SOLUTIONS

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Part 0 : Preliminaries

All too trivial. This part will be omitted.

Part I : The derivations and their elementary properties

1. $D(u/v) = (u/v)^{-1}(D(u)/u - D(v)/v)$. $D(u^n) = nu^{n-1}D(u)$. $D(u)/u$ is a homomorphism from $(K^*, \cdot)$ to $(K, +)$. (this explains the name *logarithmic derivative*).
2. Verify for monomials $P = a_i u^i$.
3. Let $G = \prod_{j \in J} Q_j$. Multiplying both sides by $Q^2 G$ and expanding, one can see in $K[X]$:

$$Q^2 \cdot \text{somepolynomial} = G(QD(P) - PD(Q)).$$

By assumption, G is square-free. Hence even though Q might divide G, Q^2 cannot. Thus $Q \mid QD(P) - PD(Q)$, and $Q \mid D(Q)$.

Assume that Q_i is prime to Q. Let $S \mid Q_i$ be an irreducible factor. Multiplying both sides by Q_i

$$Q^2(P_i + Q_i \sum_{j \neq i} \frac{P_j}{Q_j}) = Q_i(QD(P) - PD(Q)).$$

Taking quotient modulo S, noting that all Q_j except Q_i is prime to S, hence invertible modulo S:

$$Q^2 P_i = 0$$

in $K[X]/(S)$. By assumption $Q \neq 0$, therefore $P_i = 0$. So for each irreducible $S \mid Q_i$, $S \mid P_i$. Note that Q_i is square-free, this implies $Q_i \mid P_i$.

Part II : Derivation on $K(X)$ of logarithmic type

1. $D(P) = P' D(X) + P^D \in K[X]$. If the coefficient of the highest-order term in P is in K_{cst}.
2. $Q = 1$. $P = cX + g$ with $c \in K_{cst}$.
3. Let $R = A + P/Q$, with A, P, $Q \in K[X]$, $\deg P < \deg Q$. $D(A) \in K[X]$, $D(R) \in K \subset K[X]$, so $D(P/Q) \in K[X]$. But $D(P/Q) = (QD(P) - PD(Q))/Q^2$, and $\deg P < \deg Q$ implies that $D(P/Q) = 0$. The problem is reduced to the previous one.
4. By definition of Liouville sums, one may write

$$f = D(R) + \sum_{i \in I} c_i \frac{D(P_i)}{P_i},$$

Write

$$P_i = f_i \prod_{j \in J_i} Q_j^{n_j},$$

with each $f_i \in K$, Q_j monic irreducible. Note that $u \mapsto D(u)/u$ is a homomorphism,

$$f = D(R) + \sum_{i \in I} c_i \frac{D(f_i)}{f_i} + \sum_{i \in I} \sum_{j \in J_i} n_j c_i \frac{D(P_j)}{P_j}.$$

The said form can be obtained by combining terms with the same P_j's.

Now let $R = P/Q$ with Q monic irreducible, then

$$D(P/Q) = \sum_{j \in J} \frac{-d_j D(Q_j)}{Q_j} + (f - \frac{\sum_{i \in I} c_i D(f_i)/f_i}{1}$$

has the form of the proposition above, where the last term is in K by assumption. So $Q \mid D(Q)$, and $Q = 1$, which is prime to every Q_j, so $Q_j \mid D(Q_j)$, $Q_j = 1$. Thus

$$f = D(P^*) + \sum_{i \in I} c_i \frac{D(f_i)}{f_i} \in K,$$

with $P^* \in K[X]$. But the latter term is in K, hence $P^* = cX + g$, with $c \in K_{cst}$. The rest is obvious.

5. Direct verification. Note this problem demonstrates the motivation behind the definition of a Liouville sum.

Part III : Derivation on $K(X)$ of exponential type

1. Let $D(X) = fX$, $f \in K$. Then $D(P) = fXP' + P^D \in K[X]$. $\deg D(P) \leq \deg P$.

2. If $P \mid D(P)$, then $D(P) = kP$ for some $k \in K^*$. Let

$$P = \sum_{i \in I} a_i X^i,$$

where $I \subseteq \{0, 1, \ldots, n\}$ and each $a_i \neq 0$. Then

$$D(P) = \sum_{i \in I} (f \cdot i a_i + D(a_i)) X^i.$$

So one has for each $i \in I$,

$$f \cdot i a_i + D(a_i) = k a_i.$$

This is equivalent to

$$\frac{D(a_i)}{a_i} = k - if.$$

Let i, $j \in I$, by the property of logarithmic derivatives,

$$\frac{D(a_i/a_j)}{a_i/a_j} = (j - i)f = \frac{D(X^{j-i})}{X^{j-i}}.$$

So again by the said property, $\dfrac{a_i/a_j}{X^{i-j}}$ has logarithmic derivative 0, and so $a_i/a_j = gX^{i-j}$ with $g \in K(X)^*_{cst} \subseteq K^*$. But a_i, $a_j \in K^*$, and this is not possible unless $i = j$. Therefore I is a singleton.

3. Let $R = P/Q$ with $\gcd(P, Q) = 1$ and Q monic. $D(R) = (QD(P) - PD(Q))/Q^2 \in K[X]$, thus $Q \mid D(Q)$, and $Q = X^n$. Then R is a Laurent polynomial, i.e., $R = \sum_{i \in \mathbb{Z}} a_i X^i$. Note that $D(a_i X_i) \neq 0$ (because $K(X)_{cst} = K_{cst}$) and has degree i, one sees that $R \in K[X]$ and that $\deg R = \deg D(R)$.

4. If $f \in K$ is a Liouville sum in E, then f is a Liouvile sum in K. Indeed, by the proof of the problem in the previous section, one has

$$f = D(P/Q) + \sum_{i \in I} c_i \frac{D(f_i)}{f_i} + \sum_{j \in J} d_j \frac{D(Q_j)}{Q_j}$$

with Q, Q_j irreducible and $Q \mid D(Q)$, hence $Q = X$ or $Q = 1$. If $Q = 1$, every Q_j is prime to Q, so $Q_j \mid D(Q_j)$. If on the other hand $Q = X$ and Q_j is not prime to Q, then $Q_j = X$. Either way, the sum actually consists of only one term, $d\dfrac{D(X)}{X} \in K$. Therefore $D(R) \in K$, and $R = g \in K$. It follows that

$$f - d\frac{D(X)}{X} = D(g) + \sum_{i \in I} c_i \frac{D(f_i)}{f_i}$$

is a Liouville sum in K.

Part IV : Norm, trace and reduction in extension fields of finite dimension

1. $uu^{\#} = (-1)^{n-1}(\chi_u(u) - \chi_u(0))$. By Cayley-Hamilton theorem, $\chi_u(u) = 0$. $\chi_u(0) = (-1)^{n-1} \det u$, so $uu^{\#} = \det u$, and $A^{\#}$ is the transpose of cofactor matrix of A if u is invertible. This is an algebraic identity that holds on the Zariski open set of all invertible matrices, and hence on the space of all matrices.

2. $\mathrm{tr}(\lambda) = n\lambda$, $N(\lambda) = \lambda^n$ when $\lambda \in K$. Consider the case $x \neq 0$. Let $x^{\#} = N(x)/x$. For all $y \in E$, $(m_x)^{\#}(y) = \det m_x \cdot (m_x)^{-1}(y) = x^{\#} y$ as shown above. The uniqueness is direct.

3. Let A^{ij} denote the cofactor matrix of (i,j)-entry in A. For each $\sigma \in S_n$ with $\sigma(i) = j$, one can assign a $\sigma' \in S_{n-1}$, removing $\binom{i}{j}$ in the permutation, and one easily verify that $\operatorname{sgn}\sigma = (-1)^{i+j}\operatorname{sgn}\sigma'$. Also it is well-known that $\operatorname{tr}(A^T B) = \sum_{i,j} a_{ij} b_{ij}$. One now has

$$\begin{aligned}
D(\det A) &= D\left(\sum_{\sigma\in S_n} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}\right) \\
&= \sum_{\sigma} (-1)^{\operatorname{sgn}\sigma} \sum_i D(a_{i\sigma(i)}) \prod_{k\neq i} a_{k\sigma(k)} \\
&= \sum_{i,j} D(a_{ij}) \sum_{\substack{\sigma\in S_n \\ \sigma(i)=j}} (-1)^{\operatorname{sgn}\sigma} \prod_{k\neq i} a_{k\sigma(k)} \\
&= \sum_{i,j} D(a_{ij})(-1)^{i+j} \sum_{\sigma'\in S_{n-1}} (-1)^{\operatorname{sgn}\sigma'} \prod_{k=1}^{n-1} (A^{ij})_{k\sigma'(k)} \\
&= \sum_{i,j} D(a_{ij})(-1)^{i+j} \det A_{ij} \\
&= \operatorname{tr}(A^{\#} D(A)).
\end{aligned}$$

Sorry about the mess, but I did not find a more clear proof.

4. To show that Eq. (*) holds, one observes that all terms involved are linear, and verifies that

$$\begin{aligned}
(\Delta \circ m_x + m_x^D)(e_j) &= \Delta\left(\sum_i a_{ij} e_i\right) + \sum_i D(a_{ij}) e_i \\
&= \sum_i a_{ij} D(e_i) + \sum_i D(a_{ij} e_i) \\
&= D\left(\sum_i a_{ij} e_i\right) \\
&= D(m_x(e_j)) = D(x e_j) \\
&= x D(e_j) + D(x) e_j = (m_x \circ \Delta + m_{D(x)})(e_j).
\end{aligned}$$

Taking traces of both sides and note that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ yields the desired relation.

5. By above propositions,

$$\begin{aligned}
D(N(x)) &= D(\det m_x) = \operatorname{tr}(m_x^{\#} m_x^D) \\
&= \operatorname{tr}(m_{x^\#}(m_x \circ \Delta - \Delta \circ m_x + m_{D(x)})) \\
&= \operatorname{tr}(m_x \circ m_{x^\#} \circ \Delta) - \operatorname{tr}(m_{x^\#} \circ \Delta \circ m_x) + \operatorname{tr}(m_{x^\#} \circ m_{D(x)}) \\
&= \operatorname{tr}(m_{x^\#} m_{D(x)}) = \operatorname{tr}(x^{\#} D(x)).
\end{aligned}$$

Note the commutativity between m_x and $m_{x^\#}$ is neccessary: $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$ is not true in general.

To obtain the desired conclusion, note that $N(x) \in K$, tr is K-linear and substitute $x^{\#} = \dfrac{N(x)}{x}$.

6. Let E be a finite extension field of K. If $f \in K$ is a Liouville sum in E, then f is a Liouvile sum in K. Indeed, let $\dim_K(E) = n$, write

$$f = D(g) + \sum_i \frac{c_i}{D(h_i)} h_i,$$

with g, $h_i \in E$. Taking traces,

$$nf = D(\operatorname{tr}(g)) + \sum_i \frac{D(N(h_i))}{N(h_i)}$$

is a Liouville sum in K, as $\operatorname{tr}(g)$, $N(h_i) \in K$, and n is a constant. Here $\mathbb{Q} \subseteq K$ is used.

Part V : Liouville-Ostrowski Theorem

1. Let $f \in K[X]$ be the minimal polynomial of x over K. Then $f(x) = 0$, $f'(x) \neq 0$ (this uses $\mathbb{Q} \subseteq K$). Taking derivative, $f'(x)D(x) + f^D(x) = 0$, so $D(x) = -f^D(x)/f'(x) \in K(x)$, and it is easy to see that $D(K(x)) \subseteq K(D(x))$.

2. $f = D(g)$ is trivially a Liouville sum in E. Descend along the tower of field extensions.

Part VI : $\int e^{t^2} \mathrm{d}\, t$ is not an elementary function

1. One may write, like above,

$$D(P/Q) = Xf - \sum_{i \in I} c_i \frac{D(f_i)}{f_i},$$

and conclude that $Q = 1$, $|I| = 1$ and $f_i = X$. So

$$D(P) = Xf - cD(X)/X.$$

By assumption, $cD(X)/X \in K$, and the right hand side has degree 1. Hence $\deg P = 1$. Let $P = Xg+h$, g, $h \in K$. Comparing $D(P)$ with $Xf - cD(X)/X$ as polynomials shows that $D(Xg) = Xf$, $D(h) = cD(X)/X$. And

$$f = \frac{D(X)g + XD(g)}{X} = D(g) + gD(X)/X.$$

Let $g = p/q$, p, $q \in k[T]$, then

$$q^2 f = qD(p) - pD(q) + pq\frac{D(X)}{X}.$$

Hence $q \mid D(q)$. This implies $q = 1$ under the usual derivative operation.

2. Let $P \in K[X]$ be the minimal (monic) polynomial of v over K. $P(v) = 0$, so $P'(v)D(v) + P^D(v) = 0$. As $D(v) \in K$, $D(v)P' + P^D$ is a polynomial in $K[X]$, and because P is monic, it has degree lower than P, which is impossible.

Let $P = \sum_{i \in I} a_i X^i \in K[X]$ be the minimal polynomial of e^u over K. Then e^u satisfies $D(u)XP' + P^D = 0$, which leads to $D(u)ia_i + D(a_i) = D(u)a_i$. Analogous to the equivalence of $P \mid D(P)$ and $P = aX^n$, one obtains that $e^{(i-j)u} \in K^*$ for all i, $j \in I$. Thus either I is a singleton $\{n\}$ and $e^{nu} = 0$, which is impossible, or $v = e^{nu} \in K^*$ for some $n \in \mathbb{N}^*$, and it is easy to verify that $nu = \log v$.

3. If $\log(f/g) = P/Q$, namely $D(P/Q) = D(f)/f - D(g)/g$, then $Q \mid D(Q)$ and $Q = 1$, hence $f \mid D(f)$, $g \mid D(g)$, and $f = g = 1$. Then $D(P/Q) = 0$.

4. e^u is transcendental, so $e^u \mapsto X$ is an isomorphism onto $K(X)$. As shown above, the assumption implies that Xf is a Liouville sum in $\mathbb{C}(T)$, and hence $\exists g \in \mathbb{C}(T)$ such that $f = g' + gD(X)/X = g' + gu'$. It is also shown above that if f, u are polynomials, then g is a polynomial. Now $u' = (f - g')/g \in \mathbb{C}[T]$, and $\deg g' < \deg g$, so it must be that $\deg u' = \deg f - \deg g \leq \deg f$, and $\deg u \leq \deg f + 1$.

5. Take $f = 1$, $u = t^2$, then $\deg u > 1 + \deg f$.