MAT1120

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4.1 Vector spaces and subspaces

Theory

Vector Space

Definition 1 A vector space is a nonempty set V of objects, called vectors, on which are fefined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{W}$ in V and for all scalars c and d.

1. The sum of u and v, denoted by $\mathbf{u} + \mathbf{v}$, is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector $\vec{0}$ in V such that $\mathbf{u} + \mathbf{0} = u$
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = 0$
6. The scalar multiple of u by c denoted by $c\mathbf{u}$ is in V
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Definition 2 A subspace of a vector space V is a subset H of V that has three properties:

- The zero vector of V is in H
- H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}\ C\ \mathbf{v}$ is in H
- H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H

Theorem 1 *If $\vec{v_1}, \ldots, \vec{v_n} \in V$ then span$\{\vec{v_1}, \ldots, \vec{v_n}\}$ is a subspace of V*

Exercises

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Let W be the set of all vectors of the form $\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

We can split W up into a sum of other vectors.

$$\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix}$$

$$W = span \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix} \right\} \rightarrow \text{theorem 1 says } W \text{ is a subspace of } \mathbb{R}^4$$

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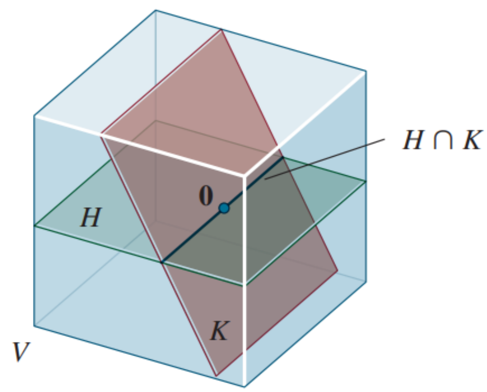
Let H and K be subspaces of a vector space V. The intersection of H and K, written $H \bigcap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace



Figur 1

If you lift the H plane, higher and would by definition 2, not be a subspace as the intersection wouldn't go through origin. We can do the same for two lines in $\mathbb{R}^2$. If they do not intersect in the origin, then their intersection can't be a subspace.

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Determine if $\mathbf{y}$ is in the subspace of $\mathbb{R}^4$ spanned by the collumns of A, where

$$\mathbf{y} = \begin{pmatrix} -4 \\ -8 \\ 6 \\ -5 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -5 & -9 \\ 8 & 7 & -6 \\ -5 & -8 & 3 \\ 2 & -2 & -9 \end{pmatrix}$$

We can divide A into three vectors where each one is a column in A. If these three vectors can form a linear combination which can create $\mathbf{y}$ (in other words, $\mathbf{y}$ is in the span of these vectors), then $\mathbf{y}$ will be in the subspace of $\mathbb{R}^4$ spanned by the columns of A. To test this we append the vector $\mathbf{y}$ to the end of A and then reduce it to row echelon form. As seen above

```
ans = 4x4
     1.0000         0         0   -0.2000
          0    1.0000         0   -0.4000
          0         0    1.0000    0.6000
          0         0         0         0
```

Figur 2: Results using matlab

the equation has a solution which means $\mathbf{y}$ is in the subspace of $\mathbb{R}^4$ spanned by the columns of A.

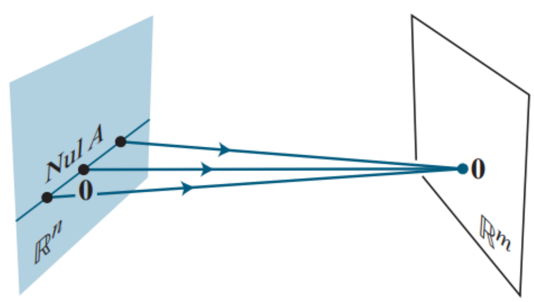
4.2 Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Theory

The Null Space of a Matrix

Definition 3 The **null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogenous equation $A\mathrm{x} = \mathbf{0}$. In set notation,**

$$\textbf{Nul } A = \{\mathbf{x} : \mathbf{xis\ in\ } \mathbb{R}^n \textbf{ and } A\mathbf{X} = \mathbf{0}\}$$



Figur 3: Visualization of the subspace (in this case a line) formed by the null space of A

Theorem 2 *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalentrly, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

The Column Space of a Matrix

Definition 4 The Column space of an $m \times n$ matrix A, written as Col A, is the set of all linear cominations of the collumns of A. If $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$, then

$$Col\ A = Span = \{\mathbf{a}_1 \dots \mathbf{a}_n\}$$

Theorem 3 *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.*

Exercises

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Find the explicit definition of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

We start by reducing it to echelon form multiplying by $\mathbf{x}$ and set as equal to the zero vector.

$$A\mathbf{x} = \mathbf{0}$$

$$rref(A) = \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Nul\ A = span \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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Let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$ and $K = Span\{\mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ -12 \\ -28 \end{pmatrix}$$

Then H and K are subspaces of $\mathbb{R}^3$. In fact, H and K are planes in $\mathbb{R}^3$ through origin, and they intersect in a line through $\mathbf{0}$. Find a nonzero vector $\mathbf{w}$ that generates that line. [Hint: $\mathbf{w}$ can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and also as $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$. To build $\mathbf{w}$, solve the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ for the unknown c_j's]