Here i am to show some solutions to selected problems from the book by Kallenberg.

1 Sets and functions, measures and integration

Problem 3

For any space S let μA denote the cardinality of a set $A \subset S$. Show that μ is a measure in $(S, 2^S)$.

Proof. We apply the definition of a measure. **1. Null empty set:** We have

$$\mu(\emptyset) = \#\emptyset = 0.$$

2. Countable additivity: Let $(A_k)_{k\geq 1} \subset 2^S$ be a countable sequence of pairwise disjoint subsets of S. Then their union is:

$$\mu\left(\bigcup_{k\geq 1} A_k\right) = \#\left(\bigcup_{k\geq 1} A_k\right).$$

Since the sets A_k are disjoint, each element in the union belongs to exactly one A_k, so the cardinality of the union is the sum of the cardinalities:

$$\#\left(\bigcup_{k\geq 1} A_k\right) = \sum_{k=1}^{\infty} \#A_k = \sum_{k=1}^{\infty} \mu(A_k).$$

Hence, μ is countably additive.

□

Problem 10 – Fubini–Tonelli with counting measure

Let $(S, \mathcal{S}, \mu)$ be a σ–finite measure space and let

$$\big(\mathbb{N},\, 2^{\mathbb{N}},\, \nu\big), \qquad \nu(A) = \#A,$$

be the counting–measure space on the natural numbers. Write $\mu \otimes \nu$ for the product measure on $\big(S \times \mathbb{N},\, \mathcal{S} \otimes 2^{\mathbb{N}}\big)$.

Theorem (Tonelli–Fubini). Let $f : S \times \mathbb{N} \to [-\infty, \infty]$ be $\mathcal{S} \otimes 2^{\mathbb{N}}$-measurable.

(i) (*Tonelli*) If $f \geq 0$, then

$$\int_{S\times\mathbb{N}} f(s,t)\,(\mu\otimes\nu)(ds,dt) \;=\; \int_S \Big(\sum_{t\in\mathbb{N}} f(s,t)\Big)\,\mu(ds) \;=\; \sum_{t\in\mathbb{N}} \int_S f(s,t)\,\mu(ds), \qquad (1)$$

allowing the value $+\infty$.

(ii) (*Fubini*) If $f \in L^1(\mu \otimes \nu)$, all three integrals in (1) are finite and equal, and both iterated integrals exist as absolutely convergent expressions.

Proof. **Step 1. Indicator rectangles.** Let $A = B \times C$ with $B \in \mathcal{S}$ and $C \subseteq \mathbb{N}$. For the indicator $\mathbf{1}_A(s,t) = \mathbf{1}_B(s)\,\mathbf{1}_C(t)$ we have

$$\int_{S\times\mathbb{N}} \mathbf{1}_A \, d(\mu\otimes\nu) = (\mu\otimes\nu)(A) = \mu(B)\,\nu(C).$$

On the other hand,

$$\int_S \sum_{t\in\mathbb{N}} \mathbf{1}_A(s,t)\,\mu(ds) = \int_S \mathbf{1}_B(s)\Big(\sum_{t\in C} 1\Big)\mu(ds) = \mu(B)\,\#C = \mu(B)\,\nu(C),$$

and the same equality holds if the order of integration and summation is reversed. Hence (1) holds for $\mathbf{1}_A$.

Step 2. Simple functions. Any non-negative simple function can be written $f = \sum_{k=1}^m c_k\,\mathbf{1}_{A_k}$ with $c_k \ge 0$ and A_k rectangles as above. By linearity of the integral and Step 1, (1) holds for every such f.

Step 3. Non-negative measurable functions. For a general $f \ge 0$ choose an increasing sequence of simple functions $(f_n)_{n\ge1}$ with $f_n \uparrow f$. Applying the Monotone Convergence Theorem on both sides of (1) and using Step 2 yields the Tonelli identity for f.

Step 4. Integrable functions. If $f \in L^1(\mu\otimes\nu)$, decompose $f = f^+ - f^-$ with $f^\pm \ge 0$ and $f^\pm \in L^1$. Apply Step 3 to f^+ and f^- separately and subtract; finiteness follows from integrability. Thus (1) holds and all integrals are finite.

Steps 1–4 establish Tonelli's theorem for $f \ge 0$ and Fubini's theorem for $f \in L^1(\mu\otimes\nu)$. □

Processes, Distributions and independence

Problem 4.
Let $X, Y \in \{0,1\}$ on an index set T.
Show::

$$X \overset{d}{=} Y \iff P\left(\sum_i x_{t_i} > 0\right) = P\left(\sum_i y_{t_i} > 0\right) \tag{2}$$

Proof. Let $S = \{0,1\}$ and equip S^T with the product σ–algebra $\mathcal{B}$.

(⇒) If $X \overset{d}{=} Y$, then for every $A \in \mathcal{B}$ we have $\mathbb{P}(X\in A) = \mathbb{P}(Y \in A)$. Taking $A = \big\{f \in S^T : \sum_{i=1}^n f_{t_i} > 0\big\}$ gives

$$\mathbb{P}\Big(\sum_{i=1}^n X_{t_i} > 0\Big) = \mathbb{P}\Big(\sum_{i=1}^n Y_{t_i} > 0\Big) \qquad (\forall n,\ t_1,\dots,t_n \in T).$$

(⇐) Assume these equalities hold for every finite set of indices. Define the class

$$\mathcal{D} := \big\{A \in \mathcal{B} : \mathbb{P}(X\in A) = \mathbb{P}(Y\in A)\big\}.$$

Claim: $\mathcal{D}$ is a dynkin–system. Indeed, (i) $S^T \in \mathcal{D}$; (ii) if $A \in \mathcal{D}$ then $S^T \setminus A \in \mathcal{D}$ by complementing the equal probabilities; (iii) if $(A_k)_{k\ge1} \subset \mathcal{D}$ are disjoint, then countable additivity gives $\mathbb{P}(X \in \bigcup_k A_k) = \mathbb{P}(Y \in \bigcup_k A_k)$. Hence $\mathcal{D}$ is a Dynkin-system.

Next, for every finite non–empty $J = \{t_1, \dots, t_m\} \subset T$ set

$$Z_J := \{ f \in S^T : f_{t_i} = 0 \ \forall i \in J \}.$$

Because Z_J is the complement of the event $\{\sum_{i=1}^m f_{t_i} > 0\}$, our hypothesis puts Z_J in $\mathcal{D}$. Let $\mathcal{C} := \{Z_J : 0 < |J| < \infty\}$.

Claim: $\mathcal{C}$ is a pi–system. For finite $J, K \subset T$, $Z_J \cap Z_K = Z_{J\cup K}$, so finite intersections stay inside $\mathcal{C}$.

Thus we have a pi–system $\mathcal{C} \subset \mathcal{D}$. By the – (Dynkin–Sierpiński) theorem,

$$\sigma(\mathcal{C}) \subset \mathcal{D}.$$

But $\sigma(\mathcal{C}) = \mathcal{B}$. For any $t \in T$ the single-coordinate event $\{f_t = 1\}$ is $Z^{\mathrm{c}}_{\{t\}} \in \sigma(\mathcal{C})$. Finite intersections of such events and their complements therefore yield every cylinder set of the form $\{f : (f_{t_1}, \dots, f_{t_n}) \in B\}$, and the cylinders generate the product -algebra.

Consequently $\mathcal{B} \subset \mathcal{D}$; i.e. $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all $A \in \mathcal{B}$. That is exactly $X \overset{d}{=} Y$. $\square$

Problem 10.
Let ξ_i i.i.d on $(S, \mathcal{S}, \mu)$ and $\tau = \{k \geq 1 : \xi_k \in A\}$. Fix $A \in \mathcal{S}$ and set $\mu(A) > 0$.
Show that the distribution of ξ_τ is $\mu(\cdot \mid A)$.

Proof. Start by noting that

$$\{\xi_\tau \in B\} = \bigcup_{k=1}^{\infty} \{\tau = k \cap \xi_k \in B\} \tag{3}$$

Next we note that the sets are disjoint, and so we use the property of a measure.

$$\begin{aligned}
P(\xi_\tau \in B) &= \sum_{k=1}^{\infty} P(\tau = k \cap \xi_k \in B) &(4)\\
&= \sum_{K=1}^{\infty} P(\xi_1 \notin A, \cdots, \xi_k \in A \cap B) &(5)\\
&= \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} P(\xi_i \notin A) \right) P(\xi_k \in A \cap B) &(6)\\
&= \sum_{k=1}^{\infty} (1 - \mu(A))^{k-1} \mu(A \cap B) &(7)\\
&= \mu(A)^{-1} \mu(B \cap A) &(8)
\end{aligned}$$

Holds $\forall B \subset \mathcal{S}$.

$\square$

Problem 11.
Let $\xi_i \in [0, 1]$ i.i.d. Show that $E\left(\prod_n \xi_n\right) = \prod_n E(\xi_n)$.

Proof. let $X_i : ([0,1], \mathcal{B}([0,1]), P) \to \mathbb{R}$ be $\mathcal{L}^1$. Define $\mu_k = P \circ X_k^{-1}$ and $\mu = \bigotimes_n \mu_n$, be independent.

$$E\left(\prod_{k=1}^{n} X_k\right) = \int_\Omega \prod_{k=1}^{n} X_k(\omega) dP(\omega) \tag{9}$$

$$\overset{Indp.}{=} \int_{[0,1]^n} \prod_{k=1}^{n} x_k d(\mu_1 \otimes \cdots \otimes \mu_n) \tag{10}$$

$$\overset{FB}{=} \int_{[0,1]} x_n d\mu_n \int_{[0,1]^{n-1}} \prod_{k=1}^{n-1} x_k d(\mu_1 \otimes \cdots \otimes \mu_{n-1}) \tag{11}$$

$$\overset{iter.}{=} \prod_{k=1}^{n} \int_{[0,1]} x_k d\mu_k \tag{12}$$

The proof can be extended to apply for probabilities by setting $X_k(\omega) = 1\{A_k(\omega)\}$. □

Problem 3.

We are F which is right cont. with bounded variation and is zero at minus infinity. We are asked to show that $E(F(\xi)) = \int P(\xi \geq t) dF(t)$.

Proof. Define the push forward / law of ξ as $\mu_\xi = P \circ \xi^{-1}$ or rather $\mu_\xi(B) = P(\xi \in B)$.

$$E[F(\xi)] = \int_\Omega \left(\int_\mathbb{R} 1_{\{s \geq t\}} \, dF(t) \right) d\mu_\xi(s) \tag{13}$$

$$\overset{\text{FT}}{=} \int_\mathbb{R} \left(\int_\Omega 1_{\{s \geq t\}} \, d\mu_\xi(s) \right) dF(t) \tag{14}$$

$$= \int_\mathbb{R} \mu_\xi\big([t, \infty)\big) \, dF(t) \tag{15}$$

$$= \int_\mathbb{R} P(\xi \geq t) \, dF(t). \tag{16}$$

□