# Hand in set 4, FYS3140 spring 2024

**Use the methods presented in the lectures.**

**Problem 1 (Residues).**
Calculate the residue of the following functions:

a)
$$f(z) = \frac{z+2}{z-2} \text{ at } z_0 = 2,$$

b)
$$f(z) = \frac{z+3}{z^3 - 4z^2 - 3z + 18} \text{ at } z_0 = -2 \text{ and } z_1 = 3,$$

c)
$$f(z) = \sin\left(\frac{1}{z}\right) \text{ at } z_0 = 0.$$

**Problem 2 (Why $b_1$?).**
In this problem we aim to get some intuition about why we define the residue as we do and why the methods of calculating the residue actually works. Let

$$C = \{z \in \mathbb{C} \ : \ |z| = 1\},$$

and recall from a previous hand in set that for integers $n$ we have that

$$\oint_C z^n dz = \begin{cases} 0, & \text{if } n \neq -1, \\ 2\pi i, & \text{if } n = -1. \end{cases}$$

a) Use this fact to show that if $f(z)$ is analytic inside and on $C$ except for at the point zero that

$$\oint_C f(z) dz = 2\pi i b_1$$

by writing $f$ in its Laurent series expansion around $z_0 = 0$.

The main idea now becomes to find clever ways to extract the coefficient $b_1$ from $f$ without necessarily computing the entire Laurent series of $f$. If we assume that $f$ has a simple pole at zero, so it has the form

$$f(z) = \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

on some annulus containing $C$, we can manipulate this equality to recover the simple pole rule as follows:

$$\begin{aligned}
f(z) &= \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n, \\
zf(z) &= b_1 + z\sum_{n=0}^{\infty} a_n z^n, \\
\lim_{z\to 0}(zf(z)) &= \lim_{z\to 0}\left(b_1 + z\sum_{n=0}^{\infty} a_n z^n\right), \\
\lim_{z\to 0}(zf(z)) &= b_1 + \lim_{z\to 0}\left(z\sum_{n=0}^{\infty} a_n z^n\right), \\
\lim_{z\to 0}(zf(z)) &= b_1.
\end{aligned}$$

b) Assume now that $f$ has a pole of order $m$ at zero. By doing a similar manipulation, prove the multiple-pole rule.

**Problem 3 (Residue theorem).**
Consider the closed contour

$$C = \{z \in \mathbb{C} \ : \ |z| = 5\}$$

for the contour integral

$$\oint_C f(z)dz.$$

Evaluate this integral using the residue theorem for the following integrand functions:

a)

$$f(z) = \frac{z+5}{z^2 - z - 6},$$

b)

$$f(z) = \frac{1}{z^3 - 4z^2 - 3z + 18},$$

c)

$$f(z) = \frac{\sin(z)}{(z-\pi)^4},$$

d)

$$f(z) = e^{1/z}.$$