Training Set 1

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Problem T1.1

a)

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r = \sqrt{2} \quad , \quad \theta = \frac{5\pi}{4}$$

$$x = \sqrt{2}\cos\frac{5\pi}{4} = -1 \quad , \quad y = \sqrt{2}\sin\frac{5\pi}{4} = -1$$

$$\underline{\underline{z = -1 - i}}$$

b)

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \arctan\frac{y}{x}$$

$$r = \sqrt{3+1} = 2 \quad , \quad \theta = \arctan\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}} = \frac{\left(\sqrt{2}e^{i\pi/4}\right)^{48}}{\left(2e^{-i\pi/6}\right)^{25}} = 2^{24-25}e^{i\pi(12+25/6)} = \frac{1}{2}e^{i\pi(12+25/6)}$$

$$x = \frac{1}{2}\cos(\pi(12+25/6))$$

$$y = \frac{1}{2}\sin(\pi(12+25/6))$$

$$\underline{\underline{z = \frac{1}{2}\cos(\pi(12+25/6)) + i\frac{1}{2}\sin(\pi(12+25/6))}}$$

c)

$$\frac{e^{1+3\pi i}}{e^{-1+i\pi/2}} = e^{1-(-1)+3\pi i - i\pi/2} = e^{2+5\pi i/2} = e^2 e^{i5\pi/2} = e^2 e^{i\pi/2}$$

$$x = 0 \quad , \quad y = 1 \Rightarrow \underline{\underline{z = ie^2}}$$

d)

$$x = -8 \quad , \quad y = 8\sqrt{3}$$

$$r = \sqrt{64+192} = 16 \quad , \quad \theta = \arctan\frac{\sqrt{3}}{-1} = \frac{2\pi}{3}$$

$$z = (8i\sqrt{3}-8)^{1/4} = \left(16e^{i2\pi/3}\right)^{1/4} = 2e^{i\pi/6}$$

$$w_0 = 2e^{i\pi/6} \quad , \quad w_1 = 2e^{i(\pi/6+2\pi/4)} = 2e^{i(\pi/6+\pi/2)} = 2e^{i2\pi/3}$$

$$w_2 = 2e^{i\pi(1/6+2/2)} = 2e^{i\pi 7/6} \quad , \quad w_3 = 2e^{i\pi(1/6+3/2)} = 2e^{i\pi 5/3}$$

e)

$$z = 8 = 8e^{i0}$$

$$z^{1/3} = w_0 = 8^{1/3} = 2 \quad , w_1 = 2e^{i2\pi/3} \quad , w_2 = 2e^{-i2\pi/3}$$

$$\sum_{i=0}^{2} w_i = 2 + 2e^{i2\pi/3} + 2e^{-i2\pi/3} = 2 + 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$$

$$2 + 4\cos 2\pi/3 = 2 + 4\left(-\frac{1}{2}\right) = 0$$

After finding one root we rotate around the unit circle by $2\pi/n$. If n is even then for each root with argument θ there is another root with argument $-\theta$. As the angles cancel out, the sum of the numbers below the real axis will be some negative number $-a$, and the numbers above will be a. The sum of all the roots will then be 0.

f)

$$z = i = e^{i\pi/2}$$

$$z^{1/3} = w_0 = e^{i\pi/6}, \quad w_1 = e^{i\pi(1/6+2/3)} = e^{i\pi 5/6}, \quad w_2 = e^{i\pi(1/6+4/3)} = e^{-i\pi 3/2}$$

g)

$$z = 2 + 2\sqrt{3}i = 4e^{i\pi/3}$$

$$w = \sqrt{z} = 2e^{i\pi/6}$$

$$\sqrt{w} = w_0 = \sqrt{2}e^{i\pi/12} \quad , w_1 = \sqrt{2}e^{i13\pi/12} = \sqrt{2}e^{-i11\pi/12}$$

$$w_0 = \sqrt{2}e^{i\pi/12} = \sqrt{2}\left(\cos\pi/12 + i\sin\pi/12\right)$$

$$w_1 = \sqrt{2}e^{-i11\pi/12} = \sqrt{2}\left(\cos(-11\pi/12) + i\sin\left(-11\pi/12\right)\right)$$

Cosine values at 180 degrees are each other negative. Sine values at 180 degrees are also each other negative.

$$w_0 + w_1 = \sqrt{2}\left(0.966 - 0.966 + i(0.259 - 0.259)\right) = 0$$

Problem T1.2

a)

Applying the convergence test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(n+1)(n+2)(z-2i)^{n+1}}{n(n+1)(z-2i)^{n}}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{(n+2)(z-2i)}{n}\right| < 1$$

$(n+2)/n$ is always positive and converges to 1.

$$|z - 2i| < 1$$

z represent a point in the complex plane and $2i$ is an offset from the origin. The inequality represents a circle with radius 1 centered at $(0, 2i)$

b)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We apply the convergence test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{n! z^{n+1}}{(n+1)! z^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{z}{n} \right| = 0 < 1$$

This holds true for all z in the complex plane.

Problem T1.3

a)

Substituting $\sin z$ with $(e^{iz} - e^{-iz}/2i)$ and $\cos z$ with $(e^{iz} + e^{-iz}/2)$:

$$2 \sin z \cos z = \frac{1}{2}\left((-e^{iz} + e^{-iz})(e^{iz} + e^{-iz})\right) = \frac{1}{2}\left(-e^{2iz} - 1 + 1 + e^{-2iz}\right)$$

$$\underline{\underline{\frac{e^{-2iz} - e^{-2iz}}{2} = \sin 2z}}$$

b)

Substituting $\sinh z$ with $(e^z - e^{-z}/2)$ and $\cosh z$ with $(e^z + e^{-z}/2)$:

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1$$

c)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{1+i\pi} = \sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!}$$

$$\underline{\underline{e^{1+i\pi} = e \cdot -1 = -e}}$$