EXAMPLE ANSWERS

Exercise 2 – Simplification of Boolean expressions for efficient design logic

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Question 1. [2 marks]

1. Determine by means of a truth table the validity of DeMorgan's theorem for three variables: (ABC)' = A' + B' + C'.

A	B	C	A'	B'	C'	AB	ABC	A'+B'	(ABC)'	A'+B'+C'	
0	0	0	1	1	1	0	0	1	1	1	same
0	0	1	1	1	0	0	0	1	1	1	same
0	1	0	1	0	1	0	0	1	1	1	same
0	1	1	1	0	0	0	0	1	1	1	same
1	0	0	0	1	1	0	0	1	1	1	same
1	0	1	0	1	0	0	0	1	1	1	same
1	1	0	0	0	1	1	0	0	1	1	same
1	1	1	0	0	0	1	1	0	0	0	same

Noting that the values in the last two columns are the same for all rows (ie. all combinations possible of 0 and 1), DeMorgan's theorem for three variables has been shown to be valid. This is an example of Perfect Inductive proof using truth tables.

2. List the truth table of a three-variable exclusive-OR (sometimes called 'odd') function: D = A x B x C, where x denotes the XOR operator.

A	B	C	AxB	AxBxC
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	0	1

Note that the values in the last (rightmost) column are 1 if the total number of 1's in the set {A,B,C} is ODD – otherwise, if the total 1's is even, the value is 0. For this reason, the three variable XOR operator is called the ODD function.

Question 2. [5 marks]

Simplify the following expressions using Boolean algebra. In each case, state the Axiom (ie. Postulate) or Theorem being applied at each step.

For the answers below, we refer to the following Postulates and Theorems (see lecture notes):

P1: Closure: There exists x,y in B such that two independent operations, . (dot) and + (plus) are defined:

$\qquad$ x + y $\qquad$ x . y

P2: Identity: There exist identity elements 0,1 in B relative to the operations + and . , such that for every x in B:

$\qquad$ $0 + x = x + 0 = x$ $\qquad$ $1 . x = x . 1 = x$

P3: Commutativity: The operations + and . are commutative for all x,y in B:

$\qquad$ $x + y = y + x$ $\qquad$ $x . y = y . x$

P4: Distributivity: Each operation + and . is distributive over the other; that is, for all x,y,z in B:

$\qquad$ $x.(y+z) = x.y + x.z$ $\qquad$ $x+(y.z) = (x+y).(x+z)$

P5: Complementation: For every element x in B there exists an element ~x, called the complement of x, satisfying:

$\qquad$ $x + {\sim}x = 1$ $\qquad$ $x.{\sim}x = 0$

P6: Existence: There exist at least two elements x,y in B such that $x \neq y$.

T2: For each X in B: $X + 1 = 1$ $\qquad$ $X . 0 = 0$

$\qquad$ a. $A + AB$

Derivative step	Postulate
A.1 + A.B	P2
A(1+B)	P4
A.1	T2
A	P2

$\qquad$ Final answer: A

$\qquad$ b. $AB + AB'$

Derivative step	Postulate
A(B+B')	P4
A.1	P5
A	P2

$\qquad$ Final answer: A

$\qquad$ c. $A'BC + AC$

Derivative step	Postulate
(A'B+A)C	P4
(A'+A)(B+A)C	P4
1.(B+A)C	P5
(B+A)C	P2

$\qquad$ Final answer: $(B+A)C = (A+B)C = C(A+B)$ etc (all similar forms are equivalent).

d. A'B + ABC' + ABC

Derivative step	Post/Thm
A'B + ABC' + ABC' + ABC	Idempotent
B(A' + AC') + AB(C' + C)	P3, P4
B(A' + AC') + AB.1	P5
B(A' + AC') + AB	P2
B(A' + AC' + A)	P4
B(A' + A + AC')	P3
B(1 + AC')	P5
B.1	T2
B	P2

Final answer: B

e. AB + A(CD + CD')

Derivative step	Postulate
AB + AC(D + D')	P4
AB + AC.1	P5
AB + AC	P2
A(B + C)	P4

Final answer: A(B + C)   NOTE: This answer is simpler than the second last line above due to the fact that the number of operations is only 2 (one OR, one AND) in the last line, versus 3 (one OR, two AND) in the second last line.

f. (BC' + A'D)(AB' + CD')

Derivative step	Postulate
BC'(AB' + CD') + A'D(AB' + CD')	P4
BC'AB' + BC'CD' + A'DAB' + A'D CD'	P4
ABB'C' + BCC'D' + AA'B'D + A'CDD'	P3
A.0.C' + B.0.D' + 0.B'D + A'C.0	P5
0 + 0 + 0 + 0	T2
0	P2

Final answer: 0   NOTE: It is interesting (to say the least) that a complicated circuit design involving many inputs <u>may</u> simplify to a very simple circuit, as in this case where the output is 0 regardless of the inputs.

Question 3. [3 marks]

1. Using DeMorgan's theorem, show that:

    a. $(A + B)'(A' + B')' = 0$

    $(A + B)'(A' + B')' = ((A + B) + (A' + B'))'$
    $= ((A + A') + (B + B'))' = (1 + 1)'$
    $= 1'$
    $= 0 \quad$ QED!

    b. $A + A'B + A'B' = 1$

    $A + A'B + A'B' = A + A'(B + B')$
    $= A + A'(B'B)'$
    $= A + A'(0)'$
    $= A + A'.1$
    $= A + A'$
    $= (A'A)'$
    $= (0)'$
    $= 1 \quad$ QED!

    NOTE: This example is notably inefficient in how a higher order theorem, such as deMorgan's theorem, is used to get the result. The same result could obviously have been achieved in fewer steps by applying the postulates directly, or simpler theorems.

2. Given the Boolean expression: $F = X'Y + XYZ'$

    Before beginning, note that $F = (X' + XZ')Y = (X'+Z')Y$ (see examples above).

    a. Derive an algebraic expression for the complement F'.

    $F' = (X'Y + XYZ')' = (X'Y)'.(XYZ')'$
    $= (X'' + Y')(X' + Y' + Z'')$
    $= (X + Y')(X' + Y' + Z)$

    Which simplifies to:
    $= (X+Y')X' + (X+Y')Y' + (X+Y')Z$
    $= X'Y' + XY' + Y' + XZ + Y'Z$
    $= Y' + XZ + Y'Z$
    $= Y' + XZ$

    It is possible to show this more directly, using:
    $F' = ((X'+Z')Y)' = (X'+Z')' + Y'$ [deMorgan] $= XZ + Y'$ [deMorgan]

b. Show that FF' = 0. (Use algebra, not truth tables)

If we start with the simplified form of F = (X'+Z')Y, then:

FF' = ((X'+Z')Y).(Y' + XZ)
     = (XZ)'(YY' + YXZ)   [deMorgan, Distributive]
     = (XZ)'Y(XZ)   [Complement, Identity]
     = 0 QED!   [Complement]

c. Show that F + F' = 1. (Use algebra, not truth tables)

If we start with the simplified form of F = (X'+Z')Y, then:

F+F' = (X'+Z')Y + (Y' + XZ)
      = (XZ)'Y + (Y' + XZ)
      = Y' + XZ + (XZ)'Y
      = Y'+ (XZ)'Y + ((XZ) + (XZ)')(XZ + Y)
      = Y'+ (XZ)' + (XZ + Y)
      = (Y+Y') + ((XZ) + (XZ)')
      = 1 + 1
      = 1   QED!