Contents

The goal of this project is to find different investment strategies using multiple asset allocation techniques. The given dataset refers to the stocks included in the S&P100 index and it contains their prices, from 11/05/2021 to 24/10/2023, capitalization and sectors.

1 Part A

1.1 Efficient Frontier under standard constraints

Our task is to compute the efficient frontier - using the Markovitz model - and then determine on it both the Minimum Variance Portfolio (MVP) and the Maximum Sharpe Ratio Portfolio (SP). First of all, we compute the logarithmic returns for the assets within our specified time range (from 11/05/2021 to 11/05/2022). Subsequently, we create a Matlab portfolio object, setting only the default constraints. This enables us to compute the efficient frontier exploiting the Matlab function `estimateFrontier`, generating 1000 portfolios.

Ultimately, we extract the two portfolios of interest by finding the indices of the ones with the minimum variance and the maximum Sharpe Ratio among the thousand simulated portfolios. In Figure 1 we show the plot of the frontier and of the two portfolios under analysis. In Table 1 there are the results (daily return, daily volatility and Sharpe Ratio) obtained for each of the two portfolios.

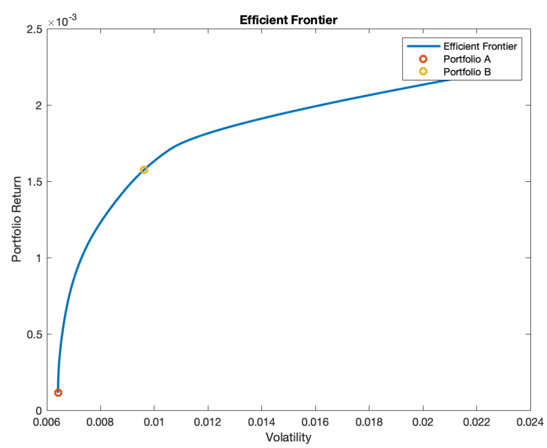

Figure 1: Portfolio (efficient) frontier - Markovitz model

Metrics	**Portfolio A**	**Portfolio B**
Daily Return (%)	0.011	0.16
Daily Volatility (%)	0.64	0.96
Sharpe Ratio	0.017	0.1637

Table 1: Performance metrics for portfolios A and B.

Portfolio A is clearly less risky, while Portfolio B is for sure a best choice since, even if it requires to take an higher risk, the highest Sharpe ratio makes it the best regarding risk-adjusted return. Looking at the metrics, it can be noticed that the returns of B are 13 times higher than the ones of A, in the face of a 50% increase in volatility.

We conclude saying that the key concept of the Modern Portfolio Theory is diversification. Markowitz argued that investors could reduce risk by choosing an optimal mix of assets that do not move in perfect correlation with each other. We can infer this from Figure 2.

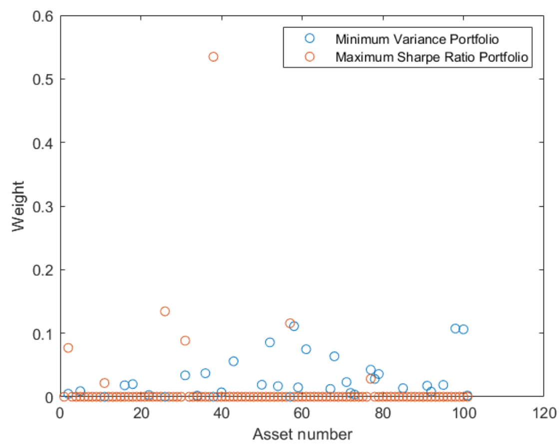

Figure 2: Weights distribution of the two portfolios

1.2 Efficient Frontier under additional constraints

In the second part, we keep the focus on the efficient frontier with its MVP and maximum SP portfolios, with the addition of more constraints. They limit both the number of assets we can choose in a sector and their exposure. In particular, we have to ensure that:

- The overall exposure of the companies belonging to the sector "Communication Services" is greater than 12%,
- The overall exposure of the companies belonging to the sector "Utilities" is less than 10%,
- The weights of the companies belonging to sectors that are composed by less than 5 companies are null.

In order to take into account the third constraint we impose to zero the upper bound for the weights of the companies belonging to sectors that are composed by less than 5 companies (using `UpperBound` function). Then, for the first and second constraints, we add the limits on the weights with `addInequality`. In Figure 3 we show the comparison between the results obtained from the analysis with standard constraints made in Question 1 and the results with the additional constraints.

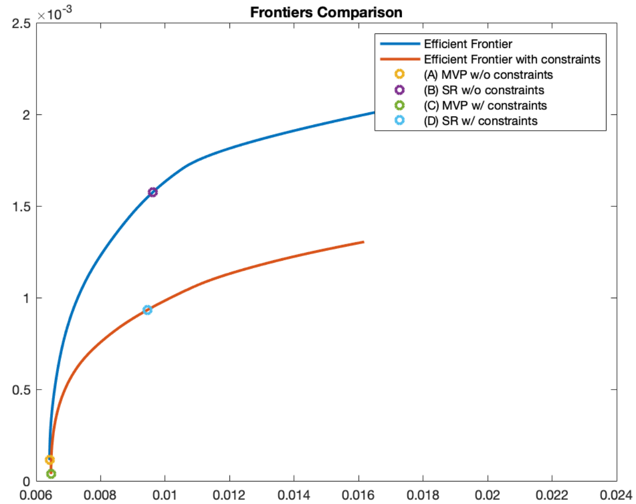

Figure 3: Frontier with standard constraints and frontier with additional constraints

We see that the frontier with the additional constraints, and so its MVP and SP portfolios behave worse than the ones with only standard constraints. This is due to the fact that the optimization problem having less degree of freedom can lead to a lower proportion between the return over the risk. This is nonetheless a more realistic analysis, which gives us portfolio C (MVP) and D (SP) with their characteristics contained in Table 2.

Metrics	**Portfolio C**	**Portfolio D**
Daily Return (%)	0.00375	0.093
Daily Volatility (%)	0.65	0.95
Sharpe Ratio	0.0058	0.098

Table 2: Performance metrics for portfolios C and D.

1.3 Robust Frontier

We are asked to compute two robust frontiers using the resampling method. To do so, we use `mvnrnd` to generate multiple samples of observations that simulate possible scenarios, based on the estimated parameters. For each of the 50 samples, we compute the efficient frontier and store the weights for each of the 100 portfolios. The robust frontier is then obtained by computing the mean of returns and risks across the different samples. From this robust frontier, we extract the indices of the Minimum Variance Portfolio (MVP) and the Maximum Sharpe Ratio (SR) Portfolio. Subsequently, we calculate the average weights for these portfolios across the different simulations.

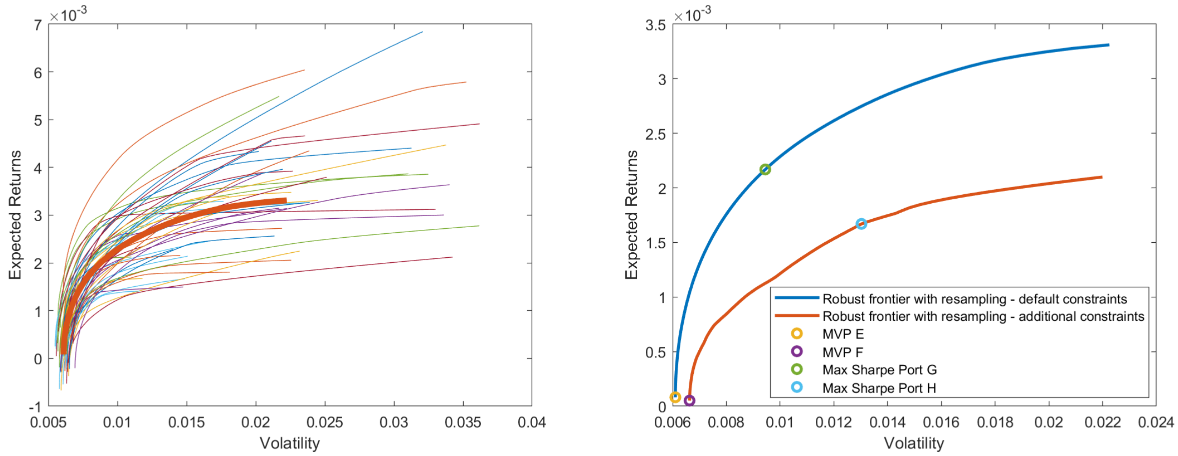

Figure 4: (Left) Efficient frontier for each sample, with the average line being thicker. (Right) Default and custom constraints frontiers with their respective portfolios.

As we can see by comparing the Question 1 results with the right plot in Figure 4, the frontiers obtained through the resampling method dominate those obtained before, showcasing its superiority in portfolio optimization. This dominance is attributed to the resampling method's ability to mitigate the impact of outliers and reduce sensitivity to estimation errors. By generating multiple samples of observations, the resampling method provides a more robust and stable estimation of the efficient frontier, minimizing the influence of extreme data points and uncertainties in parameter estimates.

1.4 Black-Litterman model

The Black-Litterman model is an asset allocation approach, based on Bayesian Statistics theory, that allows investors to incorporate subjective views into market equilibrium returns, in order to have a portfolio that is consistent with the investor's belief while also respecting market realities. In our case, the opinions regarding the asset performances are:

- The companies belonging to the sector "Industrials", which are 13, will have an annual return of 3%, thus 13 views,
- The companies belonging to the sector "Materials", which are 2, will have an annual return of 5%, thus 2 views,
- The companies belonging to the sector "Information Technology" will outperform the companies belonging to the sector "Consumer Staples" by 7%. In this case, for simplicity, we choose to consider an equal distribution, instead of taking into account all the possible couples of assets.

We have 16 views in total ($v = 16$). Our goal is to estimate the blended asset return μ_{BL} and

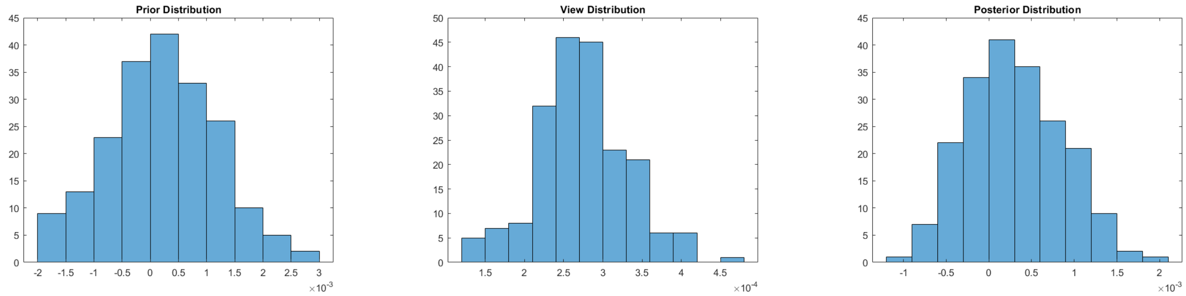

Figure 5: (Left) Prior distribution on the first asset, Apple. (Center) Distribution of the last view, the one that affects Apple. (Right) Posterior distribution of the first asset.

variance σ_{BL}; to do this, we have to follow the following path.

The first step is specifying the views distribution: the investor's beliefs on the returns are mathematically represented as a linear function of the market, $\mathbf{q} = \mathbf{P}\mu + \epsilon$, where $\epsilon \sim N(0, \Omega)$, $\mathbf{P}$ is a $v \times N$ matrix (v is the total number of views and N is the number of assets), $\mathbf{q}$ is a $v \times 1$ vector and Ω is a $v \times v$ diagonal matrix. These are the inputs for the likelihood distribution, how likely it is for the views to happen given the outcome of the market.

The second step considers the Market Implied Equilibrium Return as prior distribution: μ_{mkt} represents the expected return that would make the current prices in "equilibrium" under certain assumptions. In practise, it is the optimal portfolio that the analyst would use in the absence of additional views on the market. It is computed in the following way: $\mu_{mkt} = \lambda V w_{mkt}$, where λ is the risk adversion coefficient, V is the covariance matrix of returns and w_{mkt} is the market capitalization weight vector of assets. C represents the uncertainty in the prior and the Black-Litterman model makes the assumption that $C = \tau V$, where τ is a small constant (in our case `1/length(LogRet)`). We thus obtain the prior distribution.

The third step is the computation of the posterior distribution, the distribution of the market conditioned on the investor's view, using the Bayes theorem: $posterior \propto likelihood \times prior$. In this way we obtain μ_{BL} and Σ_{BL}. The graphical results for Apple, belonging to the sector "Information Technology", are shown in Figure 5.

Comparing the blended expected return from Black-Litterman model to the prior belief of expected return, we find that the expected return from Black-Litterman model is indeed a mixture of both prior belief and investor views. For example, the prior belief assumes returns of 4.96% for "BAUNEquity", belonging to the "Industrial" sector, but in blended expected return, the asset has a return of 3.02%, according to the first view. Again, in the blended expected return, "AAPLUWEquity" ("IT") has an higher return than "CLUNEquity" ("Consumer Staples") by 6,55%, according to the last view.

Finally, we can focus on the portfolio optimization: by combining the Bayesian posterior distribution of μ_{BL} and the model of asset returns $r \sim N(\mu_0, V)$, we obtain the posterior prediction of asset returns as $r \sim N(\mu_{BL}, V + \Sigma_{BL})$. The frontier, the Maximum Variance Portfolio and the Maximum Sharpe Ratio Portfolio can be found (Figure 6).

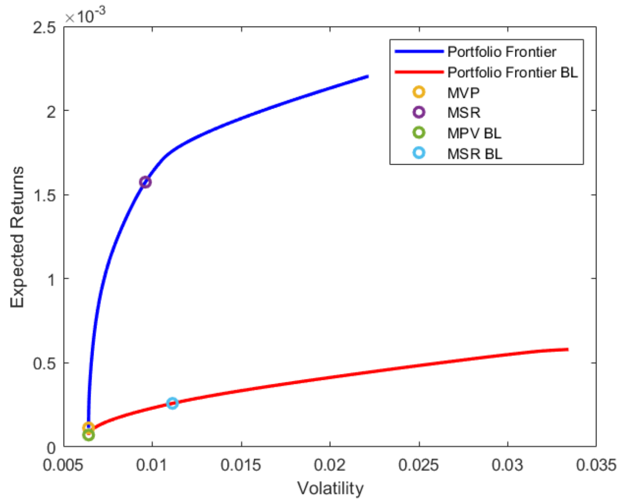

Figure 6: Mean-Variance Optimization Frontier and Black-Litterman Frontier

We observe that the Black-Litterman frontier is dominated by the standard frontier. The reason could be that the views are not compatible with the real-world data.

1.5 Maximum Diversified Portfolio and Maximum Entropy Portfolio

Our task is to compute the Maximum Diversified Portfolio and Maximum Entropy Portfolio under the following constraints:

- $\sum_{i=1}^{N} w_i = 1, \quad w_i \geq 0,$
- $0.003 \leq w_i \leq 0.01$, for companies belonging to the sector "Materials",
- $0.001 \leq w_i \leq 0.03$, for companies belonging to the sector "Energy".

We compute the maximum diversified portfolio that maximizes the diversification ratio $DR = \frac{w^T \sigma}{\sigma_p}$. Using the function `estimateCustomObjectivePortfolio` we obtain $DR = 2.4713$.
Following the same procedure we find the maximum entropy portfolio, using as objective function, the entropy measure in asset volatility $H_{vol} = \sum_{i=1}^{N} \frac{w_i^2 \sigma_i^2}{\sum_{i=1}^{N} w_i^2 \sigma_i^2} \ln \left(\frac{w_i^2 \sigma_i^2}{\sum_{i=1}^{N} w_i^2 \sigma_i^2} \right)$. We get $H_{vol} = 4.6069$ and $\eta_{vol} = e^{H_{vol}} = 100.1703$; this suggests that, on average, almost all assets are relevant in the risk space. In Figure 7, we can see the weights distribution in the two portfolios, in particular the Maximum entropy portfolio is uniform across all assets.

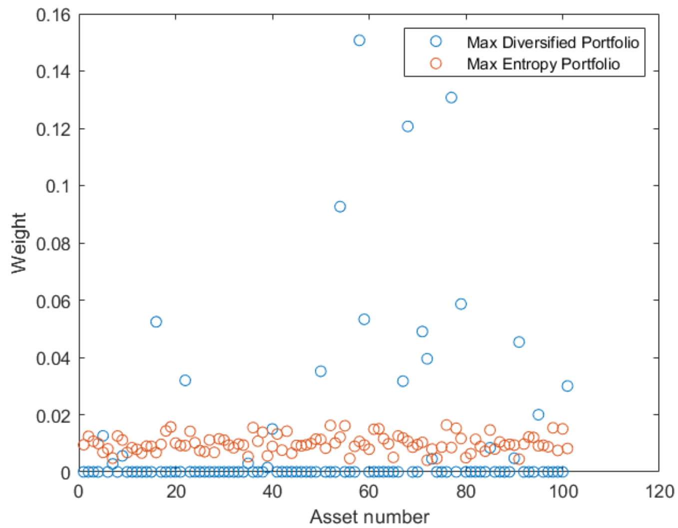

Figure 7: Weights distribution of the two portfolios.

1.6 PCA

Principal component analysis is a method to use factor models in portfolio optimization. It reduces the dimensionality of data, thus lowering the computational cost at the expense of a loss of interpretability of underlying factors. In this task we use PCA to try to explain variance in the logged returns with $k = 10$ principal components and create a portfolio under the following constraints:

- Standard constraints,
- The volatility of the portfolio is $\sigma \leq \sigma_{tgt} = 0.007$.

Since we want to be able to explain as much as possible of the original variation in the data, we prefer a large explained variance from the new principal components. In Figure 8, the cumulative explained variance is displayed and we see that the first ten principal components account for almost 70% of the explained variance. Ideally, we would like to use a few more components to get more reliable results but recognize that we would have to add many components to get a high explained variance.

By trying to reconstruct the asset returns with our components, there are returns that we are unable to explain since we drop 91 possible principal elements. The unexplained variance can however be used to obtain the covariance between the assets. Moreover, from this matrix we create an optimal portfolio with `estimateFrontierByRisk`. This works since we know that the portfolio is on the frontier and therefore we obtain the highest return when $\sigma = \sigma_{tgt}$. However, it is important to note that the frontier is not the same as in the other exercises as we only use ten principal components.

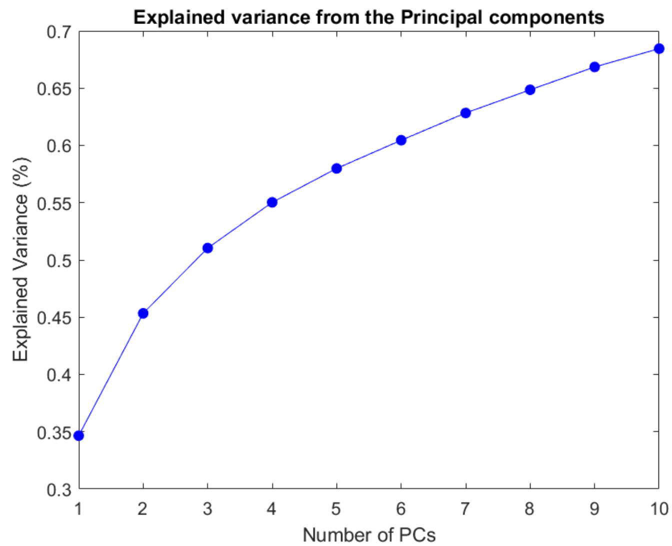

Figure 8: Explained variance by the principal components.

1.7 Maximum Expected Shortfall-modified Sharpe Ratio Portfolio

Traditional Sharpe Ratio evaluates risk-adjusted returns by considering the volatility of returns, but it treats gains and losses symmetrically. In contrast, the MSR-ES gives more weight to the downside risk by incorporating Expected Shortfall (ES), a measure that focuses on the average magnitude of losses beyond a certain confidence level. After setting the confidence level for the ES at 5% we calculate the portfolio that maximizes the MSR-ES using the optimizer `estimateCustomObjectivePortfolio`, as before. Then, plotting the return and risk for the obtained portfolio and the efficient frontier (Figure 9), we can notice that it does not belong to the frontier. This last result is not uncommon or unreasonable, since, instead of the mean-variance principle, which regulates the frontier construction, we are dealing with a metric related to the distribution of the losses.

1.8 Performance and diversification metrics

Performance metrics are shown in Table 3.

To compute them, it is necessary to calculate the Equity Curve for each portfolio. It is a graphical representation of the portfolio's value over time: it tracks the cumulative performance, taking into account gains and losses. The graph we obtain with our weights is represented in Figure 10.

We consider the equally weighted portfolio as benchmark. This is the simplest strategy since it naively allocates the same weights for each asset. In our case, this leads to bad results: the annualized returns, hence the Sharpe and Calmar ratios, are negative, instead the annualized volatility and Maximum Drawdown take large values. Thanks to the optimization techniques

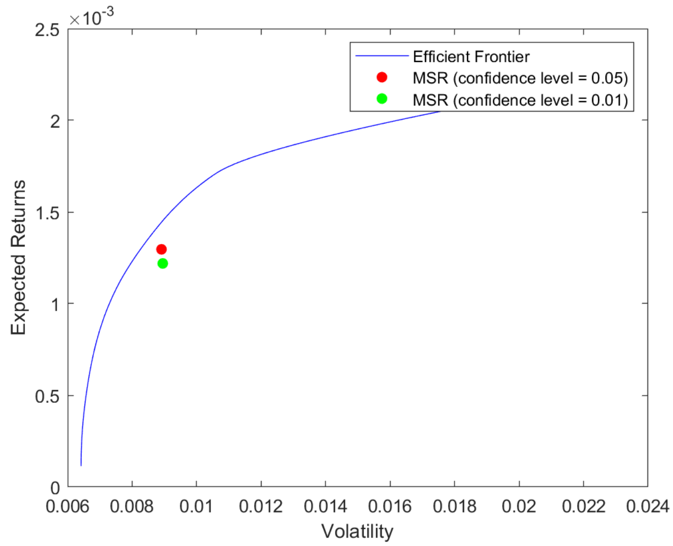

Figure 9: Maximum ES Modified Sharpe Ratio portfolios.

Portfolio	**Ann.Ret.(%)**	**Ann.Vol.(%)**	**Sharpe Ratio**	**MaxDD(%)**	**Calmar Ratio**
EW	-2.08	15.06	-0.1384	-14.86	-0.1403
A	6.31	10.11	0.6240	-5.63	1.1197
B	53.71	15.18	3.5390	-7.57	7.0956
C	4.23	10.18	0.4154	-5.74	0.7373
D	29.71	15.03	1.9763	-10.70	2.7755
E	6.22	10.19	0.0699	-5.66	-0.1403
F	3.78	10.49	0.3605	-6.34	0.5962
G	33.51	12.60	2.6585	-6.61	5.0688
H	56.93	20.99	2.7128	-10.15	5.6092
I	6.30	10.11	0.6228	-5.64	1.1174
L	6.78	17.51	0.3872	-11.33	0.5981
M	6.57	11.28	0.5826	-7.80	0.8431
N	-0.21	13.65	-0.0156	-11.47	-0.0186
P	29.44	11.31	2.6032	-5.04	5.8362
Q	43.03	14.15	3.0414	-6.97	6.1757

Table 3: Performance Metrics - from 11/05/2021 to 11/05/2022.

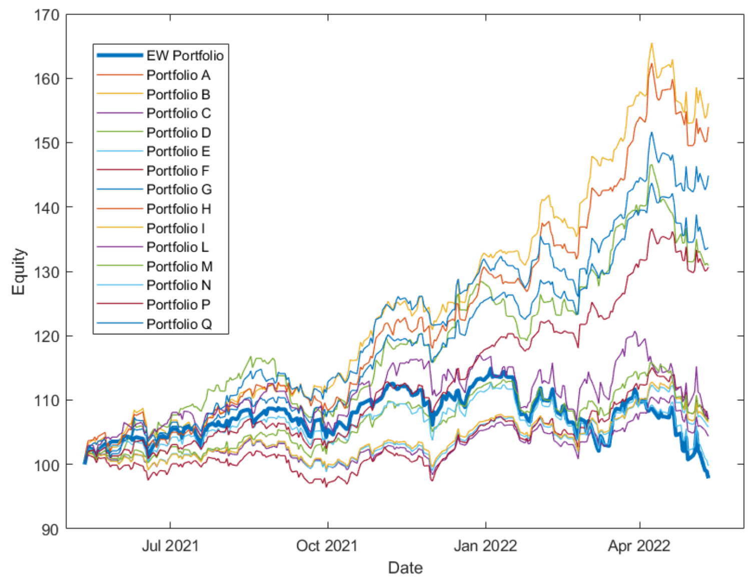

Figure 10: Portfolio equities trend.

performed in the previous points, we can obtain better outcomes.

Starting from the annualized returns, the best portfolio is H, the one that maximizes the Sharpe Ratio on the robust frontier with additional constraints. At the beginning this result could appear unexpected, but then we take into account the fact that the weights on the robust frontier are built through simulations; we also rely on the point that annualized returns could be affected on when changes occur. In general, we obtain the best values for portfolios where we try to maximize the Sharpe Ratio, as expected. On the other hand, when we are asked to optimize diversification metrics, such as entropy in portfolio N, annualized returns are lower or negative.

The annualized volatility stays among 10.11%, obtained by A, and 20.99%, obtained by H, the one with the highest return. The large difference between this last value and all the other ones could also explain the anomalous high returns for H.

To optimize the trade-off between expected return and volatility, the highest Sharpe Ratio is given by B and all the other portfolios with this scope. By comparing returns and volatility of this portfolios, we observe that maximizing the firsts is better than maximizing the seconds.

About Maximum Drawdown, the highest numbers are attained by portfolios EW, N and L, instead the best performances regard portfolios with low volatility. This suggests a correlation between these two factors.

The Calmar Ratio is an index that synthesizes the final performance with the global path, penalizing eventual shocks. This is shown, for example, by portfolio B: it has a lower return than H, but a higher Calmar Ratio, since the second portfolio has higher drawdowns in the considered interval

Portfolio	**Herf. Ind.**	H_w	**DR**	H_{vol}	H_{risk}	**Asset Max RC**	**Max RC**
EW	0.0099	4.6151	1.7911	4.6151	4.5191	'NVDAUW'	0.0223
A	0.0650	2.9853	2.0880	2.3687	2.9853	'LMTUN'	0.1111
B	0.3324	1.4494	1.5035	0.6004	1.3882	'EXCUW'	0.5302
C	0.0774	2.8302	2.0688	2.1736	2.8302	'VZUN'	0.1341
D	0.1893	1.8459	1.5584	1.3833	1.6715	'LLYUN'	0.3150
E	0.0467	3.4130	2.1076	2.5533	3.4434	'LMTUN'	0.1068
F	0.0879	2.6043	1.9367	2.1941	2.5876	'LMTUN'	0.1645
G	0.0628	3.1944	2.0105	2.1678	3.0680	'COPUN'	0.1637
H	0.3837	1.0234	1.4065	0.8612	0.9234	'LLYUN'	0.4693
I	0.0649	2.9862	2.0882	2.3690	2.9862	'LMTUN'	0.1116
L	0.1340	2.3596	1.7014	1.4902	2.4675	'DOWUN'	0.1626
M	0.0831	2.7212	2.4713	2.1360	2.7390	'PFEUN'	0.1358
N	0.0108	4.5709	1.8074	4.4549	4.5848	'BLKUN'	0.0145
P	0.0973	2.4812	1.8662	2.0660	2.4186	'EXCUW'	0.2147
Q	0.2573	1.5417	1.6812	1.0568	1.5414	'EXCUW'	0.3966

Table 4: Diversification Metrics - from 11/05/2021 to 11/05/2022.

(Figure 10).

Let us move now to diversification metrics, shown in Table 4.

Firstly, we describe the diversification of our strategies through portfolio weights: this kind of measures does not require information on risk properties but it depends only on the weights. Herfindal index is one of these: it quantifies the concentration of investments within a portfolio. It ranges from 0 to 1; higher values indicates greater market concentration, while a lower index suggests more diversification. With the same scope, there is the Shannon entropy measure in portfolio weights H_w. In this case the most diversified allocations are rewarded with higher scores. The equally weighted portfolio attains the best performances in these two metrics, thus it sets a limit for the other strategies values. The second best results are obtained by N, which is built on purpose to maximize the entropy. The difference is caused by the constraints set for the latter.

Then, we consider the metrics where the diversification is expressed in terms of risk contributions. They rely on historical evaluation of information about asset risk. Diversification Ratio evaluates the benefits of diversification in allocating assets in such a way. Dividing the weighted average volatility by the portfolio volatility gives us an index that rewards strategies that induce a negative correlation among assets. As we expected, we obtain high scores when we are asked to directly maximize this ratio (M) and when we are asked to minimize the variance. We proceed with other two entropy measures: one in asset volatility H_{vol} and the other in risk contribution H_{risk}. The first replaces the set of probabilities in Shannon's formula with the normalized contribution of each asset to the variance, not considering the correlation among them. Moreover, by taking the exponential of this index we obtain the average number of relevant assets in the risk space. In our case, for the portfolios with the highest score (EW and N) all the 101 stocks are considered important, instead for B, the one with the lower entropy, only 2 assets play a role on average. We conclude the analysis of this kind of diversification metrics with the entropy measure in risk contribution H_{risk}, where the set of probabilities is replaced by the set of risk contributions, this time taking into account the eventual correlation among assets. In general, the values of H_{vol} and H_{risk} are similar, with some exceptions (B, E, L), where the second is higher: probably more uncorrelated assets had been used.

In conclusion, we take into consideration the assets with the highest risk contribution for every portfolio. As expected, the strategies where the maximum RC is high are the ones with lower entropy measures, since they are less diversified. It is curious to notice that the maximum contributor asset that is most frequent among portfolios is "LMTUN", corresponding to the Lockheed Martin Corporation, an American aerospace, arms and defense corporation with worldwide interests. This might find a reason in the outbreak of the war in Ukraine in February 2022, within our reference period.

2 Part B

Since it is preferable to create a portfolio that performs well over a long time, a natural and more interesting continuation is to evaluate the portfolio allocations in a completely new period. Hopefully, a good portfolio will continue to perform well given its focused metric. The considered period is the consecutive year from the original period, which means from 12/05/2022 to 12/05/2023. We can see the trend of the portfolio equities in Figure 5. The results regarding the metrics are shown in Table 11, and while we still consider the equal-weighted portfolio as a benchmark, we focus on the difference in portfolio performances between the two periods.

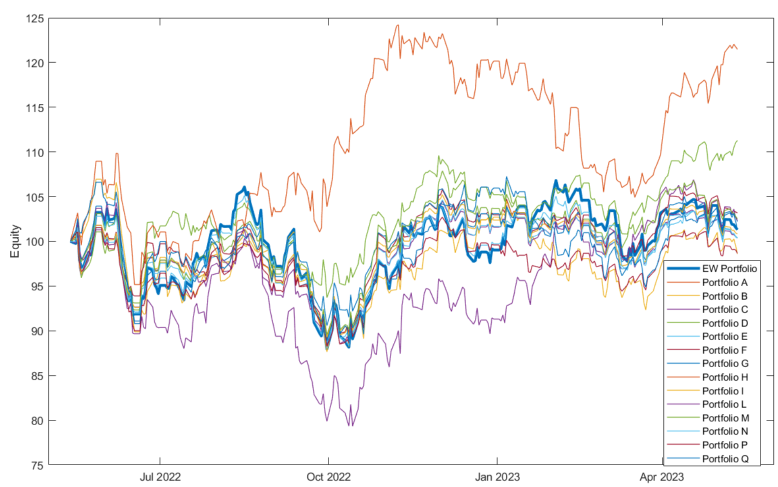

Figure 11: Portfolio equities trend.

Portfolio	**Ann.Ret.(%)**	**Ann.Vol.(%)**	**Sharpe Ratio**	**MaxDD(%)**	**Calmar Ratio**
EW	1.27	19.99	0.0633	-17.11	0.0740
A	0.81	14.30	0.0567	-12.31	0.0658
B	-1.37	19.53	-0.0699	-18.05	-0.0756
C	0.38	14.09	0.0271	-12.80	0.0298
D	10.87	16.90	0.6429	-09.42	1.1538
E	0.35	14.47	0.0239	-12.64	0.0274
F	3.08	14.13	0.2179	-12.57	0.2449
G	2.13	16.69	0.1278	-12.85	0.1660
H	20.66	22.18	0.9318	-15.55	1.3293
I	0.81	14.30	0.0568	-12.31	0.0660
L	2.25	23.37	0.0964	-24.06	0.0937
M	2.45	16.58	0.1476	-14.14	0.1731
N	0.41	18.36	0.0221	-16.24	0.0250
P	-1.29	15.05	-0.0860	-13.11	-0.0987
Q	1.24	17.79	0.0698	-16.77	0.0740

Table 5: Performance Metrics - from 12/05/2022 to 12/05/2023.

If we compare Tables 3 and 5, we see that in general, the annual returns decreases for most portfolios in the new period. This can very likely depend on a worse overall performance of the economy, hence we focus on the relative annual return within the portfolios. Given the metric, the best strategy is still H, the maximum Sharpe Ratio portfolio for the robust frontier with the additional constraints. This portfolio still decreases by a lot, but in comparison with other portfolios it performs very well and it is stable. We see that overall, there is still the trend where portfolios focused on maximizing the Sharpe Ratio are performing the best. The most remarkable aspect is that minimum variance portfolios tend to be more stable among multiple periods, as expected.

For most portfolios, the annualized volatility remains approximately at its original value or is slightly higher. In general, minimum variance portfolios still have the lowest annual volatility. The high volatility of maximum Sharpe ratio portfolios explain the large differences in returns during the two periods.

Regarding the Sharpe Ratio, we see that, since volatilities are approximately unchanged, the annual return is still the metric determining whether the SR is high or low. All the SR have drastically decreased, but the maximum SR portfolios still have the largest values due to their largest annual returns.

The maximum drawdowns for almost all portfolios are larger in this new period, but once again, this is probably due to a worse overall performance of the assets. We see that minimum variance portfolios still in general have the lowest maximum drawdown, indicating a correlation between these two metrics.

Considering the Calmar Ratio, the best performing portfolios are still the maximum Sharpe ratio portfolios. However, we also see that the worst performing portfolios is portfolio P and B respectively, suggesting that Sharpe ratio portfolios are more volatile also in this metric.

At the end, we can conclude that the purpose of each strategy is maintained in the new period. Of course, for a more accurate and robust analysis, we could consider larger intervals to better estimate parameters and finally come up with more reliable strategies for the future.