Computer Vision, Assignment 1
Elements of Projective Geometry
Theoretical Exercises

1 Instructions

In this assignment you will study the basics of projective geometry. You will study the representations of points lines and planes, as well as transformations and camera matrices.

Please see Canvas for detailed instructions on what is expected for a passing/higher grade. All theoretical exercises not marked **OPTIONAL** are "mandatory" in the sense described on Canvas.

The maximum amount of points for the theoretical exercises in Assignment 1 is 34. 50% of 34 is 17.

Your final lab submission should include:

1. Your edited notebook file ('.ipynb') (computer exercises).
2. An HTML printout of the executed notebook with all outputs visible: File → Save and export Notebook As → HTML (computer exercises).
3. A pdf report containing answers to the theoretical exercises.

2 Points in Homogeneous Coordinates.

Theoretical Exercise 1 (5 points). What are the 2D Cartesian coordinates of the points with homogeneous coordinates

$$\mathbf{x}_1 = \begin{pmatrix} -8 \\ 6 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} \text{ and } \mathbf{x}_3 = \begin{pmatrix} -3\lambda \\ 12\lambda \\ 6\lambda \end{pmatrix}, \lambda \neq 0? \tag{1}$$

What is the interpretation of the point with homogeneous coordinates

$$\mathbf{x}_4 = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}? \tag{2}$$

Is $\mathbf{x}_4$ the same point as

$$\mathbf{x}_5 = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}? \tag{3}$$

For the report: Answers are enough.

Computer Exercise 1. (See Jupyter Notebook)

3 Lines

Theoretical Exercise 2 (5 points). Compute the homogeneous coordinates of the intersection (in $\mathbb{P}^2$) of the lines

$$l_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } l_2 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}. \tag{4}$$

What is the corresponding point in $\mathbb{R}^2$?

Compute the intersection (in $\mathbb{P}^2$) of the lines

$$l_3 = \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix} \text{ and } l_4 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}. \tag{5}$$

What is the geometric interpretation in $\mathbb{R}^2$?

Compute the line that goes through the points with Cartesian coordinates

$$x_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{6}$$

Hint: Re-use the calculations from the line intersections above.

For the report: Submit a complete solution.

Theoretical Exercise 3 (3 points). The nullspace of a $m \times n$ matrix A is the set

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n ; Ax = 0\}, \tag{7}$$

that is, all the x for which the multiplication Ax gives the zero vector. Explain why the intersection point (in homogeneous coordinates) of l_1 and l_2 (from Exercise 2) is in the null space of the matrix

$$M = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \end{pmatrix}. \tag{8}$$

Are there any other $\mathbb{P}^2$-points in the null space besides the intersection point?

For the report: Full solution.

Computer Exercise 2. (See Jupyter Notebook)

4 Projective Transformations

Theoretical Exercise 4 (5 points). Let H be the projective transformation

$$H = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Compute the transformations $\mathbf{y}_1 \sim H\mathbf{x}_1$ and $\mathbf{y}_2 \sim H\mathbf{x}_2$ if

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{10}$$

Compute the lines l_1, l_2 containing $\mathbf{x}_1$,$\mathbf{x}_2$ and $\mathbf{y}_1$,$\mathbf{y}_2$ respectively.

Compute $(H^{-1})^T l_1$ and compare to l_2.

For the report: Submit the answers.

Theoretical Exercise 5 (4 points). Prove that every projective transformation H preserves lines. That is, for each line l_1 there is a corresponding line l_2 such that if x belongs to l_1 then the transformation $\mathbf{y} \sim H\mathbf{x}$ belongs to l_2. (Hint: If $l_1^T\mathbf{x} = 0$ then $l_1^T H^{-1} H\mathbf{x} = 0$.)

For the report: Submit the full (short) proof.

Theoretical Exercise 6 (6 points). Take a look at the following transformation matrices.

$$H_1 = \begin{pmatrix} 1/2 & 2 & 1 \\ -2 & 1/2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \; H_2 = \begin{pmatrix} \sqrt{3} & -1 & 2 \\ 1 & \sqrt{3} & 1 \\ 0 & 0 & 2 \end{pmatrix}, \tag{11}$$

$$H_3 = \begin{pmatrix} \sqrt{2} & -1 & 1 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } H_4 = \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \tag{12}$$

(a) Which of the transformations are projective transformations?

(b) Which are affine transformations?

(c) Which are similarity transformations?

(d) Which are Euclidean?

(e) Which of the transformations preserve lengths between points?

(f) Which map lines to lines?

(g) Which map parallel lines to parallel lines?

For the report: Answers to the questions.

5 The Pinhole Camera

Theoretical Exercise 7 (6 points). Compute the projections of the 3D points with homogeneous coordinates

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{X}_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{X}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \tag{13}$$

in the camera with camera matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{14}$$

What is the interpretation of the projection of $\mathbf{X}_2$?

Compute the camera center (position) of the camera and the principal axis.

For the report: Answers are enough.

Computer Exercise 3. (See Jupyter Notebook)

Theoretical Exercise 8. (**OPTIONAL**, 10 optional points) Consider the calibrated camera pair $P_1 = [I\ 0]$ and $P_2 = [R\ t]$. If $\mathbf{x} \in \mathbb{P}^2$ is the 2D projection in P_1 of the 3D point $\mathbf{U} \in \mathbb{P}^3$, i.e. $\mathbf{x} \sim P_1\mathbf{U}$, verify that

$$\mathbf{U} \sim \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix}, \tag{15}$$

where $s \in \mathbb{R}$. That is, for any s the point of the form $\mathbf{U}(s) = (\mathbf{x}^T, s)^T$ projects to $\mathbf{x}$. What kind of object is this collection of points? Is it possible to determine s using only information from P_1?

Assume that $\mathbf{U}$ belongs to the plane

$$\Pi = \begin{pmatrix} \pi \\ 1 \end{pmatrix}, \tag{16}$$

where $\pi \in \mathbb{R}^3$. Compute the s that makes $\mathbf{U}(s)$ belong to the plane, that is, find s such that $\Pi^T\mathbf{U}(s) = 0$.

Verify that if $\mathbf{x} \sim P_1\mathbf{U}$, $\mathbf{y} \sim P_2\mathbf{U}$ and $\Pi^T\mathbf{U} = 0$ then the homography

$$H = (R - t\pi^T), \tag{17}$$

where $P_2 = [R\ t]$, maps $\mathbf{x}$ to $\mathbf{y}$. (Hint: What is $P_2\mathbf{U}(s)$ for the s from above?)

For the report: Provide all answers, computations and verifications.

Computer Exercise 4. (**OPTIONAL**, 15 optional points) (See Jupyter Notebook)