Theorem 1 (Negative answer to Thorisson Problem 3.1)**.** *Let* $Y = (Y_k)_{k\in\mathbb{Z}}$ *be an* arbitrary *stationary* $\{0,1\}$*-valued process. Let* $(p_k)_{k\geq 0} \subset (0,1)$ *satisfy*

$$p_k \xrightarrow{k\to\infty} 0, \tag{1a}$$

$$c := \liminf_{n\to\infty} \sum_{k=n}^{2n} p_k \ > \ 0. \tag{1b}$$

Let $(J_k)_{k\in\mathbb{Z}}$ *be independent* Bernoulli(p_k) *variables, independent of* Y*, and set*

$$X_k := Y_k \oplus J_k,$$

where $\oplus$ *denotes addition modulo 2 (the XOR–operator on* $\{0,1\}$*). Denote by* θ_n *the time–shift:* $(\theta_n \boldsymbol{x})_t := x_{t+n}$*. Then*

(a) ***Setwise convergence:*** $\lim_{n\to\infty} P(\theta_n X \in A) = P(Y \in A) \quad \forall A \in \mathcal{E}^\infty$.

(b) ***No total–variation convergence:*** $\liminf_{n\to\infty} \big\| P(\theta_n X) - P(Y) \big\|_{\mathrm{TV}} \ \geq \ 1 - e^{-c} \ > \ 0.$

Hence setwise convergence of the shifted laws does not imply convergence in total variation.

Proof. Step 1 (setwise convergence). Fix a cylinder A that depends on the coordinates $t_1 < \cdots < t_m$. Because $X_k \neq Y_k$ exactly when $J_k = 1$,

$$\big|P(\theta_n X \in A) - P(Y \in A)\big| \leq \sum_{i=1}^{m} p_{n+t_i} \xrightarrow{n\to\infty} 0 \qquad \text{(by (1a))}.$$

Let $\mathcal{C}$ denote the *finite cylinder sets*; this family is a π-system since it is closed under finite intersections. Set

$$\mathcal{D} := \big\{ A \subset \{0,1\}^{\mathbb{Z}} : P(\theta_n X \in A) \xrightarrow{n\to\infty} P(Y \in A) \big\}.$$

$\mathcal{D}$ fulfills the properties of a Dynkin system, and is then one. The estimate above shows $\mathcal{C} \subset \mathcal{D}$. Hence the $\pi - \lambda$ lemma gives

$$\sigma(\mathcal{C}) = \mathcal{E}^\infty \subset \mathcal{D},$$

so $P(\theta_n X \in A) \to P(Y \in A)$ for every $A \in \mathcal{E}^\infty$; that is, setwise convergence holds.
Step 2 (failure of total variation). For fixed $n \in \mathbb{N}$ condition on the entire path $Y = y$ and put

$$\mu_y := \mathcal{L}(\theta_n X \mid Y = y), \qquad \nu_y := \delta_y.$$

Only the coordinates $n, \ldots, 2n$ can differ, so

$$\|\mu_y - \nu_y\|_{\mathrm{TV}} \ = \ P\big((J_n, \ldots, J_{2n}) \neq (0, \ldots, 0)\big) \ = \ 1 - \prod_{k=n}^{2n} (1 - p_k) \ \geq \ 1 - \exp\Big(-\sum_{k=n}^{2n} p_k\Big).$$

Using (1b) and taking $\liminf_{n\to\infty}$ we obtain

$$\liminf_{n\to\infty} \big\|P(\theta_n X) - P(Y)\big\|_{\mathrm{TV}} \ \geq \ 1 - e^{-c} \ > \ 0,$$

which proves (b). $\square$