Oblig 4

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Problem 1

a)

Finding the Residues as seen in equation (1):

$$\underbrace{\mathrm{Res}(f, z_n)}_{b_n} = \lim_{z \to z_n} (z - z_n) f(z) \tag{1}$$

$$b_0 = \lim_{z \to z_0} (z - z_0) f(z)$$

Inserting the value for z_0:

$$b_0 = \lim_{z \to 2} \cancel{(z-2)} \frac{z+2}{\cancel{(z-2)}}$$

$$\underline{\underline{b_0 = 4}}$$

b)

We rewrite the denominator as follows:

$$f(z) = \frac{z+3}{z^3 - 4z^2 - 3z + 18} = \frac{z+3}{(z+2)(z-3)^2}$$

We can see that we have a residue at $z_0 = 3$ and $z_1 = -2$. Again we find the residue for each singularity. For z_0 we must take into account that it is a pole of order 2.:

$$b_0 = \frac{1}{(2-1)!} \lim_{z \to 3} \frac{\partial}{\partial z} \cancel{(z-3)^2} \frac{z+3}{(z+2)\cancel{(z-3)^2}} = \lim_{z \to 3} \frac{\partial}{\partial z} \frac{z+3}{(z+2)}$$

$$\frac{\partial}{\partial z} \frac{z+3}{(z+2)} = \frac{(z+2) - (z+3)}{(z+2)^2} = \frac{-1}{(z+2)^2}$$

$$b_0 = \lim_{z \to 3} \frac{-1}{(z+2)^2}$$

$$\underline{\underline{b_0 = -\frac{1}{25}}}$$

For b_1 we find the residue as normal:

$$b_1 = \lim_{z \to -2} (z+2) \frac{z+3}{(z+2)(z-3)^2} = \lim_{z \to -2} \frac{z+3}{(z-3)^2} = \underline{\underline{\frac{1}{25}}}$$

c)

We use

$$b_0 = \lim_{z \to 0} (z - 0) \sin\left(\frac{1}{z}\right) = \lim_{z \to 0} \; z \sin\left(\frac{1}{z}\right)$$

For small angles $\sin(z) \approx z$. We can therefore write:

$$b_0 = \lim_{z \to 0} \; z \frac{1}{z} = 1$$

Problem 2

a)

$$\oint_C f(z)\mathrm{d}z$$

We expand the function:

$$\oint_C \left(\sum_0^\infty a_n z^n + \sum_{-\infty}^0 b_n z^n \right)$$

From the function definition we know the integral evaluates to the following:

$$\oint_C z^n \mathrm{d}z = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n = -1 \end{cases}$$

$$\therefore \underline{\underline{\oint_C f(z)\mathrm{d}z = b_1 2\pi i \operatorname{Res}(f,0) = b_1 2\pi i}}$$

b)

Problem 3

a)

We factor the denominator:

$$f(z) = \frac{z+5}{z^2 - z - 6} = \frac{z+5}{(z-3)(z+2)}$$

It's easy to see that we have two simple poles at $z_0 = 3$ and $z_1 = -2$. We can use the Residue Theorem equation (2) to evaluate the closed curve integral after finding the residues at each singularity:

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_n \underbrace{\operatorname{Res}(f, z_n)}_{b_n} \tag{2}$$

For $z_0 = 3$:

$$b_0 = \lim_{z\to 3} (z-3)\frac{z+5}{(z-3)(z+2)} = \lim_{z\to 3} \frac{z+5}{z+2}$$

$$\underline{b_0 = \frac{8}{5}}$$

For $z_1 = -2$:

$$b_1 = \lim_{z\to -2} \cancel{(z+2)}\frac{z+5}{(z-3)\cancel{(z+2)}} = \lim_{z\to -2} \frac{z+5}{z-3}$$

$$\underline{b_1 = -\frac{3}{5}}$$

Now we can use the Residue Theorem to evaluate the integral:

$$\underline{\underline{\oint_C f(z)\mathrm{d}z = 2\pi i \left(\frac{8}{5} - \frac{3}{5} \right) = 2\pi i}}$$

b)

Factoring the denominator:

$$f(z) = \frac{1}{z^3 - 4z^2 - 3z + 18} = \frac{1}{(z+2)(z-3)^2}$$

We can see that we have a simple pole at $z_0 = -2$ and a pole of order 2 at $z_1 = 3$.

$$b_0 = \lim_{z \to -2} \cancel{(z+2)} \frac{1}{\cancel{(z+2)}(z-3)^2} = \lim_{z \to -2} \frac{1}{(z-3)^2}$$

$$\underline{b_0 = \frac{1}{25}}$$

For $z_1 = 3$ we must take into account that it is a pole of order 2:

$$b_1 = \frac{1}{(2-1)!} \lim_{z \to 3} \frac{\partial}{\partial z} \cancel{(z-3)^2} \frac{1}{(z+2)\cancel{(z-3)^2}} = \lim_{z \to 3} \frac{\partial}{\partial z} \frac{1}{(z+2)} = \lim_{z \to 3} \frac{-1}{(z+2)^2}$$

$$\underline{b_1 = -\frac{1}{25}}$$

Now we can use the Residue Theorem to evaluate the integral:

$$\underline{\underline{\oint_C f(z) \mathrm{d}z = 2\pi i \left(\frac{1}{25} - \frac{1}{25} \right) = 0}}$$

c)

$$f(z) = \frac{\sin z}{(z - \pi)^4}$$

This is a pole of order 4 at $z_0 = \pi$. We calculate the residue accordingly:

$$b_0 = \frac{1}{(4-1)!} \lim_{z \to \pi} \frac{\partial^3}{\partial z^3} \cancel{(z-\pi)^4} \frac{\sin z}{\cancel{(z-\pi)^4}} = -\frac{1}{6} \lim_{z \to \pi} \cos z$$

$$\underline{b_0 = -\frac{1}{6}}$$

$$\underline{\underline{\oint_C f(z) \mathrm{d}z = 2\pi i \frac{1}{6} = \frac{\pi i}{3}}}$$

d)

$$f(z) = e^{1/z}$$

We have a simple pole at $z_0 = 0$. We expand using the Laurent series:

$$e^{1/z} = \sum_0^\infty \frac{1}{n!} \left(\frac{1}{z} \right)^n$$

The coefficient of $1/z$ (the first term) is 1 and so the residue is 1. We can now use the Residue Theorem to evaluate the integral:

$$\underline{\underline{\oint_C f(z) \mathrm{d}z = 2\pi i}}$$