MAT1120

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4.1 Vector spaces and subspaces

Theory

Vector Space

Definition 1 A vector space is a nonempty set V of objects, called vectors, on which are fefined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{W}$ in V and for all scalars c and d.

1. The sum of u and v, denoted by $\mathbf{u} + \mathbf{v}$, is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector $\vec{0}$ in V such that $\mathbf{u} + \mathbf{0} = u$
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = 0$
6. The scalar multiple of u by c denoted by $c\mathbf{u}$ is in V
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Definition 2 A subspace of a vector space V is a subset H of V that has three properties:

- The zero vector of V is in H
- H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}\ C\ \mathbf{v}$ is in H
- H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H

Theorem 1 *If $\vec{v_1}, \ldots, \vec{v_n} \in V$ then span$\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a subspace of V*

Exercises

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Let W be the set of all vectors of the form $\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

Solution

We can split W up into a sum of other vectors.

$$\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix}$$

$$W = span\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix}\right\}$$

theorem 1 says W is a subspace of $\mathbb{R}^4$

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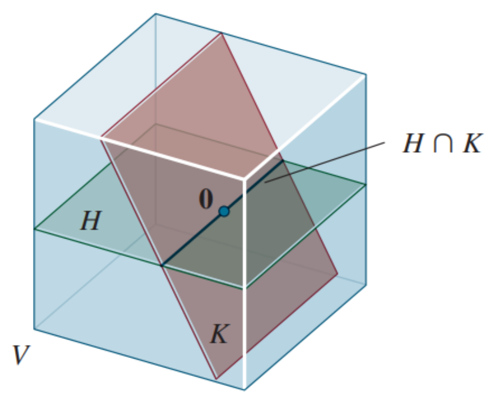
Let H and K be subspaces of a vector space V. The intersection of H and K, written $H \bigcap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace



Figur 1

Solution

The intersection of the two planes creates a line going across the x-axis which we already know is a subspace of both $\mathbb{R}^3$ and $\mathbb{R}^2$. Same goes for all straight lines going trough origin.
The union of two subspaces, (lets use the x and y-axis) will often create vectors outside the subspace, thus not being closed by addition. If we take a vector from the x-axis and a vector from the y-axis and add them together, they will create a vector outside both axis.

Solution

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Determine if y is in the subspace of $\mathbb{R}^4$ spanned by the collumns of A, where

$$\mathbf{y} = \begin{pmatrix} -4 \\ -8 \\ 6 \\ -5 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -5 & -9 \\ 8 & 7 & -6 \\ -5 & -8 & 3 \\ 2 & -2 & -9 \end{pmatrix}$$

Solution

We can divide A into three vectors where each one is a column in A. If these three vectors can form a linear combination which can create y (in other words, y is in the span of these vectors), then y will be in the subspace of $\mathbb{R}^4$ spanned by the columns of A. To test this we append the vector y to the end of A and then reduce it to row echelon form. As seen above the equation has a solution

```
ans = 4x4
    1.0000         0         0   -0.2000
         0    1.0000         0   -0.4000
         0         0    1.0000    0.6000
         0         0         0         0
```

Figur 2: Results using matlab

which means y is in the subspace of $\mathbb{R}^4$ spanned by the columns of A.

4.2 Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

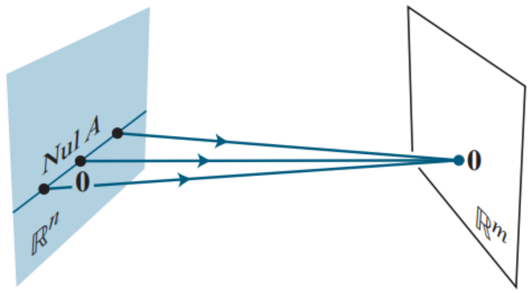
Theory

The Null Space of a Matrix

Definition 3 The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogenous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\operatorname{Nul} A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Theorem 2 ***The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.***



Figur 3: Visualization of the subspace (in this case a line) formed by the null space of A

The Column Space of a Matrix

Definition 4 The Column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$, then

$$Col\ A = Span = \{\mathbf{a}_1 \dots \mathbf{a}_n\}$$

Theorem 3 ***The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.***

Exercises

4

Find the explicit definition of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution

We start by reducing it to echelon form multiplying by x and set as equal to the zero vector.

$$A\mathbf{x} = \mathbf{0}$$

$$rref(A) = \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Nul\ A = span \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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Let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$ **and** $K = Span\{\mathbf{v}_3, \mathbf{v}_4\}$**, where**

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ -12 \\ -28 \end{pmatrix}$$

Then H **and** K **are subspaces of** $\mathbb{R}^3$**. In fact,** H **and** K **are planes in** $\mathbb{R}^3$ **through origin, and they intersect in a line through 0. Find a nonzero vector w that generates that line. [Hint: w can be written as** $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ **and also as** $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$**. To build w, solve the equation** $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ **for the unknown** c_j**'s]**

Solution

4.3 Linearly Independent Sets; Bases

Theorem 4 ***And indexed set*** $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ ***of two or more vectors, with*** $\mathrm{v}_1 \neq \mathbf{0}$***, is linearly dependent if and only if some*** v_j ***with j ¿ 1 is a linear combination of the preceding vectors*** $\mathrm{v}_1 \cdots \mathrm{v}_{j-1}$***.***

Definition 5 Let H be a subspace of a vector space V**. A set of vectors** $\mathcal{B}$ **in** V **is a basis for** H **if**

- $\mathcal{B}$ **is a linearly independent set, and**
- **the subspace spanned by** B_c **coincides with** H**; that is,**

$$H = Span\ \mathcal{B}$$

Exercises

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Find a basis for the set of vectors in $\mathbb{R}^2$ **on the line** $y = 5x$**.**

Solution

$$x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix}$$

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Find bases for $\mathrm{Nul}\,A$**,** $\mathrm{Col}\,A$ **and** $\mathrm{Row}\,A$**.**

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

Get matrix A in echelon form and you will se the rows of A and B are equivalent. This means the space spanned by the vectors making the rows of A and B (Row A) is equal. We find Col A by looking at the linearly independent columns of B. We find $\operatorname{Nul} A$ by solving the equation $A\mathbf{x} = \mathbf{0}$.

$$\operatorname{Nul} A = \left\{ \begin{pmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \operatorname{Col} A = \left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}, \quad \operatorname{Row} A = \left\{ \begin{pmatrix} 1 & 0 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 5 & 3 \end{pmatrix} \right\}$$

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In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t, \cos t\}$.

4.4 Coordinate Systems

Theory

We can use a basis in a given dimension as a tool to create a new coordinate system.

Theorem 5 (The Uniqe Representation Theorem) ***Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each x in V, there exist a unique set of scalars $c_1, c_2, \cdots, c_n$ such that***

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

Exercises

3

Find the vector $\mathbf{c}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$, and the given basis $\mathcal{B}$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -8 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ -4 \end{pmatrix} \right\} [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

Solution

We put the vectors forming the basis $\mathcal{B}$ in a matrix and reduce it to echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ -8 & -5 & 9 \\ 6 & 7 & -4 \end{bmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{73}{22} \\ 0 & 1 & 0 & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{27}{22} \end{pmatrix}$$