Theorem 1 (Negative answer to Thorisson Problem 3.1)**.** *Let $Z = (Z_k)_{k\in\mathbb{Z}}$ be an* arbitrary *stationary $\{0,1\}$-valued process. Let $(p_k)_{k\geq 0} \subset (0,1)$ satisfy*

$$p_k \xrightarrow{k\to\infty} 0, \tag{1a}$$

$$c := \liminf_{n\to\infty} \sum_{k=n}^{2n} p_k \ > \ 0. \tag{1b}$$

Let $(J_k)_{k\in\mathbb{Z}}$ be independent Bernoulli(p_k) *variables, independent of Z, and set*

$$X_k := Z_k \oplus J_k, \qquad Y_k := Z_k, \qquad k \in \mathbb{Z},$$

where $\oplus$ denotes addition modulo 2.[1] *Denote by θ_n the time–shift: $(\theta_n \boldsymbol{x})_t := x_{t+n}$. Then*

(a) ***Setwise convergence:*** $\lim_{n\to\infty} P(\theta_n X \in A) = P(Y \in A) \quad \forall A \in \mathcal{E}^\infty$.

(b) ***No total–variation convergence:*** $\liminf_{n\to\infty} \big\| P(\theta_n X) - P(Y) \big\|_{\mathrm{TV}} \ \geq \ 1 - e^{-c} \ > \ 0.$

Hence setwise convergence of the shifted laws does not imply convergence in total variation.

Proof. Step 1 (stationarity). Since $((Z_k, J_k))_{k\in\mathbb{Z}}$ is i.i.d. in k, its distribution is shift-invariant; thus so is the joint law of (X, Y).

Step 2 (setwise convergence). Fix a cylinder A depending on coordinates $t_1 < \cdots < t_m$. Because $X_k \neq Y_k$ iff $J_k = 1$,

$$\big|P(\theta_n X \in A) - P(Y \in A)\big| \leq P\big(\exists i \leq m : J_{n+t_i} = 1\big) \leq \sum_{i=1}^{m} p_{n+t_i}.$$

By (1a) this sum tends to 0. Let $\mathcal{C}$ be the π–system of cylinder sets and $\mathcal{D} = \{A : P(\theta_n X \in A) \to P(Y \in A)\}$. Then $\mathcal{C} \subset \mathcal{D}$ and $\mathcal{D}$ is a Dynkin system; the π–λ lemma yields $\sigma(\mathcal{C}) = \mathcal{E}^\infty \subset \mathcal{D}$, proving (a).

Step 3 (failure of total variation). Define the cylinder

$$B_n := \big\{\boldsymbol{x} \in \{0,1\}^{\mathbb{Z}} \ : \ \exists k \in [0,n] \text{ with } x_k \neq y_k\big\}.$$

We have $P(Y \in B_n) = 0$. For $\theta_n X$,

$$P(\theta_n X \in B_n) = 1 - \prod_{k=n}^{2n} (1 - p_k) \geq 1 - \exp\Big(-\sum_{k=n}^{2n} p_k\Big),$$

using $(1 - z) \leq e^{-z}$. By (1b), $\liminf_{n\to\infty} P(\theta_n X \in B_n) \geq 1 - e^{-c}$. Thus

$$\liminf_{n\to\infty} \big\|P(\theta_n X) - P(Y)\big\|_{\mathrm{TV}} \ \geq \ 1 - e^{-c} \ > \ 0,$$

establishing (b). $\square$

[1] $\oplus$ is the XOR–operator on $\{0,1\}$.