

Here i am to show some solutions to selected problems from the book by Kallenberg.

1 Sets and functions, measures and integration

Problem 3

For any space S let μA denote the cardinality of a set $A \subset S$. Show that μ is a measure in $(S, 2^S)$.

Proof. We apply the definition of a measure. **1. Null empty set:** We have

$$\mu(\emptyset) = \#\emptyset = 0.$$

2. Countable additivity: Let $(A_k)_{k \geq 1} \subset 2^S$ be a countable sequence of pairwise disjoint subsets of S . Then their union is:

$$\mu \left(\bigcup_{k \geq 1} A_k \right) = \# \left(\bigcup_{k \geq 1} A_k \right).$$

Since the sets A_k are disjoint, each element in the union belongs to exactly one A_k , so the cardinality of the union is the sum of the cardinalities:

$$\# \left(\bigcup_{k \geq 1} A_k \right) = \sum_{k=1}^{\infty} \#A_k = \sum_{k=1}^{\infty} \mu(A_k).$$

Hence, μ is countably additive. □

Problem 10 – Fubini–Tonelli with counting measure

Let (S, \mathcal{S}, μ) be a σ -finite measure space and let

$$(\mathbb{N}, 2^{\mathbb{N}}, \nu), \quad \nu(A) = \#A,$$

be the counting-measure space on the natural numbers. Write $\mu \otimes \nu$ for the product measure on $(S \times \mathbb{N}, \mathcal{S} \otimes 2^{\mathbb{N}})$.

Theorem (Tonelli–Fubini). Let $f : S \times \mathbb{N} \rightarrow [-\infty, \infty]$ be $\mathcal{S} \otimes 2^{\mathbb{N}}$ -measurable.

(i) (*Tonelli*) If $f \geq 0$, then

$$\int_{S \times \mathbb{N}} f(s, t) (\mu \otimes \nu)(ds, dt) = \int_S \left(\sum_{t \in \mathbb{N}} f(s, t) \right) \mu(ds) = \sum_{t \in \mathbb{N}} \int_S f(s, t) \mu(ds), \quad (1)$$

allowing the value $+\infty$.

(ii) (*Fubini*) If $f \in L^1(\mu \otimes \nu)$, all three integrals in (1) are finite and equal, and both iterated integrals exist as absolutely convergent expressions.

Proof. Step 1. Indicator rectangles. Let $A = B \times C$ with $B \in \mathcal{S}$ and $C \subseteq \mathbb{N}$. For the indicator $\mathbf{1}_A(s, t) = \mathbf{1}_B(s) \mathbf{1}_C(t)$ we have

$$\int_{S \times \mathbb{N}} \mathbf{1}_A d(\mu \otimes \nu) = (\mu \otimes \nu)(A) = \mu(B) \nu(C).$$

On the other hand,

$$\int_S \sum_{t \in \mathbb{N}} \mathbf{1}_A(s, t) \mu(ds) = \int_S \mathbf{1}_B(s) \left(\sum_{t \in C} 1 \right) \mu(ds) = \mu(B) \#C = \mu(B) \nu(C),$$

and the same equality holds if the order of integration and summation is reversed. Hence (1) holds for $\mathbf{1}_A$.

Step 2. Simple functions. Any non-negative simple function can be written $f = \sum_{k=1}^m c_k \mathbf{1}_{A_k}$ with $c_k \geq 0$ and A_k rectangles as above. By linearity of the integral and Step 1, (1) holds for every such f .

Step 3. Non-negative measurable functions. For a general $f \geq 0$ choose an increasing sequence of simple functions $(f_n)_{n \geq 1}$ with $f_n \uparrow f$. Applying the Monotone Convergence Theorem on both sides of (1) and using Step 2 yields the Tonelli identity for f .

Step 4. Integrable functions. If $f \in L^1(\mu \otimes \nu)$, decompose $f = f^+ - f^-$ with $f^\pm \geq 0$ and $f^\pm \in L^1$. Apply Step 3 to f^+ and f^- separately and subtract; finiteness follows from integrability. Thus (1) holds and all integrals are finite.

Steps 1–4 establish Tonelli's theorem for $f \geq 0$ and Fubini's theorem for $f \in L^1(\mu \otimes \nu)$. \square

Processes, Distributions and independence

Problem 4.

Let $X, Y \in \{0, 1\}$ on an index set T .

Show::

$$X \stackrel{d}{=} Y \iff P\left(\sum_i x_{t_i} > 0\right) = P\left(\sum_i y_{t_i} > 0\right) \quad (2)$$

Proof. Let $S = \{0, 1\}$ and equip S^T with the product σ -algebra \mathcal{B} .

(\Rightarrow) If $X \stackrel{d}{=} Y$, then for every $A \in \mathcal{B}$ we have $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$. Taking $A = \{f \in S^T : \sum_{i=1}^n f_{t_i} > 0\}$ gives

$$\mathbb{P}\left(\sum_{i=1}^n X_{t_i} > 0\right) = \mathbb{P}\left(\sum_{i=1}^n Y_{t_i} > 0\right) \quad (\forall n, t_1, \dots, t_n \in T).$$

(\Leftarrow) Assume these equalities hold for every finite set of indices. Define the class

$$\mathcal{D} := \{A \in \mathcal{B} : \mathbb{P}(X \in A) = \mathbb{P}(Y \in A)\}.$$

Claim: \mathcal{D} is a dynkin-system. Indeed, (i) $S^T \in \mathcal{D}$; (ii) if $A \in \mathcal{D}$ then $S^T \setminus A \in \mathcal{D}$ by complementing the equal probabilities; (iii) if $(A_k)_{k \geq 1} \subset \mathcal{D}$ are disjoint, then countable additivity gives $\mathbb{P}(X \in \bigcup_k A_k) = \mathbb{P}(Y \in \bigcup_k A_k)$. Hence \mathcal{D} is a Dynkin-system.

Next, for every finite non-empty $J = \{t_1, \dots, t_m\} \subset T$ set

$$Z_J := \{f \in S^T : f_{t_i} = 0 \ \forall i \in J\}.$$

Because Z_J is the complement of the event $\{\sum_{i=1}^m f_{t_i} > 0\}$, our hypothesis puts Z_J in \mathcal{D} . Let $\mathcal{C} := \{Z_J : 0 < |J| < \infty\}$.

Claim: \mathcal{C} is a π -system. For finite $J, K \subset T$, $Z_J \cap Z_K = Z_{J \cup K}$, so finite intersections stay inside \mathcal{C} .

Thus we have a π -system $\mathcal{C} \subset \mathcal{D}$. By the – (Dynkin–Sierpiński) theorem,

$$\sigma(\mathcal{C}) \subset \mathcal{D}.$$

But $\sigma(\mathcal{C}) = \mathcal{B}$. For any $t \in T$ the single-coordinate event $\{f_t = 1\}$ is $Z_{\{t\}}^c \in \sigma(\mathcal{C})$. Finite intersections of such events and their complements therefore yield every cylinder set of the form $\{f : (f_{t_1}, \dots, f_{t_n}) \in B\}$, and the cylinders generate the product -algebra.

Consequently $\mathcal{B} \subset \mathcal{D}$; i.e. $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all $A \in \mathcal{B}$. That is exactly $X \stackrel{d}{=} Y$. \square

Problem 10.

Let ξ_i i.i.d on (S, \mathcal{S}, μ) and $\tau = \{k \geq 1 : \xi_k \in A\}$. Fix $A \in \mathcal{S}$ and set $\mu(A) > 0$. Show that the distribution of ξ_τ is $\mu(\cdot \mid A)$.

Proof. Start by noting that

$$\{\xi_\tau \in B\} = \bigcup_{k=1}^{\infty} \{\tau = k \cap \xi_k \in B\} \quad (3)$$

Next we note that the sets are disjoint, and so we use the property of a measure.

$$P(\xi_\tau \in B) = \sum_{k=1}^{\infty} P(\tau = k \cap \xi_k \in B) \quad (4)$$

$$= \sum_{K=1}^{\infty} P(\xi_1 \notin A, \dots, \xi_K \in A \cap B) \quad (5)$$

$$= \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} P(\xi_i \notin A) \right) P(\xi_k \in A \cap B) \quad (6)$$

$$= \sum_{k=1}^{\infty} (1 - \mu(A))^{k-1} \mu(A \cap B) \quad (7)$$

$$= \mu(A)^{-1} \mu(B \cap A) \quad (8)$$

Holds $\forall B \subset \mathcal{S}$. \square

Problem 11.

Let $\xi_i \in [0, 1]$ i.i.d. Show that $E(\prod_n \xi_n) = \prod_n E(\xi_n)$.

Proof. let $X_i : ([0, 1], \mathcal{B}([0, 1]), P) \rightarrow \mathbb{R}$ be \mathcal{L}^1 . Define $\mu_k = P \circ X_k^{-1}$ and $\mu = \bigotimes_n \mu_n$, be independent.

$$E \left(\prod_{k=1}^n X_k \right) = \int_{\Omega} \prod_{k=1}^n X_k(\omega) dP(\omega) \quad (9)$$

$$\stackrel{Indp.}{=} \int_{[0,1]^n} \prod_{k=1}^n x_k d(\mu_1 \otimes \cdots \otimes \mu_n) \quad (10)$$

$$\stackrel{FB}{=} \int_{[0,1]} x_n d\mu_n \int_{[0,1]^{n-1}} \prod_{k=1}^{n-1} x_k d(\mu_1 \otimes \cdots \otimes \mu_{n-1}) \quad (11)$$

$$\stackrel{iter.}{=} \prod_{k=1}^n \int_{[0,1]} x_k d\mu_k \quad (12)$$

The proof can be extended to apply for probabilities by setting $X_k(\omega) = 1\{A_k(\omega)\}$. \square

Problem 3.

We are F which is right cont. with bounded variation and is zero at minus infinity. We are asked to show that $E(F(\xi)) = \int P(\xi \geq t) dF(t)$.

Proof. Define the push forward / law of ξ as $\mu_{\xi} = P \circ \xi^{-1}$ or rather $\mu_{\xi}(B) = P(\xi \in B)$.

$$E[F(\xi)] = \int_{\Omega} \left(\int_{\mathbb{R}} 1_{\{s \geq t\}} dF(t) \right) d\mu_{\xi}(s) \quad (13)$$

$$\stackrel{FT}{=} \int_{\mathbb{R}} \left(\int_{\Omega} 1_{\{s \geq t\}} d\mu_{\xi}(s) \right) dF(t) \quad (14)$$

$$= \int_{\mathbb{R}} \mu_{\xi}([t, \infty)) dF(t) \quad (15)$$

$$= \int_{\mathbb{R}} P(\xi \geq t) dF(t). \quad (16)$$

\square

Optimal times and martingales

Problem 17

Let $\xi_n \rightarrow \xi$ in L^1 and let \mathcal{F}_n be an increasing filtration.

Show:

$$E(\xi_n | \mathcal{F}_n) \xrightarrow{L^1} E(\xi | \mathcal{F}_\infty)$$

Proof.

$$\begin{aligned} \|E(\xi_n | \mathcal{F}_n) - E(\xi | \mathcal{F}_\infty)\| &\leq \|E(\xi_n | \mathcal{F}_n) - E(\xi | \mathcal{F}_n)\| + \|E(\xi | \mathcal{F}_n) - E(\xi | \mathcal{F}_\infty)\| \\ &= \underbrace{\|E(\xi_n - \xi | \mathcal{F}_n)\|}_{\substack{\xi_n \xrightarrow{L^1} \xi \Rightarrow 0 \\ \stackrel{th 9.24}{=} 0}} + \|E(\xi | \mathcal{F}_n) - E(\xi | \mathcal{F}_\infty)\| \end{aligned}$$

Where we have used conditioning limits, Jessen, Levy, since we have a filtration on an unbounded index set, and an $\xi \in L^1$. \square

Problem 21

Let $\tau > 0$ and let the filtration $\mathcal{F} = (\mathcal{F}_t)$ be right cont.

Show that $X_t = P(\tau < t | \mathcal{F}_t)$ has a right cont. version ($X_t = X_{t+}$) $\forall t \geq 0$

Proof.

$$\begin{aligned} E(X_t | \mathcal{F}_s) &= E(P(\tau < t | \mathcal{F}_t) | \mathcal{F}_s) \\ &= P(\tau < t | \mathcal{F}_s) \\ &= P(\tau < t |) \\ &\geq X_s \end{aligned}$$

Hence X_t is a sub-martingale. We also note that EX is right cont. since

$$\begin{aligned} E(X_t) &= E(P(\tau < t | \mathcal{F}_t)) \\ &= P(\tau < t) \end{aligned}$$

Then we apply theorem 9.28 (regularization, Doob) (ii) which states that for any \mathcal{F} sub-martingale X on \mathbb{R}_+ with restriction Y to \mathbb{Q}_+ we have that when \mathcal{F} is right cont., X has a right cont. version with left hand limits iff EX is right cont. \square