Hand in set 6, FYS3140 spring 2024

Use the methods presented in the lectures.

Problem 1 (Fermat's principle).
In an optical medium filling the region $0 < y < h$, the speed of light is given by

$$c(y) = \frac{c_0}{\sqrt{1-ky}}, \ 0 < k < 1/h.$$

a) Show that you can write the time functional T as

$$T = \int \frac{\sqrt{1-ky}}{c_0}\sqrt{1+[x'(y)]^2}dy.$$

b) Find and solve the first integral of this functional.

c) Show that if a ray of light enters the medium at $(-x_0, 0)$ and leaves it at $(x_0, 0)$ then

$$(kx_0)^2 = 4ky_0(1-ky_0)$$

where y_0 is the greatest value of y attained on the ray path.

Problem 2 (Queen Dido's problem).

Legend: According to legend, Queen Dido of Carthage faced a mathematical problem when establishing her city. She was instructed by gods to found her city by spreading out the ox hide (skin). She devised a cunning solution to maximize the area of the ox hide. Instead of simply using the hide as it was to cover an area, she cut it into thin strips and used them to mark the boundary area. Thus, by arranging them strategically, she could enclose a larger area of land within which to establish her city of Carthage.

Problem: An area A in the (x, y)-plane is enclosed by a flexible curve $y(x)$ of length l, starting at the origin and ending at a distance a from the origin on the x-axis. We want to maximize the area A for fixed l.

a) Draw an appropriate visualization of this problem and derive the functional expression for the total area A enclosed.

The total length l of the stationary curve $y_{eq}(x)$ is given by

$$\int_0^a \sqrt{1+y'_{eq}(x)^2}dx = l.$$

This is a constraint for maximising the area, A. Adding constraints to a functional in order to find the stationary condition is done by the method of Lagrange multipliers. In our case, we construct the total functional as

$$I[y, y'] = A[y, y'] + \lambda\left(\int_0^a \sqrt{1+y'(x)^2}dx - l\right)$$

where λ is a constant called the Lagrange multiplier determined by the stationarity and boundary conditions.

b) Derive the stationary condition for the functional $I[y, y']$ and solve it. Show that the solution reduces to the equation of the circle

$$(x - x_0)^2 + (y - y_0)^2 = \lambda^2.$$

Thus, the solution represents an arc segment of length l on the circle centered at (x_0, y_0) where x_0, y_0 are the integration constants and of radius $r = \lambda$.

c) Use the boundary conditions that the curve starts at the origin and ends on the x-axis at a distance a to determine the position x_0 of the circle center as function of a.

d) Determine the radius of the circle (hence the Lagrange multiplier $r = \lambda$) and its y_0 position, considering the particular case of

$$a = 2\sqrt{3}, \text{ and } l = \frac{4\pi}{3}.$$

Use these values and trigonometry to find the corresponding values for r and (x_0, y_0). Draw your final solution. This is the shape of the city of Carthage.