Here i am to show some solutions to selected problems from the book by Kallenberg.

1 Sets and functions, measures and integration

Problem 3

For any space S let μA denote the cardinality of a set $A \subset S$. Show that μ is a measure in $(S, 2^S)$.

Proof. We apply the definition of a measure. **1. Null empty set:** We have

$$\mu(\emptyset) = \#\emptyset = 0.$$

2. Countable additivity: Let $(A_k)_{k\geq 1} \subset 2^S$ be a countable sequence of pairwise disjoint subsets of S. Then their union is:

$$\mu\left(\bigcup_{k\geq 1} A_k\right) = \#\left(\bigcup_{k\geq 1} A_k\right).$$

Since the sets A_k are disjoint, each element in the union belongs to exactly one A_k, so the cardinality of the union is the sum of the cardinalities:

$$\#\left(\bigcup_{k\geq 1} A_k\right) = \sum_{k=1}^{\infty} \#A_k = \sum_{k=1}^{\infty} \mu(A_k).$$

Hence, μ is countably additive.

□

Problem 10 – Fubini–Tonelli with counting measure

Let $(S, \mathcal{S}, \mu)$ be a σ–finite measure space and let

$$\big(\mathbb{N},\, 2^{\mathbb{N}},\, \nu\big), \qquad \nu(A) = \#A,$$

be the counting–measure space on the natural numbers. Write $\mu \otimes \nu$ for the product measure on $\big(S \times \mathbb{N},\, \mathcal{S} \otimes 2^{\mathbb{N}}\big)$.

Theorem (Tonelli–Fubini). Let $f : S \times \mathbb{N} \to [-\infty, \infty]$ be $\mathcal{S} \otimes 2^{\mathbb{N}}$-measurable.

(i) (*Tonelli*) If $f \geq 0$, then

$$\int_{S\times\mathbb{N}} f(s,t)\,(\mu\otimes\nu)(ds,dt) \;=\; \int_S \Big(\sum_{t\in\mathbb{N}} f(s,t)\Big)\,\mu(ds) \;=\; \sum_{t\in\mathbb{N}} \int_S f(s,t)\,\mu(ds), \tag{1}$$

allowing the value $+\infty$.

(ii) (*Fubini*) If $f \in L^1(\mu\otimes\nu)$, all three integrals in (1) are finite and equal, and both iterated integrals exist as absolutely convergent expressions.

Proof. **Step 1. Indicator rectangles.** Let $A = B \times C$ with $B \in \mathcal{S}$ and $C \subseteq \mathbb{N}$. For the indicator $\mathbf{1}_A(s,t) = \mathbf{1}_B(s)\,\mathbf{1}_C(t)$ we have

$$\int_{S\times\mathbb{N}} \mathbf{1}_A \, d(\mu \otimes \nu) = (\mu \otimes \nu)(A) = \mu(B)\,\nu(C).$$

On the other hand,

$$\int_S \sum_{t\in\mathbb{N}} \mathbf{1}_A(s,t)\,\mu(ds) = \int_S \mathbf{1}_B(s)\Big(\sum_{t\in C} 1\Big)\mu(ds) = \mu(B)\,\#C = \mu(B)\,\nu(C),$$

and the same equality holds if the order of integration and summation is reversed. Hence (1) holds for $\mathbf{1}_A$.

Step 2. Simple functions. Any non-negative simple function can be written $f = \sum_{k=1}^m c_k\,\mathbf{1}_{A_k}$ with $c_k \geq 0$ and A_k rectangles as above. By linearity of the integral and Step 1, (1) holds for every such f.

Step 3. Non-negative measurable functions. For a general $f \geq 0$ choose an increasing sequence of simple functions $(f_n)_{n\geq 1}$ with $f_n \uparrow f$. Applying the Monotone Convergence Theorem on both sides of (1) and using Step 2 yields the Tonelli identity for f.

Step 4. Integrable functions. If $f \in L^1(\mu \otimes \nu)$, decompose $f = f^+ - f^-$ with $f^\pm \geq 0$ and $f^\pm \in L^1$. Apply Step 3 to f^+ and f^- separately and subtract; finiteness follows from integrability. Thus (1) holds and all integrals are finite.

Steps 1–4 establish Tonelli's theorem for $f \geq 0$ and Fubini's theorem for $f \in L^1(\mu \otimes \nu)$. □