogeneous ode can be determined by the successive integration method and equal to

$$y_1(t) = e^{-t}, \qquad y_2(t) = te^{-t}$$

The Green's function for the homogeneous initial condition is then

$$G(t,t') = \begin{cases} 0, & t \le t' \\ C(t')e^{-t} + D(t')te^{-t}, & t \ge t' \end{cases}$$

From the properties of $G(t,t')$ at $t = t'$:

$$\begin{aligned} Ce^{-t'} + Dt'e^{-t'} &= 0 \\ -Ce^{-t'} + D(1-t')e^{-t'} &= 1 \end{aligned}$$

with the solutions: $C = -t'e^{t'}$ and $D = e^{t'}$. Thus,

$$G(t,t') = \begin{cases} 0, & t < t' \\ -t'e^{t'-t} + te^{t'-t}, & t > t' \end{cases}$$

and the solution is

$$\begin{aligned} y(t) &= -\int_0^t dt' t' e^{t'-t} f(t') + \int_0^t dt' e^{t'-t} f(t') \\ &= \begin{cases} -\int_0^t dt' t' e^{t'-t} + \int_0^t dt' e^{t'-t}, & 0 < t < a \\ 0, & t > a \end{cases} \\ &= \begin{cases} 1 - e^{-t}(1+t), & 0 < t < a \\ 0, & t > a \end{cases} \end{aligned}$$

Lecture 17: Power series method

Some 2nd order linear ODEs might be difficult to solve with any of the previous methods for the reason that the independent solutions are not in a closed form (in terms of elementary functions), but instead are infinite series. The power series method allows us to generate special functions such as Legendre polynomials or Bessel functions which are infinite series solutions to ODEs with the similar names, and appear in many physics problems.

Let us consider the normal form of a 2nd order *homogeneous and linear* ODEs given by

$$y'' + p(x)y' + q(x)y = 0, \tag{17.1}$$

where $p(x)$ and $q(x)$ are some elementary functions of x. When $q(x)$ and $q(x)$ can be Taylor expanded around some expansion point x_0, the solution of Eq. 17.1 can also be written in the form of **power series**

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n} \tag{17.2}$$

This method can be generalized to series expansions around an arbitrary x_0 not necessarily a regular point, in which case the power series may have have *non-integer* exponents. This generalised power series is called **Frobenius series** given as

$$\boxed{y(x) = x^s \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad s \text{ is real or complex}} \tag{17.3}$$

Of course, the ODE has a valid solution on the domain of x where the series expansions are convergent. In this lecture we focus on generating the independent solutions using the power series method.

17.1 *Power series method*

THIS METHOD APPLIES when $p(x)$ and $q(x)$ are regular functions at x_0, i.e. they can be Taylor expanded around x_0. Then, the solution can also be represented as a convergent power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots .$$

with coefficients a_n determined by Eq. 17.1. The first and second derivatives of the solution $y(x)$ follow straightforwardly through term-by-term differentiations and are also power series,

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = 2a_2 + 6a_3(x - x_0) + \cdots . \end{aligned}$$

The gist of the method is that we insert these power series into Eq. 17.1, collect terms with equal powers of $(x - x_0)$ and set their corresponding coefficients to zero order by other. This way, we generate a set of recursive equations for a_n's which can be solved by the iterative method.

We will apply this technique for a 2nd order ODE which is commonly encountered in many physics problems and is called the **Legendre equation**. We will find that its independent solutions reduce to the finite power series called the **Legendre polynomials**.

17.1.1 *Legendre equation*

THE LEGENDRE EQUATION is an important ODE that arises in many physics problems that involve a spherical symmetry and is associated with the Laplace operator ($\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$) in spherical coordinates. It is used to describe the steady flow of an ideal fluid past a spherical obstacle, the electric field induced by a charged sphere, the temperature profile around a heated spherical object, the electron wavefunction for the hydrogen atom, and so on.

The Legendre equation is defined as

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0, \qquad l \in \mathcal{N}. \tag{17.4}$$

Relating it to the canonical form, we identify the coefficients $p(x)$ and $q(x)$ as

$$p(x) = -\frac{2x}{1-x^2}, \qquad q(x) = \frac{l(l+1)}{1-x^2}, \qquad \text{for } x^2 \neq 1.$$

Notice that $p(x)$ and $q(x)$ are both analytic at $x_0 = 0$ which indicates that we can use the power series expansion around this point to write the general solution as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The convergence domain of this power series is the same as the convergence domain of the Taylor series of $p(x)$ and $q(x)$ at $x_0 = 0$ and determined by the fraction

$$\frac{1}{1-x^2} = \sum_{n}^{\infty} x^{2n}$$

which is convergent for $|x| < 1$.

We differentiate the general solution term-by-term to obtain Taylor series for its first two derivatives,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Next, we insert these series expansions into Eq. (17.4)

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

By rearranging terms,

$$\textstyle\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

and absorbing the x-dependent terms coefficients inside the series

$$\textstyle\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + l(l+1)a_n]x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n = 0.$$

Equivalently, we can rewrite this as

$$\begin{aligned} & 2a_2 + l(l+1)a_0 + [6a_3 + (l(l+1)-2)a_1]x \\ + & \sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} + [l(l+1) - n(n-1) - 2n]a_n\}x^n = 0 \end{aligned}$$

For this power series to equal zero for any x, the coefficients in front of each power of x must vanish. This leads to recursion formulas for the coefficients a_n, namely

$$\begin{aligned} 2a_2 + l(l+1)a_0 &= 0 \\ 6a_3 + (l^2+l-2)a_1 &= 0 \\ (n+2)(n+1)a_{n+2} + [l^2+l-n^2-n]a_n &= 0, \qquad n \geq 2. \end{aligned}$$

which imply that

$$\begin{aligned} a_2 &= -\frac{l(l+1)}{2}a_0 \\ a_3 &= -\frac{(l-1)(l+2)}{6}a_1 \\ a_{n+2} &= -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n, \qquad n \geq 2 \end{aligned}$$

From the recursive relation, we notice that all coefficients of even powers x^{2k} are generated by a_0, namely

$$\begin{aligned} a_2 &= -\frac{l(l+1)}{2}a_0 \\ a_4 &= -\frac{(l-2)(l+3)}{3\cdot 4}a_2 = (-1)^2\frac{l(l-2)\cdot(l+1)(l+3)}{4!}a_0 \\ &\dots \\ a_{2n} &= (-1)^n\frac{l(l-2)\cdots(l-2n+2)\cdot(l+1)(l+3)\cdots(l+2n-1)}{(2n)!}a_0 \end{aligned}$$

while all coefficients of odd powers x^{2k+1} are generated by a_1,

$$\begin{aligned} a_3 &= -\frac{(l-1)(l+2)}{2\cdot 3}a_1 \\ a_5 &= -\frac{(l-3)(l+4)}{4\cdot 5}a_1 = (-1)^2\frac{(l-1)(l-3)\cdot(l+2)(l+4)}{5!}a_1 \\ &\dots \\ a_{2n+1} &= (-1)^n\frac{(l-1)(l-3)\cdots(l-2n+1)\cdot(l+2)(l+4)\cdots(l+2n)}{(2n+1)!}a_1 \end{aligned}$$

WE NOTICE THAT a_0 and a_1 are two arbitrary constants and they can be factored out in front of the corresponding power series. Thus, the general solution of Eq. (17.4) is written as a superposition of the independent solutions

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where the y_1 and y_2 are power series with uniquely defined coefficients given as

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{l(l-2)\cdots(l-2n+2)\cdot(l+1)(l+3)\cdots(l+2n-1)}{(2n)!}x^{2n} \\ y_2(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{(l-1)(l-3)\cdots(l-2n+1)\cdot(l+2)(l+4)\cdots(l+2n)}{(2n+1)!}x^{2n+1} \end{aligned}$$

Convergence interval: As expected from the structure of $p(x)$ and $q(x)$, the series should be convergent for $|x| < 1$, and we can double-check it using the ratio test.

For $y_1(x)$:

$$\rho_1 = \lim_{n\to\infty} \frac{(l-2n-4)\cdot(l-2n+1)}{(2n+1)(2n+2)}|x|^2 < 1 \to |x|^2 < 1$$

For $y_2(x)$:

$$\rho_2 = \lim_{n\to\infty} \frac{(l-2n-1)\cdot(2n+4)}{(2n+3)(2n+3)}|x|^2 < 1 \to |x|^2 < 1$$

Important point: The convergence at the end points $x = \pm 1$ depends on specific values of the parameter l. It can be shown that for a positive integer l, only one of the independent solutions is convergent at $x = \pm 1$ and becomes a polynomial of degree l which is called the **Legendre polynomial**. We distinguish two cases which we will take separately, namely when l is even or odd, respectively.

For $l = 2m$: Let us take the lowest value of the parameter $l = 0$. From the recursive relations in Eq. (17.5), we see that all coefficients depend on l thus are also zero, expect for a_0. Now let us consider the next order $l = 2$, for which only a_0 and

$$a_2 = -3a_0$$

are non-zero and all the other coefficients becomes zero because of the $l-2$ term in the product. Perhaps we start to see the pattern that for a degree $l = 2m$ only the first coefficients up to a_{2m} are non-zero and all other vanish. Hence, the infinite series becomes a polynomial of degree $2m$. Thus, in general, the products of l's in Eq. (17.5) can be written in terms of factorials, namely

$$\begin{aligned}
l(l-2)\cdots(l-2n+2) &= 2m(2m-2)\cdots(2m-2n+2) \\
&= \frac{2^n m!}{(m-n)!} \\
(l+1)(l+3)\cdots(l+2n-1) &= (2m+1)(2m+3)\cdots(2m+2n-1) \\
&= \frac{(2m+2n)!m!}{2^n(2m)!(m+n)!}
\end{aligned}$$

such that the coefficients of even powers are

$$a_{2n} = (-1)^n \frac{(m!)^2}{(2m)!} \frac{(2m+2n)!}{(m-n)!(m+n)!(2n)!} a_0 \qquad (17.5)$$

which are non-zero for $n < m$ and zero otherwise. This means that the independent solution $y_1(x)$ reduces to a finite sum given by a polynomial of degree $2m$, namely

$$\boxed{y_1(x) \equiv Q_{2m}(x) = \frac{(m!)^2}{(2m)!}\sum_{n=0}^{m}(-1)^n\frac{(2m+2n)!}{(m+n)!(m-n)!(2n)!}x^{2n}}$$

which determines the Legendre polynomial $P_n(x)$ up to some rescaling constant fixed by the boundary condition. Thus, there is a parametric family of Legendre equations for different values of even values of l parameter $l = 0, 2, \cdots, 2m$ where $y_1(x)$ is given by the Legendre polynomial of degree l. For instance, the polynomials of lowest degrees are

$$\begin{aligned} P_0(x) &= Q_0(x) = 1 \\ P_2(x) &= -\frac{1}{2}Q_2(x) = \frac{1}{2}(3x^2-1) \end{aligned}$$

Note that the other independent solution $y_2(x)$ remains an infinite series with odd powers of x, because the coefficients of x^{2n+1} are never zeroes and therefore it is divergent at $x = \pm 1$. For example, let us consider the case with $l = 0$ and see that the recursive relations for a_{2n+1} gives non-zero a_{2n+1} for any n:

$$\begin{aligned} a_3 &= -\frac{(l-1)(l+2)}{2\cdot 3}a_1 = \frac{1}{3}a_1 \\ a_5 &= -\frac{(l-3)(l+4)}{4\cdot 5}a_1 = \frac{1}{5}a_1 \\ &\cdots \\ a_{2n+1} &= -\frac{(l-2n+1)(l+2n)}{2n\cdot(2n+1)}a_{2n-1} = \frac{1}{2n+1}a_1 \end{aligned}$$

Thus,

$$y_2(x) = \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}$$

which diverges at $|x|^2 = 1$.

For $l = 2m+1$***:*** Now, let us consider the lowest odd parameter $l = 1$. From the recursive relations in Eq. (17.5), it follows that the corresponding coefficients in $y_2(x)$ are

$$a_{2n+1} = 0, \quad \text{for } n \geq 1$$

hence the second independent solution is simply

$$y_2(x) = x$$

On the other hand, the coefficients of even exponents in $y_1(x)$ are all non-zero

$$\begin{aligned} a_2 &= -\frac{l(l+1)}{2}a_0 = -a_0 \\ a_4 &= -\frac{(l-2)(l+3)}{3\cdot 4}a_2 = -\frac{1}{3}a_0 \\ &\dots \\ a_{2n} &= -\frac{(l-2n+2)(l+2n-1)}{(2n)(2n-1)}a_{2n-2} = \frac{(2n-3)}{(2n-1)}a_{2n-2} = -\frac{1}{(2n-1)}a_0 \end{aligned}$$

and

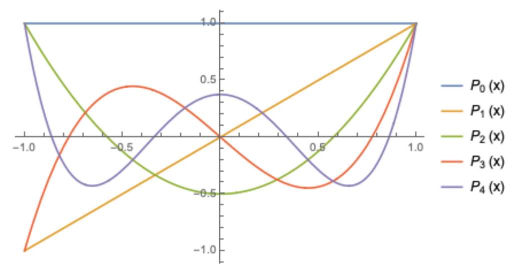
$$y_1(x) = -\sum_{n=0}^{\infty} \frac{1}{2n-1}x^{2n}$$

and diverges at $|x| = 1$.

We can generalize this to any odd parameter $l = 2m+1$ and find that, in this case, $y_1(x)$ remains an infinite power series, while $y_2(x)$ reduces to the polynomial of degree $2m+1$, $Q_{2m+1}(x)$, given as

$$\boxed{y_2(x) \equiv Q_{2m+1} = \frac{(m!)^2}{(2m+1)!}\sum_{n=0}^{m}(-1)^n \frac{(2m+2n+1)!}{(m+n)!(m-n)!(2n+1)!}x^{2n+1}}$$

The first P_n polynomials are



$$\begin{aligned} P_1(x) &= Q_1(x) = x \\ P_3(x) &= -\frac{3}{2}Q_2(x) = \frac{1}{2}(5x^3 - 3x) \end{aligned}$$

Properties of Legendre polynomials $P_n(x)$: By setting the boundary condition so that $y(x) = 1$ when $x = 1$, the coefficients a_0 and a_1 are fixed and *specific solution* of the Legendre equation is then the **Legendre polynomial of order** l $P_l(x)$. It is determined by the polynomial $Q_l(x)$ with the appropriate rescaling x. Fig.17.1.1 show the dependence of the Legendre polynomials on x for some of the lowest even or odd degrees.

The Legendre polynomials $P_n(x)$ satisfy these properties (which we list without proving them here):

- $P_n(1) = 1$, for any $n \geq 0$ positive integer
- $P_n(x) = (-1)^n P_n(-x)$, for any $n \geq 0$ positive integer
- **Legendre polynomials are orthogonal functions**

$$\int_{-1}^{1} dx P_n(x) P_m(x) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

- **Rodrigues' formula**

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$$

- **Recursion relation**

$$\boxed{nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)}$$

Generating function for $P_n(x)$ The generating function for $P_n(x)$ is given as

$$\boxed{\Phi(x,h) = (1-2xh+h^2)^{-1/2}, \qquad |h| < 1,}$$

such that $P_n(x)$ are given as the coefficients of the power series of $\Phi(x,h)$ wrt h:

$$\boxed{\Phi(x,h) = \sum_{n=0}^{\infty} P_n(x)h^n}$$

For example, for $x = 1$:

$$\Phi(1,h) = \frac{1}{1-h} = \sum_{n=0}^{\infty} h^n \to P_n(1) = 1.$$

Example 17.1 (Application to gravitational potential). The generating function $\Phi(x,h)$ is useful in expanding the potential energy associated with any inverse square force. As an example, let us consider the gravitational potential between two pointwise mass objects separated by a distance d. Since the gravitational force goes as $1/d^2$ and is also a gradient force $\vec{F} = -\nabla V$, it follows that the gravitational potential goes as $1/d$

$$V(d) = \frac{K}{d}, \qquad K \text{ appropriate constant,}$$

and the distance $d = |\vec{R} - \vec{r}|$, where $\vec{R}$ and $\vec{r}$ are the position vectors of the mass objects relative to the origin of a 3D coordinate system. We denote $R = |\vec{R}|$ and $r = |\vec{r}|$. The distance can be rewritten in terms of r/R as

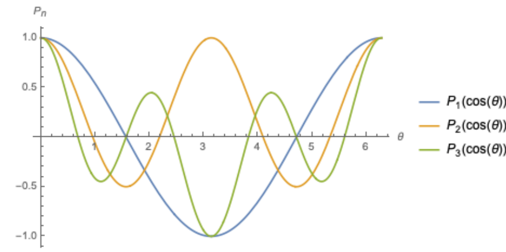
$$d = \sqrt{R^2 - 2\vec{R}\cdot\vec{r} + r^2} = \sqrt{R^2 - 2Rr\cos(\theta) + r^2} = R\sqrt{1 - 2\frac{r}{R}\cos(\theta) + \frac{r^2}{R^2}}$$

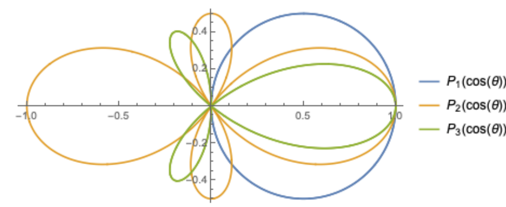
We consider the case where one object is much closer to the origin than the other, $r < R$, and use the change of variables

$$h = \frac{r}{R}, \qquad x = \cos\theta$$

such that the gravitational potential is written in terms of the generating function $\Phi(x,h)$ and can be expanded the Legendre polynomials as

$$\begin{aligned} V(x,h) &= \frac{K}{R}\Phi(x,h) = \frac{K}{R}\sum_{n=0}^{\infty} h^n P_n(x) \\ &= K\sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\theta) \end{aligned}$$





We may recognise from quantum mechanics that this form $P_n(\cos\theta)$ is part of the electron wave-function for the hydrogen atom (spherical harmonics $Y_{l,m}(\theta,\phi)$). This can be generalized to a mass distribution with density per unit volume $\rho(\vec{r})$. Let is consider a discrete configuration of mass points at positions $\vec{r}_i$ and angles θ_i from the position $\vec{R}$ of a test point mass where we want to compute the gravitational potential and the gravitational force induced by the other point masses. Then, the potential at R is a superposition of the potential induced by each point mass. In the assumption that $R > r_i$, it becomes

$$V(x,h) = \sum_i K \sum_{n=0}^{\infty} \frac{r_i^n}{R^{n+1}} P_n(\cos\theta_i)$$

In the limit of a continuous mass distribution, the sum $\sum_i$ over all point masses becomes a volume integral over the mass density

$$V = K \iiint d\vec{r}\rho(r) \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\theta)$$

where θ is the angle between $\vec{R}$ and $\vec{r}$. When $R \gg r$, the dominant term of this expansion corresponds to $n = 0$ for which:

$$V_0 \approx \frac{K}{R} \iiint d\vec{r}\rho(r) = M\frac{K}{R}$$

This is effectively the gravitational potential induced by the mass M localized at the origin or when the mass is distributed spherically symmetric around the origin. Any deviation from the spherical symmetry in the mass distribution will pick up more terms in the expansion, and the potential will depend on higher order Legendre polynomials.

This has important applications to **satellite gravimetry**, for instance when satellites orbit around Earth and measure accurately the local gravitation field induced by the Earth around a given orbit at radius R (hence $R > r$). This involves higher order Legendre polynomials since Earth is not a spherical ball. It applies not only to Earth but to any mass object floating in space which we have access to with satellites orbiting around it.

Lecture 18: Frobenius method

We consider a second order *homogeneous and linear* ODEs with normal form

$$y'' + p(x)y' + q(x)y = 0, \tag{18.1}$$

where $p(x)$ and $q(x)$ are some elementary functions of x. When $p(x)$ and $q(x)$ are not analytic at the expansion point x_0, the power series method does not apply but can be generalised to a Frobenius series expansion. This is what we will focus on in this lecture.

As a representative 2nd order linear ODE's where $x_0 = 0$ is a singular point, we consider

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0, \tag{18.2}$$

such that $b(x)$ and $c(x)$ **are** analytic at $x_0 = 0$ (they can be Taylor expanded at $x = 0$). Notice that $p(x) = \frac{b(x)}{x}$ and $q(x) = \frac{c(x)}{x^2}$. An independent solution can be represented as a *generalized power series* also known as **Frobenius series**

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n, \qquad s \text{ is real or complex}$$

where s and a_n's are determined from Eq. (18.2). We notice that power series are a special case of the Frobenius series when $s = 0$. Therefore, this is a versatile method that can be applied to find solutions in the form of series expansion around a point x_0 that may be regular or not.

The general method is the same as for the power series expansion and based on collecting terms of the same power in x and setting the corresponding coefficients to zero. Let us Taylor expand the coefficients $b(x)$ and $c(x)$ around $x_0 = 0$

$$b(x) = \sum_{m=0} b_m x^m, \qquad c(x) = \sum_{m=0} c_m x^m$$

and combine this with the Frobenius form of an independent solution and its derivatives

$$\begin{aligned} y(x) &= \sum_{n=0} a_n x^{s+n} \\ xy'(x) &= \sum_{n=0} (s+n) a_n x^{s+n} \\ x^2 y''(x) &= \sum_{n=0} (s+n)(s+n-1) a_n x^{s+n} \end{aligned}$$

Inserting these expansions into Eq. (18.2), we find

$$x^s \sum_{n=0} \left[(s+n-1)(s+n) a_n x^n + \sum_{m=0} (s+n) b_m a_n x^{n+m} + \sum_{m=0} c_m a_n x^{n+m} \right] = 0 \tag{18.3}$$

or equivalently

$$\sum_{n=0} \left[(s+n-1)(s+n) a_n x^n + \sum_{m=0} [(s+n) b_m + c_m] a_n x^{n+m} \right] = 0. \tag{18.4}$$

This implies that the coefficients in front of each power of x must vanish one by one. The zeroth order term leads to the **indicial equation** which determines s, namely

$$a_0[s(s-1) + sb_0 + c_0] = 0. \tag{18.5}$$

Since by construction, $a_0 \neq 0$, it follows that s is a solution of the quadratic equation

$$s^2 + s(b_0 - 1) + c_0 = 0. \tag{18.6}$$

For distinct roots, $s_1 \neq s_2$, the two Frobenius series are both solutions of the ODE, however they might not be the *independent* solutions. Thus, in general, this method determined at least one of them.

The higher order terms in Eq. (18.4) leads to recursive relations for the coefficients a_n. Let's take a concrete example to illustrate how this method is applied and to introduce Bessel functions which are important in many physics problems with cylindrical symmetry.

Example 18.1. Let us consider this ODE

$$xy'' + y = 0$$

where $b(x) = 0$ and $c(x) = x$. Obviously, $x_0 = 0$ is a singular point. The recurrence relations for a_n follow from Eq. (18.4) for powers larger than x^s. Adapted to this example, Eq. (18.4) reads as

$$\sum_{n=0} \left[(s+n-1)(s+n)a_n x^n + a_n x^{n+1} \right] = 0 \tag{18.7}$$

which is equivalent with

$$(s-1)sa_0 + [s(s+1)a_1 + a_0]\, x + [(s+1)(s+2)a_2 + a_1]\, x^2 + \cdots = 0 \tag{18.8}$$

Thus, the zeroth order term $n = 0$ corresponds to the indicial equation

$$s(s-1) = 0 \to s_1 = 1, \quad s_2 = 0.$$

and the recursive relations for a_n's are obtained by setting the rest of the coefficients to zero,

$$(s+n-1)(s+n)a_n + a_{n-1} = 0, \quad n \geq 1. \tag{18.9}$$

From this relation, we see that the coefficients in the Frobenius series are determined by the values of s.

Let us compute the coefficients a_n corresponding to $\underline{s_1 = 1}$:

$$a_n = -\frac{1}{n(n+1)} a_{n-1}, \qquad n \geq 1,$$

which can be solved iteratively

$$a_1 = -\frac{1}{2} a_0 \tag{18.10}$$

$$a_2 = -\frac{1}{2 \cdot 3} a_1 = (-1)^2 \frac{1}{2^2 \cdot 3} a_0 \tag{18.11}$$

$$a_3 = -\frac{1}{3 \cdot 4} a_2 = (-1)^3 \frac{1}{2^2 \cdot 3^2 \cdot 4} a_0 \tag{18.12}$$

$$a_4 = -\frac{1}{4 \cdot 5} a_3 = (-1)^4 \frac{1}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5} a_0 \tag{18.13}$$

and, in general,

$$a_n = (-1)^n \frac{1}{n! \cdot (n+1)!} a_0, \quad n \geq 0$$

where a_0 is arbitrary. Thus, one Frobenius series is the power series

$$S_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(n+1)!} x^{n+1}$$

For $\underline{s_2 = 0}$, the corresponding recursive relations are:

$$a_n = -\frac{1}{n(n-1)} a_{n-1}, \quad n \geq 2$$

Solving for the first few terms

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_1 \\
a_3 &= -\frac{1}{3\cdot 2}a_2 = (-1)^2\frac{1}{3\cdot 2^2}a_1 \\
a_4 &= -\frac{1}{4\cdot 3}a_3 = (-1)^3\frac{1}{4\cdot 2^2\cdot 3^2}a_1 \\
a_5 &= -\frac{1}{5\cdot 4}a_4 = (-1)^4\frac{1}{5\cdot 2^2\cdot 3^2\cdot 4^2}a_1
\end{aligned}$$

we notice that it generalizes to

$$a_n = (-1)^{n-1}\frac{1}{n!(n-1)!}a_1, \quad n \geq 1$$

where a_1 is arbitrary. Hence, the second Frobenius series is also a regular power series

$$S_2(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n!(n-1)!}x^n$$

In fact, by redefining the summation index $k = n-1$, we notice that the two series are the same

$$S_2(x) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!k!}x^{k+1} = S_1(x).$$

For this equation, the Frobenius method gives us only one of the two independent solutions. This solution can be expressed in terms of the **Bessel function of first kind and first order**

$$S_1(x) = \sqrt{x}J_1(2\sqrt{x}).$$

The Bessel function of first kind and of order p (integer) is an infinite power series given by

$$J_p(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!(n+p)!}\left(\frac{x}{2}\right)^{2n-p}.$$

Bessel functions J_p: The point of the Frobenius method is that it leads us to solutions of important ODEs in physics in terms of special functions, such as Bessel functions, Hermite functions, etc.. These are denoted in a short hand notation by simple letters as "elementary" functions, but are infinite series sometime with integral representations. The Frobenius method allows us to derive them and study their properties. As an example, we will consider the **Bessel equation** which arises many problems of wave propagation and static potentials when we find separable solutions to the Laplace equation in cylindrical coordinates. The normal form of the **Bessel equation** reads as

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

with p being an *integer parameter* which determines the **order** of the Bessel function of the first kind $J_p(x)$. The formalism can be generalized to non-integer p's. In applying the Frobenius method, it is easier to work with an equivalent form of the Bessel equation written as

$$x(xy')' + (x^2 - p^2)y = 0 \tag{18.14}$$

We apply the Frobenius method to find an independent solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$$

Thus,

$$xy'(x) = \sum_{n=0}^{\infty} (n+s)a_n x^{n+s}$$

and

$$x(xy'(x))' = \sum_{n=0}^{\infty} (n+s)^2 a_n x^{n+s}.$$

Inserting these series into the Eq. 18.14, we have then

$$\sum_{n=0}^{\infty} [(n+s)^2 - p^2]a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

By redefining the summation index in the second series, $n \to n+2$, we have

$$\sum_{n=0}^{\infty} [(n+s)^2 - p^2]a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

The zero coefficient in this series leads to the indicial equation

$$[s^2 - p^2]a_0 = 0$$

which implies that $s_1 = p$ and $s_2 = -p$. The next order coefficient ($n = 1$) leads to the following relation

$$[(s+1)^2 - p^2]a_1 = 0$$

from which it follows that $a_1 = 0$. For the higher order terms, we have the following recursive relations

$$[(s+n)^2 - p^2]a_n + a_{n-2} = 0, \quad n \geq 2.$$

or equivalently

$$a_n = -\frac{1}{(s+n)^2 - p^2} a_{n-2}, \quad n \geq 2.$$

For $s_1 = p$, this leads to

$$a_n = -\frac{1}{(p+n)^2 - p^2} a_{n-2} = -\frac{1}{n(n+2p)} a_{n-2}.$$

Since $a_1 = 0$, all the odd terms are also zero $a_{2n+1} = 0$. The even terms are given by

$$a_{2n} = -\frac{1}{2^2 n(n+p)} a_{2n-2}.$$

Solving iteratively for each term, we find

$$\begin{aligned} a_2 &= -\frac{1}{2^2(p+1)} a_0 \\ a_4 &= -\frac{1}{2^2 2(p+2)} a_2 = (-1)^2 \frac{1}{2^4 2(p+1)(p+2)} a_0 \\ a_4 &= (-1)^2 \frac{p!}{2^4 2!(p+2)!} a_0 \\ a_6 &= -\frac{1}{2^2 3(p+3)} a_4 = (-1)^3 \frac{p!}{2^6 3!(p+3)!} a_0 \end{aligned}$$

and in general

$$a_{2n} = (-1)^n \frac{1}{2^{2n} n!} \frac{p!}{(p+n)!} a_0$$

where a_0 is arbitrary. The factorial $p!$ is also a constant that can be absorbed into a_0. Thus, the Frobenius series is a series with only even coefficients and generates the **Bessel function** $J_p(x)$ **of order** p

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(p+n)!} \left(\frac{x}{2}\right)^{2n+p}}$$

where we have added 2^{-p} as part of the series so that we can have $x/2$ raised to $2n+p$.

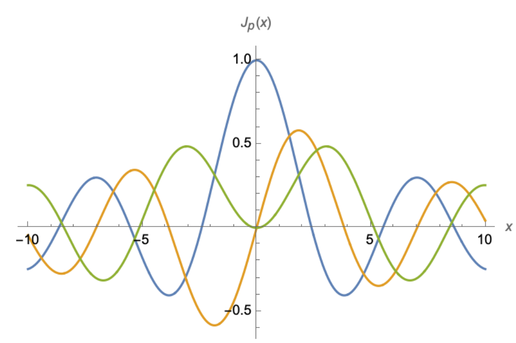


Figure 18.1: Bessel functions $J_0(x), J_1(x), J_2(x)$ as functions of x.

This Bessel function is an independent solution of the Bessel equation. Similarly, we can solve the recursive relations for the coefficients corresponding to $s_2 = -p$ and find that the coefficients in front of the odd powers also vanish by the same argument, while the coefficients in front of the even powers are generated by the corresponding recursive relation given by

$$a_{2n+2} = \begin{cases} -\frac{1}{2^2(n+1)(n-p+1)} a_{2n}, & n \geq p \\ 0, \quad n < p \end{cases}$$

so that the arbitrary coefficient is a_{2p}. Let us take $p = 1$ to check this. Then, from the main recursive relation, we see that

$$a_{2n} 2^2 n(n-1) = -a_{2n-2}.$$

This for $n = 1$, it means that $a_0 = 0$ and for arbitrary coefficient that generates the rest of the series is a_2, such that

$$\begin{aligned}
a_4 &= -\frac{1}{2^2 2} a_2 \\
a_6 &= -\frac{1}{2^2 3 \cdot 2} a_4 = -2^2(-1)^3 \frac{1}{2^6 \cdot (3!) \cdot 2} a_2 \\
a_8 &= -\frac{1}{2^2 4 \cdot 3} a_6 = -2^2(-1)^4 \frac{1}{2^8 4! 3!} a_2 \\
&\cdots \\
a_{2n} &= -\frac{1}{2^2 n \cdot (n-1)} a_{2n-2} \\
a_{2n} &= -2^2(-1)^n \frac{1}{2^{2n} n! (n-1)!} a_2.
\end{aligned}$$

Since a_2 is arbitrary, we can absorb the constant coefficient -2^2 into it. Thus, the corresponding Frobenius series defines **Bessel function $J_{-1}(x)$ of order -1**,

$$J_{-1}(x) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!(n-1)!} \left(\frac{x}{2}\right)^{2n-1}$$

By substituting $s = -1$ with $s = -p$, we obtain the **Bessel function $J_{-p}(x)$ of order $-p$** as

$$\boxed{J_{-p}(x) = \sum_{n=p}^{\infty} (-1)^n \frac{1}{n!(n-p)!} \left(\frac{x}{2}\right)^{2n-p}}$$

By relabeling the summation index $k = n - p$, we notice that $J_{-p}(x)$ maps into $J_p(x)$ up to a sign difference

$$\begin{aligned}
J_{-p}(x) &= \sum_{k=0}^{\infty} (-1)^{k+p} \frac{1}{k!(k+p)!} \left(\frac{x}{2}\right)^{2k+p} && (18.15) \\
&= (-1)^p J_p(x). && (18.16)
\end{aligned}$$

Hence, whilst $J_{-p}(x)$ is a solution of the Bessel equation, it is not independent of $J_p(x)$! So, when p is an integer, the two Bessel functions are determined from one other by this relation

$$\boxed{J_{-p}(x) = (-1)^p J_p(x).}$$

However, when p is a non-integer, then the two Bessel function are in fact independent of each other and constitute the two independent solutions of the corresponding Bessel function.

To get an intuition on the dependence of the Bessel functions on x,

Fig. 18.1 shows the plots of first few orders $J_p(x)$:

$$J_0(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n} \tag{18.17}$$

$$J_1(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1} \tag{18.18}$$

$$J_2(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!(n+2)!}\left(\frac{x}{2}\right)^{2n+2}. \tag{18.19}$$

WE NOTICE that the functions oscillates with an amplitude that is dampened with increasing $|x|$. Also, the functions are well-defined for x real, which means that the infinite series are convergent on the real axis, as you may also check by the ratio test.

Other properties: The Bessel functions of different orders can be related to each other through recursive relations or differentiations.

Recursive relations:

$$xJ_{p+1}(x) = 2xJ_p(x)xJ_p(x)]$$

Differentiation relations:

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$$

$$\frac{d}{dx}\left[x^{p}J_p(x)\right] = x^{p}J_{p-1}(x)$$

$$\frac{d}{dx}\left[J_p(x)\right] = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right]$$

Like with the Legendre polynomials, the sequence of Bessel functions $\{J_p(x)\}$ corresponds to the coefficients of a power series expansion of the *generating function*

$$g(x,t) = \sum_{p=-\infty}^{\infty} J_p(x)t^p$$

which has a closed form given by

$$g(x,t) = e^{x(t-1/t)/2}.$$

To obtained the expressions for the Bessel functions, we use the series expansion of the exponential

$$\begin{aligned} e^{xt/2}e^{-x/(2t)} &= \sum_{n=0} \frac{1}{n!}\left(\frac{xt}{2}\right)^n \sum_{m=0} \frac{(-1)^m}{m!}\left(\frac{x}{2t}\right)^n \\ &= \sum_n \sum_m \frac{(-1)^m}{n!m!}\left(\frac{x}{2}\right)^{n+m} t^{n-m} \end{aligned}$$

By substituting one of the summation indices with $p = n - m$,

$$e^{xt/2}e^{-x/(2t)} = \sum_{p=-\infty}^{\infty}\left(\sum_{m=\max(0,-p)}\frac{(-1)^m}{(p+m)!m!}\left(\frac{x}{2}\right)^{p+2m}\right)t^p.$$

Part IV

Fourier Series

Lecture 19: Fourier series
2π *periodicity*

We use periodic functions to represent phenomena and patterns that repeat themselves indefinitely with a well-defined period or frequency. Oscillations, vibrations or waves are the staples of periodic/cyclic phenomena. There are myriad of examples of *biological clocks* in living matter from the firing of neurons to the heart beatings, as well as all kinds of cyclic patterns present in Nature that can be modelled through periodic functions.

In this lecture, we focus on a periodic function $f(x)$ of a single variable x, which has the property that $f(x)$ returns the same value every time x is shifted by a whole 2π period,

$$f(x) = f(x + 2k\pi), \text{ for every } x \text{ and } k \in \mathcal{Z}$$

The most common periodic functions are the trigonometric functions $\sin(x)$ and $\cos(x)$ that have a natural 2π period. We use these trigonometric functions to expand any periodic function into a superposition of sin and cos which is known as a **Fourier series**.

Definition 19.1 (Fourier series). is an infinite series representation of periodic functions in terms of trigonometric functions

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi x}{L}\right)}$$

where L defines the edges of the **basic interval** $[-L, L]$ where $f(x)$ is defined and repeats itself indefinitely with a period of $2L$. The coefficients a_n and b_n are determined by $f(x)$ through the orthogonality condition of the Fourier modes.

19.1 *Functions with 2π periodicity*

19.1.1 *Basic interval $[-\pi, \pi]$*

To begin with, we consider functions with a period of $2L = 2\pi$, because it is the natural periodicity for the trigonometric functions

$$\sin(x) = \sin(x + 2\pi k), \qquad \cos(x) = \cos(x + 2\pi k), \qquad k \in \mathbb{Z}.$$

We also consider that the basic interval over which the function repeats itself to be $x \in [-\pi, \pi]$. *The function $f(x)$ can be quite arbitrary within the basic interval, and can have cups and discontinuities. In this respect, Fourier series can handle more "difficult" functions than power (Taylor) series which can only represent smooth, differentiable functions.*

Within this basic interval, i.e. $x \in [-\pi, \pi]$, the **Fourier modes** $\sin(x)$ and $\cos(x)$ represent the *fundamental modes* or the first harmonic, and $\sin(nx)$ and $\cos(nx)$ ($n \geq 2$) are the other *harmonics* which capture the rapidly-varying regions in $f(x)$.

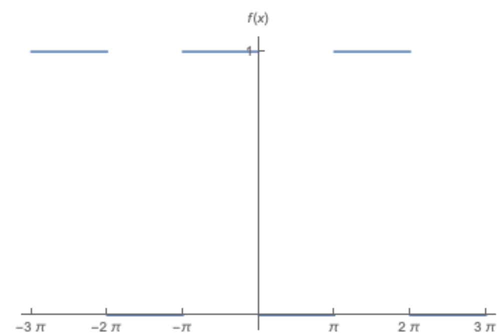


Figure 19.1: Periodic function given by Eq. 19.1.

Definition 19.2 (Fourier coefficients:). The Fourier series expansion of a function with 2π periodicity is an infinite series of the harmonic modes $\sin(nx)$ and $\cos(nx)$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$$

where the *Fourier coefficients* a_n and b_n are determined by $f(x)$ using the orthogonality condition of the Fourier modes on the basic interval $[-\pi, \pi]$.

Definition 19.3 (Orthogonality of trigonometric functions:). The sequence of Fourier modes,

$$\{\cos(nx), \sin(nx)\}_{n=0}^{\infty}$$

is a set of orthogonal functions on the basic interval $[-\pi, \pi]$. This means that their inner product is zero, unless the functions are identical. This orthogonality condition ensures that the set of Fourier modes is a complete basis for the Fourier expansion.

For the basic interval $[-\pi, \pi]$, the integral

$$\boxed{\int_{-\pi}^{\pi} dx \sin(nx)\cos(mx) = 0, \text{ for any } n, m.}$$

vanishes or any n and m. We can see this quickly since we have an odd integrand over a symmetric interval

$$2\sin(nx)\cos(mx) = \sin((n+m)x) + \sin((n-m)x).$$

On the other hand, the integral

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} dx \sin(nx)\sin(mx) = \begin{cases} \delta_{m,n}, & m, n \neq 0 \\ 0, & m = n = 0 \end{cases}}$$

is nonzero when $n = m$. We see this quickly using that

$$2\sin(nx)\sin(mx) = \cos((n-m)x) - \cos((n+m)x).$$

Similarly,

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos(nx)\cos(mx) = \begin{cases} \delta_{m,n}, & m, n \neq 0 \\ 0, & m = n = 0 \end{cases} .}$$

This orthogonality property generalises to any basic interval of length $2L$, changing the integration limits appropriately.

By taking half of the average of $f(x)$ in the basic interval $[-\pi, \pi]$, we find the coefficient a_0 as follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{a_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} dx a_n \cos(nx) + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} dx b_n \sin(nx)$$

where the other terms in the series vanish. In general, by integrating the $f(x)\cos(kx)$ over the basic interval, we extract the a_k coefficient of the Fourier series as follows:

$$
\begin{aligned}
& \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\cos(kx) \\
= & \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\cos(kx)dx + \sum_{n=1}^{\infty}\frac{a_n}{\pi}\int_{-\pi}^{\pi} dx\cos(nx)\cos(kx) + \sum_{n=1}^{\infty}\frac{b_n}{\pi}\int_{-\pi}^{\pi} dx\sin(nx)\cos(kx) \\
= & \sum_{n=1}^{\infty} a_n\delta_{n,k} \\
= & a_k.
\end{aligned}
$$

Similarly, the integral of $f(x)\sin(kx)$ over the basic interval gives the b_k coefficient of the Fourier series:

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\sin(kx) \\
&= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\sin(kx)dx + \sum_{n=1}^{\infty}\frac{a_n}{\pi}\int_{-\pi}^{\pi} dx\cos(nx)\sin(kx) + \sum_{n=1}^{\infty}\frac{b_n}{\pi}\int_{-\pi}^{\pi} dx\sin(nx)\sin(kx) \\
&= \sum_{n=1}^{\infty} b_n\delta_{n,k} \\
&= b_k
\end{aligned}
$$

To summarize, the Fourier coefficients corresponding to a function defined on the basic interval $[-\pi,\pi]$ are:

$$
\boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\cos(kx), \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\sin(kx)}
$$

Example 19.1. To illustrate the point that Fourier series can handle functions with discontinuities, tet us consider the step (Heaviside) function defined on the basic interval $[-\pi,\pi]$ with the jump at $x=0$, hence

$$
f(x) = \begin{cases} 1, -\pi < x < 0 \\ 0, 0 < x < \pi \end{cases} \tag{19.1}
$$

and $f(x)$ is then extended over the real axis with 2π-periodicity as illustrated in Fig.(19.1.1). We aim to determine its Fourier series and for this we need to determine the coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} dx f(x) && (19.2)\\ &= \frac{1}{\pi}\int_{-\pi}^{0} dx && (19.3)\\ &= 1. && (19.4) \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} dx f(x)\cos(kx) && (19.5)\\ &= \frac{1}{\pi}\int_{-\pi}^{0} \cos(kx)dx && (19.6)\\ &= \frac{1}{k\pi}\sin(kx)\Big|_{-\pi}^{0} = 0, k > 0. && (19.7) \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} dx f(x)\sin(kx) && (19.8)\\ &= \frac{1}{\pi}\int_{-\pi}^{0} \sin(kx)dx = -\frac{1}{k\pi}\cos(kx)\Big|_{-\pi}^{0} && (19.9)\\ &= -\frac{1}{k\pi}[1-(-1^k)]. && (19.10) \end{aligned}$$

Thus, the Fourier series is given by

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{2}{\pi}\sin(x) - -\frac{2}{3\pi}\sin(3x) - \frac{2}{5\pi}\sin(5x) - \cdots\\ &= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{1}{k\pi}[1-(-1)^k]\sin[kx] \end{aligned}$$

Fig. 19.1.1 shows the finite sum of this Fourier expansion for $N = 1$, $N = 2$, $N = 13$, $N = 100$.

Gibbs phenomenon: This behavior is encountered in the Fourier series of functions with discontinuities. The more terms in the sum we include, the closer the series approximates the function away from the jumps and goes through the *midpoint* of the jump. Notice also that there is an *overshoot* on either sides of a jump which appears because the Fourier series converges much slower at a discontinuity point.

Definition 19.4 (**Convergence of Fourier series:**)**.** You may wonder how we insure that the Fourier series is convergent. **Dirichlet conditions** provide us with *sufficient* conditions on $f(x)$ such that the Fourier series converges to $f(x)$:

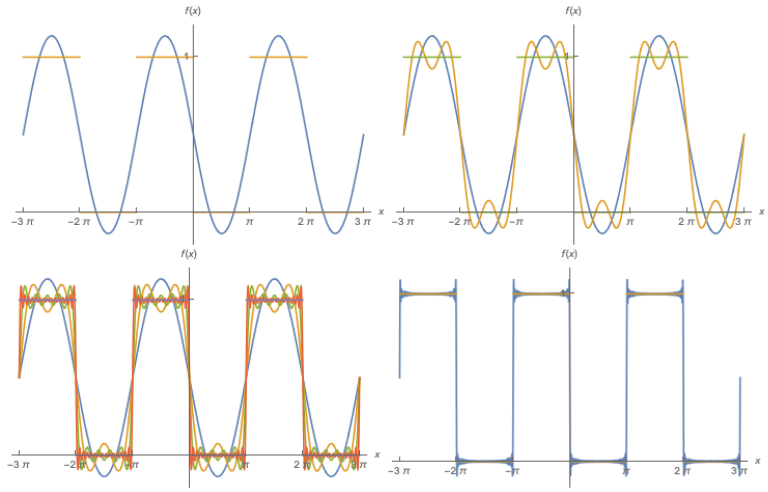


Figure 19.2: Truncated Fourier series of Eq. 19.1 for $N = 1$, $N = 2$, $N = 13$, $N = 100$.

1. $f(x)$ has finite number of extreme points (min, max) in the basic interval, e.g. $[-\pi, \pi]$

2. $f(x)$ has finite number of discontinuities (finite jumps) in the basic interval

3. $f(x)$ is bounded, $\int\limits_{-\pi}^{\pi} |f(x)| dx < 2\pi M$, where M is the maximum value of $f(x)$ in the basic interval.

At a discontinuity point x_0 in $f(x)$, the Fourier series converges to the midpoint

$$\lim_{\epsilon \to 0} \frac{1}{2} [f(x_0 + \epsilon) + f(x_0 - \epsilon)]$$

There are exceptions to this rule, i.e. there may be functions that do not fulfill the Dirichlet conditions and still have a convergent Fourier expansion.

Definition 19.5 (**Complex form of the Fourier series:**). Since the trigonometric functions have a complex representation

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

and

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}),$$

we may equivalently express the Fourier series in terms of powers of the complex exponential $e^{\pm ix}$ for the basic interval $[-\pi, \pi]$:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}}$$

where the **Fourier coefficients** are determined as

$$\boxed{c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x) e^{-ikx}}$$

using the **orthogonality relation** of the complex exponential modes

$$\int_{-\pi}^{\pi} dx e^{inx} e^{-imx} = 2\pi\delta_{n,m}.$$

Namely,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x) e^{-ikx} &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx} e^{-ikx} \\ &= \sum_n c_n \delta_{n,k} \\ &= c_k \end{aligned}$$

Example 19.2. For the function in Eq. (19.1), the zero order coefficient is

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x) = \frac{1}{2}$$

and the higher order ones are

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx f(x) e^{-ikx} \\ &= -\frac{1}{2ik\pi} e^{-ikx}\Big|_{-\pi}^{0} \\ &= -\frac{1}{2ik\pi}[1 - (-1)^k] \end{aligned}$$

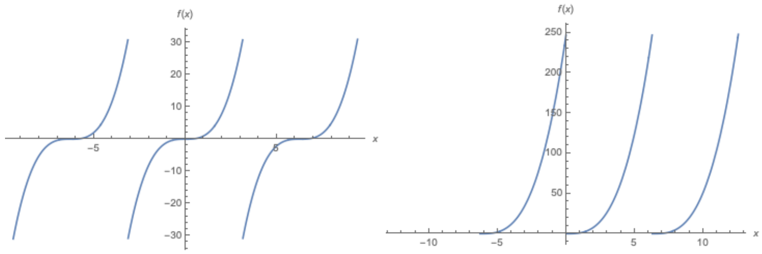


Figure 19.3: Two different periodic functions starting from $f(x) = x^3$ in different basic intervals: (left) $x \in [-\pi, \pi]$, (right) $x \in [0, 2\pi]$.

Thus, the Fourier series in the complex exponential form reads as

$$f(x) = \frac{1}{2} + \frac{i}{\pi} \sum_{k=-\infty}^{+\infty} \frac{1}{2k}[1-(-1)^k]e^{ikx}$$

Notice that the Fourier coefficients have a slow decay as $c_k \sim 1/k$, because the function has points of discontinuity. Thus, the derivative of $f(x)$ (Eq. (19.1)) cannot be represented by a convergent Fourier series. This is contrast to Taylor series which represent the function and all of its derivatives.

19.1.2 *Other basic intervals of 2π length*

We can use the same Fourier modes $\{\cos(nx), \sin(nx)\}$ or $\{e^{inx}\}$ to expand 2π-periodic functions defined on shifted basic interval, e.g. $[0, 2\pi]$, $[3\pi, 5\pi]$, etc. While the expansion is formally the same, the coefficients $\{a_n, b_n\}$ or, equivalently, $\{c_n\}$ are unique to the basic interval, since they represent different periodic functions! For instance, let us we take the function

$$f(x) = x^3$$

and make it a periodic function with a basic interval $[-\pi, \pi]$ to obtained a periodic signal as illustrated in Fig. (19.1.2). Now, let us extend the same function defined instead on the basic internal $[0, 2\pi]$. This generates a completely different periodic signal as shown in Fig. (19.1.2).

The important difference is that the orthogonality condition of the Fourier modes applies to different integration intervals. For the basic interval $[0, 2\pi]$, the orthogonality condition is

$$\int_0^{2\pi} dx e^{inx} e^{-imx} = 2\pi\delta_{n,m}$$

This implies that when $f(x)$ is defined over a basic interval $[0, 2\pi]$, the Fourier coefficients c_n are given by

$$\boxed{c_n = \frac{1}{2\pi} \int_0^{2\pi} dx f(x) e^{-inx}}$$

and, equivalently, the Fourier coefficients of the trigonometric form are

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} dx f(x)\cos(nx), \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} dx f(x)\sin(nx)}$$

Example 19.3. Let us take the $f(x) = x^3$ and derive the two Fourier series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where the different sets of c_n's are uniquely determined by the corresponding basic interval.

Basic interval $[-\pi, \pi]$*:* The zeroth order coefficient is

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^3 = 0$$

For $n \neq 0$, the integral is evaluated by successive integrations by parts and given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^3 e^{-inx} = i\frac{(-1)^n}{n^3}(\pi^2 n^2 - 6).$$

Hence, the Fourier series reads as

$$\begin{aligned} f(x) &= i\sum_{n=\pm 1}^{\infty} \frac{(-1)^n}{n^3}(\pi^2 n^2 - 6)e^{inx} \\ &= i\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}(\pi^2 n^2 - 6)\left(e^{inx} - e^{-inx}\right) \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}(\pi^2 n^2 - 6)\sin(nx) \end{aligned}$$

Basic interval $[0, 2\pi]$*:* The zeroth order coefficient is

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} dx x^3 = 2\pi^3$$

while the other coefficients for $n \neq 0$

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} dx x^3 e^{-inx} \\ &= \frac{2}{n^3}(2i\pi^2 n^2 + 3\pi n - 3i) = 2i\left(\frac{2\pi^2}{n} - \frac{3}{n^3}\right) + \frac{6\pi}{n^2} \end{aligned}$$

Hence, the Fourier series reads as

$$\begin{aligned}
f_{series}(x) &= 2\pi^3 + 2i\sum_{n=\pm 1}^{\infty}\left(\frac{2\pi^2}{n} - \frac{3}{n^3}\right)e^{inx} + 6\pi\sum_{n=\pm 1}^{\infty}\frac{1}{n^2}e^{inx} \\
&= 2\pi^3 + 2i\sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n} - \frac{3}{n^3}\right)\left(e^{inx} - e^{-inx}\right) + 6\pi\sum_{n=1}^{\infty}\frac{1}{n^2}\left(e^{inx} + e^{-inx}\right) \\
&= 2\pi^3 - 4\sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n} - \frac{3}{n^3}\right)\sin(nx) + 12\pi\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx).
\end{aligned}$$

Now, something interesting happens at function vanishes at $x = 0$, so let us have a closer look at it. The actual function x^3 vanishes at $x = 0$. However, when we evaluate $f_{series}(x = 0)$, we find a nonzero value

$$f_{series}(x = 0) = 2\pi^3 + 12\pi\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

However, from Fig. (19.1.2), we see that $x = 0$ is a discontinuity point and hence the Fourier expansion interpolates to the midpoint value

$$f_{series}(x = 0) = \frac{(2\pi)^3}{2} = 4\pi^3.$$

Combining these two expressions at $x = 0$, we find a closed expression for the infinite sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Lecture 20: Fourier series

$2L$-*periodicity*

In general, when $f(x)$ has a periodicity $2L$, we need to ensure that the Fourier modes preserve this periodicity. Namely, since

$$f(x) = f(x + 2kL), \quad k \in \mathcal{Z}$$

then each Fourier mode also needs to be $2L$ periodic. This is achieved by re-scaling x as

$$x \to \frac{\pi x}{L}$$

such that the trigonometric Fourier modes

$$\{\cos\left(n\frac{\pi x}{L}\right), \sin\left(n\frac{\pi x}{L}\right)\}$$

have a $2L$ period. That is

$$\cos\left(n\frac{\pi(x+2L))}{L}\right) = \cos\left(n\frac{\pi x}{L} + 2n\pi\right) = \cos\left(n\frac{\pi x}{L}\right) \quad (20.1)$$

$$\sin\left(n\frac{\pi(x+2L)}{L}\right) = \sin\left(n\frac{\pi x}{L} + 2n\pi\right) = \sin\left(n\frac{\pi x}{L}\right) \quad (20.2)$$

These Fourier modes form a complete basis for the Fourier expansion, given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi x}{L}\right).}$$

Similarly, the complex exponentials with $2L$-periodicity are

$$\{e^{in\frac{\pi x}{L}}\}$$

where for each n

$$e^{i\frac{n\pi}{L}(x+2L)} = e^{i\frac{n\pi}{L}} e^{i2n\pi} = e^{i\frac{n\pi}{L}}.$$

The complex exponential form of the Fourier series reads as

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}}$$

The Fourier coefficients are determined by the corresponding orthogonality relation of the Fourier modes in the basic interval of length $2L$ which defines the periodic signal.

Definition 20.1 (Basic interval $[-L, L]$:). The corresponding orthogonality condition of the complex Fourier modes is

$$\boxed{\frac{1}{2L}\int_{-L}^{L} dx e^{in\pi x/L} e^{-im\pi x/L} = \delta_{n,m}}$$

Namely for $n = m$, the integral is

$$\frac{1}{2L}\int_{-L}^{L} dx = 1$$

while for any $n \neq m$,

$$\frac{1}{2L}\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \frac{1}{2\pi i(n-m)}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] = 0.$$

From this, the Fourier coefficients c_n follow by integrating the function multiplied the appropriate Fourier mode over the basic interval,

$$\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-im\pi x/L} &= \sum_n c_n \frac{1}{2L}\int_{-L}^{L} dx e^{in\pi x/L} e^{-im\pi x/L} && (20.3)\\
&= \sum_n c_n \delta_{n,m} && (20.4)\\
&= c_m && (20.5)
\end{aligned}$$

Thus,

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-in\pi x/L}}$$

Similarly, the Fourier coefficients for the trigonometric form follow as

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} dx f(x)\cos\left(n\frac{\pi x}{L}\right), \quad b_n = \frac{1}{L}\int_{-L}^{L} dx f(x)\sin\left(n\frac{\pi x}{L}\right)}$$

Example 20.1. Let us as consider an example the function

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

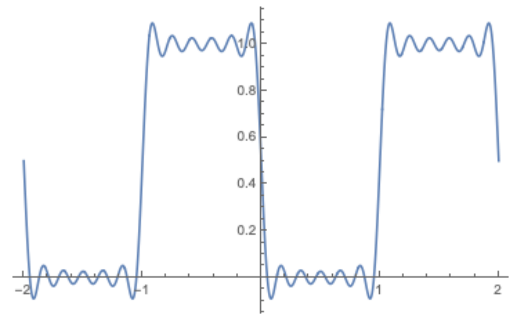


Figure 20.1: Truncated Fourier series from Eq. 20.9.

defined in the basic interval $[-1, 1]$. The function extended over the whole real axis has a periodicity of $2L = 2$. We can expand it in the basis of the Fourier modes

$$\{e^{in\pi x}\}$$

which have the same period. The complex Fourier expansion reads as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$$

with the Fourier coefficients computed as

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} dx f(x) e^{-in\pi x} && (20.6) \\ &= \frac{1}{2}\int_{-1}^{0} dx e^{-in\pi x} && (20.7) \\ &= \begin{cases} \frac{1}{2}, n = 0 \\ \frac{i}{2n\pi}[1-(-1)^n], n \neq 0 \end{cases} && (20.8) \end{aligned}$$

Thus, the Fourier expansion reduces to

$$\begin{aligned} f(x) &= \frac{1}{2} + \frac{i}{\pi}\sum_{n=\pm\text{odd}} \frac{1}{n} e^{in\pi x} \\ &= \frac{1}{2} - \frac{2}{\pi}\sum_{n=\text{odd}} \frac{\sin(n\pi x)}{n} && (20.9) \end{aligned}$$

Fig. (20) shows Fourier series truncated after the first few terms.

Definition 20.2 (Basic interval $[0, 2L]$**:).** Similarly, corresponding orthogonality condition for this basic interval is

$$\boxed{\frac{1}{2L}\int_{0}^{2L} dx e^{in\pi x/L} e^{-im\pi x/L} = \delta_{n,m}}$$

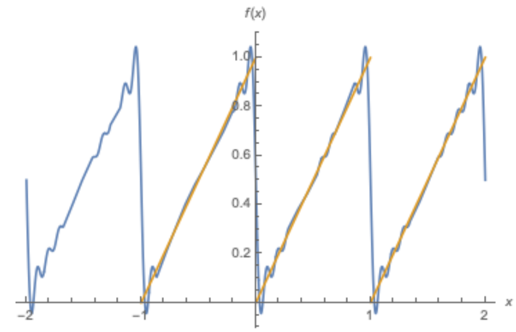


Figure 20.2: Fourier series in Eq. (20.13) truncated at $N = 10$.

Example 20.2. Let us consider another example, this time the function defined on the basic interval that is not centered at zero:

$$f(x) = x, x \in [0, 1]$$

with periodicity $2L = 1$. The corresponding Fourier modes with the same period are

$$\{e^{i2n\pi x}\}$$

since

$$e^{i2n\pi(x+1)} = e^{i2n\pi x}e^{i2n\pi} = e^{i2n\pi x}.$$

Therefore, the complex Fourier expansion reads as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x},$$

where the coefficients are computed from the orthogonality condition of the Fourier modes on the $[0, 1]$ basic interval as

$$c_n = \int_0^1 dx x e^{-2\pi i n x}$$

For $n = 0$:

$$c_0 = \int_0^1 dx x = \frac{1}{2}$$

For $n \neq 0$:

$$\begin{aligned} c_n &= \frac{i}{2n\pi}\int_0^1 dx x \frac{d}{dx} e^{-i2n\pi x} && (20.10) \\ &= \frac{i}{2n\pi}\left[x e^{-i2n\pi x}\right]_0^1 - \frac{i}{2n\pi}\int_0^1 dx e^{-i2n\pi x} && (20.11) \\ &= \frac{i}{2n\pi} e^{-i2n\pi} = \frac{i}{2n\pi}. && (20.12) \end{aligned}$$

Thus, the Fourier series becomes

$$\begin{aligned} f(x) &= \frac{1}{2} + \frac{i}{2\pi}\sum_{n\neq 0}\frac{1}{n} e^{i2n\pi x} \\ &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{n} && (20.13) \end{aligned}$$

and its truncation is plotted in Fig. (20).

20.1 Even and Odd extensions

Given a function $f(x)$ defined over a basic interval $[0, L]$, we have the freedom on how to replicate it over the entire real axis as a periodic function. Once nice thing about this freedom is that we can also incorporate even/odd symmetries on the extension to the real axis.

Even extension with a period of $2L$**:** We may construct a periodic function on the basic interval $[-L, L]$ as

$$\tilde{f}_{even}(x) = \begin{cases} f(-x), & -L < x < 0 \\ f(x), & 0 < x < L \end{cases} \tag{20.14}$$

Now, this property of being even must also hold for the Fourier expansion, which means that only the even Fourier modes have non-zero coefficients $b_n = 0$ because

$$\int_{-L}^{L} dx \tilde{f}_{even}(x) \sin\left(n\frac{\pi x}{L}\right) = 0.$$

Thus, an even function has a **cosine series**:

$$\tilde{f}_{even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right),$$

where the Fourier coefficients given by

$$a_n = \frac{1}{L}\int_{-L}^{L} dx \tilde{f}(x) \cos\left(n\frac{\pi x}{L}\right) \tag{20.15}$$

$$= \frac{2}{L}\int_{0}^{L} dx f(x) \cos\left(n\frac{\pi x}{L}\right). \tag{20.16}$$

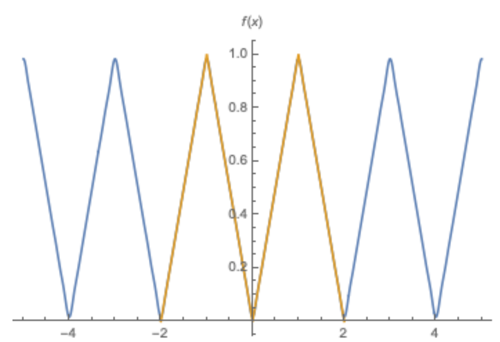


Figure 20.3: Fourier series of the even extension in Eq. (20.17).

Example 20.3. Let us consider the previous function $f(x) = x$ on the basic interval $[0, 1]$. We have computed its Fourier transform when the function is replicated over the real axis with periodicity $2L = 1$. Now, let us construct its *even* extension which is a periodic function over the basic interval $[-1, 1]$:

$$\tilde{f}_{even}(x) = \begin{cases} -x, -L < x < 0 \\ x, 0 < x < L \end{cases}$$

The coefficients of the *cosine series*:

$$a_0 = 2\int_0^1 dx x = 1$$

$$a_n = 2\int_0^1 dx x \cos(n\pi x) = -\frac{2}{n^2\pi^2}[1-(-1)^n]$$

and the resulting cosine series is then

$$\tilde{f}_{even}(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos(n\pi x) \tag{20.17}$$

which is also plotted in Fig. 20.1. Notice the fast convergence of the Fourier series since the function is smooth.

Odd extension with a period $2L$***:*** Similarly, given $f(x)$ on a basic interval $[0, L]$, we may construct its odd extension in the basic interval $[-L, L]$ as

$$\tilde{f}_{odd}(x) = \begin{cases} -f(-x), -L < x < 0 \\ f(x), 0 < x < L \end{cases}$$

which has a *sine series expansion* :

$$\tilde{f}_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

with the Fourier coefficients

$$b_n = \frac{1}{L}\int_{-L}^{L} dx \tilde{f}_{odd}(x)\sin\left(n\frac{\pi x}{L}\right) \tag{20.18}$$

$$= \frac{2}{L}\int_0^L dx f(x)\sin\left(n\frac{\pi x}{L}\right) \tag{20.19}$$

Fourier coefficient in front of the even Fourier modes are $a_n = 0$ because

$$\int_{-L}^{L} dx \tilde{f}_{odd}(x)\cos\left(n\frac{\pi x}{L}\right) = 0$$

due to the odd symmetry of the integrand.

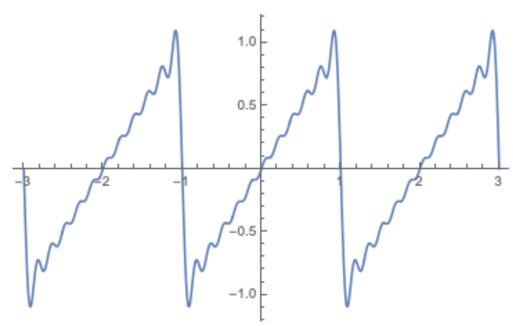


Figure 20.4: Truncated sin series from Eq. (20.20).

Example 20.4. Let us take the same function $f(x) = x$ with the basic interval $[0, 1]$. But now, we construct its *odd* extension which is a periodic function over the basic interval $[-1, 1]$:

$$\tilde{f}(x) = \begin{cases} x, -L < x < 0 \\ x, 0 < x < L \end{cases}$$

The corresponding odd Fourier modes are $\{\sin(n\pi x)\}$, such that they also have $2L$-periodicity. The resulting coefficients of the *sin series* are computed as

$$b_n = 2\int_0^1 dx x \sin(n\pi x) = -\frac{2}{n\pi}(-1)^n$$

such that the sin series reads as

$$\tilde{f}(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) \tag{20.20}$$

and plotted in Fig. 20.1.

20.2 Parseval's theorem: Completeness relation

Another important property of the Fourier series is the Parseval relation which follows from having a complete Fourier basis.

Theorem 20.5 (Parseval's theorem). *Suppose we have a function $f(x)$ defined on the basic interval $[-L, L]$ and which is replicated indefinitely on the real axis. The corresponding set of Fourier modes $\{\cos(n\pi x/L), \sin(n\pi x/L)\}$ or equivalently $\{e^{in\pi x/L}\}$ is a complete basis for the Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cos\left(n\frac{\pi x}{L}\right)$$

or, equivalently,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

*This implies that the average of $|f(x)|^2$ over the basic interval $[-L, L]$ is determined to Fourier coefficients through the **Parseval's identity**, also known as the **completeness relation***

$$\frac{1}{2L}\int_{-L}^{L} dx |f(x)|^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |b_n|^2$$

or, equivalently,

$$\frac{1}{2L}\int_{-L}^{L} dx |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof: The proof relies on using the orthogonality relation to evaluate the integral over the Fourier modes. For the complex representation of the Fourier series, this follows as

$$\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} dx |f(x)|^2 &= \frac{1}{2L}\int_{-L}^{L} dx \sum_n \sum_m c_n c_m^* e^{in\pi x/L} e^{-im\pi x/L} \\
&= \sum_n \sum_m c_n c_m^* \frac{1}{2L}\int_{-L}^{L} dx e^{i(n-m)\pi x/L} \\
&= \sum_n \sum_m c_n c_m^* \delta_{n,m} \\
&= \sum_n |c_n|^2
\end{aligned}$$

SIMILARLY, for the trigonometric representation of the Fourier series, the only non-vanishing terms are

$$\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} dx|f(x)|^2 = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \frac{1}{2L}\int_{-L}^{L} dx \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\
+ \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m \frac{1}{2L}\int_{-L}^{L} dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \\
= \quad \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} a_n^2 \frac{1}{2L}\int_{-L}^{L} dx \cos^2\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n^2 \frac{1}{2L}\int_{-L}^{L} dx \sin^2\left(\frac{n\pi x}{L}\right) \\
= \quad \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.
\end{aligned}$$

Example 20.6. For instance, the Fourier basis $\{\sin(2nx), \cos(2nx)\}$ is incomplete for a function $f(x)$ with a 2π period. The fundamental modes ($n = 1$) have a wavelength that is π instead of 2π. Hence the associated Fourier coefficients will not satisfy the completeness relation. The complete basis for the 2π-periodicity is $\{\sin(nx), \cos(nx)\}$.

Closed form for Infinite series : Parseval's theorem has a number of useful applications, one of them being that we may use it to find the sum of infinite series.

Example 20.7. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution: We notice that the terms in infinite series correspond to the Fourier coefficients of the periodic function $f(x) = x$ on the basic interval $x \in [-1, 1]$. The corresponding Fourier modes that also have a period of 2 are $\{e^{in\pi x}\}$ and their corresponding coefficients follow as

$$c_n = \frac{1}{2}\int_{-1}^{1} dx x e^{-in\pi x}.$$

For $n = 0$, this implies

$$c_0 = \frac{1}{2}\int_{-1}^{1} dx x = 0.$$

And for $n \neq 0$, this is

$$\begin{aligned} c_n &= \frac{i}{2n\pi} \int_{-1}^{1} dx x \frac{d}{dx} e^{-in\pi x} \\ &= \frac{i}{2n\pi} \left[x e^{-in\pi x} \right]_{-1}^{1} - \frac{i}{2n\pi} \int_{-1}^{1} dx e^{-in\pi x} \\ &= \frac{i}{2n\pi} \left[e^{in\pi} + e^{-in\pi} \right] \\ &= \frac{i}{n\pi} \cos(n\pi) = (-1)^n \frac{i}{n\pi}. \end{aligned}$$

Thus, the Fourier expansion is

$$f(x) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{i}{n\pi} e^{in\pi x}.$$

We can then find a closed form of the infinite series through Parseval's identity, namely

$$\frac{1}{2} \int_{-1}^{1} dx |x|^2 = \sum_{n \neq 0} \frac{i^2}{n^2 \pi^2} \tag{20.21}$$

$$= \frac{1}{\pi^2} \sum_{n \neq 0} \frac{1}{n^2} \tag{20.22}$$

$$= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{20.23}$$

where the integral is simply

$$\frac{1}{2} \int_{0}^{1} dx x^2 = \frac{1}{6} |x|^3 |_{-1}^{1} = \frac{1}{3}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

RECOMMENDED PROBLEMS: 11.5-9

Part V

Integral transforms

Lecture 21: Fourier transform

AN INTEGRAL TRANSFORM is by definition a representation of a function $f(x)$ as an integral over a complete set of orthogonal functions. The most common integral transformations are Fourier transform (FT) and Laplace transform (LT) and are often applied to solving differential equations. These are linear transformations which map a function $f(x)$ from one *function space* to another function space by integration. The map goes both ways, so there is also an *inverse integral transform* to the original function space.

In this lecture, we focus on the Fourier transform and its properties.

21.1 *Fourier transform*

As a transformation, Fourier transform takes a function $f(x)$ defined in the position x-space and maps into another function $\hat{f}(k)$ defined in the wavenumber k-space.

$$g(k) \equiv \mathcal{F}[f(x)] \text{ is the Fourier transform of } f(x)$$

and

$$f(x) \equiv \mathcal{F}^{-1}[g(k)] \text{ is the inverse Fourier transform of } g(k),$$

so they are a pair of conjugate functions.

As an integral representation of a function, Fourier transform is an extension of the Fourier series to aperiodic functions. We have seen that a periodic $f(x)$ has a Fourier series decomposition as a sum of **discrete** Fourier modes which form a complete orthogonal set determined by the basic interval. This idea can be generalized to non-periodic functions whereby we need a **continuous** spectrum of Fourier modes. Recall that the Fourier modes $\{e^{ik_n x}\}$ corresponding to a periodic function have discrete wave-numbers $k_n = n\pi/L$ (where n is integer). However, when $f(x)$ is aperiodic, the basic interval extends over the whole real axis $L \to \infty$ and we need a *continuum spectrum* of Fourier modes to represent $f(x)$.

Theorem 21.1 (Fourier transform theorem). *If $f(x)$ satisfies the Dirichlet's conditions on every finite interval, i.e.*

- *the function has finite number of cups or jumps*
- *its norm $\int\limits_{-\infty}^{\infty} |f(x)|^2 dx$ is finite,*

then $f(x)$ has a well-defined Fourier transform given by

$$\boxed{f(x) = \int\limits_{-\infty}^{\infty} dk g(k) e^{ikx}, \qquad g(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dx f(x) e^{-ikx}} \tag{21.1}$$

where

$$g(k) = \mathcal{F}[f(x)]$$

*is called the **Fourier transform** of $f(x)$, and*

$$f(x) = \mathcal{F}^{-1}[g(k)]$$

*is the **inverse Fourier transform** of $g(k)$. These form a pair of conjugate functions which can be Fourier transformed into another. In equivalent formulations, there may be a different prefactor in front of the integrals for $f(x)$ and $g(x)$.* The important things is that the product of prefactors has to be 2π. *Thus, equivalent transformations are*

$$\boxed{f(x) = \sqrt{\frac{1}{2\pi}} \int\limits_{-\infty}^{\infty} dk g(k) e^{ikx}, \qquad g(k) = \sqrt{\frac{1}{2\pi}} \int\limits_{-\infty}^{\infty} dx f(x) e^{-ikx}.} \tag{21.2}$$

The Dirichlet conditions are sufficient conditions and good rule of thumb. However, there also functions which can be Fourier transformed even though they do not satisfy these conditions. The Dirac delta function is one such example.

Heuristic proof: This proof is based on the idea of extending the basic interval of $f(x)$ over the whole real axis, i.e. the function has an infinite periodicity. For a set of discrete wavenumbers

$$k_n = n\frac{\pi}{L}$$

where n is an integer, the wavenumber increment is constant and given by

$$\Delta k \equiv k_{n+1} - k_n = \frac{\pi}{L}$$

In the limit where the length of the basic interval diverges, $L \to \infty$, the distance between successive wavenumbers becomes infinitesimally small, i.e. $\Delta k \to 0$. Let us use the definition of the coefficients c_n of the complex Fourier modes corresponding to a periodic function defined on the basic interval $[-L, L]$:

$$c_n = \frac{1}{2L}\int_{-L}^{L} du f(u)e^{-ik_n u} = \frac{\Delta k}{2\pi}\int_{-L}^{L} du f(u)e^{-ik_n u}.$$

Inserting this into the Fourier series of $f(x)$, we find that

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \\ &= \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi}\left[\int_{-L}^{L} du f(u)e^{ik_n(x-u)}\right] \end{aligned}$$

In the limit of $L \to \infty$, the sum has infinitely many terms and approaches an integral over the continuous variable k

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left[\int_{-\infty}^{\infty} du f(u)e^{ik(x-u)}\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left[\int_{-\infty}^{\infty} du f(u)e^{-iku}\right] e^{ikx} \\ &= \int_{-\infty}^{\infty} dk g(k)e^{ikx} \end{aligned}$$

where

$$\begin{aligned} g(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du f(u)e^{-iku} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x)e^{-ikx}. \end{aligned}$$

Notice that in this derivation, we have the choice how we want to split the prefactor $\frac{1}{2\pi}$ between the two Fourier integrals. This leads to the two equivalent formulations. It is important to be aware of this and be consistent with your choice.

Example 21.2. Let us compute the Fourier transform of the Gaussian $f(x) = e^{-\alpha x^2}$.

Solution: First, we may check that $f(x) = e^{-\alpha x^2}$ indeed satisfies all the Dirichlet's conditions: it is smooth and square integrable with

$$\int_{-\infty}^{\infty} dx e^{-2\alpha x^2} = \sqrt{\frac{\pi}{2\alpha}}.$$

Thus, the Gaussian has a Fourier transform which is given by

$$\begin{aligned} g(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-\alpha x^2} e^{-ikx} \end{aligned}$$

which can be integrated by completing the square in the exponent as

$$\begin{aligned} g(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-(\alpha x^2 + ikx + (ik)^2)/(4\alpha)} e^{-k^2/(4\alpha)} \\ &= \frac{1}{2\pi} e^{-k^2/(4\alpha)} \int_{-\infty}^{\infty} dx e^{-(\sqrt{\alpha}x + ik/(2\sqrt{\alpha}))^2} \\ &= \frac{1}{2\pi} e^{-k^2/4} \sqrt{\frac{\pi}{\alpha}} \\ &= \sqrt{\frac{1}{4\pi\alpha}} e^{-k^2/(4\alpha)} \end{aligned}$$

Thus, the Fourier transform of a Gaussian is also a Gaussian in the Fourier space.

Example 21.3. Let us now compute the Fourier transform of the exponential $f(x) = e^{-|x|}$.

Solution: $f(x) = e^{-|x|}$ satisfies all the Dirichlet's conditions: it is smooth and square integrable (decays sufficiently fast at $\pm\infty$):

$$\int_{-\infty}^{\infty} dx e^{-|x|} = 2.$$

Thus, its Fourier transform exists and is given by

$$\begin{aligned}
g(k) = \mathcal{F}\left(e^{-|x|}\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-|x|} e^{-ikx} \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} dx e^{-(ik-1)x} + \int_{0}^{\infty} dx e^{-(ik+1)x}\right] \\
&= \frac{1}{2\pi}\left[-\frac{1}{ik-1} + \frac{1}{ik+1}\right] \\
&= \frac{1}{\pi}\frac{1}{1+k^2}
\end{aligned}$$

The inverse Fourier transform is

$$e^{-|x|} = \mathcal{F}^{-1}\left(\frac{1}{1+k^2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{1}{1+k^2} e^{ikx}$$

Fourier transform of Dirac δ-function: Let us know compute the Fourier transform of the Dirac Delta $\delta(x-a)$.

Solution: Let us recall the properties of the δ-function

$$\int_{-\infty}^{\infty} dx \delta(x-a) = 1$$

and

$$\int_{-\infty}^{\infty} dx \delta(x-a) f(x) = f(a)$$

Thus, the Fourier transform of $\delta(x-a)$ is:

$$\mathcal{F}[\delta(x-a)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \delta(x-a) e^{-ikx} = \frac{1}{2\pi} e^{-ika} \qquad (21.3)$$

and the inverse Fourier transform is

$$\boxed{\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-a)}.}$$

21.1.1 *Fourier transform of derivatives*

Let $f(x)$ and $\mathcal{F}[f(x)]$ be a pair of Fourier transforms. Then the Fourier transform of the derivative $f'(x)$ is determined by the Fourier transform of $f(x)$ by the following transformation

$$\boxed{\mathcal{F}[f'(x)] = ik\mathcal{F}[f(x)]}$$

In general, the Fourier transform of the n'th derivative of $f(x)$ is

$$\boxed{\mathcal{F}[f^{(n)}(x)] = (ik)^n\mathcal{F}[f(x)].}$$

Proof: Let us compute the Fourier transform of $f'(x)$:

$$\begin{aligned}
\mathcal{F}[f'(x)] &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx f'(x)e^{-ikx} \\
&= ik\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx f(x)e^{-ikx} \\
&= ik\mathcal{F}[f(x)]
\end{aligned}$$

where we used integration by parts and that $f(x)$ decays to zero at $\pm\infty$ (from the Dirichlet condition of being square integrable). For the second order derivative, we use again integration by parts to move each d/dx on e^{-ikx}:

$$\begin{aligned}
\mathcal{F}[f''(x)] &= ik\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx f'(x)e^{-ikx} \\
&= ik\mathcal{F}[f'(x)] \\
&= (ik)^2\mathcal{F}[f(x)]
\end{aligned}$$

In general, the FT of the n'th derivative relates to the FT of function itself as

$$\begin{aligned}
\mathcal{F}[f^{(n)}(x)] &= ik\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx f^{(n-1)}(x)e^{-ikx} \\
&= ik\mathcal{F}[f^{(n-1)}(x)] \\
&= (ik)^n\mathcal{F}[f(x)]
\end{aligned}$$

These properties are particularly useful in solving differential equations by transforming them into algebraic equations in the k-space. We will go through examples on this later.

21.1.2 *Fourier transform of symmetric functions*

In general, we can write the Fourier transforms in terms of trigonometric functions using the Euler's formula as

$$g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x)\cos(kx) - i\frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x)\sin(kx)$$

and the inverse transform

$$f(x) = \int_{-\infty}^{\infty} dk g(k)\cos(kx) + i\int_{-\infty}^{\infty} dk g(k)\sin(kx).$$

Thus, $g(k)$ can be expressed in terms of the cosine and sine transforms of $f(x)$ as

$$g(k) = g^c(k) - ig^s(k)$$

where,

$$g^c(k) = \mathcal{F}^c[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x)\cos(kx)$$

and

$$g^s(k) = \mathcal{F}^s[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x)\sin(kx)$$

Similarly, $f(x)$ can also be expressed in terms of the cosine and sine inverse transforms of $g(k)$ as

$$f(x) = f^c(x) + if^s(x)$$

where

$$f^c(x) = \int_{-\infty}^{\infty} dk g(k)\cos(kx)$$

and

$$f^s(x) = \int_{-\infty}^{\infty} dk g(k)\sin(kx)$$

This representation becomes particularly useful when the function is either even or odd, in which case the Fourier transforms reduce either to a pair of cosine transforms or sine transforms, respectively.

When $f(x)$ is even,

$$f(x) = f(-x),$$

the Fourier transform picks up only the even Fourier modes. Consequently, the Fourier transform $\mathcal{F}[f(x)]$ is also an even function given in terms of cosine,

$$\begin{aligned} g(k) &= g^c(k) \\ &= \frac{1}{\pi}\int_0^{\infty} dx f(x)\cos(kx) \end{aligned}$$

and the sine transform vanishes because the integrand is odd. Thus, the cosine transforms are

$$f(x) = f^c(x) = 2\int_0^\infty dk g(k)\cos(kx), \qquad g(k) = g^c(k) = \frac{1}{\pi}\int_0^\infty dx f(x)\cos(kx)$$

Example 21.4. Determine the Fourier transform of the *non-periodic* $f(x)$ defined as

$$f(x) = \begin{cases} 1, -\pi < x < \pi \\ 0, |x| > \pi \end{cases}$$

Solution: We notice that the function is even in $[-\pi, \pi]$ where it is non-zero. The function also bounded, thus satisfies Dirichlet's conditions. Hence, we can use the cos transform

$$\begin{aligned} g(k) &= \frac{1}{\pi}\int_0^\infty dx f(x)\cos(kx) \\ &= \frac{1}{\pi}\int_0^\pi dx \cos(kx) \\ &= \frac{\sin(k\pi)}{\pi k} \end{aligned}$$

And

$$\begin{aligned} f(x) &= 2\int_0^\infty dk g(k)\cos(kx) \\ &= \frac{2}{\pi}\int_0^\infty dk \frac{\sin(k\pi)}{k}\cos(kx). \end{aligned}$$

Notice that this expression allows us to find a close form of the integral for a particular value of x. For instance, when $x = 1 \rightarrow f(1) = 1$, and

$$\int_0^\infty dk \frac{\sin(k\pi)}{k}\cos(k) = \frac{\pi}{2}$$

Similarly, when $f(x)$ is *odd*

$$f(x) = f(-x),$$

then the Fourier transforms pick up only the odd Fourier modes and reduce to sine transforms.

$$f(x) = if^s(x) = 2i\int_0^\infty dk g(k)\sin(kx), \qquad g(k) = -ig^s(k) = -i\frac{1}{\pi}\int_0^\infty dx f(x)\sin(kx)$$

THUS, we could equivalently use the pair of sine transforms as

$$f(x) = 2\int_0^\infty dk g^s(k)\sin(kx), \qquad g^s(k) = \frac{1}{\pi}\int_0^\infty dx f(x)\sin(kx)$$

Example 21.5. Let us determine the Fourier transform of the non-periodic $f(x)$ defined as

$$f(x) = \begin{cases} -1, -\pi < x < 0 \\ 1, 0 < x < \pi \\ 0, |x| > \pi \end{cases}$$

Solution: The function is odd in the interval $[-\pi, \pi]$ and zero everywhere else. It is bounded, thus satisfies Dirichlet's conditions. Hence, we can use the Fourier sine transform

$$\begin{aligned} g^s(k) &= \frac{1}{\pi}\int_0^\infty dx f(x)\sin(kx) \\ &= \frac{1}{\pi}\int_0^\pi dx \sin(kx) \\ &= \frac{\cos(\pi k) - 1}{\pi k} \end{aligned}$$

And

$$\begin{aligned} f(x) &= 2\int_0^\infty dk g^s(k)\sin(kx) \\ &= \frac{2}{\pi}\int_0^\infty dk \frac{\cos(\pi k) - 1}{k}\sin(kx) \end{aligned}$$

Similar to the previous example, we can use this expression to find close forms of the integral for a particular value of x. For instance, $x = 1 \to f(1) = 1$, implies that

$$\int_0^\infty dk \frac{\cos(\pi k) - 1}{\pi k}\sin(k) = \frac{\pi}{2}$$

21.1.3 *Parseval's relation*

Theorem 21.6 (Parseval's theorem:). *A pair of Fourier transforms*

$$f(x) = \int_{-\infty}^{\infty} dk g(k) e^{ikx}, \qquad g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

satisfy the completeness relation

$$\int_{-\infty}^{\infty} dx|f(x)|^2 = 2\pi \int_{-\infty}^{\infty} dk|g(k)|^2$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx|f(x)|^2 &= \int_{-\infty}^{\infty} dx\bar{f}(x)f(x) \\
&= \int_{-\infty}^{\infty} dx\bar{f}(x)\left[\int_{-\infty}^{\infty} dkg(k)e^{ikx}\right] \\
&= \int_{-\infty}^{\infty} dkg(k)\left[\int_{-\infty}^{\infty} dx\bar{f}(x)e^{ikx}\right] \\
&= 2\pi \int_{-\infty}^{\infty} dkg(k)\bar{g}(k) \\
&= 2\pi \int_{-\infty}^{\infty} dk|g(k)|^2
\end{aligned}$$

Alternatively, for the pair of Fourier transformations

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dkg(k)e^{ikx}, \qquad g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dxf(x)e^{-ikx}$$

the corresponding completeness relation is

$$\int_{-\infty}^{\infty} dx|f(x)|^2 = \int_{-\infty}^{\infty} dk|g(k)|^2$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 &= \int_{-\infty}^{\infty} \bar{f}(x)f(x) = \int_{-\infty}^{\infty} \bar{f}(x)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkg(k)e^{ikx}\right] \\
&= \int_{-\infty}^{\infty} dkg(k)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(x)e^{ikx}\right] = \int_{-\infty}^{\infty} dkg(k)\bar{g}(k) \\
&= \int_{-\infty}^{\infty} dk|g(k)|^2
\end{aligned}$$

Lecture 22: Laplace transform

Another important integral transformation is the Laplace transform which maps a function $f(t)$ defined on the (time) t-space (positive-defined domain) into a function $F(s)$ defined in the (frequency) s-space. This is particularly useful for solving initial value problems.

In this lecture, we focus on the Laplace transform and some of its properties.

22.1 Laplace transform

Theorem 22.1. *Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and bounded by an exponential*

$$|f(t)| \leq Me^{\gamma t}, \text{ for all } t \geq 0$$

for same constants γ and M. Then, there exits a unique Laplace transform *of $f(t)$*

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^\infty dt f(t)e^{-st}}$$

This integral is finite when the integrand decays sufficiently fast at infinity. This condition determines the domain of s. Using the triangle inequality and using the upper bound of $f(t)$, this implies

$$\left| \int_0^\infty dt f(t)e^{-st} \right| \leq \int_0^\infty dt |f(t)|e^{-st} \leq M \int_0^\infty dt e^{-(s-\gamma)t}$$

which is finite when $s > \gamma$.

The function $f(t)$ is also the *inverse Laplace transform* of $F(s)$ and determined formally as

$$\boxed{f(t) = \mathcal{L}^{-1}[F(s)]}.$$

In this lecture, we will the properties of the Laplace transform to also determine the inverse transform without relying on its integral representation. For the *inverse Laplace transform integral* read the section at the end of this (supplementary).

Example 22.2. The Laplace transform of a constant $f(t) = 1$ is

$$\begin{aligned}\mathcal{L}[1] &= \int_0^\infty dt e^{-st} \\ &= \frac{1}{s}, \qquad s \geq 0\end{aligned}$$

Example 22.3. The Laplace transform of the exponential $f(t) = e^{at}$ is

$$\begin{aligned}\mathcal{L}\left[e^{at}\right] &= \int_0^\infty dt e^{-(s-a)t} \\ &= \frac{1}{s-a}, \qquad s > a.\end{aligned}$$

Linearity of the Laplace transform: Like the Fourier transform, the Laplace transform is a linear operation and thus satisfies the additive property

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

This can be useful when finding Laplace transforms as exemplified below.

Example 22.4. The Laplace transform (LT) of

$$f(t) = \cos(at) = \frac{1}{2}\left(e^{iat} + e^{-iat}\right)$$

can be evaluated more elegantly from the LT of the exponential using the additive property as

$$\begin{aligned}\mathcal{L}[\cos(at)] &= \frac{1}{2}\mathcal{L}\left[e^{iat}\right] + \frac{1}{2}\mathcal{L}\left[e^{-iat}\right] \\ &= \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia} \\ &= \frac{s}{s^2+a^2}, \qquad s > 0\end{aligned}$$

Example 22.5. When the Laplace transform is a fraction

$$F(s) = \frac{1}{(s-a)(s-b)}, \qquad a \neq b$$

we can use the partial fraction decomposition to reduce it simple fractions

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

to determine the inverse Laplace transform using the additive property as

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s-a)(s-b)}\right] \\ &= \frac{1}{a-b}\left(\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-b}\right]\right) \\ &= \frac{1}{a-b}\left(e^{at} - e^{bt}\right) \end{aligned}$$

22.1.1 *Laplace transform of derivatives*

Let $f(t)$ and $\mathcal{L}[f(t)]$ be a pair of Laplace transforms. Then the Laplace transform of the derivative $f'(t)$ is determined by the Laplace transform of $f(t)$ by the following transformation

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

and

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

For the n'th derivative:

$$\boxed{\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots f^{(n-1)}(0)} \tag{22.1}$$

Proofs:

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty dte^{-st}f'(t) = f(t)e^{-st}]_0^\infty + s\int_0^\infty dte^{-st}f(t) \\ &= -f(0) + s\mathcal{L}[f(t)] \end{aligned}$$

$$\begin{aligned} \mathcal{L}[f''(t)] &= \int_0^\infty dte^{-st}f''(t) = f'(t)e^{-st}]_0^\infty + s\int_0^\infty dte^{-st}f'(t) \\ &= -f'(0) + s\mathcal{L}[f'(t)] \\ &= -f'(0) - sf(0) + s^2\mathcal{L}[f(t)] \end{aligned}$$

$$\begin{aligned} \mathcal{L}[f^{(n)}(t)] &= \int_0^\infty dte^{-st}f^{(n)}(t) = f^{(n-1)}(t)e^{-st}]_0^\infty + s\int_0^\infty dte^{-st}f^{(n-1)}(t) \\ &= -f^{(n-1)}(0) + s\mathcal{L}[f^{(n-1)}(t)] \\ &= -f^{(n-1)}(0) - sf^{(n-2)}(0)\cdots - s^{n-1}f(0) + s^n\mathcal{L}[f(t)] \end{aligned}$$

22.1.2 *Derivative of Laplace transform*

Let us assume that $f(t)$ has a Laplace transform $F(s)$. Then the derivative of its transform, $F'(s)$, can be found by differentiating with respect to s inside the integral and relates to the LT of $tf(t)$

$$F'[s] = \frac{d}{ds}\int_0^\infty dt f(t)e^{-st} = -\int_0^\infty dt[tf(t)]e^{-st}$$

This provides us useful formulas for this Laplace transform

$$\boxed{\mathcal{L}[tf(t)] = -F'(s)}$$

and its inverse transform

$$\boxed{\mathcal{L}^{-1}[F'(s)] = -tf(t)}$$

Example 22.6. To illustrate this point, let us consider

$$f(t) = \sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}).$$

Its Laplace transform is computed as follows

$$\begin{aligned} F(s) = \mathcal{L}[\sin\omega t] &= \frac{1}{2i}\int_0^\infty ds(e^{i\omega t} - e^{-i\omega t})e^{-st} \\ &= \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\ &= \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

Thus, its derivative is

$$\begin{aligned} F'(s) = -\mathcal{L}[t\sin\omega t] &= \frac{d}{ds}\frac{\omega}{s^2+\omega^2} \\ &= -\frac{2\omega s}{(s^2+\omega^2)^2} \end{aligned}$$

from which we can quickly evaluate the inverse Laplace transform integral

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right] = \frac{t}{2\omega}\sin\omega t.$$

The point of all these acrobatics is that we are not explicitly solving these integrals which are quite complicated to solve. We take advantage of mapping back and forth these integrals from one space to another and use the linearity property to evaluate these integrals in closed forms <u>without</u> performing integration.

Example 22.7. Let us take another example on the same theme. Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{4}{(s+1)^2}\right].$$

First, we notice that Laplace transform is the derivative of another the Laplace transform:

$$\frac{4}{(s+1)^2} = F'(s) \to F(s) = -\frac{4}{s+1}$$

with the inverse Laplace transform given by

$$\mathcal{L}^{-1}\left[-\frac{4}{s+1}\right] = -4\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = -4e^{-t}$$

Thus,

$$\mathcal{L}^{-1}\left[\frac{4}{(s+1)^2}\right] = 4te^{-t}$$

22.1.3 *Shifting theorems*

The shifting theorems are useful to determine how the function with shifted coordinates in the s-space or t-space transforms in the corresponding reciprocal space.

Theorem 22.8. ***s-shift*** *If $f(t)$ has the Laplace transform $F(s)$ where $s > \gamma$, then*

$$\boxed{\mathcal{L}\left[e^{at}f(t)\right] = F(s-a), \qquad F(s) = \mathcal{L}[f(t)]}$$

This is often used in finding the inverse Laplace transform as

$$\mathcal{L}^{-1}\left[F(s-a)\right] = e^{at}\mathcal{L}^{-1}[F(s)] = e^{at}f(t)$$

Proof:

$$\begin{aligned} F(s-a) &= \int_0^\infty dte^{-(s-a)t}f(t) \\ &= \int_0^\infty dt\left[e^{at}f(t)\right]e^{-st} \\ &= \mathcal{L}\left[e^{at}f(t)\right] \end{aligned}$$

Theorem 22.9. ***t-shift*** *If $f(t)$ has the Laplace transform $F(s)$ where $s > \gamma$, then*

$$\boxed{\mathcal{L}\left[f(t-a)H(t-a)\right] = e^{-as}F(s), \qquad F(s) = \mathcal{L}[f(t)]}$$

and thus the inverse Laplace transform is

$$\mathcal{L}^{-1}\left[e^{-as}F(s)\right] = f(t-a)H(t-a)$$

where the Heaviside function is

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Proof:

$$\begin{aligned} e^{-as}F(s) &= e^{-as}\int_0^\infty d\tau e^{-s\tau}f(\tau) \\ &=_{t=\tau+a} \int_a^\infty dt f(t-a)e^{-st} \\ &= \int_0^\infty dt H(t-a)f(t-a)e^{-st} \\ &= \mathcal{L}\left[H(t-a)f(t-a)\right] \end{aligned}$$

Example 22.10. The Laplace transform of the Heaviside function is

$$\mathcal{L}[H(t-a)] = e^{-as}\mathcal{L}[1] = \frac{e^{-as}}{s}$$

Example 22.11. Let us find the inverse Laplace transform $\mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s^3}\right]$.

For this, we first find the inverse LT of $F(s) = 1/s^3$ which is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] \\ &= -\frac{1}{2}t\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \\ &= \frac{1}{2}t^2\mathcal{L}^{-1}\left[\frac{1}{s}\right] \\ &= \frac{1}{2}t^2 \end{aligned}$$

Thus, by the shifting theorem, it follows that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s^3}\right] = H(t-3)f(t-3) = \frac{1}{2}(t-3)^2H(t-3)$$

22.1.4 *Applications to initial value problems*

Laplace transform is often used in solving linear differential equations with initial conditions. To exemplify this method, we consider a generic 2nd order ODE with non-homogeneous initial conditions given as

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, y'(0) = K_1. \tag{22.2}$$

and go through the main steps in the calculation.

Step 1 : is to Laplace transform the differential equation and write it in terms of LT of $y(t)$

$$Y(s) = \mathcal{L}[y(t)]$$

and the LT of the forcing term

$$R(s) = \mathcal{L}[r(t)]$$

Using the general formula for the LT of the derivatives in Eq. 22.1, we find that Eq. 22.2 maps into

$$[s^2 Y(s) - sy(0) - y'(0)] + a[sY(s) - y(0)] + bY(s) = R(s)$$

which can be rewritten as

$$Y(s)(s^2 + as + b) = (s + a)K_0 + K_1 + R(s)$$

Step 2 : is to solve the algebraic equation satisfied by $Y(s)$. In our case, this leads to

$$Y(s) = [(s+a)K_0 + K_1]Q(s) + R(s)Q(s), \qquad Q(s) = \frac{1}{s^2 + as + b}$$

Step 3 : is to apply the inverse LT to find $y(t)$ by inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

We cannot easily do this for the generic case. Therefore, we will now consider a concrete 2nd order ode and go through these steps in the calculation again.

Example 22.12. Let us find the particular solution of this initial value problem

$$y'' - y = t, \qquad y(0) = y'(0) = 1$$

Step 1 : We compute the Laplace transform of the forcing term

$$\mathcal{L}[t] \quad = \quad \int_0^\infty dt t e^{-st} = \frac{1}{s^2}$$

and Laplace transform the ODE:

$$s^2 Y - 1 - s - Y = \frac{1}{s^2} \to (s^2 - 1)Y = s + 1 + \frac{1}{s^2}$$

Step 2 : We solve for the LT of $y(t)$ which is given by

$$\begin{aligned} Y(s) &= \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} \\ &= \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}. \end{aligned}$$

Step 3 : We apply the inverse LT to $Y(s)$. We do this for each term the expression of $Y(s)$.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] &= \int_0^\infty ds \frac{1}{s-1}e^{st} \\ &= e^t\end{aligned}$$

Using the partial fraction decomposition and additive properties, we find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \sinh t$$

The last term has the inverse LT given by

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t.$$

Thus, adding up these three inverse LTs, we find the solution as

$$y(t) = \mathcal{L}^{-1}[Y(s)] = e^t + \sinh t - t.$$

22.2 *Inverse Laplace transform integral (Supplementary)*

You may wonder what is the integral representation of the inverse Laplace transform. Recall that for the LT integral we used the upper bound of $f(t)$ to determine the domain in the s-space where the LT integral is finite, i.e. $F(s)$ is well-defined. $F(s)$ is intimately related to the Fourier transform when we let s be complex. It turns out that the inverse LT integral needs to actually be extended into the complex s plane to recover a well-defined $f(t)$ for $t > 0$.

To see this, let us start from a complex variable $s = s_1 + is_2$ and write the LT integral in these coordinates

$$F(s) = \int_0^\infty dt f(t) e^{-s_1 t} e^{-is_2 t}$$

which is well-defined when $s_1 \equiv Re(s) > \gamma$. This integral can be extended to the whole axis by using the Heaviside function $H(t)$ in the integrand

$$F(s) = \int_{-\infty}^\infty dt \left[H(t) f(t) e^{-s_1 t}\right] e^{-is_2 t}.$$

We notice that this is the Fourier transform of $H(t)f(t)e^{-s_1 t}$. Thus, we can evaluate this function by the inverse Fourier transform integral

$$H(t)f(t)e^{-s_1 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds_2 F(s_1 + is_2)e^{is_2 t}.$$

For $t > 0$, the Heaviside function equals unity, therefore it follows that the function $f(t)$ has the integral representation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds_2 F(s_1 + is_2)e^{(s_1 + is_2)t},$$

which is the line integral in the complex plane (s_1, s_2) along a line parallel to the imaginary axis, i.e. for some constant $s_1 = c$. Using the line parameterization, $s = c + is_2$ with $ds = ids_2$, we re-write the integral as a complex line integral representing the *inverse Laplace transform*

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma} ds F(s)e^{st},$$

where Γ is the vertical line $s_1 = c > \gamma$ in the (s_1, s_2) complex plane. Often this is equivalently represented by the integration limits

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds F(s)e^{st}.$$

This is also known as the *Bromwich integral* and is typically evaluated using the contour integration and residue theorem.

Lecture 22: Convolutions

In this lecture, we focus on applications of the Fourier and Laplace transforms to solving linear ordinary differential equations and learn how these integral transforms are intimately connected with the Green's functions and convolution integrals.

23.1 *Integral transform of a convolution*

From the Green's function method, the particular solution of a linear, non-homogeneous differential equation follows as a convolution integral of the Green function with the sourcing term. Using the integral transforms, we can determine the Green function in the conjugate space (k-space for FT or s-space of LT) where the corresponding convolution integral takes a simpler form. Before we go into concrete examples, let us see how the convolution integral transforms under Fourier or Laplace transformation.

Convolution Theorem for Fourier transforms: Let us consider two pairs of *Fourier transforms*

$$F(k) = \mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}, \quad f(x) = \mathcal{F}^{-1}[F(k)] = \int_{-\infty}^{\infty} dk F(k) e^{ikx}$$

$$G(k) = \mathcal{F}[g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx g(x) e^{-ikx}, \quad g(x) = \mathcal{F}^{-1}[G(k)] = \int_{-\infty}^{\infty} dk G(k) e^{ikx}$$

Given the product function $H(k) = F(k) \cdot G(k)$ in the k-space, there exits a function $h(x)$ which is the *convolution of* $f(x)$ *and* $g(x)$ defined as:

$$h(x) \equiv \mathcal{F}^{-1}[2\pi H(k)] \equiv (f * g)(x) = \int_{-\infty}^{\infty} du f(u) g(x-u)$$

Equivalently,

$$\mathcal{F}[h(x)] = \mathcal{F}[f(x) * g(x)] = 2\pi H(k) = 2\pi F(k) \cdot G(k)$$

where we use the " $*$ " symbol as short-hand notation of the convolution integral.

Proof: We know that in the Fourier space, $H(k)$ is the product of the Fourier transforms:

$$\begin{aligned} H(k) &= F(k) \cdot G(k) \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{-iku} f(u) \right) \cdot \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-ikw} g(w) \right) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du dw e^{-ik(u+w)} f(u) g(w) \end{aligned}$$

In the integral over w, we can use the change of variable $x = w + u$ such that

$$\begin{aligned} H(k) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx du e^{-ikx} f(u) g(x-u) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} du f(u) g(x-u) \\ &= \frac{1}{2\pi} \mathcal{F}[h(x)] \end{aligned}$$

Application to boundary value problem: The Fourier transform is an elegant method of solving linear differential equations under homogeneous boundary conditions. We will illustrate this method for ordinary differential equations.

Let us consider as an example a generic equation for a forced and damped harmonic oscillator given by

$$y'' + \gamma y' + \omega^2 y = f(x),$$

where $\gamma > 0$ is the constant damping rate, ω is the internal frequency of the oscillator and $f(x)$ is some external driving force. We are seeking out for the solution $y(x)$ which decays sufficiently fast to zero at $\pm\infty$. This solution is square integrable and thus has a well-defined Fourier transform. Therefore, these boundary conditions are also called the *Dirichlet boundary conditions*. The Fourier transform method of finding the specific solution has three main steps:

Step 1 is to apply the Fourier transform on the differential equation and find the corresponding algebraic equation satisfied by $\hat{y}(k) = \mathcal{F}[f(x)]$. In our case, this results in

$$\mathcal{F}[y''(x) + \gamma y'(x) + \omega^2 y(x)] = \hat{f}(k)$$

where $\hat{f}(k) = \mathcal{F}[f(x)]$ is the FT of the source term. FT is a linear transformation just like the Laplace transform. Thus, using the additive property when γ, ω are constants, the Fourier transform can be applied on each term on the left hand side. This leads to

$$(-k^2 + i\gamma k + \omega)\hat{y}(k) \quad = \quad \hat{f}(k).$$

Step 2 is to solve for the FT $\hat{y}(k)$. For the harmonic oscillator this implies that

$$\begin{aligned} \hat{y}(k) &= \frac{1}{-k^2 + i\gamma k + \omega}\mathcal{F}[f(x)] \\ &= 2\pi\hat{\chi}(k)\hat{f}(k) \end{aligned}$$

Notice that the solution in the Fourier space is a product of two Fourier transforms. One is the external force $\hat{f}(k)$. The other one

$$\hat{\chi}(k) = (2\pi)^{-1}(-k^2 + i\gamma k + \omega)^{-1}$$

which corresponds to the *response function* that tells us how the oscillator responds to an impulse perturbation. Examples of response functions from statistical mechanics are susceptibilities which describe how internal state variables change due to changes in the conjugate applied field. The familiar example of this is the magnetic susceptibility which relates changes in the internal magnetization due to changes in the applied magnetic field. This idea can be extended also to dynamical variables and dynamical response functions. Mathematically, the response functions are the Green functions of the corresponding differential equations. Thus, it is quite neat that we can quickly find the Green function in the Fourier space and write the solution as a convolution with the force term in the position space.

Step 3 is to apply the inverse Fourier transform and obtain the solution in the position space.

$$y(x) = 2\pi \int_{-\infty}^{\infty} dk\hat{\chi}(k)\hat{f}(k)e^{ikx} \tag{23.1}$$

Equivalently, we can write this as a convolution integral in the position space using $\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' f(x')e^{-ikx'}$, thus

$$y(x) = \int_{-\infty}^{\infty} dx'\chi(x - x')f(x)$$

where $\chi(x) = \int\limits_{-\infty}^{\infty} dk\hat{\chi}(k)e^{ikx}$ is the response function in the position space. Thus, we have two routes for finding the solution and can choose which integral is most convenient to solve.

Theorem 23.1 (Convolution Theorem of Laplace transforms:). *Consider two functions with* Laplace transforms

$$F(s) = \mathcal{L}[f(t)], \qquad f(t) = \mathcal{L}^{-1}[F(s)]$$

$$G(s) = \mathcal{L}[g(t)], \qquad g(t) = \mathcal{L}^{-1}[G(s)]$$

Given $H(s) = F(s)G(s)$, there exits a function $h(t)$ which is the convolution of $f(t)$ and $g(t)$ *defined as:*

$$h(t) \equiv \mathcal{L}^{-1}[H(s)] \equiv (f * g)(t) = \int_0^t d\tau f(\tau)g(t-\tau)$$

where $*$ is a shorthand operator notation for the convolution.

Proof: We prove this by using the t-shifting theorem:

$$\begin{aligned} e^{-s\tau}G(s) &= \mathcal{L}[g(t-\tau)H(t-\tau)] \\ &= \int_0^\infty dte^{-st}g(t-\tau)H(t-\tau) \\ &= \int_\tau^\infty dte^{-st}g(t-\tau) \end{aligned}$$

in evaluating the product of the two Laplace transforms:

$$\begin{aligned} H(s) &= F(s)\cdot G(s) \\ &= \left(\int\limits_0^\infty d\tau e^{-s\tau}f(\tau)\right)\cdot G(s) = \int\limits_0^\infty d\tau f(\tau)\cdot\left(e^{-s\tau}G(s)\right) \\ &= \int\limits_0^\infty d\tau f(\tau)\cdot\left(\int\limits_\tau^\infty dte^{-st}g(t-\tau)\right) \\ &= \int\limits_0^\infty d\tau\int\limits_\tau^\infty dte^{-st}f(\tau)\cdot g(t-\tau) \end{aligned}$$

Notice that the double integral is the area integral of the upper triangle bounded by $t = \tau$ in the first quadrant of the (t,τ)-plane. Thus, we can write the area integral equivalently as

$$\int\limits_0^\infty d\tau\int\limits_\tau^\infty dt = \int\limits_0^\infty dt\int\limits_0^t d\tau$$

Hence,

$$\begin{aligned} H(s) &= \int\limits_0^\infty dt \int\limits_0^t d\tau e^{-st} f(\tau)\cdot g(t-\tau) \\ &= \int\limits_0^\infty dt e^{-st} \int\limits_0^t d\tau f(\tau)\cdot g(t-\tau) \\ &= \int\limits_0^\infty dt e^{-st} h(t) = \mathcal{L}[h(t)] \end{aligned}$$

Thus,

$$h(t) = \mathcal{L}^{-1}[G(s)] = \int\limits_0^t d\tau f(\tau)\cdot g(t-\tau)$$

OBS: $f * g \equiv g * f$ meaning that we can interchange the arguments τ and $t-\tau$ between the two functions.

Example 23.2. Given

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t)$$

we want to find the inverse LT $L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$. By the convolution theorem for the product of the Laplace transforms, it follows that the inverse LT is the the convolution of the two sin functions

$$\begin{aligned} L^{-1}\left[\frac{1}{(s^2+1)^2}\right] &= L^{-1}\left[\frac{1}{s^2+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right] \\ &= \int_0^t d\tau \sin\tau\cdot\sin(t-\tau) \\ &= -\frac{1}{4}\int_0^t d\tau (e^{i\tau}-e^{-i\tau})(e^{i(t-\tau)}-e^{-i(t-\tau)}) \\ &= -\frac{1}{4}\int_0^t d\tau (e^{it}+e^{-it}-e^{i(t-2\tau)}-e^{-i(t-2\tau)}) \\ &= -\frac{1}{4}\int_0^t d\tau (2\cos t - 2\cos(t-2\tau)) \\ &= \frac{1}{2} t\cos t + \frac{1}{2}\sin(t). \end{aligned}$$

Application to initial value problem: Consider the generic ODE with homogeneous initial conditions

$$y''(t) + ay'(t) + by(t) = r(t), \qquad y(0) = y'(0) = 0$$

which has the solution given as an inverse Laplace transform $y(t) = \mathcal{L}^{-1}[Y(s)]$. The Laplace transform function $Y(s)$ is obtained by the LT of the ode and reduces to

$$Y(s) = R(s)Q(s),$$

with $R(s) = \mathcal{L}[r(t)]$ being the LT of the forcing term, and

$$Q(s) = \frac{1}{s^2 + as + b}$$

which represents the Laplace transform of the Green's function denoted as $q(t) = \mathcal{L}^{-1}[Q(s)]$. Then by convolution theorem, the solution given by the inverse LP transform of the solution in the s-space is the same as the convolution of the Green's function with the forcing term

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] && (23.2) \\ &= \int_0^t d\tau q(\tau) r(t-\tau) && (23.3) \end{aligned}$$

23.2 Boundary value problem

We apply the Fourier transform method to few concrete ODE's.

Example 23.3. Consider the following damped and forced harmonic oscillator

$$y'' + 2y' + y = \delta(x)$$

The force term which is the Dirac delta function gives straightforwardly the constant value in Fourier space

$$\mathcal{F}[\delta(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \delta(x) e^{-ikx} = \frac{1}{2\pi}$$

Thus, the solution fron Eq. 23.1 is given by

$$\begin{aligned} y(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{1}{-k^2 + 2ik + 1} e^{ikx} \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{(k-i)^2} \\ &= -\frac{1}{2\pi} I \end{aligned}$$

We can solve this I integral using complex analysis using Jordan's lemma. For $x > 0$, the integral equals the contour integral in the upper half plane where e^{izx} is an exponentially decaying function. The integrand has a pole of order 2 at $z = i$, and by residue theorem,

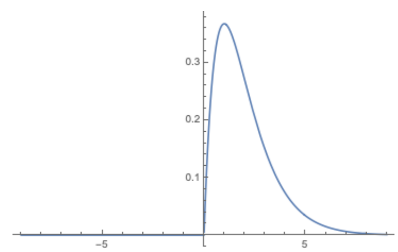


Figure 23.1: The solution $y(x)$ as function of x for the damped harmonic oscillator with a delta function impulse at $x = 0$.

the contour integral is

$$\begin{aligned} x > 0 : I &= \oint_{C_+} dz \frac{e^{izx}}{(z-i)^2} \\ &= 2\pi i \text{Res}\left[\frac{e^{izx}}{(z-i)^2}, z = -i\right] \\ &= 2\pi i(ixe^{-x}) = -2\pi x e^{-x} \end{aligned}$$

For $x < 0$, e^{izx} is exponentially decaying ($e^{-xIm(z)}$) in the lower half-plane, and by Cauchy's theorem

$$x \geq 0 : I = \oint_{C_-} dz \frac{e^{izx}}{(z-i)^2} = 0$$

Thus, the solution is

$$y(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

This we can also verify using the Green's function method (check!). Fig. 23.1 is a graphic illustration of the solution.

Example 23.4. Let now take a similar ODE with a different sourcing term

$$y''(x) + 2y'(x) + y(x) = e^{-x}H(x)$$

The Fourier transform of this equation leads to

$$(-k^2 + 2ik + 1)\hat{y}(k) = \mathcal{F}[e^{-x}H(x)]$$

which implies that the FT of the Green function is

$$\hat{\chi}(k) = \frac{1}{2\pi}\frac{1}{-k^2 + 2ik + 1} = \frac{1}{2\pi}\frac{1}{(ik+1)^2} = -\frac{1}{2\pi}\frac{1}{(k-i)^2}$$

and the solution in the k space is

$$\hat{y}(k) = 2\pi\hat{\chi}(k)\mathcal{F}[e^{-x}H(x)]$$

The Green function in the position space is thus

$$\begin{aligned} \chi(x) &= \mathcal{F}^{-1}\left[\hat{\chi}(k)\right] && (23.4) \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{(k-i)^2} = xe^{-x}H(x) && (23.5) \end{aligned}$$

Since both the forcing term and the Green's function are non-zero for $x > 0$, the solution will also be non-zero for $x > 0$. By the convolution theorem, the solution for $x > 0$ is

$$\begin{aligned} y(x) &= \int_{-\infty}^{\infty} du \chi(x-u) e^{-u} H(u) \\ &= \int_0^{\infty} du u e^{-u} e^{-x-u} H(x-u) = \int_0^x du u e^{-u} e^{-x+u} \\ &= e^{-x} \int_0^x du u = \frac{1}{2} x^2 e^{-x} \end{aligned}$$

Thus, for any x:

$$y(x) = \frac{1}{2} x^2 e^{-x} H(x)$$

23.3 Initial value problem

Let us now consider few concrete examples of odes with initial conditions where we can apply the Laplace transform method.

Example 23.5. Let us consider this ode

$$y'' - y = H(t-a),$$

and homogeneous initial condition $y(0) = y'(0) = 0$ and where the Heaviside function is

$$H(t-a) = \begin{cases} 1, t > a \\ 0, t < a \end{cases} .$$

Recall from the previous lecture that the Laplace transform of the Heaviside function is

$$\begin{aligned} R(s) &= \mathcal{L}[H(t-a)] && (23.6) \\ &= \int_0^{\infty} H(t-a) e^{-st} = \int_a^{\infty} e^{-st} && (23.7) \\ &= \frac{e^{-as}}{s} && (23.8) \end{aligned}$$

Inserting this into the Laplace transform of the ode and using the additive property of the LT, we find that the Laplace transform of the solution $Y(s) = \mathcal{L}[y(t)]$ is given by

$$(s^2 - 1) Y(s) = \frac{e^{-as}}{s} \rightarrow Y(s) = \frac{e^{-as}}{s(s^2-1)}$$

By partial fraction decomposition, we can rewrite it equivalently as

$$Y(s) = e^{-as} \left(\frac{1}{2(s+1)} + \frac{1}{(s-1)} - \frac{1}{s} \right) = e^{-as} F(s)$$

Using the inverse LT of the simple fractions

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \qquad \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s-1}\right)\right] = \cosh(t)$$

we then find that the solution is

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[e^{-as}F(s)]$$

can be straightforwardly computed using the t-shifting theorem and given by

$$\begin{aligned} y(t) &= H(t-a)\left[\cosh(t-a)-1\right] \\ &= \begin{cases} 0, & t<a \\ \cosh(t-a)-1, & t>a \end{cases} \end{aligned}$$

Example 23.6. A harmonic oscillator with linear damping, initially at rest and experiencing a sharp kick at $t = a$ is described by the following ode

$$y'' + 3y' + 2y = \delta(t-a), \qquad y(0) = y'(0) = 0$$

We find the solution $y(t)$ by the Laplace transform method. For this, we compute the LT of

$$\mathcal{L}\left[\delta(t-a)\right] = \int_0^\infty \delta(t-a)e^{-st} = e^{-as}$$

and insert into the LT of the ode,

$$s^2Y + 3sY + 2Y = e^{-as} \rightarrow Y(s) = \frac{e^{-as}}{s^2+3s+2} = \frac{e^{-as}}{(s+1)(s+2)}$$

which implies that

$$Y(s) = e^{-as}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = e^{-as}F(s)$$

Using that the inverse LT of simple fractions

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

we then find the solution by applying the t-shifting theorem as

$$y(t) = \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = H(t-a)\left[e^{-(t-a)} - e^{-2(t-a)}\right]$$

Equivalently, this can be written as

$$y(t) = \begin{cases} 0, & t<a \\ e^{-(t-a)} - e^{-2(t-a)}, & t>a \end{cases}$$

(check that the Green function method leads to the same result)

Example 23.7. Consider this ODE with homogeneous initial conditions

$$y''(t) + 3y'(t) - 4y(t) = e^{3t}, \qquad y(0) = y'(0) = 0$$

Its Laplace transform is

$$(s^2 + 3s - 4)Y(s) = \mathcal{L}[e^{3t}],$$

which implies that

$$Y(s) = \frac{1}{s^2 + 3s - 4} L[e^{3t}] = \frac{1}{(s+4)(s-1)} L[e^{3t}]$$

We use the partial fraction decomposition to evaluate this inverse LT,

$$\mathcal{L}^{-1}\left[\frac{1}{(s+4)(s-1)}\right] = \frac{1}{5}(e^t - e^{-4t})$$

Thus, since the LT function $Y(s)$ is a product of two LT, it follows that the specific solution given by inverse LT can be obtained directly from the convolution of the inverse Laplace transform functions when we already computed, hence

$$\begin{aligned} y(t) &= \frac{1}{5}\int_0^t d\tau (e^{\tau} - e^{-4\tau}) \cdot e^{3(t-\tau)} \\ &= \frac{1}{5} e^{3t} \int_0^t d\tau (e^{-2\tau} - e^{-7\tau}) \\ &= \frac{1}{5} e^{3t} \left[\frac{5}{14} - \frac{1}{2} e^{-2t} + \frac{1}{7} e^{-7t}\right] \end{aligned}$$

Part VI

Partial differential equations

Lecture 24: Separation of variable method Cartesian coordinates

In physics, we reply on partial differential equations to describe *evolution laws* of spatially-expended physical systems. Such evolution laws include transport equations of mass, momentum and energy in both space and time. Important examples are the mass conservation law, the Navier-Stokes momentum balance equations for fluid flow, heat diffusion, wave equations, etc..

In this lecture, we will introduce some basic definitions and notations for scalar functions of several variables and their derivatives. We focus on the separation of variable method to then solve linear second order pde's with *homogeneous boundary conditions*.

24.1 *Definitions*

Let us start by considering a scalar function $u(x_1, x_2, \cdots, x_n)$ of n **independent** variables $x_1, \cdots, x_n$. Equivalently, we can also collect the coordinates into a vector form $\vec{x}$ and denote the scalar field as $u(\vec{x})$.

Definition 24.1. A *partial derivative* of $u(\vec{x})$ with respect to one of its variables x_i is the derivative of u with respect to x_i keeping all the other variables fixed:

$$\frac{\partial u}{\partial x_i} \equiv \lim_{dx_i \to 0} \frac{u(x_1, \cdots, x_i + dx_i, \cdots x_n) - u(x_1, \cdots, x_i, \cdots x_n)}{dx_i}$$

This can be generalized to any n'th order partial derivative, with the difference that one can also have mixed terms such as

$$\frac{\partial^2 u}{\partial x_i^2}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Definition 24.2. A *partial differential equation* (PDE) for the function $u(\vec{x})$ is an equation that relates the function u with its derivatives:

$$F[u, D^1 u, D^2 u, \cdots D^n u] = g(\vec{x})$$

where $D^n u$ is the n'th derivative with respect to any of the coordinates.

Definition 24.3. The *order* of a PDE is determined by the **highest** derivatives in the equation. For example, diffusion equation is a 2^{nd} *order* PDEs for $u(t, x)$. It has second order derivatives with respect to spatial coordinates and a first order derivative in time. The diffusion equation with one spatial coordinate is

$$\text{1D Diffusion eq.:} \qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Similarly, the wave equation is also a 2^{nd} *order* PDE. In this case, we have second order derivatives both with respect to time and spatial coordinates. The wave equation with one spatial coordinate reads as

$$\text{1D Wave eq.:} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Definition 24.4 (Linear 2nd order PDE). A *linear* PDE has a normal form in which the coefficients in front of each derivative are either constants or depend *only* on the variables $\vec{x}$. We write the normal form of a *linear 2nd order PDE with constant coefficients* as

$$a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_k \frac{\partial u}{\partial x_k} + cu = g(\vec{x})$$

where a_{ij}, b_k, c are constant coefficients. We use Einstein summation convention that repeated indices are summed over. From the commutation of derivatives for well-defined functions, it follows that the coefficients are symmetric $a_{ij} = a_{ji}$ and can be collected into a $n \times n$ symmetric matrix A also called the coefficient matrix. The coefficients b_k in front of the first order derivatives can also be collected into the vector $B = \{b_k\}$.

24.2 *Separation of variable method*

The separation of variable method is often used to solve **homogeneous and linear PDE's** under appropriate boundary conditions and where the dependence of u on its variables can be somehow factorized. We will present this method for functions of two variables. One variable is typically time and other is one spatial coordinate in which case the pde describes the time evolution of one-dimensional (1D) systems, i.e. elastic vibration of a string, pressure waves in a one dimensional wire, heat diffusion in a thin bar, etc... We then extend the method to functions of more than two variations, which typically represent fields in higher spatial dimensions.

24.2.1 *Function of two variables $u(t, x)$*

We consider a function $u(t, x)$ of two independent variables where t is the time variable and x is the space variable. The function satisfies a *homogeneous* linear, 2nd order PDE with normal form given as

$$a_{11}\frac{\partial^2 u}{\partial t^2} + 2a_{12}\frac{\partial^2 u}{\partial t \partial x} + a_{22}\frac{\partial^2 u}{\partial x^2} + b_1\frac{\partial u}{\partial t} + b_2\frac{\partial u}{\partial x} + cu = 0 \qquad (24.1)$$

We set out to find the specific solution of this PDE under **homogeneous boundary conditions** in a finite domain

$$u(x_1, t) = 0, \quad u(x_2, t) = 0, \quad \text{for every } t$$

and arbitrary initial conditions. The separation of variable method follows these basic steps:

Step 1: Separate variables We seek for a solution where the function can be written as a product of single-variable functions as in this form

$$u(t, x) = F(x)G(t),$$

where the functions $F(x)$ and $G(t)$ are determined from the PDE. By inserting this ansatz in Eq. 24.1, we can reduce the PDE to a set of two ODE which are satisfied by $F(x)$ and $G(t)$, respectively.

Step 2: Spectrum of independent solutions We determine the *independent* solutions for $F(x)$ and $G(t)$ by solving the corresponding ODE's that they satisfy and use them together with the appropriate boundary conditions to construct the infinite set of **orthogonal functions** $\{u_n(t, x)\}$ which form a complete basis in which the specific solution can be expanded.

Step 3: General solution We determine the general solution of the PDE as a linear superposition of the independent solution $\{u_n(t, x)\}$

$$u(x, t) = \sum_n B_n u_n(x, t)$$

where the arbitrary coefficients B_n which are fixed by the initial condition. We will see shortly, that B_n represent the Fourier coefficients in the Fourier transform of the initial profile.

THESE GENERAL STEPS become straightforward when we apply them to specific equations. To exemplify this, we consider two representative homogeneous PDE's, namely the *wave equation* and the *diffusion equation* in 1D.

Wave equation in 1D: The wave equation in one dimension is a canonical example of a *hyperbolic* equation. In physics, we use it to model the elastic vibrations of a thin string. It reads as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ represents the local displacement of the string relative to its equilibrium at position x and at a given time t, and the constant c represents the speed of sound in the string. Let us go through the steps of applying the separation of variable method. Suppose we have a string of length L and that its displacement satisfies the homogeneous boundary conditions

$$u(0,t) = u(L,t) = 0 \text{ for any } t$$

which means that the string has fixed positions at its ends. These are also called the Dirichlet's boundary conditions.

Step 1: We construct the ansatz solution as a product of two functions $u(t,x) = F(x)G(t)$ and insert it into the 1D wave equation, to obtain the following relation between the functions $F(x)$ and $G(t)$ and their derivatives

$$F''(x)G(t) = c^2 F(x)\ddot{G}(t) \to \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)},$$

where we denote $F''(x) = \frac{d^2F}{dx^2}$ and $\ddot{G} = \frac{d^2G}{dt^2}$. Since the two functions are independent of each other the only way the two ratios are equal with each other is then they equal the same constant. Let us denote the proportionality constant as $-k^2$ (this particular choice will make more sense when we use boundary conditions) such that

$$\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = -k^2$$

which means that we can reduce the wave equation to these two ODE's

$$\begin{aligned} F''(x) &= -k^2 F(x) \\ \ddot{G}(t) &= -k^2 c^2 G(t) \end{aligned} \tag{24.2}$$

Step 2: We can straightforwardly find the *independent* solutions of $F(x)$ and $G(t)$ corresponding to Eq. (24.2), namely as

$$F_1(x) = \cos(kx), \quad F_2(x) = \sin(kx)$$

giving as the spatial oscillations of wavelengths $2\pi/k$ and

$$G_1(t) = \cos(ckt), \quad G_2(t) = \sin(ckt)$$

corresponding to temporal oscillations with frequency $\omega = ck$. The general solutions for $F(x)$ and $G(t)$ will then be linear superpositions of the corresponding independent functions with coefficients determined by the boundary conditions and initial conditions, respectively.

The boundary conditions act on the function

$$F(x) = a\sin(kx) + b\cos(kx)$$

and imply that

$$F(0) = F(L) = 0$$

and consequently that

$$\begin{aligned} b &= 0 \\ a\sin(kL) + b\cos(kL) &= 0 \end{aligned} \tag{24.3}$$

The second equation reduces to $\sin(kL) = 0$ from which we determine that the constant k must take discrete values

$$k = n\frac{\pi}{L}, \quad n \in \mathcal{Z}$$

This also implies that we have a discrete yet infinite spectrum of independent solutions for the pde. This is also known as the eigenfunction spectrum and given by

$$\left\{\sin\left(n\frac{\pi x}{L}\right)\cos\left(n\frac{\pi ct}{L}\right), \sin\left(n\frac{\pi x}{L}\right)\sin\left(n\frac{\pi ct}{L}\right)\right\}_n$$

This is also an orthogonal set since each independent solution is a product of trigonometric functions.

Step 3: We can now construct the general solution $u(x,t)$ as a series expansion in the basis of these eigenfunctions, as

$$u(x,t) = \sum_n \left[A_n\cos\left(n\frac{\pi ct}{L}\right) + B_n\sin\left(n\frac{\pi ct}{L}\right)\right]\sin\left(n\frac{\pi x}{L}\right)$$

where the coefficients A_n and B_n are determined by imposing appropriate initial conditions on $u(x,t)$ at $t = 0$. In fact, for a given initial profile $u(x,0) = f(x)$, the general series expansion reduces to a Fourier expansion

$$f(x) = \sum_n A_n \sin\left(n\frac{\pi x}{L}\right)$$

from which it follows that the A_n are the Fourier coefficients of the sine series of the initial profile and thus are determined from the corresponding integrals

$$A_n = \frac{2}{L}\int_0^L dx f(x) \sin\left(n\frac{\pi x}{L}\right)$$

Now, since we have a second order derivatives in time, we need another initial condition which is applied on the first derivative. For an initial velocity profile

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

the general series expansion reduces to a cosine expansion of $g(x)$

$$g(x) = \sum_n \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

where the B_n are the Fourier coefficients determined by

$$B_n = \frac{2}{n\pi c}\int_0^L dx g(x) \sin\left(\frac{n\pi x}{L}\right)$$

Thus, the specific solution for **homogeneous** boundary conditions and some initial conditions $u(x,0) = f(x), \quad \partial_t u(x,0) = g(x)$ reads as

$$\begin{aligned} u(x,t) &= \sum_n \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \\ A_n &= \frac{2}{L}\int_0^L dx f(x) \sin\left(\frac{n\pi x}{L}\right), \quad B_n = \frac{2}{n\pi c}\int_0^L dx g(x) \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Diffusion equation in 1D: Now we will consider the diffusion equation in one dimension. This is another important pde which typically describes heat or mass transport and reads as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ would now represent the temperature (or mass concentration) field which spreads out in space and time with a rate set by the diffusivity coefficient c^2.

Step 1: Using the separation of variables $u(t,x) = F(x)G(t)$, we reduce the diffusion equation to this relation

$$\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{\dot{G}(t)}{G(t)} = -k^2$$

where k is some proportionality constant determined by the boundary conditions. This implies that the $F(x)$ and $G(t)$ satisfy the following ode's

$$F''(x) = -k^2 F(x) \tag{24.4}$$

$$\dot{G}(t) = -c^2 k^2 G(t) \tag{24.5}$$

Step 2: The corresponding independent solutions of $F(x)$ and $G(t)$ are then

$$F_1(x) = \cos(kx), \quad F_2(x) = \sin(kx)$$

and

$$G_1(t) = e^{-c^2k^2t}$$

Using the homogeneous boundary conditions,

$$u(0,t) = u(L,t), \text{ for any } t \to F(0) = F(L) = 0$$

we then find by the same arguments as for the wave equation (the equation for $F(x)$ is the same) that

$$k = \frac{\pi}{L}n, \quad n \in \mathcal{Z}.$$

This implies that the independent solutions of the diffusion equations are given by the eigenfunction spectrum

$$\{\sin\left(n\frac{\pi x}{L}\right)e^{-\lambda_n^2 t}\}$$

where $\lambda_n = n\pi c/L$.

Step 3: The general solution $u(x,t)$ follows then as a series expansion in this basis

$$u(x,t) = \sum_n A_n e^{-\lambda_n^2 t}\sin(n\pi x/L)$$

with the coefficients A_n determined by the initial condition. For a given initial profile $u(x,0) = f(x)$, the expansion series reduces to the sine series of the initial profile

$$f(x) = \sum_n A_n \sin\left(n\frac{\pi x}{L}\right)$$

with the Fourier coefficients determined as

$$A_n = \frac{2}{L}\int_0^L dx f(x)\sin\left(n\frac{\pi x}{L}\right).$$

Thus, the solution for homogeneous boundary conditions and an initial condition $u(x,0) = f(x)$ is given by

$$\begin{aligned} u(x,t) &= \sum_n A_n e^{-\lambda_n^2 t}\sin(n\pi x/L) \\ A_n &= \frac{2}{L}\int_0^L dx f(x)\sin(n\pi x/L) \end{aligned}$$

24.2.2 *Diffusion equation in 2D Cartesian coordinates*

The separation of variables can also be applies to functions of more than two variables. We will apply this method on few specific examples and start with the heat diffusion in a rectangular spatial domain. The corresponding pde is given by

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

supplemented with the Dirichlet B.C:

$$u(x,y,t) = 0, \quad (x,y) \text{ on } [0,L_x]\times[0,L_y], \text{ for any } t \geq 0$$

and an initial condition set by a profile function

$$u(x,y,0) = f(x,y).$$

Step 1: By separation of variables, we write a solution as a product of three function of single variables,

$$u(t,x,y) = F(x)H(y)G(t)$$

which satisfy the following relations

$$\frac{F''(x)}{F(x)} + \frac{H''(y)}{H(y)} = \frac{1}{c^2}\frac{\dot{G}(t)}{G(t)}.$$

By the same arguments as in 1D diffusion case, it follows that

$$\frac{F''(x)}{F(x)} = -k^2, \quad \frac{H''(y)}{H(y)} = -p^2, \quad \frac{1}{c^2}\frac{\dot{G}(t)}{G(t)} = -(k^2+p^2)$$

where k and p are some unknown constants fixed by the BCs.

Step 2: We then find the *independent* solutions of $F(x)$, $H(y)$ and $G(t)$. The independent solutions for the functions of the spatial coordinates $F(x)$, $H(y)$ are given in terms of the sin and cos functions, just like in the 1D case. Furthermore, the homogeneous boundary conditions pick out only the sin solutions, namely

$$F''(x) + k^2 F(x) = 0, \quad \text{with } F(0) = F(L_x) = 0 \to F(x) = \sin\left(n\frac{\pi x}{L_x}\right), \quad k = n\frac{\pi}{L_x}$$

and similarly

$$H''(y) + p^2 F(y) = 0, \quad \text{with } H(0) = H(L_y) = 0 \to H(y) = \sin\left(m\frac{\pi y}{L_y}\right), \quad p = m\frac{\pi}{L_y}$$

The corresponding equation for $G(t)$ is the same as in the 1D case

$$\dot{G}(t) + \lambda_{mn}^2 G(t) = 0 \to G(t) = e^{-\lambda_{mn}^2 t}$$

where

$$\lambda_{mn} = \frac{n^2\pi^2c^2}{L_x^2} + \frac{n^2\pi^2c^2}{L_y}.$$

Hence the spectrum of independent functions for the heat diffusion in 2D is given by

$$\{\sin\left(n\frac{\pi x}{L_x}\right)\sin\left(m\frac{\pi y}{L_y}\right)) e^{-\lambda_{mn}^2 t}\}$$

Step 3: General solution $u(x,t)$ is then given as

$$u(x,y,t) = \sum_n \sum_m A_{nm} e^{-\lambda_{nm}^2 t} \sin\left(n\frac{\pi x}{L_x}\right)\sin\left(m\frac{\pi y}{L_y}\right)$$

where the coefficients A_{nm} are fixed by the two-dimensional Fourier transform of the initial profile

$$f(x,y) = \sum_n \sum_m A_{nm} \sin\left(n\frac{\pi x}{L_x}\right)\sin\left(m\frac{\pi y}{L_y}\right)$$

and determined as

$$A_{nm} = \frac{2}{L_x}\frac{2}{L_y}\int_0^{L_x} dx \int_0^{L_y} dy f(x,y) \sin\left(n\frac{\pi x}{L_x}\right)\sin\left(m\frac{\pi y}{L_y}\right)$$

LET US CONSIDER as a rectangular domain is $[0,\pi]\times[0,\pi]$, i.e. $L_x = L_y = \pi$ and a constant initial profile $f(x,y) = 1$. Then, the field at time t starting from this constant profile and homogeneous boundary conditions is given as

$$u(x,y,t) = \sum_n \sum_m A_{nm} e^{-(n^2+m^2)t} \sin(nx)\sin(my)$$

where

$$A_{nm} = \frac{4}{\pi^2}\int_0^{\pi} dx \sin(nx) \int_0^{\pi} dy \sin(my) \tag{24.6}$$

$$= \frac{4^2}{\pi^2}\frac{1}{nm} \tag{24.7}$$

Hence,

$$u(x,y,t) = \frac{4^2}{\pi^2}\sum_n \sum_m \frac{1}{mn} e^{-(n^2+m^2)t} \sin(nx)\sin(my).$$

THIS FORMULA can be generalized to 3D as

$$u(x,y,z,t) = \frac{4^3}{\pi^3}\sum_n \sum_m \sum_q \frac{1}{mnq} e^{-(n^2+m^2+q^2)t} \sin(nx)\sin(my)\sin(qz)$$

Lecture 25: Separation of variable method Non-Cartesian coordinates

25.1 Separation of variables method

In the previous lecture, we have applied the separation of variable method to **homogeneous** wave and heat equations with **homogeneous** boundary conditions on a finite interval in 1D. This method can also be generalized to higher dimensions using Cartesian or non-Cartesian coordinates. Often, the symmetries of the spatial domain are also reflected in the solution of the linear PDE, e.g. problems defined on domains with spherical or radial symmetries (invariant to rotations) have also solutions with the same invariance. Symmetries are often guiding us in finding solutions and simplifying the problem.

We explore this idea to solve PDE's defined on a disk or on a sphere. Along the way, we will also discover how special ODE's - such as Euler-Cauchy equation, Legendre equation and Bessel equation which we studied before - arise using this the separation of variable method. *Look up in the lecture notes 14, 17, and 18 on these ODE's whenever needed to remind yourself how to solve them and some of their properties!*

25.1.1 Laplace equation: spherical coordinates

Let us consider a scalar function of three independent variables $u = u(r, \phi, \theta)$ corresponding to the spherical coordinates to represent a point in a three-dimensional space, i.e. radial distance r to origin, inclination angle $\theta \in [0, \pi]$ from the z-axis and azimuth angle $\phi \in [0, 2\pi]$.

A PDE satisfied by this function contains spatial derivatives which in the Cartesian (x, y, z) coordinates are

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}$$

and of higher order. However, when the natural variables are spherical coordinates as in this case, we need to express the differential operators in curvilinear coordinates using the transformation of variables

$$x = r \cos\phi \, \sin\theta, \tag{25.1}$$

$$y = r \sin\phi \, \sin\theta, \tag{25.2}$$

$$z = r \cos\theta. \tag{25.3}$$

The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ along the coordinate axes in the (x, y, z)-space transform into the spherical coordinate system into the unit vectors

$$\boldsymbol{r}_0 = \mathbf{i} \sin\theta \cos\phi + \mathbf{j} \sin\theta \sin\phi + \mathbf{k} \cos\theta \tag{25.4}$$

$$\boldsymbol{\theta}_0 = \mathbf{i} \cos\theta \cos\phi + \mathbf{j} \cos\theta \sin\phi - \mathbf{k} \sin\theta \tag{25.5}$$

$$\boldsymbol{\phi}_0 = -\mathbf{i} \sin\phi + \mathbf{j} \cos\phi. \tag{25.6}$$

A point in space can be uniquely identified as the intersection of three mutually perpendicular planes. The normal vectors to these planes relate to an orthogonal coordinate system. For instance, we can locate a point in the (x,y,z) coordinates by the intersection between the plane $x = constant$, the plane $y = constant$ and the plane of $z = constant$. The same point can also be located in spherical coordinates by the interactions of the three planes $r = constant$, $\theta = constant$, $\phi = constant$. The differential operators transform from one coordinate system to another by how the function changes in each of these sets of planes.

Using these coordinate transformations, one can show that the gradient vector

$$\nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

can be expressed in spherical coordinates as

$$\nabla u = \boldsymbol{r}_0 \frac{\partial u}{\partial r} + \boldsymbol{\theta}_0 \frac{1}{r}\frac{\partial u}{\partial \theta} + \boldsymbol{\phi}_0 \frac{1}{r \sin\theta}\frac{\partial u}{\partial \theta}$$

The Laplacian operator defined as the divergence of the gradient field $\nabla \cdot \nabla$ is expressed in Cartesian coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

In spherical coordinates, the Laplacian of a function $u(r, \phi, \theta)$ takes the form

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

By setting the Laplacian of u equal to zero, we obtain the so-called *Laplace equation*

$$\nabla^2 u = 0$$

which is an important PDE in physics and often arises from heat or wave equations when the problem is independent of time (steady-state). Thus, the Laplace equation often describes the stationary or steady-state solutions under certain boundary conditions.

Spherical symmetry: We seek to find a spherically-symmetric solution $u = u(r, \theta)$, that means u is independent of the azimuth angle ϕ thus

$$\frac{\partial u}{\partial \phi} = 0.$$

For such problems with spherical symmetry, the 3D Laplace equation simplifies to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0. \tag{25.7}$$

In addition, we use radial boundary conditions given by

$$u(r = R, \theta) = f(\theta), \qquad \lim_{r\to\infty} u(r, \theta) = 0$$

meaning that on the surface of a sphere of radius R, the function u equals a given profile $f(\theta)$ that may depend on the angle θ, and as the radius becomes infinite, the function decays to zero. In concrete applications, $u(r, \theta)$ could represent for instance the gravitational field induced by a massive spherical object that is empty inside. We know how the gravitational field is distributed on the surface of this object, and that the field vanishes infinitely far away, as if this ball were the only mass in the Universe. We want to use the separation of variable method to find what is the gravitational field inside the ball and outside it.

By the separation of variables, we construct an ansatz solution as a product of two functions

$$u(r, \theta) = G(r) \cdot H(\theta)$$

and insert this into Eq. (25.7) to obtain the corresponding relation satisfied by them

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) \equiv k$$

where k is some proportionality constant determined by the boundary conditions.

Looking at the 2nd order ODE in r,

$$\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = kG \rightarrow r^2G'' + 2rG' - kG = 0,$$

we may recognize that this is the **Euler-Cauchy equation**. Without loss of generality, we take $k = n(n+1)$ with $n = 0, 1, 2, \cdots$ such that the above equation becomes

$$r^2G'' + 2rG' - n(n+1)G = 0$$

We seek for independent solutions of the form r^λ with λ determined by the characteristic equation:

$$\lambda(\lambda-1) + 2\lambda - n(n+1) = 0 \rightarrow \lambda_1 = n, \quad \lambda_2 = -(n+1)$$

Hence, the two independent solutions are parameterised by n and given by:

$$G_n(r) = r^n, \qquad \tilde{G}_n(r) = r^{-(n+1)}$$

Let us know look at the ODE in θ given by

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + n(n+1)H = 0. \tag{25.8}$$

We use the change of variable

$$\omega = \cos\theta \rightarrow \sin^2\theta = 1 - \omega^2$$

such that the derivative with respect to θ transforms as

$$\frac{d}{d\theta} = \frac{d\omega}{d\theta}\frac{d}{d\omega} = -\sin\theta\frac{d}{d\omega},$$

and Eq. (25.8) can be rewritten in ω as

$$-\frac{d}{d\omega}\left(-\sin\theta^2\frac{dH}{d\omega}\right) + n(n+1)H = 0 \rightarrow \tag{25.9}$$

$$\frac{d}{d\omega}\left((1-\omega^2)\frac{dH}{d\omega}\right) + n(n+1)H = 0, \tag{25.10}$$

which is the **Legendre equation**

$$(1-\omega^2)H'' - 2\omega H' + n(n+1)H = 0,$$

defined in the interval $\omega \in [-1, 1]$.

Recall from Lecture 17 that we solved this equation using the power series expansion method. The solution which is finite and equal to 1 at the boundary points $\omega = \pm 1$ is given by the Legendre polynomial of order n:

$$H(\omega) = P_n(\omega) \rightarrow H(\theta) = P_n(\cos\theta).$$

In our case, $n = 0, 1, \cdots$ and labels the infinite number of independent solutions of the Laplace equation are given by

$$u_n(r, \theta) = r^n P_n(\cos\theta), \qquad \tilde{u}_n(r, \theta) = r^{-(n+1)} P_n(\cos\theta).$$

Thus, the general solution can be written as a series expansion in terms of these eigenfunctions, namely

$$u(r, \theta) = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+1)} \right] P_n(\cos\theta), \qquad (25.11)$$

with constants A_n and B_n determined by the boundary conditions and the orthogonality relation of the Legendre polynomials. Recall that

$$\int_{-1}^{1} d\omega P_n(\omega) P_m(\omega) = \frac{2}{2n+1} \delta_{n,m}$$

or equivalently using the θ variable

$$\int_{0}^{\pi} d\theta \sin\theta P_n(\cos\theta) P_m(\cos\theta) = \frac{2}{2n+1} \delta_{n,m}$$

Application to potential fields: Let us apply this solution to find the electrostatic potential $u(r, \theta)$ inside and outside of a surface of radius R for a known profile of the electrostatic potential on the surface namely

$$u(R, \theta) = f(\theta).$$

Outside solution: The boundary condition

$$\lim_{r \to \infty} u(r, \theta) = 0$$

implies that the field is zero at infinity, hence

$$A_n = 0.$$

Thus, the solution from Eq. (25.11) simplifies to

$$u(r, \theta) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta), \qquad r \geq R$$

The coefficients B_n are determined from the orthogonality condition applied to the radial boundary condition at $r = R$

$$u(R, \theta) = f(\theta).$$

Namely,

$$\begin{aligned}\int_0^\pi d\theta \sin\theta u(R,\theta) P_m(\cos\theta) &= \sum_{n=0}^{\infty} B_n R^{-(n+1)} \int_0^\pi d\theta \sin\theta P_n(\cos\theta) P_m(\cos\theta) \\ &= B_m R^{-(m+1)} \frac{2}{2m+1}\end{aligned}$$

Hence, the coefficients are computed as

$$B_n = R^{(n+1)} \frac{(2n+1)}{2} \int_0^\pi d\theta \sin\theta f(\theta) P_n(\cos\theta).$$

Inside solution: For the inside solution to be finite at the origin, the coefficients in front of the terms $r^{-(n+1)}$ terms that diverge at origin must vanish, hence

$$B_n = 0, \quad r \le R,$$

such that the inside solution is expanded as

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \qquad r \le R$$

where the coefficients A_n are determined by the profile on the surface of radius R

$$\begin{aligned}\int_0^\pi d\theta \sin\theta u(R,\theta) P_m(\cos\theta) &= \sum_{n=0}^{\infty} A_n R^n \int_0^\pi d\theta \sin\theta P_n(\cos\theta) P_m(\cos\theta) \\ &= A_m R^m \frac{2}{2m+1}\end{aligned}$$

Hence,

$$A_n = R^{-n} \frac{(2n+1)}{2} \int_0^\pi d\theta \sin\theta f(\theta) P_n(\cos\theta)$$

For a constant field on the surface $f(\theta) = 1$, the integral that determines the coefficients A_n and B_n reduces to the integral over the Legendre polynomials,

$$\int_{-1}^{1} d\omega P_n(\omega) = 2\delta_{n,0},$$

all the odd polynomials integrate to zero from symmetry, while for the even polynomials only $P_0(\omega) = 1$ leads to a nonzero integral.

The solution of the electrostatic potential for this constant distribution on the spherical surface is

$$u(r,\theta) = \begin{cases} \frac{R}{r}, & r \leq R \\ 1, & r \leq R. \end{cases}$$

25.1.2 *Diffusion equation in polar coordinates*

Another useful example of a curvilinear domain with radial symmetry is the disk of radius R. Linear PDE's that are defined on a disk have solutions that are also radially-symmetric. The geometry of the spatial domain helps us choose the natural spatial coordinates. For the disk geometry, we may want to choose the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

The 2D Laplace operator in polar coordinates reads as

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

where $u = u(r,\theta)$. Radially-symmetric solutions in the plane correspond are isotropic solutions, i.e.

$$\frac{\partial u}{\partial \theta} = 0,$$

and the Laplace of u contains only to the radial derivatives.

Let us consider the diffusion of a (temperature) field $u(r,t)$ in a disk given by

$$\frac{\partial u}{\partial t} = c^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

We can solve this equation by the separation of variable method for the *homogeneous* boundary condition on the circumference of the disk

$$u(r=a,t) = 0, \text{ on the boundary of the disk, for any } t \geq 0 \quad (25.12)$$

and for a radially-symmetric initial profile

$$u(r,0) = f(r).$$

BY INSERTING the ansatz solution

$$u(t,r) = F(r)G(t)$$

into the diffusion equation, we have that

$$\frac{1}{F}\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = \frac{1}{c^2}\frac{1}{G}\frac{dG}{dt} = -k^2,$$

where the proportionality constant is taken as $-k^2$ for reasons that become clearer as we apply the boundary conditions. The evolution in time is a first order ode

$$\frac{\dot{G}(t)}{G(t)} = -k^2c^2t$$

with the independent solution given by the exponential decay

$$G_1(t) = e^{-k^2c^2t}.$$

The radial equation reads as

$$\frac{d}{dr}\left(r\frac{dF}{dr}\right) - k^2rF = 0$$

which we may recognize as the **Bessel** equation:

$$x\frac{d}{dx}\left(x\frac{dy}{dr}\right) + (x^2 - n^2)y = 0$$

for $n = 0$ and $x = kr$.

In Lecture 18, we have solved this equation using the Frobenius method and found that one of independent solutions is given by the *Bessel* function of the first kind of order n $J_n(kr)$. In our case, $n = 0$. It turns out that the other independent solution is singular at the origin, thus does not contribute to the general solution expansion, i.e. the eigenfunctions depend only on zero order Bessel function

$$J_0(kr) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{kr}{2}\right)^{2n},$$

which is also plotted in Fig. (25.1). The homogeneous condition on the boundary of the disk at $r = a$, implies that the Bessel function J_0 must vanish $J_0(ka) = 0$. However, from Fig. (25.1), we notice that J_0 has infinitely many zeros at positions μ_n where $n = 0, 1, 2, \cdots$. This determines the allowed values of the proportionality constant k from the position of the zeros, namely that

$$k = \frac{\mu_n}{a}, \quad n \in \mathcal{N}.$$

Thus the solution is a series expansion

$$u(r,t) = \sum_{n=0}^{\infty} A_n J_0\left(\mu_n\frac{r}{a}\right)e^{-\gamma_n t}$$

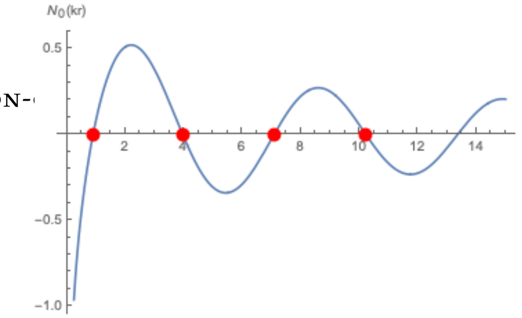


Figure 25.1: Zeros of the Bessel function $J_0(x)$.

where $\gamma_n = \frac{\mu_n^2 c^2}{a^2}$ is the decay rate.

Applying this expression to the initial profile $u(r,t) = f(r)$, we find that the $f(r)$ is represented as an infinite series expansion

$$f(r) = \sum_{n=0}^{\infty} A_n J_0\left(\mu_n \frac{r}{a}\right).$$

The Bessel function of order zero J_0 satisfies this orthogonality relation

$$\int_0^a dr r J_0\left(\mu_n \frac{r}{a}\right) J_0\left(\mu_m \frac{r}{a}\right) = \frac{a^2}{2} J_1^2\left(\mu_n\right) \delta_{n,m}$$

where $J_1(\mu_n)$ is the first order Bessel function evaluated at the position of the n-th zero μ_n of the zeroth order Bessel function J_0. J_1 has its own set of zeros that are different from those of J_0, thus $J_1(\mu_n) \neq 0$. We use this property to determine the coefficients A_n as

$$\begin{aligned} \int_0^a dr r J_0\left(\mu_m \frac{r}{a}\right) f(r) &= \sum_n A_n \int_0^a dr r J_0\left(\mu_m \frac{r}{a}\right) J_0\left(\mu_n \frac{r}{a}\right) \\ &= \frac{a^2}{2} A_m J_1^2\left(\mu_m\right) \end{aligned}$$

which implies that

$$A_m = \frac{2}{a^2} \frac{1}{J_1^2\left(\mu_m\right)} \int_0^a dr r J_0\left(\mu_m \frac{r}{a}\right) f(r).$$

For a uniform initial profile $f(r) = 1$ on a disk of radius $a = 1$, the corresponding coefficients are

$$\begin{aligned} A_m &= 2\frac{1}{J_1^2(\mu_m)} \int_0^1 dr r J_0\left(\mu_m r\right) \\ &= \frac{2}{\mu_m} \frac{1}{J_1^2(\mu_m)} J_1(\mu_m) \\ &= \frac{2}{\mu_m} J_1(\mu_m) \end{aligned}$$

and the solution at radius r and t is then

$$u(r,t) = \sum_{m=0}^{\infty} \frac{2}{\mu_m} \frac{J_0(\mu_m r)}{J_1(\mu_m)} e^{-\mu_m^2 t},$$

where $c = 1$. In practice, this could represent the temperature field (in some rescaled units) inside the disk starting from a uniform value and fixing the temperature to zero on the circumference of the disk. What happens is that the heat is diffused out of the system and as result the temperature inside the disk will gradually lower with time.

Lecture 26:
Integral transform method

Although the separation of variable method is a powerful tool, it cannot handle all types of linear PDE's. When the PDE has a non-zero forcing or the boundary conditions are non-homogeneous and not consistent with the separation of variable ansatz, then we need other tools. An alternative approach that is powerful in such cases is based on the integral transform method. The Fourier or Laplace transforms come to rescue and help us find solutions to boundary value problems and initial value problems, respectively. The Green function method is often used as an equivalent method for solving non-homogeneous, linear PDE's.

In this lecture, we will highlight how to apply integral transforms and the Green function method for concrete examples of PDE's.

26.1 *Laplace transform method*

The Laplace transform method is tailored to linear initial value problems, i.e. when we know that the function has initial conditions $t = 0$ (often *homogeneous*) and non-homogeneous boundary conditions, and want to know the solution at a later time t.

Transport equation: Let us consider a homogeneous first order PDE given by

$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the **homogeneous initial condition**

$$u(x,0) = 0$$

and the non-homogeneous boundary condition at $x = 0$

$$u(0, t) = t, \quad t \geq 0$$

Solution:

Let us denote the Laplace transform of the field $u(x, t)$ as

$$\mathcal{L}[u(x, t)] = U(x, s).$$

Using the homogeneous initial condition, it follows that the L.T. of the time derivative of $u(x, t)$ is then

$$\mathcal{L}[\partial_t u(x, t)] = sU(x, s).$$

The first step in solving this PDE is to Laplace transform the equation with respect to time t, which leads to this differential equation

$$sxU + \frac{\partial U}{\partial x} = 0$$

Since there are no derivatives with respect to s, this is essentially a first order ODE in x, which we can solve straightforwardly by integrating the differential form

$$\frac{dU}{U} = -sxdx$$

and obtain the general solution

$$U(x, s) = c(s)e^{-sx^2/2}$$

with an integration factor $c(s)$ that may depend on s. This is fixed by the boundary condition at $x = 0$ where the Laplace transform function $U(0, s) = c(s)$ must equal the Laplace transform of t $\mathcal{L}[t] = s^{-2}$, namely

$$c(s) = \frac{1}{s^2}.$$

Hence, the solution in the s-space is

$$U(x, s) = \frac{1}{s^2}e^{-sx^2/2}$$

and its inverse Laplace transform gives us the specific solution to this problem,

$$u(x, t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-sx^2/2}\right].$$

Using the t-shifting theorem (see Lecture 22)

$$\mathcal{L}^{-1}\left[F(s)e^{-sa}\right] = f(t - a)H(t - a),$$

with $a = x^2/2$ and $F(s) = \frac{1}{s^2} = \mathcal{L}[t]$, we then obtain the final solution given as

$$u(x,t) = \left(t - \frac{x^2}{2}\right) H\left(t - \frac{x^2}{2}\right)$$

where $H(x)$ is the Heaviside function. Equivalently, this can be written as

$$u(x,t) = \begin{cases} 0, & t < \frac{x^2}{2} \\ t - \frac{x^2}{2}, & t \geq \frac{x^2}{2} \end{cases} \tag{26.1}$$

We may notice that the final solution is not consistent with the ansatz solution from the separation of variable method, and seeking to solve the problem using that ansatz would only give us the trivial solution (zero).

1D wave equation: Let us consider again the 1D string equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with *homogeneous initial conditions*

$$u(x,0) = 0, \quad \partial_t u(x,0) = 0$$

and the boundary conditions on the string, which extends from $x = 0$ to $x \to \infty$, given by:

$$u(0,t) = \sin(t)H(2\pi - t) = \begin{cases} \sin(t), & 0 \leq t \leq 2\pi \\ 0, & \text{otherwise} \end{cases}, \qquad \lim_{x\to\infty} u(x,t) = 0, \qquad t \geq 0.$$

These boundary conditions mean that the end of the string at $x = 0$ oscillates in time for one period and then it is fixed at 0 for the remaining time while the other end at infinity is fixed at all times. In plain words, we set up a wave at one end and see how it travels across the string to the other end. Our intuition may tells us that the solution may be some kind of a travelling wave.

Solution:

Again, we denote the Laplace transform of the deformation field u with respect to time t as

$$U(x,s) = \mathcal{L}[u(x,t)]$$

The homogeneous initial conditions simplify the expressions of the L.T. of the second derivative in time, such that the L.T. of the wave equation reads as

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U = 0$$

which is a second order ODE in x with the solution

$$U(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c},$$

where the integration coefficients $A(s)$ and $B(s)$ may depend on s and are fixed by the boundary conditions at x. Namely, using the boundary condition at infinity of zero displacement,

$$\lim_{x\to\infty} U(x,s) = 0 \to A(s) = 0,$$

we find that

$$U(x,s) = B(s)e^{-sx/c}.$$

Evaluating this solution at $x = 0$, we find that $U(0,s) = B(s)$ which must equal the Laplace transform of the boundary condition profile at $x = 0$, hence

$$B(s) = \mathcal{L}[u(0,t)]$$

By the t-shifting theorem, the inverse Laplace transform gives us the specific solution as

$$u(x,t) = u\left(0, t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right),$$

Using the initial profile, we can rewrite the solution as

$$u(x,t) = \sin\left(t - \frac{x}{c}\right) H\left(2\pi + \frac{x}{c} - t\right) H\left(t - \frac{x}{c}\right),$$

which is equivalent to

$$u(x,t) = \begin{cases} \sin\left(t - \frac{x}{c}\right), & \frac{x}{c} \le t \le 2\pi + \frac{x}{c} \\ 0, & \text{otherwise} \end{cases}$$

It represents traveling a sin-wave moving to the right with speed c as shown in Fig. (26.1)

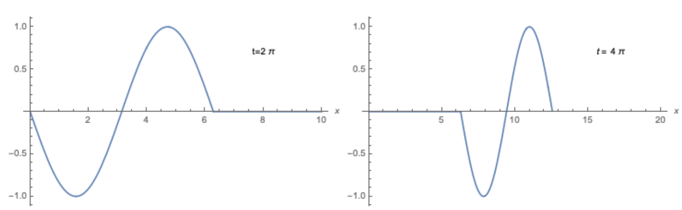


Figure 26.1: Travelling wave solution.

26.2 *Fourier transform method*

We now exemplify how to apply the Fourier transform method for a boundary value problem, where the function decays sufficiently fast at $\pm\infty$ (homogeneous boundary conditions in an infinite system). Let us consider the diffusion equation in one spatial dimension,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with a non-homogeneous initial condition

$$u(x,0) = b\delta(x)$$

and **homogeneous boundary conditions at** $\pm\infty$ (Dirichlet's conditions):

$$\lim_{x\to\pm\infty} u(x,t) = 0, \qquad t \geq 0.$$

Solution:

Let us denote the Fourier transform with respect to x of the function itself by

$$\mathcal{F}[u(x,t)] = \hat{u}(k,t),$$

such that the F.T. of the second order derivatives in x is then (see Lecture 21)

$$\mathcal{F}\left[\frac{\partial u(x,t)}{\partial x^2}\right] = -k^2\hat{u}(k,t).$$

We apply the Fourier transform of the diffusion equation and arrive at a first order ODE in time of the Fourier transform function,

$$\frac{d\hat{u}}{dt} + c^2k^2\hat{u} = 0,$$

which has the solution given by an exponential

$$\hat{u}(k,t) = A(k)e^{-c^2k^2t}$$

with the integration coefficient $A(k)$ that may depend on k through the initial condition. Namely, setting $t = 0$ in the above equation and equating it with the Fourier transform of the initial profile, we have

$$A(k) = \mathcal{F}[b\delta(x)] = \frac{b}{2\pi},$$

thus the solution is a Gaussian function of k in the Fourier space,

$$\hat{u}(k,t) = \frac{b}{2\pi}e^{-c^2k^2t}.$$

By applying the inverse Fourier transform on $\hat{u}(k,t)$, we then find the solution in the x variable given by

$$u(x,t) = \frac{b}{2\pi}\mathcal{F}^{-1}\left[e^{-c^2k^2t}\right] \tag{26.2}$$

$$= \frac{b}{2\pi}\int_{-\infty}^{\infty} dk e^{-c^2k^2t}e^{ikx}. \tag{26.3}$$

By completing the square in the exponent and using the Gaussian integral formula, we arrive at

$$u(x,t) = \frac{b}{2\pi}e^{-x^2/(4c^2t)}\int_{-\infty}^{\infty} dk e^{-(c\sqrt{t}k+ix/\sqrt{4c^2t})^2} \tag{26.4}$$

$$= \frac{b}{\sqrt{4\pi c^2t}}e^{-x^2/(4c^2t)}. \tag{26.5}$$

which is a Gaussian distribution centered at origin and with a time-dependent width that broadens over time. This represents the diffusion process which spreads out the field in space and time. We can see this more clearly when we write the solution equivalently as

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} e^{-x^2/(2\sigma^2(t))}. \tag{26.6}$$

where the width $\sigma(t) = \sqrt{2c^2 t}$ increases as square root of time. Notice that the Gaussian distribution is normalized at all times,

$$\int_{-\infty}^{+\infty} dx u(x,t) = 1, \tag{26.7}$$

which follows straightforwardly using the Gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}.$$

THE FOURIER TRANSFORM is a powerful method to apply also for higher spatial dimensions. Then, we perform a Fourier transform with respect to each spatial component. For a 3D position $\vec{r}$ space in Cartesian coordinates (x, y, z), the Fourier vector $\vec{k}$ space corresponds to components (k_x, k_y, k_z). Thus, the Fourier transform of a field $u(\vec{r}, t)$ is another field $\hat{u}(\vec{k}, t)$ in the $\vec{k}$-space and given as a three dimensional integral

$$\hat{u}(\vec{k},t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz u(x,y,z,t) e^{-ik_x x} e^{-ik_y y} e^{-ik_z z} \tag{26.8}$$

$$= \frac{1}{(2\pi)^3} \iiint d^3\vec{r} u(\vec{r},t) e^{-i\vec{k}\cdot\vec{r}}. \tag{26.9}$$

When the problem has a *spherical symmetry* whereby the field depends only on the radius $r = |\vec{r}|$, i.e. $u(\vec{r}, t) = u(r, t)$, this symmetry carries over to the Fourier space, i.e. $\hat{u}(\vec{k}, t) = \hat{u}(k, t)$. We can make use the transformation to spherical variables to integrate over the azimuthal angles to obtain

$$\hat{u}(k,t) = \frac{1}{(2\pi)^2}\int_0^{+\infty} dr r^2 u(r,t) \int_0^{\pi} d\theta \sin\theta e^{-ikr\cos\theta} \quad (26.10)$$

$$= \frac{1}{(2\pi)^2}\int_0^{+\infty} dr r^2 u(r,t) \int_{-1}^{1} dt e^{-ikrt} \quad (26.11)$$

$$= \frac{1}{(2\pi)^2}\int_0^{+\infty} dr r^2 u(r,t) \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right) \quad (26.12)$$

$$= \frac{1}{2\pi^2 k}\int_0^{+\infty} dr r u(r,t) \sin(kr). \quad (26.13)$$

where we used the substitution $t = \cos(\theta)$ is the zeroth order Bessel function.

26.3 *Green's function method*

The Green's function method is tailored for non-homogeneous, linear differential equations. As an example, we apply it to solve the Poisson equation in three dimensional space with Cartesian coordinates. The 3D Poisson equation satisfied by a scalar function $u(x,y,z)$ reads as

$$\nabla^2 u(x,y,z) = f(x,y,z), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and we consider it together with the homogeneous boundary conditions at $\pm\infty$ for x, y, z:

$$\lim_{x\to\pm\infty} u(x,y,z) = 0, \forall y, z,$$

$$\lim_{y\to\pm\infty} u(x,y,z) = 0, \forall x, z,$$

$$\lim_{z\to\pm\infty} u(x,y,z) = 0, \forall x, y.$$

along each principal axis.

Solution:

The corresponding equation satisfied by the Green's function $G(x,y,z,x',y',z')$ is obtained by replacing the forcing term with a product of Dirac delta functions for each variable which written equivalently as a Dirac delta function in 3D

$$\delta(x - x') \cdot \delta(y - y') \cdot \delta(z - z') \equiv \delta^3(\vec{r} - \vec{r'})$$

Thus,

$$\nabla^2 G(x,y,z,x',y',z') = \delta^3(\vec{r}-\vec{r'})$$

The Green's function is also called the *fundamental solution*. By the convolution theorem, the solution of the Poisson equation with a source $f(x,y,z)$ is then determined as a convolution integral of the fundamental solution with the source function,

$$u(x,y,z) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' G(x,y,z,x',y',z') f(x',y',z')$$

The idea is that we can solve the equation satisfied by the Green's function using the Fourier transform method since it quickly takes care of the δ function and spatial derivatives. We denote the 3D Fourier transform of the Green function as

$$\begin{aligned} \hat{G}(k_x,k_y,k_z,x',y',z') &= \mathcal{F}[G] \\ &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz G(x,y,z,x',y',z') e^{-i(k_x x + k_y y + k_z z)}. \end{aligned}$$

We also need to Fourier transform the product of the delta functions

$$\mathcal{F}[\delta^3(\vec{r}-\vec{r'})] = \frac{1}{(2\pi)^3} \iiint d^3\vec{r} \delta^3(\vec{r}-\vec{r'}) e^{-i\vec{k}\cdot\vec{r}} \tag{26.14}$$

$$= \frac{1}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r'}} \tag{26.15}$$

The Fourier transforms of the spatial derivatives are obtained by successive integration by parts and given as

$$\mathcal{F}[\partial_x^2 G] = -k_x^2 \hat{G}, \qquad \mathcal{F}[\partial_y^2 G] = -k_y^2, \qquad \mathcal{F}[\partial_z^2 G] = -k_z^2$$

Putting it all together, we find that the Poisson equation for the Green's function in the Fourier space reduces to an algebraic equation

$$-(k_x^2 + k_y^2 + k_z^2)\hat{G} = \frac{1}{(2\pi)^3} e^{-i(k_x x' + k_y y' + k_z z')}$$

which determines $\hat{G}$ as

$$\hat{G} = -\frac{1}{(2\pi)^3} \frac{1}{k^2} e^{-i\vec{k}\cdot\vec{r'}}$$

By applying the inverse F.T. on $\hat{G}$ and writing the triple integral over $\vec{k}$ in spherical coordinates, we find an expression for the Green function in the real space given as

$$\begin{aligned} G(\vec{r},\vec{r}') &= \iiint d^3\vec{k}\hat{G}(\vec{k},\vec{r}')e^{i\vec{k}\cdot\vec{r}} \\ &= -\frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk k^2 \frac{1}{k^2} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \\ &= -\frac{2\pi}{(2\pi)^3}\int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk k^2 \frac{1}{k^2} e^{ik|\vec{r}-\vec{r}'|\cos\theta} \end{aligned}$$

where we integrated out the azimuths angle ϕ and aligned the $\vec{k}$-coordinate system so that the vector $\vec{r}-\vec{r}'$ is along the k_z direction such that the inner product of $\vec{k}$ with $\vec{r}-\vec{r}'$ is determined by the inclination angle θ. Let us now denote by $a = |\vec{r}-\vec{r}'| > 0$ as the magnitude of this vector. We can perform the integral over the inclination angle θ, by using the substitution $t = \cos\theta$, namely

$$\begin{aligned} G(a) &= -\frac{1}{(2\pi)^2}\int_0^{\infty} dk \int_{-1}^{1} dt e^{ikat} \\ &= -\frac{1}{(2\pi)^2}\int_0^{\infty} dk \frac{1}{ika}\left[e^{ika} - e^{-ika}\right] \\ &= -\frac{1}{(2\pi)^2}\frac{2}{a}\int_0^{\infty} dk \frac{\sin(ka)}{k}. \end{aligned}$$

Since the integrand is even, we rewrite it in terms of the integral from $\pm\infty$ as

$$\begin{aligned} G(a) &= -\frac{1}{(2\pi)^2}\frac{1}{a}\int_{-\infty}^{\infty} dk \frac{\sin(ka)}{k} \\ &= -\frac{1}{(2\pi)^2}\frac{1}{a}\int_{-\infty}^{\infty} dk \frac{1}{2ik}\left[e^{ika} - e^{-ika}\right]. \end{aligned}$$

This integrand has a singular behavior at $k = 0$. We can deal with it when extend the integral into the complex plane in the complex variable w such that $Re(w) = k$ and apply the principal value method. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} dk \frac{\sin(ka)}{k} &= \frac{1}{2i}P.V.\int_{-\infty}^{\infty} dk \frac{e^{ika}}{k} - \frac{1}{2i}P.V.\int_{-\infty}^{\infty} dk \frac{e^{-ika}}{k} \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi. \end{aligned}$$

Hence, the Green's function of the Poisson equation takes this simple form

$$G(\vec{r}, \vec{r}') = -\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r'}|}.$$

It is spherically-symmetric and depends only on the distance between two points $G(|\vec{r} - \vec{r}'|)$. The integral solution to the Poisson equation with a given source $f(\vec{r})$ is then the convolution integral

$$u(\vec{r}) = \iiint d^3\vec{r'} G(|\vec{r} - \vec{r}'|) f(\vec{r'}).$$

This is the typical integral solution for the gravitational field or electrostatic field induced by a given source in 3D space.