Mandatory Assignment 1

Oskar Idland

Dirichlet Problem

The Dirichlet problem is given by:

$$u(x, y, t) = \sin(k_x x) \sin\big(k_y y\big) \cos(\omega t).$$

Using that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

we get:

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \sin(k_x x) \sin\big(k_y y\big) \cos(\omega t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -c^2 \left(-k_x^2 \sin(k_x x) \sin\big(x_y y\big) \cos(\omega t) - k_y^2 \sin(k_x x) \sin\big(x_y y\big) \cos(\omega t) \right)$$

$$\omega^2 = c^2 (k_x^2 + k_y^2) \implies \omega = c\sqrt{k_x^2 + k_y^2}.$$

Exact Solutions

To find the relation between ω, k_x and k_y we use the exact solution:

$$u(x, y, t) = e^{i(k_x x + k_y y - \omega t)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} \right) u$$

$$\frac{\partial^2 u}{\partial x^2} = -k_x^2 e^{i(k_x x + k_y y - \omega t)}$$

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 e^{i(k_x x + k_y y - \omega t)}.$$

This shows their relationship to be:

$$\omega^2 = c^2 (k_x^2 + k_y^2) \implies \omega = c\sqrt{k_x^2 + k_y^2}.$$

Dispersion Coefficient ω

If we assume that $k_x = k_y = k$, and $m_x = m_y$, we get the following discreet solution:

$$u_{ij}^n = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}.$$

Using equation 1.3 from the assignment, we get:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2} \right).$$

Adding this to the left-hand side of the equation for u_{ij}^n, we get:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = \frac{e^{ikh(i+j)}}{\Delta^2}\left(e^{i\tilde{\omega}(n+1)\Delta t} - 2e^{-i\tilde{\omega}} + e^{-i\tilde{\omega}(n-1\Delta t}\right)$$

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = \frac{e^{ikh(i+j)}}{\Delta^2}\left(e^{i\tilde{\omega}\Delta t} - 2 + e^{-i\tilde{\omega}\Delta t}\right).$$

Doing the same for the right-hand side of the equation for u_{ij}^n, we get:

$$\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} = \frac{e^{i\tilde{\omega}n\Delta t}}{h^2}\left(e^{ikh(i+1+j)} - 2e^{ikh(i+j)} + e^{ikh(i-1+j)}\right)$$

$$\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} = \frac{e^{i(kh(i+j)-\tilde{\omega}n\Delta t)}}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right).$$

Combining these results, we get:

$$\frac{e^{i(kh(i+j)-\tilde{\omega}n\Delta t)}}{\Delta^2}\left(e^{i\tilde{\omega}\Delta t} - 2 + e^{-i\tilde{\omega}\Delta t}\right) = c^2\frac{e^{-i(\tilde{\omega}n\Delta t-kh(i+j))}}{h^2}\left(e^{ikh} - 2 + e^{-ikh} + e^{ikh} - 2 + e^{-ikh}\right).$$

Simplifying this, we get:

$$\frac{e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t}}{\Delta t^2} = 2c^2\frac{e^{ikh} - 2 + e^{-ikh}}{h^2}.$$

Using the CLF number $C = c\Delta t/h = 1/\sqrt{2}$:

$$e^{i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} = 2C\left(e^{ikh} - 2 + e^{-ikh}\right)$$

$$2\cos(\tilde{\omega}\Delta t) - 2 = 2C^2\left(2\cos(kh) - 2\right)$$

$$\cos(\tilde{\omega}\Delta t) = 1 + 2C^2\cos(kh) - 2C^2$$

$$\cos(\tilde{\omega}\Delta t) = \cos(kh)$$