Computer Vision, Assignment 3
Epipolar Geometry

1 Instructions

In this assignment, you will study epipolar geometry. You will use the fundamental matrix and the essential matrix for simultaneously reconstructing the 3D structure and the camera motion from two images.

Please see Canvas for detailed instructions on what is expected for a passing/higher grade. All exercises that are not marked as **OPTIONAL** are “mandatory” in the sense described on Canvas.

The maximum amount of points for the theoretical exercises in Assignment 3 is 18. 50% of 18 is 9.

2 The Fundamental Matrix

Theoretical Exercise 1 (3 points).

If $P_1 = [I\ 0]$ and

$$P_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

Compute the fundamental matrix.

Suppose the point $x = (0, 1)$ is the projection of a 3D-point $\mathbf{X}$ into P_1. Compute the epipolar line in the second image generated from x.

Which of the points $(1, 0)$, $(0, 1)$ and $(1, 1)$ could be a projection of the same point $\mathbf{X}$ into P_2?

For the report: Answers are enough.

Theoretical Exercise 2 (3.5 points).

If $P_1 = [I\ 0]$ and

$$P_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{2}$$

Compute the epipoles, by projecting the camera centers.

Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $F e_1 = 0$.

For the report: Complete solution.

Theoretical Exercise 3 (**OPTIONAL**, 12 optional points).

For a general camera pair $P_1 = [I\ 0]$ and $P_2 = [A\ t]$. Compute the epipoles, by projecting the camera centers. (You may assume that A is invertible.)

Verify that for the fundamental matrix $F = [t]_\times A$ the epipoles will always fulfill $e_2^T F = 0$ and $F e_1 = 0$.

Given the above result explain why the fundamental matrix has to have determinant 0.

For the report: Complete solution.

Theoretical Exercise 4 (1 point). When computing the fundamental matrix F using the 8-point algorithm it is recommended to use normalization. Suppose the image points have been normalized using

$$\tilde{\mathbf{x}}_1 \sim N_1 \mathbf{x}_1 \text{ and } \tilde{\mathbf{x}}_2 \sim N_2 \mathbf{x}_2. \tag{3}$$

If $\tilde{F}$ fulfills $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ what is the fundamental matrix F that fulfills $\mathbf{x}_2^T F \mathbf{x}_1 = 0$ for the original (un-normalized) points?

For the report: Answer is enough.

Computer Exercise 1. (See Jupyter Notebook)

Theoretical Exercise 5 (3.5 points).

Consider the fundamental matrix

$$F = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$

Compute the projections of the scene points $X = (0, 3, 1)^T$ and $Y = (-1, 2, 0)^T$ in two cameras $P_1 = [I\ 0]$ and $P_2 = [[e_2]_\times F\ e_2]$. For each scene point, verify that the projections fulfill the epipolar constraint ($x_2^T F x_1 = 0$). What is the camera center of P_2?

For the report: Complete solution.

3 The Essential Matrix

Theoretical Exercise 6 (**OPTIONAL**, 13 optional points).

The goal of this exercise is to show that an essential matrix has two nonzero identical singular values.

Suppose the 3×3 skew symmetric matrix $[t]_\times$ has a singular value decomposition

$$[t]_\times = USV^T, \tag{4}$$

where U, V are orthogonal and S diagonal with non-negative elements. Show that the eigenvalues of $[t]_\times^T [t]_\times$ are the squared singular values. *HINT:* Show that $S^T S = S^2$ diagonalizes $[t]_\times^T [t]_\times$, see your linear algebra book.

Verify that if w satisfies

$$-t \times (t \times w) = \lambda w. \tag{5}$$

for some λ, then w is an eigenvector of $[t]_\times^T [t]_\times$ with eigenvalue λ.

In "Linjär Algebra" by Sparr (on page 96) we find the formula

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w. \tag{6}$$

Show that $w = t$ is an eigenvector to $[t]_\times^T [t]_\times$ with eigenvalue 0 and that any w that is perpendicular to t is an eigenvector with eigenvalue $||t||^2$. Are these all of the eigenvectors?

Show that the singular values of $[t]_\times$ are $0, ||t||$ and $||t||$.

If $E = [t]_\times R$ and $[t]_\times$ has the SVD in (4), state an SVD of E. What are the singular values of E?

For the report: A complete solution.

Computer Exercise 2. (See Jupyter Notebook)

Theoretical Exercise 7 (7 points).

Suppose that an essential matrix E has the singular value decomposition

$$E = U \mathrm{diag}([1\ 1\ 0]) V^T \tag{7}$$

where

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

Verify that $\det(UV^T) = 1$.

Compute the essential matrix and verify that $x_1 = (1, 0)$ (in camera 1) and $x_2 = (0, 1)$ (in camera 2) is a plausible correspondence.

If x_1 is the projection of $\mathbf{X}$ in $P_1 = [I\ 0]$ show that $\mathbf{X}$ must be one of the points

$$\mathbf{X}(s) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ s \end{pmatrix}. \tag{9}$$

For each of the solutions

$$P_2 = [UWV^T\ u_3] \text{ or } [UWV^T\ -u_3] \text{ or } [UW^TV^T\ u_3] \text{ or } [UW^TV^T\ -u_3], \tag{10}$$

where

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

and u_3 is the third column of U, compute s such that $\mathbf{X}(s)$ projects to x_2.

For what choice of P_2 is the camera pair (P_1, P_2) valid, such that the 3D point $\mathbf{X}(s)$ is in front of both cameras? *HINT:* For a calibrated camera $P = [R, t]$, unscaled such that $\det R = 1$, and for a finite 3D point $\mathbf{X}$ in homogeneous coordinates, unscaled such that $\mathbf{X}_4 = 1$, $P\mathbf{X}$ yields the Cartesian coordinates of the 3D point in the camera coordinate frame.

For the report: Complete solution.

Computer Exercise 3. (See Jupyter Notebook)