Exercises Week 34

Exercise 1

a)

- An unitary matrix U has the property that $U^{-1} = U^\dagger \Rightarrow UU^\dagger = U^\dagger U = I$.
- An orthogonal matrix Q has rows and columns which are orthogonal vectors. It has the properties that $Q^T = Q^{-1} \Rightarrow QQ^T = I$.

b)

A Hermitian matrix has real eigenvalues and real values on its diagonal. It is also symmetric in that $H_{ij} = \overline{H}_{ji} = H_{ji}$

c)

We begin by our transformation:

$$\left|\psi_p\right\rangle = U \left|\phi_\lambda\right\rangle$$

We check for orthonormality.

$$\left\langle\psi_p\middle|\psi_p\right\rangle = \left\langle\phi_\lambda\right| UU^\dagger \left|\phi_\lambda\right\rangle = \left\langle\phi_\lambda\middle|\phi_\lambda\right\rangle = 1$$

d)

$$O\left|\phi_\lambda\right\rangle = o\left|\phi_\lambda\right\rangle$$

$$UO\left|\phi_\lambda\right\rangle = oU\left|\phi_\lambda\right\rangle$$

$$UO\left|\phi_\lambda\right\rangle = o\left|\psi_p\right\rangle$$

$$UOU^\dagger U\left|\phi_\lambda\right\rangle = o\left|\psi_p\right\rangle$$

$$UOU^\dagger\left|\psi_p\right\rangle = o\left|\psi_p\right\rangle$$

If $\left|\phi_\lambda\right\rangle$ exist in the Hilbert space $\mathcal{H}_\lambda$ and $\left|\psi_p\right\rangle$ in $\mathcal{H}_p$, then $U^\dagger : \mathcal{H}_p \mapsto \mathcal{H}_\lambda$, letting us act with $O : \mathcal{H}_\lambda \mapsto \mathcal{H}_\lambda$ on $\left|\psi_p\right\rangle$. This is converted back into $\mathcal{H}_p$, as $U : \mathcal{H}_\lambda \mapsto \mathcal{H}_p$. The vectors therefore share eigenvalues.

Exercise 2

a)