Oblig 5

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Problem 1

a)

$$I = \int_{-\infty}^{\infty} \frac{x+3}{x^4+1}\,\mathrm{d}x$$

We rewrite this to a complex integral along a semicircle path in the upper half-plane with radius R.

$$\oint_C \frac{z+3}{z^4+1}\,\mathrm{d}z$$

This we can evaluate using the residue theorem. We have a fourth degree pole at $z_0 = -1$. The roots are $r_1 = e^{i\pi/4}$, $r_2 = e^{3i\pi/4}$, $r_3 = e^{5i\pi/4}$ and $r_4 = e^{7i\pi/4}$. We only use the roots in the upper half-plane, so we get r_1 and r_2. Now to find the residues at these poles.

$$\mathrm{Res}(f, r_1) = \lim_{z \to r_1}$$

b)

$$I = \int_0^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x$$

We have a poles at $z = \pm i$. As the function is symmetric we use this to expand the integral to an infinite one.

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\oint_C \frac{\cos z}{z^2+1}\,\mathrm{d}z$$

$$2I = i2\pi\,\mathrm{Res}(f, i)$$

$$I = i\pi\cosh(1)/2i = \frac{\pi}{2}\cosh(1)$$

c)

$$I = \int_0^{2\pi} \frac{\cos(2\theta)}{\sin(\theta)+5} d\theta$$

We use that $z = e^{i\theta}$ and $\frac{\mathrm{d}z}{\mathrm{d}\theta} = ie^{i\theta}$, so $\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$.

$$I = \int_C \frac{\cos(2z)}{\sin(z)+5}\frac{\mathrm{d}z}{iz}$$

We only have a pole at 0. Again we use the residue theorem.

$$I = i2\pi\,\mathrm{Res}(f, 0) = i2\pi\cos(0)/5 = \frac{2\pi}{5}$$

d)

$$I = \int_{-\infty}^{\infty} \frac{(x-1)\sin(8x-7)}{x^2-2x+5}\ \mathrm{d}x$$

First we find the roots of the denominator to be $r_1 = 1+2i$ and $r_2 = 1-2i$.

$$I = \frac{1}{1i}\left[\int_{-\infty}^{\infty} \frac{(x-1)e^{i(8x-7)}}{x^2-2x+5}\ \mathrm{d}x - \int_{-\infty}^{\infty} \frac{(x-1)e^{-i(8x-7)}}{x^2-2x+5}\ \mathrm{d}x\right]$$

$$I = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{x-1}{x^2-2x+5e^{i(8x-7)}}\ \mathrm{d}x\right)$$

Then rewrite the integral to a complex integral along a semicircle path in the upper half-plane with radius R.

$$I = \mathrm{Im}\left(\oint_C \frac{z-1}{z^2-2z+5e^{i(8z-7)}}\ \mathrm{d}z\right)$$

I would then use the residue theorem to solve the integral, but I did not get the time.

$$I = \mathrm{Im}\left(\pi Res\left(f(x)\right)\right)$$

Problem 2

a)

Laplace's equation is given by:

$$\nabla^2 f(x) = 0$$

For complex functions, we have:

$$\nabla^2 f(z) = \nabla^2(u(x,y) + iv(x,y)) = \nabla^2 u(x,y) + i\nabla^2 v(x,y)$$

$$\nabla^2 f(z) = \partial_x^2 u + \partial_y^2 u + i(\partial_x^2 v + \partial_y^2 v) = 0$$

We use the following relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\underline{\underline{\nabla^2 f(z) = \partial_x\partial_y v - \partial_y\partial_x v + i(\partial_x\partial_y u - \partial_y\partial_x u) = 0}}$$